Mem. Differential Equations Math. Phys. 6(1995), 130–133

Z. Došla and I. Kiguradze

ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

(Reported on December 11, 1995)

In the present note we give sufficient conditions of boundedness and vanishing at infinity solutions of the differential equation

$$u'' + (l(t) + p(t))u = q(t), \tag{1}$$

where $l : [a, +\infty[\to]0, +\infty[$ is the function with bounded variation on every finite interval, and p and $q : [a, +\infty[\to R$ are measurable functions such that

$$\int_a^{+\infty} \frac{|p(t)|}{\sqrt{l(t)}}\, dt < +\infty \tag{2}$$

and

$$\int_a^{+\infty} \frac{|q(t)|}{\sqrt{l(t)}}\, dt < +\infty. \tag{3}$$

The use will be made of the following notation and definitions.

M is the set of functions $l : [0, +\infty[\to]0, +\infty[$ admitting the representation

$$l(t) = l_0(t) + \lambda(t), \tag{4}$$

where $l_0 : [0, +\infty[\to]0, +\infty[$ is the nondecreasing function and $\lambda : [0, +\infty[\to R$ is a locally absolutely continuous function such that

$$\lim_{t\to\infty} \frac{\lambda(t)}{l_0(t)} = 0, \quad \int_0^{+\infty} \frac{|\lambda'(t)|}{l_0(t)}\, dt < +\infty. \tag{5}$$

$$M^{\infty} = \{l \in M : \lim_{t\to+\infty} l(t) = +\infty\}.$$

$\dim X$ is the dimension of a linear space X.

We say that a function $l : [0, +\infty[\to]0, +\infty[$ belongs to the set H if it tends monotonically to $+\infty$ as $t \to +\infty$, and there exists $\varepsilon > 0$ such that for any increasing unbounded sequence of positive numbers $(t_k)_{k=1}^{\infty}$ satisfying the conditions

---

1991 *Mathematics Subject Classification.* 34D05.

Key words and phrases. Second order linear differential equation, bounded solution, vanishing at infinity solution.

$$\pi - \varepsilon < \lim_{k\to+\infty} \int\limits_{t_{2k}}^{t_{2k+1}} \sqrt{l(t)}\, dt < \pi$$

and

$$0 < \liminf_{k\to\infty} \sqrt{l(t_{2k})}\,(t_{2k} - t_{2k-1}) \le \limsup_{k\to\infty} \sqrt{l(t_{2k})}\,(t_{2k} - t_{2k-1}) < \varepsilon,$$

the equality

$$\sum_{k=1}^{\infty} \Big[\lg l(t_{2k+1}) - \lg l(t_{2k})\Big] = +\infty.$$

holds.

M_H is the set of functions $l : [0,+\infty[\to]0,+\infty[$ admitting the representation (4), where $l_0 \in H$, and $\lambda : [0,+\infty[\to R$ is a locally absolutely continuous function satisfying (5).

The solution u of equation (1) is said to be bounded if

$$\sup\Big\{|u(t)| : 0 \le t < +\infty\Big\} < +\infty,$$

and vanishing at infinity if

$$\lim_{t\to+\infty} u(t) = 0.$$

Along with (1), we consider linear homogeneous equations

$$u'' = l(t)u \tag{6}$$

and

$$u'' = (l(t) + p(t))u, \tag{7}$$

whose spaces of vanishing at infinity solutions will be denoted by $Z(l)$ and $Z(l+p)$, respectively.

Theorem -3. *If $l \in M$ and the conditions* (2) *and* (3) *are satisfied, then every solution of equation* (1) *is bounded.*

Corollary 1 (Z. Opial [9]). *If $l \in M$ and the condition* (2) *is satisfied, then every solution of equation* (7) *is bounded.*

H. Milloux [7] and Z. Opial [8] have proved that if $l \in M^\infty$, then

$$\dim Z(l) \ge 1.$$

The question, whether the dimension of the space $Z(l)$ is invariant with respect to the perturbation of p satisfying (2), remained open.

The following theorem answers this question.

