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ON THE QUESTION OF SOLVABILITY OF THE PERIODIC BOUNDARY VALUE PROBLEM FOR A SYSTEM OF LINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Let ω be a positive number, $A=(a_{ik})_{i,k=1}^{n}:R\to R^{n\times m}$ and $g=(g_i)_{i=1}^{n}:R\to R^{n}$ be a matrix function and a vector function from $BV_\omega^{n\times m}$ and BV_ω^{n}, respectively.

We consider the ω-periodic boundary value problem

$$dx(t)=dA(t)\cdot x(t)+dg(t),\quad x(0)=x(\omega). \tag{1}$$

The use will be made of the following notation and definitions: $R=]-\infty,+\infty[$; $R^{n\times m}$ is a set of all real $n\times m$-matrices; I is the identity $n\times n$-matrix; $R^n=R^{n\times 1}$. $BV_\omega^{n\times m}$ is the set of all matrix functions $X:R\to R^{n\times m}$ such that $X(t+\omega)=X(t)+X(\omega)$ for $t\in R$, and the restriction on $[0,\omega]$ of every its components has bounded total variation; $X(t-)$ and $X(t+)$ are the left and the right limits of X at the point $t\in R$; $d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

If $g:R\to R$ is nondecreasing, $x:R\to R$ and $s<t$, then

$$\int_s^t x(\tau)dg(\tau)=\int_{]s,t[} x(\tau)dg(\tau)+x(t)d_1g(t)+x(s)d_2g(s),$$

where $\int_{]s,t[}x(\tau)dg(\tau)$ is the Lebesque–Stieltjes integral over the open interval $]s,t[$ with respect to the measure μ_g corresponding to g, (if $s=t$, then $\int_s^t x(\tau)dg(\tau)=0$).

$$L_\omega(g)=\Big\{p\in BV_\omega:\ \int_0^\omega |p(t)|dg(t)<\infty\Big\}.$$

A vector function $x=(x_i)_{i=1}^{n}\in BV_\omega^{n}$ is a solution of the problem (1) if it is ω-periodic and

$$x_i(t)=x_i(s)+\sum_{k=1}^{n}\int_s^t x_k(\tau)da_{ik}(\tau)\quad\text{for}\quad s\le t\ \ (i=1,\dots,n).$$

Let natural numbers m and $n_1,\dots,n_m$ $(0=n_0<n_1<\dots<n_m=n)$, nondecreasing functions $c_{lj}:[0,\omega]\to R$ $(l=1,2;\ j=1,\dots,m)$, functions $\alpha_{lj}\in L_\omega(c_{lj})$ $(l=1,2;\ j=1,\dots,m)$ and matrix functions $P_{lj}=(p_{ljik})_{i,k=1}^{n}$ $(l=1,2;\ j=1,\dots,m)$, $p_{ljik}\in L_\omega(c_{lj})$

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$(i,k=n_{j-1}+1,\dots,n_j)$ be such that $a_{ik}(t)\equiv 0$ $(i=n_{j-1}+1,\dots,n_j;\ k=n_j+1,\dots,n;$ $j=1,\dots,m-1)$,

$$a_{ik}(t)-\frac{1}{2}\Big(\sum_{0<\tau\le t}\ \sum_{\sigma=n_{j-1}+1}^{n_j} d_1a_{\sigma i}(\tau)\cdot d_1a_{\sigma k}(\tau)-$$

$$-\sum_{0\le\tau<t}\ \sum_{\sigma=n_{j-1}+1}^{n_j} d_2a_{\sigma i}(\tau)\cdot d_2a_{\sigma k}(\tau)\Big)=$$

$$=b_{1jik}(t)-b_{2jik}(t)\quad\text{for}\quad t\in[0,\omega]\quad (i,k=n_{j-1}+1,\dots,n_j;\ \ j=1,\dots,m),$$

$$(-1)^{l+1}\sigma_j\sum_{i,k=n_{j-1}+1}^{n_j} p_{ljik}(t)x_ix_k\ge\alpha_{lj}(t)\sum_{i=n_{j-1}+1}^{n_j}x_i^2$$

for $\mu_{c_{lj}}$ almost everywhere $t\in[0,\omega]$, $(x_i)_{i=1}^n\in R^n$ $(l=1,2;\ \ j=1,\dots,m)$,

where $\sigma_j\in\{-1,1\}$, $b_{ljik}(t)\equiv\int_0^t p_{ljik}(\tau)dc_{lj}(\tau)$ $(i\ne k)$ and b_{ljii} is such that

$$(-1)^{l+1}\sigma_j(b_{ljii}(t)-b_{ljii}(s)-\int\limits_s^t p_{ljii}(\tau)dc_{lj}(\tau))\ge 0\quad\text{for}\quad 0\le s\le t\le\omega.$$

Then we shall say that

$$A\in Q_\omega\Big(m,(n_j;c_{1j},c_{2j};\alpha_{1j},\alpha_{2j};P_{1j},P_{2j})_{j=1}^m\Big). \tag{2}$$

Theorem. *Let there exist natural numbers m and $n_1,\dots,n_m$ $(0=n_0<n_1<\dots<n_m=n)$, functions c_{lj} and α_{lj} $(l=1,2;\ j=1,\dots,m)$ and matrix functions $P_{lj}=(p_{ljik})_{i,k=1}^n$ such that* (2) *holds. Let, moreover,*

$$\det(I+(-1)^kd_kA(t))\ne 0,\quad (1+\sigma_j)d_1c_j(t)+(1-\sigma_j)d_2c_j(t)<2,$$
$$(1-\sigma_j)d_1c_j(t)+(1+\sigma_j)d_2c_j(t)\ne -2$$

and

$$\exp\Big(c_j(\omega)-\sum_{0<\tau\le\omega}d_1c_j(\tau)-\sum_{0\le\tau<\omega}d_2c_j(\tau)\Big)>$$

$$>\frac{1}{2}\Big[(1+\sigma_j)\prod_{0<\tau\le\omega}\big(1-d_1c_j(\tau)\big)\prod_{0\le\tau<\omega}\big(1+d_2c_j(\tau)\big)^{-1}+$$

$$+(1-\sigma_j)\prod_{0<\tau\le\omega}\big(1+d_1c_j(\tau)\big)^{-1}\prod_{0\le\tau<\omega}\big(1-d_2c_j(\tau)\big)\Big],$$

for every $t\in[0,\omega]$ and $j\in\{1,\dots,m\}$, where

$$c_j(t)\equiv 2\sum_{l=1}^2\int\limits_0^t\alpha_{lj}(\tau)dc_{lj}(\tau).$$

Then the problem (1) *has one and only one solution.*

The analogous question has been considered in [1] for a system of linear ordinary differential equations.

References

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