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ON ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Consider the equation

$$u^{(n)}(t)+\sum_{i=1}^{m}\int_{\tau_i(t)}^{\sigma_i(t)}u(s)d_sr_i(s,t)=0, \tag{1}$$

where $n\geq 2$, $m\in N$, $\tau_i;\sigma_i\in C(R_+;R_+)$, $\tau_i(t)\leq\sigma_i(t)\leq t$ for $t\in R_+$, $\lim_{t\to+\infty}\tau_i(t)=+\infty$, $(i=1,\dots,m)$, the functions $r_i(s,t)$ are measurable, and $r_i(\cdot,t)$ is nondecreasing $(i=1,\dots,m)$.

Let $t_0\in R_+$. A function $u:[t_0,+\infty[\to R$ is called a proper solution of the equation (1) if it is absolutely continuous along with its derivatives up to the $n-1$-th order inclusively,

$$\sup\Big\{|u(s)|:s\in[t,+\infty[\Big\}>0\quad\text{for}\quad t\geq t_0,$$

and there exists $\overline{u}\in C(R_+;R)$ such that $\overline{u}(t)\equiv u(t)$ for $t\in[t_0,+\infty[$ and

$$\overline{u}^{(n)}(t)+\sum_{i=1}^{m}\int_{\tau_i(t)}^{\sigma_i(t)}\overline{u}(s)d_sr_i(s,t)=0$$

almost everywhere on $[t_0,+\infty[$. A proper solution $u:[t_0,+\infty[\to R$ is said to be oscillatory if it has a sequence of zeroes tending to $+\infty$. Otherwise the solution is said to be nonoscillatory.

Definition. We say that the equation (1) has the property A if each of its proper solutions is oscillatory when n is even, and either is oscillatory or satisfies $|u^{(i)}|\downarrow 0$ for $t\uparrow+\infty$ when $(i=0,\dots,n-1)$ is odd.

Theorem. *Let for some $i_0\in\{1,\dots,m\}$ there exist a nondecreasong $\delta\in C(R_+;R_+)$ such that $\tau_{i_0}(t)\leq\delta(t)\leq\sigma_{i_0}(t)$ for $t\in R_+$,*

$$\varliminf_{t\to+\infty}\int_{\delta(t)}^{t}\int_{\tau_{i_0}(s)}^{\delta(s)}\xi^{n-1}d_\xi r_{i_0}(\xi,s)ds>0,$$

$$\operatorname{vrai\,sup}\Big\{t\int_{\tau_{i_0}(t)}^{\delta(t)}s^{n-1}d_sr_{i_0}(s,t):t\in R_+\Big\}<+\infty$$

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and

$$\overline{\lim_{t\to+\infty}} \frac{\ln t}{\ln \tau_i(t)} < +\infty \quad (i=1,\dots,m). \tag{2}$$

Then the condition

$$\inf\Big\{ \underline{\lim_{t\to+\infty}} \ln^{\lambda} t \int\limits_{t}^{+\infty} \sum_{i=1}^{m} \int\limits_{\tau_i(s)}^{\sigma_i(s)} \xi^{n-1} \times$$

$$\times \ln^{-\lambda} \xi \, d_{\xi} r_i(\xi,s) ds : \lambda \in]0,k] \Big\} > (n-1)! \text{ for all } k \in N \tag{3}$$

is sufficient for the equation (1) *to have the property A.*

Corollary 1. *Let* $c_i \in]0,+\infty[$, $\alpha_i; \overline{\alpha}_i \in]0,1]$, *and* $\alpha_i < \overline{\alpha}_i$ $(i=1,\dots,m)$. *Then for the equation*

$$u^{(n)}(t) + \sum_{i=1}^{m} \frac{c_i}{t} \int\limits_{t^{\alpha_i}}^{t^{\overline{\alpha}_i}} \frac{u(s)}{s^n \ln^2 s} ds = 0$$

to have the property A, it is sufficient that

$$\inf\Big\{ \frac{1}{\lambda(\lambda+1)} \sum_{i=1}^{m} c_i(\alpha_i^{-\lambda-1} - \overline{\alpha}_i^{-\lambda-1}) : \lambda \in]0,+\infty[\Big\} > (n-1)!.$$

Corollary 2. *Let* $c_i \in]0,+\infty[$, $\alpha_i \in]0,1]$, *and* $\alpha_{i_0} < 1$ *for some* $i_0 \in \{1,\dots,m\}$. *Then for the equation*

$$u^{(n)}(t) + \frac{1}{t\ln t} \sum_{i=1}^{m} \frac{c_i}{t^{\alpha_i(n-1)}} u(t^{\alpha_i}) ds = 0$$

to have the property A, it is sufficient that

$$\inf\Big\{ \frac{1}{\lambda} \sum_{i=1}^{m} c_i \alpha_i^{-\lambda} : \lambda \in]0,+\infty[\Big\} > (n-1)!.$$

In the case where the condition

$$\overline{\lim_{t\to+\infty}} \frac{t}{\tau_i(t)} < +\infty \quad (i=1,\dots,m)$$

holds instead of (2), analogous questions are considered in [1].

Remark. Note that the inequality (2) cannot be replaced by

$$\inf\Big\{ \underline{\lim_{t\to+\infty}} \ln^{\lambda} t \int\limits_{t}^{+\infty} \sum_{i=1}^{m} \int\limits_{\tau_i(s)}^{\sigma_i(s)} \xi^{n-1} \times$$

$$\times \ln^{-\lambda} \xi \, d_{\xi} r_i(\xi,s) ds : \lambda \in]0,k] \Big\} > (n-1)! - \varepsilon$$

for any whatever small ε.

References

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