Theorem -2. *If $l \in M^\infty$ and the condition* (2) *is satisfied, then*

$$\dim Z(l+p) = \dim Z(l).$$

Generalizing earlier known results on vanishing at infinity solutions of equation (6) (see [1, 6, 10, 11, 12]), P. Hartman [3] and T. Chanturia [2] have respectively proved that

$$l \in H \Longrightarrow \dim Z(l) = 2$$

and

$$l \in M_H \Longrightarrow \dim Z(l) = 2.$$

Therefore from Theorem 2 it follows

Corollary 2. *If $l \in M_H$ and the condition* (2) *is satisfied, then*

$$\dim Z(p+l) = 2.$$

Corollary 3 (Kiguradze–Chanturia [4]). * *Let $(m_j)_{j=1}^{\infty}$ be a sequence of natural numbers and $(r_j)_{j=1}^{\infty}$ be a nondecreasing sequence of positive numbers such that*

$$\lim_{j\to\infty} r_j = \infty \ \textit{and} \ t_j = \pi \sum_{i=1}^{j-1} \frac{m_i}{r_i} \to \infty \ \textit{for} \ j \to \infty.$$

In addition, let

$$l(t) = r_j^2 \ \textit{for} \ t_j \le t < t_{j+1} \quad (j = 1, 2, \dots),$$

where $t_1 = 0$, and let (2) *be satisfied. Then*

$$\dim Z(l+p) = 1.$$

Theorem -1. *If*

$$l \in M^{\infty}, \ \dim Z(l) = 2$$

and the conditions (2) *and* (3) *are satisfied, then every solution of equation* (1) *is vanishing at infinity.*

Corollary 4. *If*

$$l \in M_H$$

and the conditions (2) *and* (3) *are satisfied, then every solution of equation* (1) *is vanishing at infinity.*

References

1. G. Armellini, Sopra un'equazione differenziale della Dinamica. *Rend. Accd. Naz. Lincei* **21**(1935), 111–116.

2. T. Chanturia, On the asymptotic behavior of oscillatory solutions of second order ordinary differential equations. *Differentsial'nye Uravneniya* **11**(1975), 1232–1245.

3. P. Hartman, The existence of large or small solutions of linear differential equations. *Duke Math. J.* **28**(1961), 421–429.

4. I. T. Kiguradze and T. A. Chanturia, A remark on the asymptotic behavior of solutions of the equation $u'' + f(t)u = 0$. *Differentsial'nye Uravneniya* **6**(1970), 1115–1117.

5. I. T. Kiguradze and T. A. Chanturia, Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations. *Kluwer Academic Publishers, Dordrecht-Boston-London* 1993, 432.

6. J. Kurzweil, Sur l'équation $x'' + f(t)x = 0$. *Časopis Pěst. Mat.* **82**(1957), 218–226.

7. H. Milloux, Sur l'equation différentielle $x'' + A(t)x = 0$. *Prace Mat. Fiz.* **41**(1934), 39–54.

8. Z. Opial, Sur l'equation différentielle $u'' + a(t)u = 0$. *Ann. Polon. Math.* **5**(1958), 77–93.

9. Z. Opial, Nouvelles remarques sur l'équation différentielle $u'' + a(t)u = 0$. *Ann. Polon. Math.* **6**(1959), 75–81.

10. G. Sansone, Sopra il comportamento asintotico delle soluzioni di un'equatzione differenziale della dinamica. *Scritti Matematici offerti a Luigi Berzolari, Pavia*, 1936, 385-403.

*See also [6], Theorem 4.10.

11. G. Sansone, Equazioni differentiali nel campo reale II. *Zanichelli, Bologna*, 1949.
12. L. Tonelli, Estratto di lettera al prof. Giovanni Sansone. *Scritti Matematici offerti a Luigi Berzolari, Pavia*, 1936, 404-405.

Authors' addresses:

Z. Došla
Department of Mathematics
Masaryk University
Janáčkovo nám. 2a
66 295 Brno
Czech Republic

I. Kiguradze
A. Razmadze Mathematical Institute
1, M. Aleksidze St.
Tbilisi 380093
Georgia