Memoirs on Differential Equations and Mathematical Physics
Volume 6, 1995, 1–57

ON THE STABILITY OF SOLUTIONS OF THE MULTIPOINT BOUNDARY VALUE PROBLEM FOR THE SYSTEM OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

The boundary value problem

$$dx(t) = dA(t) \cdot f\big(t, x(t)\big), \quad x_i(t_i) = \varphi_i(x) \quad (i = 1, \dots, n) \tag{1}$$

is considered, where $A : [a, b] \to R^{n \times n}$ is a matrix-function with of components bounded variation, f is a vector-function belonging to the Carathedory class corresponding to A; $t_1, \dots, t_n \in [a, b]$, $x = (x_i)_{i=1}^n$ and $\varphi_1, \dots, \varphi_n$ are the continuous functionals, in general nonlinear, given on the space of all vector-functions of bounded variation. The sequence of problems

$$dx(t) = dA_m(t) \cdot f_m\big(t, x(t)\big), \quad x_i(t_{im}) = \varphi_{im}(x) \quad (i = 1, \dots, n) \quad (1_m)$$
$$(m = 1, 2, \dots)$$

is considered along with (1).

Sufficient conditions are given which guarantee both solvability of the problem (1_m) for any sufficiently large m and convergence of its solutions as $m \to +\infty$ to the solution of the problem (1), provided this problem is solvable. Difference schemes of numerical solutions for the multipoint differential and difference problems are constructed.

Multipoint boundary value problem, system of generalized differential equations, correct problem, system of ordinary differential and difference equations.

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$$dx(t) = dA(t) \cdot f(t, x(t)), \quad x_i(t_i) = \varphi_i(x) \quad (i = 1, \dots, n), \tag{1}$$

სადაც $A : [a, b] \to R^{n \times n}$ არის სასრული ვარიაციის კომპონენტებიანი მატრიცული ფუნქცია, f არის A-ს შესაბამისი კარათეოდორის კლასის ფუნქცია; $t_1, \dots, t_n \in [a, b]$, $x = (x_i)_{i=1}^n$, ხოლო $\varphi_1, \dots, \varphi_n$ არიან სასრული ვარიაციის ვექტორ-ფუნქციათა სივრცეზე განსაზღვრული საზოგადოდ არაწრფივი უწყვეტი ფუნქციონალები. (1)-ის პარალელურად განხილულია სასაზღვრო ამოცანების მიმდევრობა

$$dx(t) = dA_m(t) \cdot f_m(t, x(t)), \quad x_i(t_{im}) = \varphi_{im}(x) \quad (i = 1, \dots, n) \quad (1_m)$$
$$(m = 1, 2, \dots).$$

მოყვანილია საკმარისი პირობები, რომლებიც უზრუნველყოფენ როგორც (1_m) ამოცანის ამოხსნადობას საკმარისად დიდი m-თვის, აგრეთვე მათი ამონახსნების კრებადობას (1) ამოცანის ამონახსნისკენ, როცა $m \to +\infty$, თუკი ეს ამოცანა ამოხსნადია. მრავალწერტილოვანი დიფერენციალური და სხვაობიანი ამოცანების რიცხვითი ამონახსნების საპოვნელად აგებულია სხვაობიანი სქემები.

§ 1. Statement of the Problem and Formulation of Main Results

Let $t_1, \dots, t_n \in [a,b]$; $A_0 = (a_{ik0})_{i,k=1}^n : [a,b] \to R^{n\times n}$ be a matrix-function with bounded variation components; $a_{ik0}(t) \equiv a_{1ik}(t) - a_{2ik}(t)$, where a_{jik} is a function nondecreasing on every interval $[a, t_i[$ and $]t_i, b]$ for $j \in \{1,2\}$ and $i,k \in \{1,\dots,n\}$; let $A^{(j)}(t) \equiv \big(a_{jik}(t)\big)_{i,k=1}^n$ $(j = 1,2)$; $f_0 = (f_{i0})_{i=1}^n : [a,b] \times R^n \to R^n$ be a vector-function belonging to the Caratheodory class corresponding to the matrix-function A_0, and let $\varphi_i : BV_S\big([a,b], R^n\big) \to R$ $(i = 1,\dots,n)$ be continuous functionals, in general nonlinear.

For the system of generalized ordinary differential equations

$$dx(t) = dA_0(t) \cdot f_0\big(t, x(t)\big), \tag{1.1}$$

consider the multipoint boundary value problem

$$x_i(t_i) = \varphi_i(x) \quad (i = 1,\dots,n). \tag{1.2}$$

Consider a sequence of matrix-functions of bounded variation $A_m : [a,b] \to R^{n\times n}$ $(m = 1,2,\dots)$, a sequence of vector-functions $f_m = (f_{im})_{i=1}^n : [a,b] \times R^n \to R^n$ $(m = 1,2,\dots)$ belonging to the Caratheodory class corresponding to the matrix-function A_m, a sequences of points $t_{1m}, \dots, t_{nm} \in [a,b]$ $(m = 1,2,\dots)$ and sequence of continuous functionals $\varphi_{1m}, \dots, \varphi_{nm} : BV_S\big([a,b], R^n\big) \to R$ $(m = 1,2,\dots)$.

In this paper, sufficient conditions are given guaranteeing both solvability of the problem

$$dx(t) = dA_m(t) \cdot f_m\big(t, x(t)\big), \tag{1.1_m}$$

$$x_i(t_{im}) = \varphi_{im}(x) \quad (i = 1,\dots,n) \tag{1.2_m}$$

for any sufficiently large m and the convergence of its solutions as $m \to +\infty$ to the solution of the problem (1.1), (1.2), provided this problem is solvable.Moreover, a method of construction of the solution of the problem (1.1), (1.2) is considered.

Analogous results can be found in [1] for the Cauchy-Nicolletti's boundary value problem ($\varphi_i(x) = c_i$, $c_i = \text{const}$) and in [2–4, 13–15] for the multipoint boundary value problems for the systems of ordinary differential, functional differential and difference equations.

The theory of generalized ordinary differential equations enables one to investigate ordinary differential and difference equations from the common standpoint. Moreover, convergence conditions for the difference schemes corresponding to the boundary value problems for the systems of ordinary differential equations can be deduced from correctness results of appropriate boundary value problems for systems of generalized ordinary differential equations [1, 5–11].

Throughout this paper, the use will be made of the following notation and definitions.

$R =]-\infty, +\infty[$, $R_+ = [0, +\infty[$; $[a, b]$ $(a, b \in R)$ is a closed segment.

$R^{n\times m}$ is the space of all real $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|;$$

$$R_+^{n\times m} = \big\{(x_{ij})_{i,j=1}^{n,m} : x_{ij} \geq 0 \quad (i = 1, \dots, n;\ j = 1, \dots, m)\big\}.$$

If $X = (x_{ij})_{i,j=1}^{n,m} \in R^{n\times m}$, then

$$|X| = \big(|x_{ij}|\big)_{i,j=1}^{n,m}, [X]_+ = \Big(\frac{|x_{ij}| + x_{ij}}{2}\Big)_{i,j=1}^{n,m}.$$

$R^n = R^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^{n}$; $R_+^n = R_+^{n\times 1}$.

If $X \in R^{n\times n}$, then X^{-1} and $\det(X)$ are, respectively, the matrix inverse to X and the determinant of X; I_n is the identity $n \times n$-matrix; O_n is the zero $n \times n$-matrix; δ_{ij} is the Kroneker symbol, i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$ $(i, j = 1, \dots, n)$.

$\overset{b}{\underset{a}{\vee}}(X)$ is the total variation of the matrix-function $X : [a, b] \to R^{n\times m}$, i.e., the sum of total variations of the latter's components x_{ij} $(i = 1, \dots, n;\ j = 1, \dots, m)$; $V(X)(t) = \big(v(x_{ij})(t)\big)_{i,j=1}^{n,m}$, where $v(x_{ij})(a) = 0$ and $v(x_{ij})(t) = \overset{t}{\underset{a}{\vee}}(x_{ij})$ for $a < t \leq b$ $(i = 1, \dots, n;\ j = 1, \dots, m)$.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X : [a, b] \to R^{n\times m}$ at the point t;[1]

$$d_1X(t) = X(t) - X(t-), \quad d_2X(t) = X(t+) - X(t);$$
$$\|X\|_S = \sup\big\{\|X(t)\| : t \in [a, b]\big\}.$$

$BV([a, b], R^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a, b] \to R^{n\times m}$ (i.e., such that $\overset{b}{\underset{a}{\vee}}(X) < +\infty$, $BV_v([a, b], R^n)$ is the Banach space $(BV([a, b], R^n), \|\cdot\|_v)$ with the norm

$$\|x\|_v = \|x(a)\| + \overset{b}{\underset{a}{\vee}}(x);$$

$BV_S\big([a, b], R^n\big)$ is the normed space $(BV([a, b], R^n), \|\cdot\|_S)$;

$$BV_S\big([a, b], R_+^n\big) = \big\{x \in BV_S([a, b], R^n) : x(t) \in R_+^n \quad \text{for} \quad t \in [a, b]\big\}.$$

If $y \in BV\big([a, b], R^n\big)$ and $r \in]0, +\infty[$, then

$$U_n(y; r) = \big\{x \in BV_S([a, b], R^n) : \|x - y\|_S < r\big\};$$

$D_n(y; r)$ is the set of all $x \in R^n$ such that $\inf\big\{\|x - y(\tau)\| : \tau \in [a, b]\big\} < r$.

[1] We will, if necessary, assume that $X(t) = X(a)$ and $X(t) = X(b)$, respectively, for $t \leq a$ and $t \geq b$.

If $D \subset R$ is an interval, then $C(D, R^n)$ is the set of all continuous vector-functions $x : D \to R^n$;

$$C(D, R_+^n) = \big\{x \in C(D, R^n) : x(t) \in R_+^n \quad \text{for} \quad t \in [a, b]\big\}.$$

If $\alpha \in BV([a, b], R)$ has not more than a finite number of discontinuity points and $m \in \{1, 2\}$, then $D_{\alpha m} = \{t_{\alpha m_1}, \dots, t_{\alpha m n_{\alpha m}}\}$ $(t_{\alpha m_1} < \dots < t_{\alpha m n_{\alpha m}})$ is the set of all points $t \in [a, b]$ for which $d_m\alpha(t) \neq 0$;

$$\mu_{\alpha m} = \max\big\{d_m\alpha(t) : t \in D_{\alpha m}\big\} \quad (m = 1, 2);$$

$$\nu_{\alpha m\beta j} = \max\big\{d_j\beta(t_{\alpha ml}) + \\ + \sum_{t_{\alpha m, l+1-m} < \tau < t_{\alpha m, l+2-m}} d_j\beta(\tau) : l = 1, \dots, n_{\alpha m}\big\}$$

for $\beta \in BV\big([a, b], R\big)$ $(j, m = 1, 2)$; here $t_{\alpha 1 n_{\alpha 1}} + 1 = b + 1$, $t_{\alpha 20} = a - 1$.

If $\beta \in BV([a, b], R)$, then

$$\mu_{\beta ji}(t) = (-1)^j\big[\beta(t) - \beta(t_i)\big] - d_j\beta(t_i) \quad \text{for} \quad t \in [a, b] \\ (j = 1, 2; \quad i = 1, \dots, n).$$

If $g : [a, b] \to R$ is a nondecreasing function, $x : [a, b] \to R$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)dg(\tau) = \int_{]s,t[} x(\tau)dg(\tau) + x(t)d_1g(t) + x(s)d_2g(s),$$

where $\int_{]s,t[} x(\tau)dg(\tau)$ is the Lebesgue-Stieltjes integral over the open interval $]s, t[$ with respect to the measure μ_g corresponding to the function g (if $s = t$, then $\int_s^t x(\tau)dg(\tau) = 0$);

$L^p([a, b], R; g)$ $(1 \le p < +\infty)$ is the space of all μ_g-measurable functions $x : [a, b] \to R$ such that $\int_a^b |x(t)|^p dg(t) < +\infty$ with the norm

$$\|x\|_{p,g} = \bigg(\int_a^b |x(t)|^p dg(t)\bigg)^{\frac{1}{p}};$$

$L^{+\infty}\big([a, b], R; g\big)$ is the space of all μ_g-measurable essentially bounded functions $x : [a, b] \to R$ with the norm

$$\|x\|_{+\infty,g} = \operatorname{ess\,sup}\big\{|x(t)| : t \in [a, b]\big\}.$$

$s_k : BV\big([a,b],R\big) \to BV\big([a,b],R\big)$ $(k=0,1,2)$ are the operators defined by

$$s_1(x)(a) = s_2(x)(a) = 0,$$
$$s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau), \ s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \quad \text{for} \quad t\in(a,b];$$
$$s_0(x)(t) \equiv x(t) - s_1(x)(t) - s_2(x)(t).$$

A matrix-function is said to be nondecreasing if every its components are such.

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to R^{l\times n}$ is a nondecreasing matrix-function and $D \subset R^{n\times m}$, then $L\big([a,b],D;G\big)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to D$ such that $x_{kj} \in L^1\big([a,b],R;g_{ik}\big)$ $(i=1,\dots,l;\ k=1,\dots,n;\ j=1,\dots,m)$;

$$\int_s^t dG(\tau)\cdot X(\tau) = \bigg(\sum_{k=1}^n \int_s^t x_{kj}(\tau)d\,g_{ik}(\tau)\bigg)_{i,j=1}^{l,m} \quad \text{for} \quad a\le s\le t\le b.$$
$$S_j(G)(t) \equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n} \quad (j=0,1,2).$$

If $D_1 \subset R^n$ and $D_2 \subset R^{n\times m}$, then $K\big([a,b]\times D_1,D_2;G\big)$ is the Caratheodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that for each $i\in\{1,\dots,l\}$, $j\in\{1,\dots,m\}$ and $k\in\{1,\dots,n\}$: (a) the function $f_{kj}(\cdot,x) : [a,b] \to D_2$ is μ_{gik}-measurable for every $x\in D_1$; (b) the function $f_{kj}(t,\cdot) : D_1 \to D_2$ is continuous for $\mu_{g_{ik}}$-almost everywhere $t\in[a,b]$, and

$$\sup\big\{|f_{kj}(\cdot,x)| : x\in D_0\big\} \in L\big([a,b],R;g_{ik}\big)$$

for every compact $D_0 \subset D_1$.

If $G_j : [a,b] \to R^{l\times n}$ $(j=1,2)$ are nondecreasing matrix-functions, $G = G_1 - G_2$ and $X : [a,b] \to R^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau) = \int_s^t dG_1(\tau)\cdot X(\tau) - \int_s^t dG_2(\tau)\cdot X(\tau) \text{ for } a\le s\le t\le b,$$
$$S_k(G) = S_k(G_1) - S_k(G_2) \quad (k=0,1,2),$$
$$K\big([a,b]\times D_1,D_2;G\big) = \bigcap_{j=1}^2 K\big([a,b]\times D_1,D_2;G\big).$$

If $B \in BV\big([a,b],R^n\big)$, then $M\big([a,b]\times R_+,R_+^n;B\big)$ is the set of all vector-functions $\omega \in K\big([a,b]\times R_+,R_+^n;B\big)$ such that $\omega(t,\cdot)$ is nondecreasing and $\omega(t,\cdot)=0$ for $t\in[a,b]$.

Inequalities between both vectors and matrices are understood componentwise.

If B_1 and B_2 are normed spaces, then the operator $\varphi : B_1 \to B_2$ is called positively homogeneous if

$$\varphi(\lambda x) = \lambda \varphi(x)$$

for every $\lambda \in R_+$ and $x \in B_1$.

An operator $\varphi : BV_S\big([a,b], R^n\big) \to R^n$ is called nondecreasing if for every $x, y \in BV_S\big([a,b], R^n\big)$ such that $x(t) \le y(t)$ for $t \in [a,b]$, the inequality

$$\varphi(x)(t) \le \varphi(y)(t)$$

is fulfilled for $t \in [a,b]$.

A vector-function $x \in BV\big([a,b], R^n\big)$ is said to be a solution of the system (1.1) if

$$x(t) = x(s) + \int_s^t d\,A_0(\tau) \cdot f_0\big(\tau, x(\tau)\big) \quad \text{for} \quad a \le s \le t \le b.$$

Under a solution of the system of generalized ordinary differential inequalities

$$dx(t) \le d\,A_0(t) \cdot f_0\big(t, x(t)\big)$$

we understand a vector-function $x \in BV\big([a,b], R^n\big)$ such that

$$x(t) \le x(s) + \int_s^t d\,A_0(\tau) \cdot f_0\big(\tau, x(\tau)\big) \quad \text{for} \quad a \le s \le t \le b.$$

A solution of the system

$$dx(t) \ge d\,A_0(t) \cdot f_0\big(t, x(t)\big)$$

is defined analogously.

Let $l : BV_S\big([a,b], R^n\big) \to R^n$ be a linear continuous operator, and let $l_0 : BV_S\big([a,b], R^n\big) \to R^n_+$ be a positive homogeneous continuous operator. We will say that a matrix-function $P : [a,b] \times R^n \to R^{n\times n}$ satisfies the Opial condition with respect to the triplet $(l, l_0; A_0)$ if

(a) $P \in K\big([a,b] \times R^n, R^{n\times n}; A_0\big)$, and there exists a matrix-function $\Phi \in L\big([a,b], R^{n\times n}_+; A_0\big)$ such that

$$|P(t,x)| \le \Phi(t) \quad \text{on} \quad [a,b] \times R^n;$$

(b) for every $B \in BV\big([a,b], R^{n\times n}\big)$,

$$\det\big(I_n + (-1)^j d_j B(t)\big) \ne 0 \quad \text{for} \quad t \in [a,b] \quad (j = 1,2)$$

and the problem

$$dx(t) = dB(t) \cdot x(t), \quad |l(x)| \le l_0(x)$$

has only trivial solution provided there exists a sequence $y_k \in BV\big([a,b],R^n\big)$ $(k=1,2,\dots)$ such that

$$\lim_{k\to+\infty}\int_a^t d\,A_0(\tau)\cdot P\big(\tau,y_k(\tau)\big)=B(t)\quad\text{uniformly on}\quad [a,b].$$

Let x^0 be a solution of the problem (1.1), (1.2) and let r be a positive number.

The solution x^0 is said to be strongly isolated in the radius r if there exist $P\in K\big([a,b]\times R^n,R^{n\times n};\ A_0\big)$, $q\in K\big([a,b]\times R^n,$ $R^n;A_0\big)$, a linear continuous operator $l: BV_S\big([a,b],R^n\big)\to R^n$, a positive homogeneous continuous operator $l_0: BV_S\big([a,b],R^n\big)\to R^n$ and a continuous operator $\widetilde{l}: BV_S\big([a,b],R^n\big)\to R^n$ such that

(a) $f_0(t,x)=P(t,x)x+q(t,x)$ for $t\in[a,b]$, $\|x-x^0(t)\|<r$ and the equality $h(x)=l(x)+\widetilde{l}(x)$ is fulfilled on $u_n(x^0;r)$;

(b) the vector-functions $\alpha(t,\rho)=\max\big\{|q(t,x)|:\|x\|\le\rho\big\}$ and $\beta(\rho)=\sup\big\{[|\widetilde{l}(x)|-l_0(x)]_+:\|x\|_S\le\rho\big\}$ satisfy

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\int_a^b d\big(A^{(1)}(t)+A^{(2)}(t)\big)\cdot\alpha(t,\rho)=0,\qquad \lim_{\rho\to+\infty}\frac{\beta(\rho)}{\rho}=0;$$

(c) the problem

$$dx(t)=dA_0(t)\cdot\big[P(t,x(t))x(t)+q\big(t,x(t)\big)\big],$$

$$l(x)+\widetilde{l}(x)=0$$

has no solution differing from x^0;

(d) the matrix-function P satisfies the Opial condition with respect to the triplet $(l,l_0;A_0)$.

Let $h(x)=\big(h_i(x)\big)_{i=1}^n$, $h_i(x)=x_i(t_i)-\varphi_i(x)$ $(i=1,\dots,n)$ and $h_m(x)=\big(h_{im}(x)\big)_{i=1}^n$, $h_{im}(x)=x_i(t_{im})-\varphi_{im}(x)$ $(i=1,\dots,n;\ m=1,2,\dots)$ for $x=(x_l)_{l=1}^n\in BV\big([a,b],R^n\big)$. By $W_r(A_0,f_0,h;\,x^0)$ we denote the set of all sequences (A_m,f_m,h_m) $(m=1,2,\dots)$ such that

(a) $\displaystyle\lim_{m\to+\infty}\int_a^t d\,A_m(\tau)\cdot f_m(\tau,x)=\int_a^t d\,A_0(\tau)\cdot f_0(\tau,x)$ uniformly on $[a,b]$ for every $x\in D_n(x^0;r)$;

(b) $\displaystyle\lim_{m\to+\infty}h_m(x)=h(x)$ uniformly on $U_n(x^0;r)$;

(c) there exists a sequence $\omega_m \in M\big([a,b]\times R_+, R_+^n;\ A_m\big)$ $(m=1,2,\dots)$ such that

$$\sup\left\{\Big\|\int_a^b dV(A_m)(t)\cdot\omega_m(t,r)\Big\| : m=1,2,\dots\right\} < +\infty, \qquad (*)$$

$$\lim_{s\to 0+}\sup\left\{\Big\|\int_a^b dV(A_m)(t)\cdot\omega_m(t,s)\Big\| : m=1,2,\dots\right\} = 0 \qquad (**)$$

and

$$|f_m(t,x)-f_m(t,y)| \le \omega_m\big(t,\|x-y\|\big) \quad \text{on} \quad [a,b]\times D_n(x^0;r) \quad (m=1,2,\dots).$$

Remark 1.1. If for every natural k there exists a positive number ρ_k such that

$$\omega_m(t,k\rho) \le \rho_k\omega_m(t,\rho) \quad \text{for} \quad \rho>0, t\in[a,b] \quad (m=1,2,\dots),$$

then the condition (*) follows from the condition (**). In particular, this is the case for the sequence of vector-functions

$$\omega_m(t,\rho) = \max\big\{|f_m(t,x)-f_m(t,y)| : \|x\|\le\|x^0\|_S + r,$$
$$\|y\|\le\|x^0\|_S + r, \quad \|x-y\|\le\rho\big\} \quad (m=1,2,\dots).$$

The problem (1.1), (1.2) is said to be $(x^0;r)$- correct if for every $\varepsilon\in]0,r[$ and $\big((A_m,f_m,h_m)\big)_{m=1}^{+\infty}\in W_r(A_0,f_0,h;\ x^0)$ there exists a natural m_0 such that the problem (1.1_m), (1.2_m) has at least one solution contained in $U_n(x^0;r)$, and every such solution belongs to the ball $U_n(x^0;\varepsilon)$ for any $m\ge m_0$.

The problem (1.1), (1.2) is said to be correct if it has a unique solution x^0, and it is $(x^0;r)$-correct for every $r>0$.

We say that the pair $((c_{il})_{i,l=1}^n;\ (\varphi_{oi})_{i=1}^n)$ consisting of a matrix-function $(c_{il})_{i,l=1}^n\in BV\big([a,b],\ R^{n\times n}\big)$ and a positively homogeneous nondecreasing operator $(\varphi_{oi})_{i=1}^n : BV_S\big([a,b],\ R_+^n\big)\to R_+^n)$ belongs to the set $U(t_1,\dots,t_n)$ if the functions c_{il} $(i\ne l;\ i,l=1,\dots,n)$ are nondecreasing on $[a,b]$ and continuous at the point t_i,

$$d_jc_{ii}(t)\ge 0 \quad \text{for} \quad t\in[a,b] \quad (j=1,2;\ i=1,\dots,n) \qquad (1.3)$$

and the problem

$$\begin{gathered}\Big[dx_i(t)-\operatorname{sign}(t-t_i)\sum_{l=1}^n x_l(t)dc_{il}(t)\Big]\operatorname{sign}(t-t_i)\le 0 \quad (i=1,\dots,n),\\ (-1)^jd_jx_i(t_i)\le x_i(t_i)d_jc_{ii}(t_i) \quad (j=1,2;\ \ i=1,\dots,n);\end{gathered} \qquad (1.4)$$

$$x_i(t_i)\le\varphi_{oi}\big(|x_1|,\dots,|x_n|\big) \quad (i=1,\dots,n) \qquad (1.5)$$

has no nontrivial non-negative solution.

Consider one method of construction of the solution of the problem (1.1), (1.2). We take an arbitrary vector-function $(x_{i0})_{i=1}^n \in BV\big([a,b],\, R^n\big)$ as the zero approximation of the solution of the problem (1.1), (1.2). If the $(m-1)$-th approximation has been constructed, then as the m-th approximation we take a vector-function $(x_{im})_{i=1}^n \in BV\big([a,b],\, R^n\big)$ whose i-th component is the solution of the Cauchy problem

$$dx_{im}(t) = \sum_{l=1}^n f_{lo}\big(t, x_{1m-1}(t), \dots, x_{i-1m-1}(t), x_{im}(t), \\ x_{i+1m-1}(t), \dots, x_{nm-1}(t)\big)\, da_{ilo}(t) \quad (i = 1, \dots, n), \tag{1.6}$$

$$x_{im}(t_i) = \varphi_i(x_{1m-1}, \dots, x_{nm-1}) \quad (i = 1, \dots, n). \tag{1.7}$$

Let the conditions

$$(-1)^{\sigma+1} f_{k0}(t, x_1, \dots, x_n)\,\mathrm{sign}\,\big[(t-t_i)x_i\big] \le \sum_{l=1}^n p_{\sigma ikl}(t)|x_l| + q_k\big(t, \|x\|\big)$$

$$\text{for } \mu_{a_{\sigma ik}} \text{-almost everywhere } t \in [a,b]\setminus\{t_i\} \quad (i,k = 1, \dots, n) \tag{1.8}$$

and

$$\Big[(-1)^{\sigma+j+1} f_{k0}(t_i, x_1, \dots, x_n)\,\mathrm{sign}\, x_i - \sum_{l=1}^n \alpha_{\sigma ikjl}|x_l| - \\ -q_k\big(t_i, \|x\|\big)\Big] d_j a_{\sigma ik}(t_i) \le 0 \quad (j = 1.2;\; i,k = 1, \dots, n) \tag{1.9}$$

be fulfilled on R^n for every $\sigma \in \{1,2\}$, and let the inequalities

$$|\varphi_i(x_1, \dots, x_n)| \le \varphi_{oi}\big(|x_1|, \dots, |x_n|\big) + \gamma\Big(\sum_{l=1}^n \|x_l\|_s\Big) \\ (i = 1, \dots, n) \tag{1.10}$$

be fulfilled on $BV\big([a,b], R^n\big)$, where $\alpha_{\sigma ikjl} \in R$ $(j,\sigma = 1,2;\, i,k,l = 1, \dots, n)$; $\big(p_{\sigma ikl}\big)_{k,l=1}^n \in L\big([a,b], R^{n\times n};\, A^{(\sigma)}\big)$ $(\sigma = 1,2;\; i = 1, \dots, n)$, $q = (q_k)_{k=1}^n \in K\big([a,b] \times R_+, R_+^n;\, A^{(\sigma)}\big)$ $(\sigma = 1,2)$ is a vector-function nondecreasing in the second variable, $\gamma \in C(R_+, R_+)$ and

$$\lim_{\rho\to+\infty} \frac{1}{\rho} \int_a^b d\big(A^{(1)}(t) + A^{(2)}(t)\big) \cdot q(t,\rho) = 0, \quad \lim_{\rho\to+\infty} \frac{\gamma(\rho)}{\rho} = 0. \tag{1.11}$$

Moreover, let there exist a matrix-function $(c_{il})_{i,l=1}^n \in BV\big([a,b],\ R^{n\times n}\big)$ *such that*

$$\big((c_{il})_{i,l=1}^n;(\varphi_{oi})_{i=1}^n\big)\in U(t_1,\dots,t_n), \tag{1.12}$$

$$\sum_{\sigma=1}^2\sum_{k=1}^n\int\limits_s^t p_{\sigma ikl}(\tau)\,da_{\sigma ik}(\tau)\le c_{il}(t)-C_{il}(s)$$
$$for\quad a\le s<t<t_i\quad and\quad t_i<s<t\le b\quad (i,l=1,\dots,n) \tag{1.13}$$

and

$$\sum_{\sigma=1}^2\sum_{k=1}^n\alpha_{\sigma ikjl}d_ja_{\sigma ik}(t_i)\le\delta_{il}d_jc_{ii}(t_i)$$
$$(j=1,2;\ i,l=1,\dots,n). \tag{1.14}$$

If the problem (1.1), (1.2) *has no more than one solution, then it is correct.*

Let the conditions (1.3), (1.11), (1.13), (1.14) *and*

$$|c_{il}(t)-c_{il}(s)|\le\int\limits_s^t h_{il}(\tau)d\alpha_l(\tau)\quad for\quad a\le s\le t\le b$$
$$(i,l=1,\dots,n) \tag{1.15}$$

hold, and let the conditions (1.8), (1.9) *be fulfilled on* R^n *for every* $\sigma\in\{1,2\}$, *where* $\alpha_{\sigma ikjl}\in R$ $(j,\sigma=1,2;\ i,k,l=1,\dots,n)$, $(p_{\sigma ikl})_{k,l=1}^n\in L\big([a,b],R^{n\times n};\ A^{(\sigma)}\big)$ $(\sigma=1,2;\ i=1,\dots,n)$, $q=(q_k)_{k=1}^n\in K\big([a,b]\times R_+,R_+^n;\ A\big)$ *is the vector-function nondecreasing in the second variable,* $\gamma\in C(R_+,R_+)$, c_{il} $(i\ne l;\ i,l=1,\dots,n)$ *are the functions, nondecreasing on* $[a,b]$ *and continuous at the point* t_i, $c_{ii}\in BV\big([a,b],R\big)$ $(i=1,\dots,n)$; α_l $(l=1,\dots,n)$ *are the functions nondecreasing on* $[a,b]$ *and having not more than a finite number of discontinuity points,* $h_{il}\in L^\mu\big([a,b],R_+;\ \alpha_l\big)$ $(i,l=1,\dots,n)$, $1\le\mu\le+\infty$. *Moreover, let the inequalities*

$$\big|\varphi_i(x_1,\dots,x_n)\big|\le\sum_{\sigma=0}^2\sum_{k=1}^n l_{\sigma ik}\|x_k\|_{\nu,s_\sigma(\alpha_k)}+\gamma\Big(\sum_{k=1}^n\|x_k\|_S\Big)$$
$$(i=1,\dots,n) \tag{1.16}$$

be fulfilled on $BV\big([a,b],R^n\big)$ *and the module of every characteristic value of* *the $3n\times 3n$-matrix $\mathcal{H}=\big(\mathcal{H}_{j+1,\sigma+1}\big)_{j,\sigma=0}^{2}$ be less than 1, where*

$$l_{\sigma ik}\in R_{+}\quad (\sigma=0,1,2;\, i,k=1,\dots,n),\quad \frac{1}{\mu}+\frac{2}{\nu}=1,$$
$$\mathcal{H}_{j+1,\sigma+1}=\big(\xi_{ij}l_{\sigma ik}+\lambda_{k\sigma ij}\|h_{ik}\|_{\mu,s_{\sigma}(\alpha_i)}\big)_{i,k=1}^{n}\quad (j,\sigma=0,1,2),$$
$$\xi_{ij}=\big[s_j(\alpha_i)(b)-s_j(\alpha_i)(a)\big]^{\frac{1}{\nu}}\quad (j=0,1,2;\, i=1,\dots,n),$$
$$\lambda_{k0k0}=(2\pi^{-1})^{\frac{2}{\nu}}\xi_{k0}^{2}\quad (k=1,\dots,n),$$
$$\lambda_{k\sigma ij}=\xi_{k\sigma}\xi_{ij}\ \textit{ for }\ \sigma^2+j^2+(i-k)^2>0,\ \ \sigma j=0$$
$$(j,\sigma=0,1,2;\ i,k=1,\dots,n),$$
$$\lambda_{k\sigma ij}=\bigg(\frac{1}{4}\mu_{\alpha_k\sigma}\nu_{\alpha_k\sigma\alpha_i j}\sin^{-2}\frac{\pi}{4n_{\alpha_k\sigma}+2}\bigg)^{\frac{1}{\nu}}\quad (j,\sigma=1,2;\ i,k=1,\dots,n).$$

If the problem (1.1), (1.2) has no more than one solution, then it is correct.

Let the inequalities (1.14) hold for $i\neq l$ $(i,l=1,\dots,n)$, let there exist σ, $\sigma_1\in\{1,2\}$ such that $\sigma+\sigma_1=3$, let the conditions (1.8), (1.9),

$$(-1)^{\sigma_1+1}f_{k0}(t,x_1,\dots,x_n)\,\mathrm{sign}\,\big[(t-t_i)x_i\big]\le\sum_{l=1}^{n}\eta_{il}|x_l|+q_k\big(t,\|x\|\big)$$
$$\textit{for}\ \ \mu_{a_{\sigma_1 ik}}\textit{-almost everywhere}\ \ t\in[a,b]\backslash\{t_i\}\quad (i,k=1,\dots,n),$$
$$\bigg[(-1)^{\sigma_1+j+1}f_{k0}(t_i,x_1,\dots,x_n)\,\mathrm{sign}\,x_i-\sum_{l=1}^{n}\alpha_{\sigma_1 ikjl}|x_l|-$$
$$-q_k\big(t_i,\|x\|\big)\bigg]d_j a_{\sigma_1 ik}(t_i)\le 0\quad (j=1,2;\ k=1,\dots,n)$$

be fulfilled on R^n and let the inequalities

$$\big|\varphi_i(x_1,\dots,x_n)\big|\le c_i\big|x_i(\tau_i)\big|+\gamma\Big(\sum_{k=1}^{n}\|x_k\|_s\Big)\quad (i=1,\dots,n)\qquad (1.17)$$

be fulfilled on $BV\big([a,b],R^n\big)$, where $\alpha_{\sigma ikjl}\in R$ and $\alpha_{\sigma_1 ikjl}\in R$ $(j=1,2;\, i,k=1,\dots,n)$, $(p_{\sigma ikl})_{k,l=1}^{n}\in L\big([a,b],R_{+}^{n\times n};\ A^{(\sigma)}\big)$ $(i=1,\dots,n)$, $\eta_{il}\in R_{+}$ $(i\neq l;\, i,l=1,\dots,n)$, $\eta_{ii}<0$ $(i=1,\dots,n)$, $a_{\sigma_1 ik}$ $(i,k=1,\dots,n)$ are the functions nondecreasing and continuous on every interval $[a,t_i[$ and $]t_i,b]$ and satisfying the condition

$$\sum_{k=1}^{n}d_j a_{\sigma_1 ik}(t_i)\le 0\quad (j=1,2;\ i=1,\dots,n),\qquad (1.18)$$

$q=(q_k)_{k=1}^{n}\in K\big([a,b]\times R_{+},R_{+}^{n};A_0\big)$ is the vector-function nondecreasing the second variable, $\gamma\in C(R_{+},R_{+})$ and the conditions (1.11) hold in $c_i\in$

R_+ and $\tau_i \in [a,b]$, $\tau_i \neq t_i$ $(i=1,\dots,n)$ are such that

$$c_i\gamma_{ij}\zeta_{ij} < 1 \quad \text{if} \quad (-1)^j(\tau_i - t_i) > 0 \quad (j=1,2;\ i=1,\dots,n), \tag{1.19}$$

$\gamma_{ij} = 1 + \eta_{ii} d_j \alpha_i(t_i) + \sum\limits_{k=1}^{n} p_{\sigma iki}(t_i) d_j a_{\sigma ik}(t_i)$, $\zeta_{ij} = \exp\big(\eta_{ii}\mu_{\alpha_i ji}(\tau_i)\big)$ $(j=1,2;\ i=1,\dots,n)$, $\alpha_i(t) \equiv \sum\limits_{k=1}^{n} a_{\sigma_1 ik}(t)$ $(i=1,\dots,n)$. *Moreover, let*

$$g_{iij} < 1 \quad \text{if} \quad (-1)^j(\tau_i - t_i) > 0 \quad (j=1,2;\ i=1,\dots,n) \tag{1.20}$$

and the real part of every characteristic value of the matrix $(\xi_{il})_{i,l=1}^n$ be negative, where

$$\xi_{il} = \eta_{il}\big[\delta_{il} + (1-\delta_{il})h_{ij}\big] - \eta_{ii} g_{ilj}$$
$$\text{for} \quad (-1)^j(\tau_i - t_i) > 0 \quad (j=1,2;\ i,l=1,\dots,n),$$
$$g_{ilj} = c_i\gamma_{ij}(1 - c_i\gamma_{ij}\zeta_{ij})^{-1}\mu_{\beta_{il}ji}(\tau_i) + \max\big\{\mu_{\beta_{il}1i}(a), \mu_{\beta_{il}2i}(b)\big\},$$
$$\beta_{il}(t) \equiv \sum_{k=1}^{n}\int_a^t p_{\sigma ikl}(\tau)\,da_{\sigma ik}(\tau), \quad h_{ij} = 1 \quad \text{for} \quad c_i\gamma_{ij} \le 1 \quad \text{and}$$
$$h_{ij} = 1 + (c_i\gamma_{ij} - 1)(1 - c_i\gamma_{ij}\zeta_{ij})^{-1} \quad \text{for} \quad c_i\gamma_{ij} > 1.$$

If the problem (1.1), (1.2) *has no more than one solution, then it is correct.*

Let the conditions

$$(-1)^{\sigma+1}\big[f_{k0}(t,x_1,\dots,x_n) - f_{k0}(t,y_1,\dots,y_n)\big]\operatorname{sign}\big[(t-t_i)(x_i-y_i)\big] \le$$
$$\le \sum_{l=1}^{n} p_{\sigma ikl}(t)|x_l - y_l|$$

$$\text{for} \quad \mu_{a_\sigma ik}\text{-almost everywhere} \quad t \in [a,b]\setminus\{t_i\} \quad (i,k=1,\dots,n), \tag{1.21}$$

$$\Big\{(-1)^{\sigma+j+1}\big[f_{k0}(t_i,x_1,\dots,x_n) - f_{k0}(t_i,y_1,\dots,y_n)\big]\operatorname{sign}(x_i - y_i) -$$
$$-\sum_{l=1}^{n}\alpha_{\sigma ikjl}|x_l - y_l|\Big\} d_j a_{\sigma ik}(t_i) \le 0 \quad (j=1,2;\ i,k=1,\dots,n) \tag{1.22}$$

be fulfilled on R^n for every $\sigma \in \{1,2\}$, and let the inequalities

$$\big|\varphi_i(x_1,\dots,x_n) - \varphi_i(y_1,\dots,y_n)\big| \le$$
$$\le \varphi_{oi}\big(|x_1-y_1|,\dots,|x_n-y_n|\big) \quad (i=1,\dots,n) \tag{1.23}$$

be fulfilled on $BV\big([a,b],R^n\big)$, where $\alpha_{\sigma ikjl} \in R$ $(j,\sigma=1,2;\ i,k,l=1,\dots,n)$, $(p_{\sigma ikl})_{k,l=1}^n \in L\big([a,b],R^{n\times n};\ A^{(\sigma)}\big)$ $(\sigma=1,2;\ i=1,\dots,n)$. Moreover, let *there exist a matrix-function* $(c_{il})_{i,l=1}^{n} \in BV\big([a,b], R^{n\times n}\big)$ *such that the conditions* (1.12)–(1.14) *hold. Then the problem* (1.1), (1.2) *is correct.*

Let the conditions (1.3), (1.13)–(1.15) *hold and let the conditions* (1.21) *and* (1.22) *be fulfilled on* R^n *for every* $\sigma \in \{1,2\}$, *where* $\alpha_{\sigma ikjl} \in R$ $(j,\sigma = 1,2;\ i,k,l = 1,\dots,n)$, $(p_{\sigma ikl})_{k,l=1}^{n} \in L\big([a,b], R^{n\times n};\ A^{(\sigma)}\big)$ $(\sigma = 1,2;\ i = 1,\dots,n)$, c_{il} $(i \neq l;\ i,l = 1,\dots,n)$ *are the functions, nondecreasing on* $[a,b]$ *and continuous at the point* $t_i, c_{ii} \in BV\big([a,b], R\big)$ $(i = 1,\dots,n)$ α_l $(l = 1,\dots,n)$ *are the functions, nondecreasing on* $[a,b]$ *and having not more than a finite number of discontinuity points,* $h_{il} \in L^{\mu}\big([a,b], R_+;\ \alpha_l\big)$ $(i,l = 1,\dots,n)$, $1 \le \mu \le +\infty$. *Moreover, let the inequalities*

$$|\varphi_i(x_1,\dots,x_n) - \varphi_i(y_1,\dots,y_n)| \le$$
$$\le \sum_{\sigma=0}^{2}\sum_{k=1}^{n} l_{\sigma ik}\|x_k - y_k\|_{\nu, s_\sigma(\alpha_k)} \quad (i,k = 1,\dots,n) \tag{1.24}$$

be fulfilled on $BV\big([a,b], R^n\big)$, *where* $l_{\sigma ik} \in R_+$ $(\sigma = 0,1,2;\ i,k = 1,\dots,n)$, $\dfrac{1}{\mu} + \frac{2}{\nu} = 1$ *and the module of every characteristic value of the* $3n\times 3n$*-matrix* $\mathcal{H} = (\mathcal{H}_{j+1,\sigma+1})_{j,\sigma=0}^{2}$ *appearing in Corollary* 1.1 *be less than* 1. *Then the problem* (1.1), (1.2) *is correct.*

Let the inequalities (1.14) *hold for* $i \neq l$ $(i,l = 1,\dots,n)$, *let there exist* $\sigma, \sigma_1 \in \{1,2\}$ *such that* $\sigma + \sigma_1 = 3$, *and let the conditions* (1.21), (1.22) *and*

$$(-1)^{\sigma_1+1}\big[f_{k0}(t,x_1,\dots,x_n) - f_{k0}(t,y_1,\dots,y_n)\big]\operatorname{sign}\big[(t-t_i)(x_i-y_i)\big] \le$$
$$\le \sum_{l=1}^{n}\eta_{il}|x_l - y_l|$$
$$\text{for } \ \mu_{a_{\sigma_1 ik}}\text{-almost everywhere } \ t \in [a,b]\backslash\{t_i\} \quad (i,k = 1,\dots,n),$$
$$\big\{(-1)^{\sigma_1+j+1}[f_{k0}(t_i,x_1,\dots,x_n) - f_{k0}(t_i,y_1,\dots,y_n)]\operatorname{sign}(x_i - y_i) -$$
$$-\sum_{l=1}^{n}\alpha_{\sigma_1 ikjl}|x_l - y_l|\big\}d_j a_{\sigma_1 ik}(t_i) \le \quad (j = 1,2;\ i,k = 1,\dots,n)$$

be fulfilled on R^n, *where* $\alpha_{\sigma ikjl} \in R$ *and* $\alpha_{\sigma_1 ikjl} \in R$ $(j = 1,2;\ i,k,l = 1,\dots,n)$, $(p_{\sigma ikl})_{k,l=1}^{n} \in L\big([a,b], R_+^{n\times n};\ A^{(\sigma)}\big)$ $(i = 1,\dots,n)$, $\eta_{il} \in R_+$ $(i \neq l;\ i,l = 1,\dots,n)$, $\eta_{ii} < 0$ $(i = 1,\dots,n)$, $a_{\sigma_1 ik}(i,k = 1,\dots,n)$ *are the functions, nondecreasing and continuous on every interval* $[a,t_i[$ *and* $]t_i,b]$ *and satisfying the condition* (1.18). *Moreover, let* $c_i \in R_+$ *and* $\tau_i \in [a,b]$, $\tau_i \neq t_i$ $(i = 1,\dots,n)$ *be such that the conditions* (1.19), (1.20) *hold and the real part of every characteristic value of the matrix* $(\xi_{il})_{i,l=1}^{n}$ *is negative, where* γ_{ij}, ζ_{ij}, ξ_{il}, g_{ilj} $(j = 1,2;\ i,l = 1,\dots,n)$ *and* $\alpha_i(t)$ $(i = 1,\dots,n)$ *are* *respectively the numbers and the functions appearing in Corollary* 1.2. *Then the problem* (1.1),

$$x_i(t_i) = \lambda_i x_i(\tau_i) + \gamma_i \quad (i = 1, \dots, n) \tag{1.25}$$

is correct for every $\lambda_i \in [-c_i, c_i]$ *and* $\gamma_i \in R$ $(i = 1, \dots, n)$.

Let the conditions of Theorem 1.2 *be fulfilled and*

$$d_j c_{ii}(t) < 1 \quad \textit{for} \quad (-1)^j (t - t_i) < 0 \quad (j = 1, 2;\, i = 1, \dots, n). \tag{1.26}$$

Then for every $(x_{i0})_{i=1}^n \in BV\big([a,b], R^n\big)$, *there exists a unique sequence* $(x_{im})_{i=1}^n \in BV\big([a,b], R^n\big)$ $(m = 1, 2, \dots)$ *such that the vector-function* $(x_{im})_{i=1}^n$ *is the solution of the problem* (1.6), (1.7) *for every natural* m, *and*

$$\sum_{i=1}^n |x_i(t) - x_{im}(t)| \le r_0 \delta^m \quad \textit{for} \quad t \in [a,b] \quad (m = 1, 2, \dots), \tag{1.27}$$

where $(x_i)_{i=1}^n$ *is the solution of the problem* (1.1), (1.2), *and* $r_0 > 0$ *and* $\delta \in]0,1[$ *are numbers independent of* m.

Let the conditions of Corollary 1.3 *and the condition* (1.26) *hold. Then the conclusion of Theorem* 1.3 *is true.*

Let the conditions of Corollary 1.4 *hold and*

$$|\eta_{ii}| d_j \alpha_i(t) + d_j \beta_i(t) < 1 \quad \textit{for} \quad (-1)^j (t - t_i) < 0 \quad (j = 1, 2;\, i = 1, \dots, n);$$

moreover, let $\lambda_i \in [-c_i, c_i]$ $(i = 1, \dots, n)$, $\gamma_i \in R$ $(i = 1, \dots, n)$, *where* η_{ii}, c_i *and* α_i, β_i *are respectively the numbers and the functions appearing in Corollary* 1.2. *Then for every* $(x_{i0})_{i=1}^n \in BV\big([a,b]\, R^n\big)$, *there exists a unique sequence* $(x_{im})_{i=1}^n \in BV\big([a,b], R^n\big)$ $(m = 1, 2, \dots)$ *such that the vector-function* $(x_{im})_{i=1}^n$ *is a solution of the system* (1.6) *satisfying the condition*

$$x_{im}(t_i) = \lambda_i x_{im-1}(\tau_i) + \gamma_i \quad (i = 1, \dots, n)$$

for every natural m, *and the estimates* (1.27) *hold, where* $(x_i)_{i=1}^n$ *is the solution of the problem* (1.1), (1.25), *and* $r_0 > 0$ *and* $\delta \in]0,1[$ *are the numbers independent of* m.

Remark 1.2. The $3n \times 3n$-matrix $\mathcal{H}$ appearing in Corollaries 1.1, 1.3 and 1.5 may be replaced by the $n \times n$-matrix

$$\bigg(\max \Big\{ \sum_{j=0}^2 (\xi_{ij} l_{\sigma ik} + \lambda_{k\sigma ij} \|h_{ik}\|_{\mu, s_\sigma(\alpha_k)}) : \sigma = 0, 1, 2 \Big\} \bigg)_{i,k=1}^n .$$

Remark 1.3. The above-described process of construction of the solution of the problem (1.1), (1.2) is stable in a definite sense (see Section 3 below).

§ 2. Auxiliary Propositions

If the problem (1.1), (1.2) *has a solution* x^0 *which is strongly isolated in the radius* $r>0$, *then it is* $(x^0;r)$*-correct.*

Proof of this lemma is given in [10, Theorem 1.1].

Let the condition (1.12) *hold. Then there exists a number* $\delta\in]0,1[$ *such that*

$$\big(\widetilde{c}_{i,l}\big)_{i,l=1}^{n};(\widetilde{\varphi}_{0i})_{i=1}^{n}\big)\in U(t_1,\dots,t_n), \tag{2.1}$$

where

$$\widetilde{c}_{il}(t)=\frac{1}{\delta}c_{il}(t)\quad\textit{for}\quad i\neq l,\quad \widetilde{c}_{ii}(t)=c_{ii}(t),$$

$$\widetilde{\varphi}_{0i}(y_1,\dots,y_n)=\frac{1}{\delta}\varphi_{0i}(y_1,\dots,y_n). \tag{2.2}$$

Proof. According to Lemma 2.5 from [11], there exists a positive number ρ_* such that every solution of the problem

$$\bigg[d|x_i(t)|-\operatorname{sign}(t-t_i)\Big(\sum_{l=1}^{n}|x_l(t)|dc_{il}(t)+$$
$$+du_i(t)\Big)\bigg]\operatorname{sign}(t-t_i)\le 0\quad (i=1,\dots,n), \tag{2.3}$$
$$(-1)^j d_j|x_i(t_i)|\le |x_i(t_i)|d_jc_{ii}(t_i)+d_ju_i(t_i)\quad (j=1,2;\ i=1,\dots,n);$$

$$|x_i(t_i)|\le\varphi_{0i}\big(|x_1|,\dots,|x_n|\big)+\gamma\quad (i=1,\dots,n) \tag{2.4}$$

admits the estimate

$$\sum_{i=1}^{n}\|x_i\|_S\le\rho_*\big[\gamma+\frac{1}{n}\|u(\cdot)-u(a)\|_S\big], \tag{2.5}$$

where $\gamma\in R_+$ and $u=(u_i)_{i=1}^n\in BV\big([a,b],R_+^n\big)$ are an arbitrary number and a vector-function, respectively.

Let

$$u_{0i}(t)=\sum_{l\neq i,l=1}^{n}\big(c_{il}(t)-c_{il}(t_i)\big)\quad (i=1,\dots,n),\quad \gamma_0=\sum_{k=1}^{n}\varphi_{ok}(1,\dots,1),$$

let $\delta\in]0,1[$ be a number satisfying the inequality

$$\frac{1-\delta}{\delta}\rho_*\bigg[\gamma_0+\frac{1}{n}\sum_{i=1}^{n}\big(u_{0i}(b)-u_{0i}(a)\big)\bigg]<\frac{1}{2}, \tag{2.6}$$

and let $\widetilde{c}_{il}$ and $\widetilde{\varphi}_{0i}$ $(i,l=1,\dots,n)$ be respectively the functions and the functionals given by (2.2).

Consider an arbitrary nonnegative solution $(x_i)_{i=1}^n$ of the problem

$$\left[dx_i(t) - \operatorname{sign}(t-t_i)\sum_{l=1}^n x_l(t)d\widetilde{c}_{il}(t)\right]\operatorname{sign}(t-t_i) \le 0$$
$$(i=1,\dots,n),$$
$$(-1)^j d_j x_i(t_i) \le x_i(t_i) d_j \widetilde{c}_{ii}(t_i) \quad (j=1,2;\ i=1,\dots,n); \tag{2.7}$$
$$x_i(t_i) \le \widetilde{\varphi}_{0i}(x_1,\dots,x_n) \quad (i=1,\dots,n). \tag{2.8}$$

It is not difficult to verify that $(x_i)_{i=1}^n$ will be the solution of the problem (2.3), (2.4), where

$$u_i(t) = \frac{1-\delta}{\delta} u_{0i}(t)\sum_{l=1}^n \|x_l\|_S \ (i=1,\dots,n),\ \gamma = \frac{1-\delta}{\delta}\gamma_o \sum_{l=1}^n \|x_l\|_S.$$

By our choice of ρ_*, the estimate (2.5) holds which, in view of (2.6), implies

$$\sum_{i=1}^n \|x_i\|_S \le \frac{1}{2}\sum_{i=1}^n \|x_i\|_S.$$

Consequently, $x_i(t) \equiv 0$ $(i=1,\dots,n)$. The lemma is proved. ■

Let the conditions (1.12) *and* (1.26) *hold. Then there exist numbers* $\rho \in]0,+\infty[$ *and* $\delta \in]0,1[$ *such that for an arbitrary* $(y_{io})_{i=1}^n \in BV\big([a,b],R_+^n\big)$ *and any sequences of numbers* $\gamma_m \in R_+$ $(m=1,2,\dots)$, *the vector-functions* $(y_{im})_{i=1}^n \in BV\big([a,b],R_+^n\big)$ $(m=1,2,\dots)$ *and nondecreasing vector-functions* $u_m = (u_{im})_{i=1}^n \in BV\big([a,b],R^n\big)$ $(m=1,2,\dots)$, *satisfying the inequalities*

$$\Big\{dy_{im}(t) - \operatorname{sign}(t-t_i)\big[y_{im}(t)dc_{ii}(t) + \sum_{l\ne i,l=1}^n y_{lm-1}(t)dc_{il}(t) +$$
$$+du_{im}(t)\big]\Big\}\operatorname{sign}(t-t_i) \le 0 \quad (i=1,\dots,n),$$
$$(-1)^j d_j y_{im}(t_i) \le y_{im}(t_i)d_j c_{ii}(t_i) + d_j u_{im}(t_i) \quad (j=1,2;\ i=1,\dots,n); \tag{2.9}$$
$$y_{im}(t_i) \le \varphi_{0i}(y_{1m-1},\dots,y_{nm-1}) + \gamma_m \quad (i=1,\dots,n) \tag{2.10}$$

for every natural m, *the estimates*

$$\sum_{i=1}^n \|y_{im}\|_S \le \rho\Big[\sum_{k=1}^m \delta^{m-k}\big(\gamma_k + \frac{1}{n}\|u_k(b)-u_k(a)\|\big) +$$
$$+\delta^m \sum_{i=1}^n \|y_{io}\|_S\Big] \quad (m=1,2\dots) \tag{2.11}$$

hold.

Proof. By Lemma 2.2, there exists a number $\delta \in]0,1[$ such that the functions and the functionals $\widetilde{c}_{il}$ and $\widetilde{\varphi}_{0i}$ respectively, $(i,l=1,\dots,n)$, given by (2.2) satisfy the condition (2.1).

On $BV([a,b],R_+^n)$ we introduce the operator

$$h_i(z_1,\dots,z_n)(t) = q_i(z_1,\dots,z_n)(t) + e_i(t)\Big[\widetilde{\varphi}_{0i}(z_1,\dots,z_n) - \\ - \int_{t_i}^{t} q_i(z_1,\dots,z_n)(\tau)de_i^{-1}(\tau)\Big]$$

for every $i \in \{1,\dots,n\}$, where

$$q_i(z_1,\dots,z_n)(t) = \sum_{l\neq i,l=1}^{n} \Big|\int_{t_i}^{t} z_l(\tau)d\widetilde{c}_{il}(\tau)\Big|,$$
$$b_i(t) = [c_{ii}(t) - c_{ii}(t_i)]\,\mathrm{sign}(t-t_i)$$

and $e_i(t)$ is a solution of the problem

$$de_i(t) = e_i(t)db_i(t), \quad e_i(t_i) = 1.$$

By (1.3) and (1.26), the latter problem has the unique solution differing from zero on whole $[a,b]$ (see [8]). Owing to the variation of constants formula (see [7, p. 120]), we have

$$h_i(z_1,\dots,z_n)(t) = \widetilde{\varphi}_{0i}(z_1,\dots,z_n) + q_i(z_1,\dots,z_n)(t) + \\ + \int_{t_i}^{t} h_i(z_1,\dots,z_n)(\tau)db_i(\tau)$$
$$\text{for} \quad t\in[a,b],\ (z_k)_{k=1}^n \in BV([a,b],R_+^n) \quad (i=1,\dots,n). \tag{2.12}$$

Let

$$z_{i0}(t) \equiv 1 \quad (i=1,\dots,n), \quad \eta = \sum_{i=1}^{n} \|e_i\|_S(1+n\|e_i^{-1}\|_S)$$

and $(z_{im})_{i=1}^n$ $(m=1,2,\dots)$ be a sequence of vector-functions defined by

$$z_{im}(t) = h_i(z_{1m-1},\dots,z_{nm-1})(t) + \eta \quad (i=1,\dots,n). \tag{2.13}$$

It is easy to show that

$$h_i(z_1,\dots,z_n)(t)=e_i(t)\bigg[\widetilde{\varphi}_{0i}(z_1,\dots,z_n)+$$
$$+\sum_{l\neq i,l=1}^{n}\bigg(\bigg|\int\limits_{t_i}^{t}e_i^{-1}(\tau)z_l(\tau)d\widetilde{c}_{il}(\tau)\bigg|+$$
$$+\sum_{\tau\in(\tau_{*i}(t),\tau_i^*(t)]}e_i^{-1}(\tau-)z_l(\tau)d_1\widetilde{c}_{il}(\tau)+$$
$$+\sum_{\tau\in[\tau_{*i}(t),\tau_i^*(t))}e_i^{-1}(\tau+)z_l(\tau)d_2\widetilde{c}_{il}(\tau)\bigg)\bigg]\quad(i=1,\dots,n)$$

on $[a,b]\times BV\big([a,b],R_+^n\big)$, where $\tau_{*i}(t)=\min\{t_i,t\}$. $\tau_i^*(t)=\max\{t_i,t\}$ $(i=1,\dots,n)$. Clearly, the operator $h_i(\cdot)(t):BV\big([a,b],R_+^n\big)\to BV\big([a,b],R_+^n\big)$ is nondecreasing for $i\in\{1,\dots,n\}$ and $t\in[a,b]$. Therefore

$$1\le z_{im-1}(t)\le z_{im}(t)\quad\text{for}\quad t\in[a,b]\quad(i=1,\dots,n;\ m=1,2,\dots).$$

Consequently,

$$\rho_m=\sum_{i=1}^{n}\|x_{im}\|_S\quad(m=1,2,\dots)$$

is a nondecreasing sequence of positive numbers. Let us show that

$$\rho=\lim_{m\to\infty}\rho_m<+\infty. \tag{2.14}$$

Assuming on the contrary that $\rho_m\to+\infty$ as $m\to+\infty$, put

$$x_{im}(t)=\frac{1}{\rho_m}z_{im}(t),\quad \overline{x}_{im}(t)=h_i(x_{1m-1},\dots,x_{nm-1})(t),$$
$$\eta_m=\frac{\eta}{\rho_m}\quad(m=1,2\dots).$$

Then

$$\lim_{m\to+\infty}\eta_m=0, \tag{2.15}$$

$$\sum_{i=1}^{n}\|x_{im}\|_S=1\quad(m=1,2,\dots). \tag{2.16}$$

Taking into account Helly's choice theorem and Lemma 3 from [9], it is not difficult to verify that

$$\lim_{m\to+\infty}\sup\overline{x}_{im}(t)=\overline{x}_i(t)\quad\text{uniformly on}\quad[a,b]\quad(1,\dots,n), \tag{2.17}$$

where $(\overline{x}_i)_{i=1}^n$ is a vector-function from $BV\big([a,b],R_+^n\big)$. On the other hand, from (2.13) we have

$$x_{im}(t)\le\overline{x}_{im}(t)+\eta\quad\text{for}\quad t\in[a,b]\quad(i=1,\dots,n;\ m=1,2,\dots)$$

and

$$\overline{x}_{im}(t) \le h_i(\overline{x}_{1m-1} + \eta_{m-1}, \dots, \overline{x}_{nm-1} + \eta_{m-1})(t)$$
$$\text{for} \quad t \in [a,b] \quad (i = 1, \dots, n;\ m = 1, 2, \dots).$$

By (2.15)–(2.17), from the latter inequalities it follows that

$$\sum_{i=1}^{n} \|\overline{x}_i\|_S > 1$$

and

$$\overline{x}_i(t) \le x_i(t) \quad \text{for} \quad t \in [a,b] \quad (i = 1, \dots, n),$$

where

$$x_i(t) = h_i(\overline{x}_1, \dots, \overline{x}_n)(t) \quad (i = 1, \dots, n).$$

Therefore, using (2.12), we obtain

$$x_i(t) - x_i(s) - \sum_{l=1}^{n} \int_s^t x_l(\tau)\, d\widetilde{c}_{il}(\tau) =$$

$$= \frac{1}{\delta} \sum_{l \ne i, l=1}^{n} \int_s^t \big(\overline{x}_l(\tau) - x_l(\tau)\big)\, dc_{il}(\tau) \le 0 \quad \text{for} \quad t_i < s \le t \quad (i = 1, \dots, n),$$

$$x_i(t) - x_i(s) + \sum_{l=1}^{n} \int_s^t x_l(\tau)\, d\widetilde{c}_{il}(\tau) =$$

$$= -\frac{1}{\delta} \sum_{l \ne i, l=1}^{n} \int_s^t \big(\overline{x}_l(\tau) - x_l(\tau)\big)\, dc_{il}(\tau) \ge 0 \quad \text{for} \quad s \le t < t_i \quad (i = 1, \dots, n),$$

$$(-1)^j d_j x_i(t_i) = x_i(t_i)\, d_j \widetilde{c}_{ii}(t_i) \quad (j = 1, 2;\ i = 1, \dots, n)$$

and

$$x_i(t_i) = \widetilde{\varphi}_{0i}(\overline{x}_1, \dots, \overline{x}_n) \le \widetilde{\varphi}_{0i}(x_1, \dots, x_n) \quad (i = 1, \dots, n).$$

Hence $(x_i)_{i=1}^n$ is a nontrivial solution of the problem (2.7), (2.8). But this contradicts the condition (2.1). The obtained contradiction proves the inequality (2.14).

Let

$$(y_{io})_{i=1}^n \in BV\big([a,b], R_+^n\big), \quad \zeta_o = \sum_{i=1}^{n} \|y_{io}\|_S > 0$$

and $\gamma_m \in R_+$, $(y_{im})_{i=1}^n \in BV\big([a,b],R_+^n\big)$, $u_m=(u_{im})_{i=1}^n \in BV\big([a,b],R^n\big)$ $(m=1,2,\dots)$ be arbitrary sequences satisfying (2.9), (2.10) for every natural m. Put

$$\zeta_m=\sum_{k=1}^m \delta^{m-k}\big(\gamma_k+\frac{1}{n}\|u_k(b)-u_k(a)\|\big)+\delta^m\sum_{i=1}^n\|y_{io}\|_S,$$

$$\overline{y}_{io}(t)\equiv 1,\ \overline{y}_{im}(t)=\frac{y_{im}(t)}{\zeta_m}\quad (i=1,\dots,n).$$

With regard to the inequalities

$$\zeta_m\geq\delta\zeta_{m-1},\quad \zeta_m>\gamma_m,\quad \zeta_m>\frac{1}{n}\ |u_m(b)-u_m(a)\|\quad (m=1,2,\dots)$$

from (2.9) and (2.10), we discover that

$$\Big\{d\overline{y}_{im}(t)-\operatorname{sign}(t-t_i)\big[\overline{y}_{im}(t)dc_{ii}(t)+\sum_{l\neq i,l=1}^n\overline{y}_{lm-1}(t)d\widetilde{c}_{il}(t)+$$

$$+d\overline{u}_{im}(t)\big]\Big\}\operatorname{sign}(t-t_i)\leq 0\quad (i=1,\dots,n),$$

$$(-1)^j d_j\overline{y}_{im}(t_i)\leq\overline{y}_{im}(t_i)d_jc_{ii}(t_i)+d_j\overline{u}_{im}(t_i) \qquad (2.18)$$

$$(j=1,2;\ i=1,\dots,n);$$

$$\overline{y}_{im}(t_i)\leq\widetilde{\varphi}_{0i}(\overline{y}_{1m-1},\dots,\overline{y}_{nm-1})+1\quad (i=1,\dots,n) \qquad (2.19)$$

for every natural m, where $\overline{u}_{im}(t)=\frac{1}{\zeta_m}u_{im}(t)$. Let now

$$q_{im}^*(t)=q_i(\overline{y}_{1m-1},\dots,\overline{y}_{nm-1})(t)+|\overline{u}_{im}(t)-\overline{u}_{im}(t_i)|$$
$$\text{for}\quad t\in[a,b]\quad (i=1,\dots,n;\ m=1,2,\dots)$$

and

$$y_{im}^*(t)=\overline{y}_{im}(t)-q_{im}^*(t)\quad\text{for}\quad t\in[a,b]\quad (i=1,\dots,n;\ m=1,2,\dots)$$

Then by (2.18), (2.19) and the equalities

$$d_jb_i(t_i)=(-1)^jd_jc_{ii}(t_i),\quad d_jq_{im}^*(t_i)=(-1)^jd_j\overline{u}_{im}(t_i)$$
$$(j=1,2;\ i=1,\dots,n)$$

we have

$$\Big[dy_{im}^*(t)-\big(y_{im}^*(t)+q_{im}^*(t)\big)db_i(t)\Big]\operatorname{sign}(t-t_i)\leq 0\quad (i=1,\dots,n),$$

$$(-1)^j\Big[d_jy_{im}^*(t_i)-\big(y_{im}^*(t_i)+q_{im}^*(t_i)\big)d_jb_i(t_i)\Big]\leq 0\quad (j=1,2;\ i=1,\dots,n)$$

and

$$y_{im}^*(t_i)\leq c_{io}\quad (i=1,\dots,n)$$

for every natural m, where

$$c_{io} = \widetilde{\varphi}_{0i}(\overline{y}_{1m-1}, \dots \overline{y}_{nm-1}) + 1 \quad (i = 1, \dots, n;\ m = 1, 2, \dots).$$

Therefore, according to Lemma 2.4 (see below),

$$y^*_{im}(t) \le x^*_{im}(t) \quad \text{for} \quad t \in [a, b] \quad (i = 1, \dots, n;\ m = 1, 2, \dots), \tag{2.20}$$

where $x^*_{im}(t)$ is the solution of the problem

$$dx^*_{im}(t) = \big(x^*_{im}(t) + q^*_{im}(t)\big)\, db_i(t),$$
$$x^*_{im}(t_i) = c_{io}.$$

Owing to the condition (1.26), the latter problem has the unique solution and it is given by the formula

$$x^*_{im}(t) = \int\limits_{t_i}^{t} q^*_{im}(\tau) db_i(\tau) + e_i(t)\Bigg\{ c_{i0} -$$
$$- \int\limits_{t_i}^{t} \Bigg(\int\limits_{t_i}^{\tau} q^*_{im}(s) db_i(s) \Bigg) de_i^{-1}(\tau) \Bigg\} \quad \text{for} \quad t \in [a, b]$$

(see [7, Theorem III.2.22]). Using the variation of constants formula, the latter equality implies

$$x^*_{im}(t) = e_i(t)\Bigg\{ c_{i0} + \int\limits_{t_i}^{t} q^*_{im}(\tau) e_i^{-1}(\tau) db_i(\tau) - \sum_{\tau \in (t_i, t]} q^*_{im}(\tau) d_1 b_i(\tau) d_1 e_i^{-1}(\tau) +$$
$$+ \sum_{\tau \in [t_i, t)} q^*_{im}(\tau) d_2 b_i(\tau) d_2 e_i^{-1}(\tau) \Bigg\} = e_i(t)\Bigg\{ c_{i0} + \int\limits_{t_i}^{t} q^*_{im}(\tau) d\Bigg[e_i^{-1}(\tau) b_i(\tau) -$$
$$- \int\limits_{t_i}^{\tau} b_i(s) de_i^{-1}(s) + \sum_{s \in (t_i, \tau]} d_1 b_i(s) d_1 e_i^{-1}(s) - \sum_{s \in [t_i, \tau)} d_2 b_i(s) d_2 e_i^{-1}(s) \Bigg] -$$
$$- \sum_{\tau \in (t_i, t]} q^*_{im}(\tau) d_1 b_i(\tau) d_1 e_i^{-1}(\tau) + \sum_{\tau \in [t_i, t)} q^*_{im}(\tau) d_2 b_i(\tau) d_2 e_i^{-1}(\tau) \Bigg\} =$$
$$= e_i(t) c_{i0} + \int\limits_{t_i}^{t} q^*_{im}(\tau) d\Bigg[b_i(\tau) e_i^{-1}(\tau) \int\limits_{t_i}^{t} b_i(s) de_i^{-1}(s) \Bigg] \quad \text{for} \quad t > t_i.$$

Consequently, by the equality

$$e_i^{-1}(\tau) = 1 - e_i^{-1}(\tau) b_i(\tau) + \int\limits_{t_i}^{t} b_i(s) d\, e_i^{-1}(s) \quad \text{for} \quad \tau \in [a, b]$$

(see [7, Proposition III.2.15]) we have

$$x_{im}^*(t) = e_i(t)\Big[c_{io} - \int\limits_{t_i}^{t} q_{im}^*(\tau)de_i^{-1}(\tau)\Big]$$
$$(i = 1,\dots,n;\ m = 1,2,\dots) \tag{2.21}$$

for $t > t_i$.

The inequality (2.21) for $t < t_i$ can be verified analogously.

By the definition of y_{im}^*, q_{im}^*, h_i and η, from (2.20) and (2.21) it follows that

$$\overline{y}_{im}(t) \le q_{im}^*(t) + e_i(t)\Big[c_{io} - \int\limits_{t_i}^{t} q_{im}^*(\tau)d\,e_i^{-1}(\tau)\Big] =$$
$$= h_i\big(\overline{y}_{1m-1},\dots,\overline{y}_{nm-1}\big)(t) + e_i(t) + |\overline{u}_{im}(t) - \overline{u}_{im}(t_i)| -$$
$$-e_i(t)\int\limits_{t_i}^{t}|\overline{u}_{im}(\tau) - \overline{u}_{im}(t_i)|de_i^{-1}(\tau) = h_i\big(\overline{y}_{1m-1},\dots,\overline{y}_{nm-1}\big)(t) + e_i(t) +$$
$$+e_i(t)\Big\{\Big|\int\limits_{t_i}^{t} e_i^{-1}d\,s_0\big(\overline{u}_{im}\big)(\tau)\Big| + \sum_{\tau\in(\tau_{*i}(t),\tau_i^*(t)]} e_i^{-1}(\tau-)d_1\overline{u}_{im}(\tau) +$$
$$+\sum_{\tau\in[\tau_{*i}(t),\tau_i^*(t))} e_i^{-1}(\tau+)d_2\overline{u}_{im}(\tau)\Big\} \le h_i\big(\overline{y}_{1m-1},\dots,\overline{y}_{nm-1}\big)(t) +$$
$$+\|e_i\|_S + \|e_i\|_S\|e_i^{-1}\|_S\Big[\int\limits_a^b d\,s_0(\overline{u}_{im})(\tau) + \sum_{\tau\in(a,b]} d_1\overline{u}_{im}(\tau) +$$
$$+\sum_{\tau\in[a,b)} d_2\overline{u}_{im}(\tau)\Big] \le h_i(\overline{y}_{1m-1},\dots,\overline{y}_{nm-1})(t) +$$
$$+\|e_i\|_S\big(1 + \|e_i^{-1}\|_S[\overline{u}_{in}(b) - \overline{u}_{im}(a)]\big) \quad \text{for} \quad t\in[a,b].$$

Therefore

$$\overline{y}_{im}(t) \le h_i\big(\overline{y}_{1m-1},\dots,\overline{y}_{nm-1}\big)(t) + \eta \quad \text{for} \quad t\in[a,b]$$
$$(i = 1,\dots,n;\ m = 1,2,\dots).$$

This, according to (2.13), implies

$$\overline{y}_{im}(t) \le z_{im}(t) \quad \text{for} \quad t\in[a,b] \quad (i = 1,\dots,n;\ m = 1,2\dots)$$

and

$$\sum_{i=1}^{n}\|\overline{y}_{im}\|_S \le \sum_{i=1}^{n}\|z_{im}\|_S = \rho_m \le \rho \quad (m = 1,2,\dots).$$

Consequently, the estimates (2.11) are valid. Since ρ does not depend on $(y_{i0})_{i=1}^{n}$, these estimates will be valid in the case where $y_{i0}(t) \equiv 0$ $(i = 1,\dots,n)$, too. The lemma is proved. ■

Let $\alpha \in BV\big([a,b],R\big)$, $\varphi \in K\big([a,b]\times R,R;\ \alpha\big)$, $t_0 \in [a,b]$, $c_0 \in R$ and every solution of the problem

$$dx(t) = \varphi(t,x(t))d\alpha(t), \tag{2.22}$$

$$x(t_0) = c_0 \tag{2.23}$$

is continuable on $[a,b]$. Moreover, let the function $z \in BV\big([a,b],R\big)$ be such that

$$\big[d\,z(t) - \varphi\big(t,z(t)\big)d\,\alpha(t)\big]\,\mathrm{sign}(t-t_0) \le 0 \quad \textit{for} \quad t\in[a,b], \tag{2.24}$$

$$z(t) + (-1)^j\,d_j z(t) \le x + (-1)^j\varphi(t,x)d_j\alpha(t)$$
$$\textit{for} \quad (-1)^j(t-t_0) > 0, \quad x \ge z(t) \quad (j=1,2), \tag{2.25}$$

$$(-1)^j[d_j z(t_0) - \varphi(t_0,c_0)d_j\alpha(t_0)] \le 0 \quad (j=1,2) \tag{2.26}$$

and

$$z(t_0) \le c_0. \tag{2.27}$$

Then the problem (2.22), (2.23) *has a solution x satisfying the inequality*

$$z(t) \le x(t) \quad \textit{for} \quad t \in [a,b]. \tag{2.28}$$

Proof. Put

$$\chi(t,x) = \begin{cases} x & \text{for} \quad x \ge z(t), \\ z(t) & \text{for} \quad x < z(t); \end{cases}$$

$$\widetilde{\varphi}(t,x) = \varphi\big(t,\chi(t,x)\big).$$

Consider the equation

$$dx(t) = \widetilde{\varphi}\big(t,x(t)\big)d\alpha(t). \tag{2.29}$$

The problem (2.29), (2.23) has a solution x defined on some interval $I_0 \subset [a,b]$, $t_0 \in I_0$.

Assume

$$z(t_*) > x(t_*) \tag{2.30}$$

for some $t_* \in [a,t_0] \cap I_0$ and put

$$t^* = \sup\Big\{t : z(s) > x(s) \quad \text{for} \quad t_* \le s \le t < t_0\Big\}.$$

If $z(t^*) > x(t^*)$, then by (2.27) we have $t^* < t_0$ and $z(t^*+) \le x(t^*+)$. Hence

$$d_2 z(t^*) < d_2 x(t^*). \tag{2.31}$$

On the other hand, it follows from (2.24) and the definition of $\widetilde{\varphi}$ that

$$d_2 z(t^*) \geq \varphi(t^*, z(t^*))d_2\alpha(t^*) = \widetilde{\varphi}(t^*, x(t^*))d_2\alpha(t^*) = d_2 x(t^*).$$

But this contradicts (2.31). Therefore

$$z(t^*) \leq x(t^*). \tag{2.32}$$

From the definition of the point t^* there follows the inequality

$$z(t^*-) \geq x(t^*-).$$

Moreover, by (2.25) and (2.32) for $t^* < t_0$ and by (2.26) and (2.27) for $t^* = t_0$, we have

$$z(t^*-) \leq x(t^*) - \varphi(t^*, x(t^*))d_1\alpha(t^*) = x(t^*-).$$

Consequently,

$$z(t^*-) = x(t^*-). \tag{2.33}$$

Let now ε be an arbitrarily small positive number. Then in view of (2.24) and the definition of $\widetilde{\varphi}$ and χ, we have

$$x(t_*) = x(t^* - \varepsilon) + \int\limits_{t^*-\varepsilon}^{t_*} \varphi(s, z(s))d\,\alpha(s) \geq$$
$$\geq x(t^* - \varepsilon) + z(t_*) - z(t^* - \varepsilon).$$

Passing in the latter inequality to limit as $\varepsilon \to 0$, by (2.33) we get

$$x(t_*) \geq z(t_*).$$

The contradiction obtained by (2.30) shows that

$$z(t) \leq x(t) \quad \text{for} \quad t \in [a, t] \cap I_0.$$

Analogously we can show the latter inequality for $t \in [a, b]$. Therefore, according to the definition of $\widetilde{\varphi}$, the function $x : I_0 \to R$ will be the solution of the problem (2.22), (2.23) satisfying the condition (2.28) and, moreover, $I_0 = [a, b]$. The lemma is proved. ■

(Wirtinger's inequalities). Let α and β be nondecreasing functions from $[a, b]$ into R, and let the function α have not more than a finite number of discontinuity points. Then the estimates

$$\int\limits_a^b \bigg(\int\limits_{t_0}^t v(\tau)d\,s_0(\alpha)(\tau)\bigg)^2 d\,s_0(\alpha)(t) \leq \gamma_0 \int\limits_a^b v^2(t)d\,s_0(\alpha)(t) \tag{2.34}$$

and

$$\int_a^b \Big(\int_{t_0}^t v(\tau)d\,s_m(\alpha)(\tau)\Big)^2 d\,s_j(\beta)(t) \le \gamma_{mj}\int_a^b v^2(t)d\,s_m(\alpha)(t)$$

$$(j,m=1,2) \tag{2.35}$$

hold for every $v \in BV\big([a,b],R\big)$ *and* $t_0 \in [a,b]$, *where*

$$\gamma_0 = \Big[\frac{2}{\pi}\big(s_0(\alpha)(b)-s_0(\alpha)(a)\big)\Big]^2, \quad \gamma_{mj} = \frac{1}{4}\mu_{\alpha m\beta j}\sin^{-2}\frac{\pi}{4n_{\alpha m}+2}$$

$$(j,m=1,2).$$

In addition, these estimates are unimprovable.

Proof. Obviously, it suffices to verify the conditions (2.34) and (2.35) for $t_0 = a$ and $t_0 = b$. Assume $t_0 = b$. Let us show (2.34). Without loss of generality we may assume that

$$s_0(\alpha)(t) < s_0(\alpha)(b) \quad \text{for} \quad a \le t < b.$$

Put

$$u(t) \equiv \int_b^t v(\tau)d\,s_0(\alpha)(\tau) \quad \text{and} \quad \widetilde{\alpha}(t) \equiv \frac{1}{\sqrt{\gamma_0}}\big[s_0(\alpha)(t)-s_0(\alpha)(b)\big].$$

Let ε be a small positive number. It is easily seen that

$$\int_a^{t_0-\varepsilon} \big(u(t)\operatorname{ctg}\widetilde{\alpha}(t)-\sqrt{\gamma_0}v(t)\big)^2 d\,s_0(\alpha)(t) = -\int_a^{t_0-\varepsilon} u^2(t)d\,s_0(\alpha)(t) +$$

$$+\gamma_0\int_a^{t_0-\varepsilon} v^2(t)d\,s_0(\alpha)(t) - \sqrt{\gamma_0}u^2(t_0-\varepsilon)\operatorname{ctg}\widetilde{\alpha}(t_0-\varepsilon).$$

Consequently,

$$\int_a^{t_0-\varepsilon} u^2(t)d\,s_0(\alpha)(t) \le \gamma_0\int_a^{t_0-\varepsilon} v^2(t)d\,s_0(\alpha)(t) - \sqrt{\gamma_0}u^2(t_0-\varepsilon)\operatorname{ctg}\widetilde{\alpha}(t_0-\varepsilon).$$

Passing in the latter inequality to the limit as $\varepsilon \to \infty$, we obtain (2.34).

Let us show (2.35). We have

$$\int\limits_a^b \Big(\int\limits_b^t v(\tau) d\,s_m(\alpha)(\tau)\Big)^2 d\,s_j(\beta)(t) = \sum_{a\le t\le b} \Big(\int\limits_b^t v(\tau) d\,s_m(\alpha)(\tau)\Big)^2 d_j\beta(t) =$$
$$= \sum_{l=1}^{n_{\alpha m}} \Big(\sum_{k=l}^{n_{\alpha m}} v\big(t_{\alpha mk}\big) d_m\alpha\big(t_{\alpha mk}\big)\Big)^2 \Big(d_j\beta(t_{\alpha m,l+m-2}) +$$
$$+ \sum_{t_{\alpha m,l-1}<t<t_{\alpha ml}} d_j\beta(t)\Big) \le \nu_{\alpha m\beta j} \sum_{l=1}^{n_{\alpha m}+1} \omega_m^2(l) \quad (j,m=1,2), \tag{2.36}$$

where

$$\omega_m(l) = \sum_{k=l}^{n_{\alpha m}} v(t_{\alpha mk}) d_m\alpha(t_{\alpha mk}) \quad (l=1,\dots,n_{\alpha m}),$$
$$W_m(n_{\alpha m}+1) = 0 \quad (m=1,2).$$

Moreover, according to the discrete analogue of Wirtinger's inequality [12,13], we obtain

$$\sum_{l=1}^{n_{\alpha m}+1} \omega_m^2(l) \le \frac{1}{4}\sin^{-2}\frac{\pi}{4n_{\alpha m}+2} \sum_{l=2}^{n_{\alpha m}+1} \big(\omega_m(l)-\omega_m(l-1)\big)^2 \le$$
$$\le \frac{1}{4}\sin^{-2}\frac{\pi}{4n_{\alpha m}+2} \sum_{l=2}^{n_{\alpha m}+1} \big(v(t_{\alpha m,l-1}) d_m\alpha(t_{\alpha m,l-1})\big)^2 \le$$
$$\le \frac{1}{4}\mu_{\alpha m}\sin^{-2}\frac{\pi}{4n_{\alpha m}+2} \int\limits_a^b v^2(t) d\,s_m(\alpha)(t) \quad (m=1,2).$$

Using this, from (2.36) we deduce (2.35). The proof of (2.34) and (2.35) is similar for $t_0 = a$.

Finally, it should be noted that the inequality (2.34) transforms into the equality for $t_0 = a$ and $v(t) = \gamma_0^{-1}\cos\big(\gamma_0\big[s_0(\alpha)(t) - s_0(\alpha)(a)\big]\big)$. As for the inequality (2.35), it transforms into the equality if $a = t_0 = 0$, $b = m$, $\alpha(t) = \beta(t) = k-1$ for $k-1 \le t < k$ $(k=1,\dots,m)$, $\alpha(m) = \beta(m) = m$, $v(k) = \sin\frac{\pi k}{2m+1} - \sin\frac{\pi(k-1)}{2m+1}$ $(k=1,\dots,m)$ and $v(t) = 0$ for $t \in [0,m]\backslash\{1,\dots,m\}$. The lemma is proved. ■

Let the conditions (1.3) *and* (1.15) *hold, where* c_{il} $(i \ne l;\ i,l = 1,\dots,n)$ *are functions nondecreasing on* $[a,b]$ *and continuous at the point* t_i, $c_{ii} \in BV\big([a,b],R\big)$ $(i=1,\dots,n)$, α_l $(l=1,\dots,n)$ *are functions nondecreasing on* $[a,b]$ *and having not more than a finite number of discontinuity points,* $h_{il} \in L^\mu\ \big([a,b],R_+;\alpha_l\big)$ $(i,l=1,\dots,n)$, $1 \le \mu \le +\infty$. *Moreover,* *let*

$$\varphi_{0i}(x_1,\dots,x_n)=\sum_{\sigma=0}^{2}\sum_{k=1}^{n} l_{\sigma ik}\|x_k\|_{\nu,s_\sigma(\alpha_k)},\, l_{\sigma ik}\in R_+$$
$$(\sigma=0,1,2;\, i,k=1,\dots,n),$$

$\dfrac{1}{\mu}+\dfrac{2}{\nu}=1$ *and let the module of every characteristic value of the* $3n\times 3n$*-matrix* $\mathcal{H}=(\mathcal{H}_{j+1,\sigma+1})_{j,\sigma=0}^{2}$ *appearing on Corollary* 1.1 *be less than* 1. *Then the condition* (1.12) *holds.*

Proof. Let $(x_i)_{i=1}^n$ be an arbitrary nonnegative solution of the problem (1.4), (1.5). By (1.15) and Hölder's inequality, we have

$$x_i(t)\le\sum_{\sigma=0}^{2}\sum_{k=1}^{n}\Big(l_{\sigma ik}\|x_k\|_{\nu,s_\sigma(\alpha_k)}+\|h_{ik}\|_{\mu,s_\sigma(\alpha_k)}\Big|\int_{t_i}^{t}|x_k(\tau)|^{\frac{\nu}{2}}ds_\sigma(\alpha_k)(\tau)\Big|^{\frac{2}{\nu}}\Big)$$
$$\text{for}\quad t\in[a,b]\quad(i=1,\dots,n). \tag{2.37}$$

This, in view of Minkowski's inequality, implies

$$\|x_i\|_{\nu,s_j(\alpha_i)}\le\sum_{\sigma=0}^{2}\sum_{k=1}^{n}\Big(l_{\sigma ik}\big[s_j(\alpha_i)(b)-s_j(\alpha_i)(a)\big]^{\frac{1}{\nu}}\|x_k\|_{\nu,s_\sigma(\alpha_k)}+$$
$$+\|h_{ik}\|_{\mu,s_\sigma(\alpha_k)}\Big[\int_a^b\Big|\int_{t_i}^{t}|x_k(\tau)|^{\frac{\nu}{2}}ds_\sigma(\alpha_k)(\tau)\Big|^2 ds_j(\alpha_i)(t)\Big]^{\frac{1}{\nu}}\Big)$$
$$(j=0,1,2;\, i=1,\dots,n). \tag{2.38}$$

On the other hand, by virtue of Hölder's inequality in the case $\sigma^2+j^2+(i-k)^2>0$, $j=0$ and by (2.34) and (2.35) in the other cases, we have

$$\Big[\int_a^b\Big|\int_{t_i}^{t}|x_k(\tau)|^{\frac{\nu}{2}}ds_\sigma(\alpha_k)(\tau)\Big|^2 ds_j(\alpha_i)(t)\Big]^{\frac{1}{\nu}}\le$$
$$\le\lambda_{k\sigma ij}\Big[\int_a^b\big|x_k(\tau)\big|^{\nu}ds_\sigma(\alpha_k)(\tau)\Big]^{\frac{1}{\nu}}\quad(j,\sigma=1,2;\, i,k=1,\dots,n).$$

This and (2.38) yield

$$\|x_i\|_{\nu,s_j(\alpha_i)}\le\sum_{\sigma=0}^{2}\sum_{k=1}^{n}\big(\xi_{ij}l_{\sigma ik}+\lambda_{k\sigma ij}\|h_{ik}\|_{\mu,s_{\sigma(\alpha_k)}}\big)\|x_k\|_{\nu,s_{\sigma(\alpha_k)}}$$
$$(j=0,1,2;\, i=1,\dots,n). \tag{2.39}$$

Therefore

$$\big(I_{3n} - \mathcal{H}\big)r \le 0, \tag{2.40}$$

where $r \in R^{3n}$ is the vector with the components

$$r_{i+nj} = \|x_i\|_{\nu, s_j(\alpha_i)} \quad (j = 0, 1, 2;\ i = 1, \dots, n).$$

From (2.40), we obtain that $r = 0$ since the module of characteristic values of the matrix $\mathcal{H}$ is less than 1. Using (2.37), we see that $x_i(t) \equiv 0$ $(i = 1, \dots, n)$. Consequently, the problem (1.4), (1.5) has no nontrivial nonnegative solution. The lemma is proved. ■

Let the conditions (1.3), (1.19),

$$\begin{gathered}\varphi_{0i}(x_1, \dots, x_n) = c_i x_i(\tau_i) \quad (i = 1, \dots, n),\\ c_{il}(t) - c_{il}(s) \le \eta_{il}\big[\alpha_i(t) - \alpha_i(s)\big] + \beta_{il}(t) - \beta_{il}(s)\\ \textit{for} \quad a \le s < t < t_i \quad \textit{and} \quad t_i < s \le t \le b \quad (i, l = 1, \dots, n)\end{gathered} \tag{2.41}$$

and

$$d_j \alpha_i(t_i) \le 0 \quad (j = 1, 2;\ i = 1 \dots, n) \tag{2.42}$$

hold, where $\eta_{il} \in R_+$ $(i \ne l;\ i, l = 1, \dots, n)$, $\eta_{ii} < 0$ $(i = 1, \dots, n)$, $c_{ii} \in BV([a, b], R)$ $(i = 1, \dots, n)$, β_{il} $(i \ne l)$ *and* α_i $(i, l = 1, \dots, n)$ *are functions from* $[a, b]$ *into* R *respectively nondecreasing and nondecreasing continuous; on every interval* $[a, t_i[$ *and* $]t_i, b]$, c_{il} $(i \ne l)$ *and* β_{ii} $(i, l = 1, \dots, n)$ *are functions from* $[a, b]$ *into* R *respectively nondecreasing continuous at the point* t_i *and nondecreasing, respectively;* $c_i \in R_+$ *and* $\tau_i \in [a, b]$, $\tau_i \ne t_i$ $(i = 1, \dots, n)$, $\gamma_{ij} = 1 + \eta_{ii} d_j \alpha_i(t_i) + d_j \beta_{ii}(t_i)$, $\zeta_{ij} = \exp(\eta_{ii}\mu_{\alpha_i ji}(\tau_i))$ $(j = 1, 2;\ i = 1, \dots, n)$. *Moreover, let the condition* (1.20) *hold and let the real part of every characteristic value of the matrix* $(\xi_{il}^n)_{i,l=1}$ *be negative, where*

$$\begin{gathered}\xi_{il} = \eta_{il}\big[\delta_{il} + (1 - \delta_{il})h_{ij}\big] - \eta_{ii} g_{ilj}\\ \textit{for} \quad (-1)^j(\tau_i - t_i) > 0 \quad (j = 1, 2;\ i, l = 1, \dots, n),\\ g_{ilj} = c_i\gamma_{ij}(1 - c_i\gamma_{ij}\zeta_{ij})^{-1}\mu_{\beta_{il}ji}(\tau_i) + \max\big\{\mu_{\beta_{il}1i}(a), \mu_{\beta_{il}2i}(b)\big\},\end{gathered}$$

$h_{ij} = 1$ *for* $c_i\gamma_{ij} \le 1$ *and* $h_{ij} = 1 + \big(c_i\gamma_{ij} - 1\big)\big(1 - c_i\gamma_{ij}\zeta_{ij}\big)^{-1}$ *for* $c_i\gamma_{ij} > 1$. *Then the condition* (1.12) *holds.*

Proof. Let $(x_i)_{i=1}^n$ be an arbitrary nonnegative solution of the problem (1.4), (1.5). Let $i \in \{1, \dots, n\}$ be fixed. By (2.41),

$$\Big[dz_i(t) - \big(z_i(t) + g_i(t)\big)da_i(t)\Big]\operatorname{sign}(t - t_i) \le 0 \quad \text{for} \quad t \in [a, b], \tag{2.43}$$

where

$$z_i(t) = x_i(t) - g_i(t), \quad a_i(t) = \eta_i\big[\alpha_i(t) - \alpha_i(t_i)\big]\,\mathrm{sign}(t - t_i),$$

$$g_i(t) = \Big[\int\limits_{t_i}^{t} x_i(\tau)d\beta_{ii}(\tau) + \sum_{l\neq i, l=1}^{n} r_l\big(c_{il}(t) - c_{il}(t_i)\big)\Big]\,\mathrm{sign}(t - t_i),$$

$$r_l = \sup\big\{\|x(t)\| : t \in [a, t_l[\cup]t_l, b]\big\} \quad (l = 1, \dots, n).$$

Hence the function z_i satisfies the condition (2.24), where $\varphi(t,z) \equiv z + g_i(t)$, $\alpha(t) \equiv a_i(t)$, $t_0 = t_i$. On the other hand, by (1.4), (1.5), (2.41)–(2.43) it is not difficult to verify that z_i satisfies the conditions (2.25)–(2.27), where $c_0 = c_i x_i(\tau_i)$, since a_i is continuous on $[a, t_i[\cup]t_i, b]$ and c_{il} $(i \neq l;\, l = 1, \dots, n)$ are continuous at the point t_i. Moreover, the Cauchy problem

$$dy(t) = \big(y(t) + g_i(t)\big)da_i(t), \quad y(t_i) = c_i x_i(\tau_i)$$

has the unique solution

$$y_i(t) = \int\limits_{t_i}^{t} g_i(s)da_i(s) + \lambda_i(t)\Big[c_i x_i(\tau_i) - \\ - \int\limits_{t_i}^{t}\Big(\int\limits_{t_i}^{\tau} g_i(s)da_i(s)\Big)d\lambda_i^{-1}(\tau)\Big] \tag{2.44}$$

(see [7, p. 120]), where the function

$$\lambda_i(t) = \big(1 + (-1)^j d_j a_i(t_i)\big)\exp\big(a_i(t) - (-1)^j d_j a_i(t_i)\big) \\ \text{for} \quad (-1)^j(t - t_i) > 0 \quad (j = 1, 2)$$

is the solution of the problem

$$d\lambda(t) = \lambda(t)da_i(t), \quad \lambda(t_i) = 1.$$

Therefore, according to Lemma 2.4 and the formula of integration by parts (see [7, p. 48]), we have

$$x_i(t) \leq g_i(t) + c_i\lambda_i(t)x_i(\tau_i) + \delta_i(t)\varphi_i(t) \quad \text{for} \quad t \neq t_i,\, t \in [a, b], \tag{2.45}$$

where

$$\varphi_i(t) = \lim_{\varepsilon\to 0+} \int\limits_{t_i + \varepsilon\,\mathrm{sign}(t - t_i)}^{t} \delta_i^{-1}(s)g_i(s)da_i(s)$$

and

$$\delta_i(t) = \exp\big(a_i(t) - (-1)^j d_j a_i(t_i)\big) \quad \text{for} \quad (-1)^j(t - t_i) > 0 \quad (j = 1, 2).$$

By the definition of g_i and the condition (1.19), it follows from (2.45) that

$$x_i(t) \le c_i\big(\lambda_i(t) + \delta_i(t) d_j \beta_{ii}(t)\big) x_i(\tau_i) + \delta_i(t)\varphi_i(t) \quad for \quad (-1)^j (t - t_i) > 0 \quad (j = 1, 2), \tag{2.46}$$

since β_{ii} is a nondecreasing function. With regard to (1.19), this implies

$$x_i(\tau_i) \le c_i \gamma_{ij} \delta_i(\tau_i)\varphi_i(\tau_i)\big(1 - c_i\gamma_{ij}\delta_i(\tau_i)\big)^{-1} \quad for \quad (-1)^j(\tau_i - t_i) > 0 \quad (j = 1, 2). \tag{2.47}$$

On the other hand, taking into account (2.41), we have

$$\varphi_i(t) \le \sum_{l \ne i, l=1}^{n} r_l \frac{\eta_{il}}{|\eta_{ii}|}\big(\delta_i^{-1}(t) - 1\big) + \lim_{\varepsilon \to 0+} \sum_{l=1}^{n} r_l \left| \int\limits_{t_i + \varepsilon\, \mathrm{sign}(t - t_i)}^{t} \delta_i^{-1}(s) d\beta_{il}(s) \right| \quad for \quad t \ne t_i,\ t \in [a, b]. \tag{2.48}$$

On the basis of (2.46)–(2.48),

$$r_i \le \frac{h_i}{|\eta_{ii}|} \sum_{l \ne i, l=1}^{n} \eta_{il} r_l + \sum_{l=1}^{n} g_{ilj} r_l \quad \text{for} \quad (-1)^j(\tau_i - t_i) > 0 \quad (j = 1, 2).$$

Hence, in view of (1.20),

$$r_i \le \sum_{l \ne i, l=1}^{n} \frac{\xi_{il}}{|\xi_{ii}|} r_l,$$

i.e., the vector $r = (r_i)_{i=1}^n$ satisfies the inequality (2.40), where $\mathcal{H} = (h_{il}^*)_{i,l}^n$, $h_{ii}^* = 0$ $(i = 1, \dots, n)$, $h_{il}^* = \frac{\xi_{il}}{|\xi_{ii}|}$ $(i \ne l;\ i, l = 1, \dots, n)$. From the conditions imposed upon the matrix $(\xi_{il})_{i,l=1}^n$ it follows that the module of every characteristic value of the matrix $\mathcal{H}$ is less than 1. Therefore, by (2.43), we have $r = 0$, i.e., $x(t) = 0$ for $t \ne t_i$ $(i = 1, \dots, n)$. Moreover, the inequalities

$$x_i(t_i) \le c_i x_i(\tau_i), \quad \tau_i \ne t_i \quad (i = 1, \dots, n)$$

imply that $x_i(t_i) = 0 \quad (i = 1, \dots, n)$. The lemma is proved. ■

§ 3. Proof of Main Results

Proof of Theorem 1.1. Let the conditions of Theorem 1.1 be fulfilled and let the problem (1.1), (1.2) have the unique solution $x^0 = (x_i^0)_{i=1}^n$. Consider the auxiliary problem

$$dz(t) = d\,A^*(t) \cdot f^*(t, z(t)), \tag{3.1}$$

$$h^*(z) = 0, \tag{3.2}$$

where

$$A^*(t) = \begin{pmatrix} A_0(t), & O_n \\ O_n, & B(t) \end{pmatrix}$$

$$f^*(t, z_1, \dots, z_{2n}) = (f^*_{i0}(t, z_1, \dots, z_{2n}))_{i=1}^{2n} \quad \text{on} \quad [a, b] \times R^{2n},$$
$$h^*(z) = (h^*_i(z))_{i=1}^{2n} \quad \text{for} \quad z \in BV([a, b], R^{2n}),$$

$B(t)$ is a diagonal matrix-function with the diagonal elements

$$b_i(t) = [s_0(c_{ii})(t) - s_0(c_{ii})(t_i)] \operatorname{sign}(t - t_i) \quad \text{for} \quad t \in [a, b] \quad (i = 1, \dots, n);$$
$$f^*_{i0}(t, z_1, \dots, z_{2n}) = f_{i0}(t, z_1 + z_{n+1}, \dots, z_n + z_{2n}),$$
$$f^*_{n+io}(t, z_1, \dots, z_{2n}) = 0 \quad \text{on} \quad [a, b] \times R^{2n} \quad (i = 1, \dots, n),$$
$$h^*_i(z_1, \dots, z_{2n}) = z_i(t_i) + z_{n+i}(t_i) - \varphi_i(z_1 + z_{n+1}, \dots, z_n + z_{2n}),$$
$$h^*_{n+i}(z_1, \dots, z_{2n}) = z_{n+i}(t_i) \quad \text{on} \quad BV([a, b], R^{2n}).$$

It is clear that the problem (3.1), (3.2) has the unique solution $z^0 = (z_i^0)_{i=1}^{2n}$, where

$$z_i^0 = x_i^0, \quad z_{n+i}^0 = 0 \quad (i = 1, \dots, n).$$

Let us show that z^0 is strongly isolated in the arbitrary radius $r > 0$.

By Lemma 2.5 from [11], there exists $\rho_* \in]0, +\infty[$ such that

$$\sum_{i=1}^{n} \|x_i\|_S \le \rho_* [\gamma + \frac{1}{n} \|u(\cdot) - u(a)\|_S] \tag{3.3}$$

for every solution $(x_i)_{i=1}^n$ of the problem

$$\left[d_j |x_i(t)| - \operatorname{sign}(t - t_i) \left(\sum_{l=1}^{n} |x_l(t)| dc_{il}(t) + d\, u_i(t) \right) \right] \operatorname{sign}(t - t_i) \le 0$$
$$(i = 1, \dots, n),$$
$$(-1)^j d_j |x_i(t_i)| \le |x_i(t_i)| d_j c_{ii}(t_i) + d_j u_i(t_i) \quad (i = 1, \dots, n); \tag{3.4}$$
$$|x_i(t_i)| \le \varphi_{oi}(|x_1|, \dots, |x_n|) + \eta \quad (i = 1, \dots, n), \tag{3.5}$$

where $\eta \in R_+$ and $u = (u_i)_{i=1}^n \in BV([a, b], R^n)$ are the arbitrary number and the vector-function, respectively. On the other hand, in view of (1.11) there exists a number

$$\rho_0 > nr + \sum_{i=1}^{n} \|x_i^0\|_S$$

such that

$$\rho_* \left[\gamma(\rho) + \frac{1}{n} \left\| \int_a^b d(A^{(1)}(t) + A^{(2)}(t)) \cdot q(t, \rho) \right\| \right] < \rho \quad \text{for} \quad \rho \ge \rho_0. \tag{3.6}$$

Let

$$\chi(s) = \begin{cases} 1 \quad \text{for} \quad |s| \leq \rho_0, \\ 2 - \dfrac{|s|}{\rho_0} \quad \text{for} \quad \rho_0 < |s| < 2\rho_0, \\ 0 \quad \text{for} \quad |s| \geq 2\rho_0; \end{cases}$$

$$\widetilde{\varphi}_i(x_1, \dots, x_n) = \chi\Big(\sum_{k=1}^{n} \|x_k\|_S \Big) \varphi_i(x_1, \dots, x_n) \quad (i = 1, \dots, n)$$

and let $l^*, \widetilde{l}^*$ and l_0^* be the operators from $BV([a,b], R^{2n})$ into R^{2n} defined by

$$l^*(z_1, \dots, z_{2n}) = \begin{pmatrix} (z_i(t_i) + z_{n+i}(t_i))_{i=1}^n \\ (z_{n+i}(t_i))_{i=1}^n \end{pmatrix},$$

$$\widetilde{l}^*(z_1, \dots, z_{2n}) = -\begin{pmatrix} (\widetilde{\varphi}_i(z_1 + z_{n+1}, \dots, z_n + z_{2n}))_{i=1}^n \\ 0 \end{pmatrix},$$

$$l_0^*(z_1, \dots, z_{2n}) = 0.$$

It is not difficult to see that

$$f^*(t,z) = P^*(t)z + q^*(t,z) \quad \text{for} \quad t \in [a,b], \|z - z^0(t)\| < r,$$
$$h^*(z) = l^*(z) + \widetilde{l}^*(z) \quad \text{for} \quad z \in BV([a,b], R^{2n}),$$
$$\alpha^*(t,\rho) < +\infty \quad \text{and} \quad \beta^*(\rho) < +\infty \quad \text{for} \quad t \in [a,b], \rho \in R_+,$$

where

$$P^*(t) = \begin{pmatrix} O_n & O_n \\ I_n & I_n \end{pmatrix}, \quad q^*(t,z) = \chi\bigg(\sum_{k=1}^{n} |z_k + z_{n+k}| \bigg) \times$$

$$\times \begin{pmatrix} f_0(t, z_1 + z_{n+1}, \dots, z_n + z_{2n}) \\ -(z_i + z_{n+i})_{i=1}^n \end{pmatrix} \quad \text{for} \quad t \in [a,b], \quad z = (z_i)_{i=1}^{2n} \in R^{2n};$$

$$\alpha^*(t,\rho) = \max\big\{ |q^*(t,z)| : \|z\| \leq \rho \big\} \quad \text{for} \quad t \in [a,b], \rho \in R_+$$

and

$$\beta^*(\rho) = \sup\big\{ [|\widetilde{l}^*(z)| - l_0^*(z)]_+ : \|z\|_S \leq \rho \big\} \quad \text{for} \quad \rho \in R_+.$$

Consequently, the conditions (a) and (b) of Definition 1.2 are fulfilled for $f^*, P^*, q^*, h^*, l^*, \widetilde{l}^*, l_0^*, \alpha^*$, and β^*. Moreover, the matrix-function P^* satisfies the Opial condition with respect to the triplet $(l^*, l_0^*; A)$ as B^* is a continuous matrix-function.

Let now $z = (z_i)_{i=1}^{2n}$ be an arbitrary solution of the problem

$$dz(t) = d\,A^*(t) \cdot [P^*(t) z^*(t) + q^*(t, z(t))], \tag{3.7}$$

$$l^*(z) + \widetilde{l}^*(z) = 0. \tag{3.8}$$

Then the function $u(t)=(u_i(t))_{i=1}^n$, $u_i(t)=z_i(t)+z_{n+i}(t)$ $(i=1,\dots,n)$ will be the solution of the problem

$$du(t)=d\,B(t)\cdot u(t)\Big[1-\chi\Big(\sum_{k=1}^n|u_k(t)|\Big)\Big]+$$
$$+dA(t)\cdot\chi\Big(\sum_{k=1}^n|u_k(t)|\Big)f(t,u(t)),^2$$
$$u_i(t_i)=\widetilde{\varphi}_i(u)\quad(i=1,\dots,n).$$

But every solution of the latter problem is also a solution of the problem (1.1), (1.2), and the estimate

$$\sum_{m=1}^n\|u_m\|_S\le\rho_0 \tag{3.9}$$

is valid (see the proof of Theorem 1.1 from [11]). Consequently,

$$u(t)\equiv x^0(t), \tag{3.10}$$

since the problem (1.1), (1.2) has the unique solution x^0. On the other hand, by (3.9), (3.10) and the definition of the function χ, from (3.7), (3.8) we have

$$z(t)\equiv z_0(t),$$

i.e., the problem (3.7), (3.8) has no solution differing from z^0. Hence z^0 is strongly isolated in the radius $r>0$.

According to Lemma 2.1, the problem (3.1), (3.2) is $(z^0;r)$-correct.

Let us show that the problem (1.1), (1.2) is $(x^0;r)$-correct.

Let $\varepsilon\in]0,r[$ and

$$\big((A_m,f_m,h_m)\big)_{m=1}^{+\infty}\in W_r\big(A_0,f_0,h;x^0\big).$$

For every natural m, we put

$$A_m^*(t)=\begin{pmatrix}A_m(t), & O_n\\ O_n, & B(t)\end{pmatrix},$$

$$f_m^*(t,z_1,\dots,z_{2n})=(f_{im}^*(t,z_1,\dots,z_{2n}))_{i=1}^{2n}\quad\text{on}\quad[a,b]\times R^{2n},$$
$$h_m^*(z_1,\dots,z_{2n})=(h_{im}^*(z_1,\dots,z_{2n}))_{i=1}^{2n}\quad\text{for}\quad z=(z_i)_{i=1}^{2n}\in BV([a,b],R^{2n}),$$

and

$$\omega_m^*(t,\rho)=(\omega_{im}^*(t,\rho))_{i=1}^{2n}\quad\text{on}\quad[a,b]\times R_+,$$

[2] A vector-function from $BV\big([a,b],R^n\big)$ is said to be a solution of this system if it satisfies the corresponding integral equality.

where

$$
\begin{gathered}
f^*_{im}(t,z_1,\dots,z_{2n}) = f_{im}(t,z_1+z_{n+1},\dots,z_n+z_{2n}),\\
f^*_{n+im}(t,z_1,\dots,z_{2n}) = 0 \quad (i=1,\dots,n),\\
h^*_{im}(z_1,\dots,z_{2n}) = h_{im}(z_1+z_{n+1},\dots,z_n+z_{2n}),\\
h^*_{n+im}(z_1,\dots,z_{2n}) = z_{n+i}(t_i) \quad (i=1,\dots,n),\\
\omega^*_{im}(t,\rho) = \omega_{im}(t,\rho), \omega^*_{n+im}(t,\rho) = 0 \quad (i=1,\dots,n)
\end{gathered}
$$

and $\omega_m = (\omega_{im})_{i=1}^n \in M\big([a,b]\times R_+, R_+^n; A_m\big)$ is the vector-function appearing in the definition of the set $W_r(A_0,f_0,h;x^0)$.

We can easily conclude that

$$
\big((A^*_m,f^*_m,h^*_m)\big)_{m=1}^{+\infty} \in W_r(A^*,f^*,h^*;z^0).
$$

Consequently, by $(z^0;r)$-correctness of the problem (3.1), (3.2) there exists a natural number m_0 such that the problem

$$
d\,z(t) = d\,A^*_m(t)\cdot f^*_m(t,z(t)), \tag{3.1_m}
$$

$$
h^*_m(z) = 0 \tag{3.2_m}
$$

has at least one solution in $U_{2n}(z^0;r)$ and every such solution belongs to the ball $U_{2n}(z^0;\varepsilon)$ for any $m\ge m_0$. On the other hand, if $z=(z_i)_{i=1}^{2n}$ is a solution of the problem(3.1_m), (3.2_m), then $x=(z_i)_{i=1}^n$ will be the solution of the problem (1.1_m), (1.2_m) and conversely, if $x=(x_i)_{i=1}^n$ is a solution of the problem (1.1_m), (1.2_m), then the vector-function $z=(z_i)_{i=1}^{2n}$, where $z_i=x_i$, $z_{n+i}=0$ $(i=1,\dots,n)$ will be a solution of the problem (3.1_m), (3.2_m). Moreover,

$$
\|z-z^0\|_S = \|x-x^0\|_S.
$$

Therefore the problem (1.1), (1.2) is $(x^0;r)$-correct for every $r>0$, too. In view of the arbitrariness of r and Definition 1.4, the problem (1.1), (1.2) is correct. Thus the theorem is proved. ■

Proof of Theorem 1.2. This theorem immediately follows from Theorem 1.1 and from Theorem 1.2 in [11].

Proof of Theorem 1.3. According to Theorem 1.2, the problem (1.1), (1.2) has the unique solution $(x_i)_{i=1}^n$.

Let us show that the Cauchy problem

$$
\begin{gathered}
d\,u(t) = \sum_{k=1}^n f_{k0}(t,z_1(t),\dots,z_{i_0-1}(t),u(t),\\
z_{i_0+1}(t),\dots,z_n(t))\,d\,a_{i_0k0}(t),
\end{gathered} \tag{3.11}
$$

$$
u(t_{i_0}) = c_0 \tag{3.12}
$$

has the unique solution defined on the whole $[a,b]$ for every $i_0 \in \{1,\dots,n\}$, $c_0 \in R$ and $z_k \in BV\big([a,b],R\big)$ $(k \neq i_0; k=1,\dots,n)$. In fact, the latter problem may be written in the form

$$d\overline{u}(t) = d\overline{A}_0(t) \cdot \overline{f}_0\big(t,\overline{u}(t)\big), \tag{3.13}$$

$$\overline{u}_i(t_i) = \overline{\varphi}_i(\overline{u}) \quad (i=1,\dots,n), \tag{3.14}$$

where

$$\overline{u}(t) = \big(\overline{u}_i(t)\big)_{i=1}^n, \quad \overline{u}_i(t) = \delta_{i_0 i} u(t) \quad (i=1,\dots,n),$$
$$\overline{A}_0(t) = \big(\overline{a}_{ik0}(t)\big)_{i,k=1}^n, \quad \overline{a}_{ik0}(t) = \delta_{i_0 i} a_{ik0}(t) \quad (i,k=1,\dots,n);$$
$$\overline{f}_0(t,x_1,\dots,x_n) = \big(\overline{f}_{k0}(t,x_1,\dots,x_n)\big)_{k=1}^n,$$
$$\overline{f}_{k0}(t,x_1,\dots,x_n) = f_{k0}\big(t,z_1(t),\dots,z_{i_0-1}(t),x_{i_0},z_{i_0+1}(t),\dots,z_n(t)\big)$$
$$(k=1,\dots,n),$$
$$\overline{\varphi}_i(\overline{u}) = \delta_{i_0 i} c_0 \quad (i=1,\dots,n).$$

It is easily seen that $\overline{A}_0$, $\overline{f}_0$ and $\overline{\varphi}_i$ $(i=1,\dots,n)$ satisfy conditions of Theorem 1.2. Therefore, by this theorem the problem (3.13), (3.14) has the unique solution. This in its turn implies that the Cauchy problem (3.11), (3.12) has the unique solution defined on $[a,b]$. Consequently, for every $(x_{i0})_{i=1}^n \in BV\big([a,b],R^n\big)$ there exists the unique sequence $(x_{im})_{i=1}^n \in BV\big([a,b],R^n\big)$ $(m=1,2,\dots)$ such that x_{im} is the solution of the problem (1.6), (1.7) for every natural m and $i \in \{1,\dots,n\}$. By virtue of Lemma 2.2 from [11] and (1.13), (1.14), (1.21)–(1.23), the functions

$$y_{im}(t) = \big|x_i(t) - x_{im}(t)\big| \quad (i=1,\dots,n)$$

satisfy inequalities (2.9) and (2.10), where $u_{im}(t) \equiv 0$ and $\gamma_m = 0$ for every m. Therefore, according to Lemma 2.3, the estimates (1.27) are valid, where $r_0 \in]0,+\infty[$ and $\delta \in]0,1[$ are numbers independent of m. Thus the theorem is proved. ■

The Corollaries 1.1–1.6 follow immediately from Theorems 1.1–1.3 with regard to Lemmas 2.6 and 2.7. It should only be noted that we take the following functions as c_{il} $(i,l=1,\dots,n)$:

$$c_{il}(t_i) = 0 \quad (i,l=1,\dots,n),$$
$$c_{il}(t) = |\eta_{il}|\big(\alpha_i(t) - \alpha_i(t_i) - (-1)^j d_j \alpha_i(t_i)\big) + \beta_{il}(t) -$$
$$-\beta_{il}(t_i) - (-1)^j d_j \beta_{il}(t_i) +$$
$$+\delta_{il} \sum_{\sigma=1}^{2} \sum_{k=1}^{n} \big|\alpha_{\sigma ikjl} d_j a_{\sigma ik}(t_i)\big|$$
$$\text{for} \quad (-1)^j (t-t_i) > 0 \quad (j=1,2;\ i,l=1,\dots,n)$$

in Corollaries 1.2, 1.4 and 1.6.

Let us consider Remark 1.3. Let the conditions of Theorem 1.3 be fulfilled. Then for every $(\overline{x}_{i0})_{i=1}^{n} \in BV([a,b],R^n)$, $(\overline{u}_{im})_{i=1}^{n} \in BV([a,b],R^n)$ $(m=1,2,\dots)$ and $(\overline{\gamma}_{im})_{i=1}^{n} \in R^n$ $(m=1,2,\dots)$ there exists a unique sequence of vector-functions $(\overline{x}_{im})_{i=1}^{n} \in BV([a,b],R^n)$ $(m=1,2,\dots)$ such that the vector-function $(\overline{x}_{im})_{i=1}^{n}$ is a solution of the following Cauchy problem

$$d\overline{x}_{im}(t) = \sum_{k=1}^{n} f_{k0}\big(t, \overline{x}_{1m-1}(t), \dots, \overline{x}_{i-1m-1}(t), \overline{x}_{im}(t), \overline{x}_{i+1m-1}(t),$$
$$\dots, \overline{x}_{nm-1}(t)\big)\, da_{ik0}(t) + d\overline{u}_{im}(t) \quad (i=1,\dots,n),$$
$$\overline{x}_{im}(t_i) = \varphi_i\big(\overline{x}_{1m-1}, \dots, \overline{x}_{nm-1}\big) + \overline{\gamma}_{im} \quad (i=1,\dots,n)$$

for every natural m. On the other hand, by Lemma 2.2 from [11] and (1.13), (1.14), (1.21)–(1.23) the functions

$$y_{im}(t) = \big|x_{im}(t) - \overline{x}_{im}(t)\big| \quad (i=1,\dots,n)$$

satisfy the inequalities (2.9) and (2.10), where

$$u_{im}(t) = \bigvee_a^t(\overline{u}_{im}) \quad (i=1,\dots,n), \quad \gamma_m = \sum_{i=1}^{n} |\overline{\gamma}_{im}|$$

for every m. Therefore, according to Lemma 2.3,

$$\sum_{i=1}^{n} \|x_{im} - \overline{x}_{im}\|_S \le \rho \sum_{k=0}^{n} \eta_k \delta^{m-k} \quad (m=1,2,\dots), \tag{3.15}$$

where $(x_{im})_{i=1}^{n}$ $(m=1,2,\dots)$ is the sequence appearing in Theorem 1.3,

$$\eta_0 = \sum_{i=1}^{n} \|x_{i0} - \overline{x}_{i0}\|_S, \quad \eta_m = \sum_{i=1}^{n} \left(|\overline{\gamma}_{im}| + \frac{1}{n} \bigvee_a^b (\overline{u}_{im})\right) \quad (m=1,2,\dots)$$

and $\rho \in]0,+\infty[$ and $\delta \in]0,1[$ are numbers independent of m, $\overline{u}_{im}$ and $\overline{\gamma}_{im}$. Presuppose now that

$$\lim_{m\to+\infty} \eta_m = 0.$$

Then for every positive number ε there exists a natural number m_ε such that

$$\delta^m < \varepsilon \quad \text{and} \quad \eta_m < \varepsilon \quad \text{for} \quad m \ge m_\varepsilon.$$

This implies

$$\sum_{k=0}^{m} \eta_k \delta^{m-k} = \sum_{k=0}^{m_\varepsilon} \eta_k \delta^{m-k} + \sum_{k=m_\varepsilon+1}^{m} \eta_k \delta^{m-k} < \rho_0 \sum_{k=0}^{m_\varepsilon} \delta^{m-k} +$$
$$+\varepsilon \sum_{k=m_\varepsilon+1}^{m} \delta^{m-k} \le \varepsilon \frac{\rho_0 + 1}{1-\delta} \quad \text{for} \quad m > 2m_\varepsilon,$$

where ρ_0 is a number such that

$$\eta_k < \rho_0 \quad (k = 1, 2, \dots).$$

Consequently, by (3.15)

$$\lim_{m\to+\infty} \|x_{im} - \overline{x}_{im}\|_S = 0 \quad (i = 1, \dots, n).$$

§ 4. On the Numerical Solution of the Multipoint Boundary Value Problem for a System of Ordinary Differential Equations

Let $E(N_m, R^n)$ and $E(\widetilde{N}_m, R^n)$ be the spaces of all vector-functions $y : N_m \to R^n$ and $y : \widetilde{N}_m \to R^n$ with the norms

$$\|y\|_m = \max\big\{\|y(k)\| : k \in N_m\big\}$$

and

$$\|y\|_{\widetilde{m}} = \max\big\{\|y(k)\| : k \in \widetilde{N}_m\big\},$$

respectively, where $N_m = \{1, \dots, m\}$, $\widetilde{N}_m = \{0, \dots, m\}$ $(m = 1, 2, \dots)$.

$$E(\widetilde{N}_m, R^n_+) = \big\{y \in E(\widetilde{N}_m, R^n) : y(k) \in R^n_+ \quad \text{for} \quad k \in \widetilde{N}_m\big\}.$$

Let Δ_1 and Δ_2 be the first order difference operator defined by the equalities

$$\Delta_1 y(k) = y(k) - y(k-1) \quad \text{for} \quad y \in E(\widetilde{N}_m, R^n)\ (k = 1, \dots, m)$$

and

$$\Delta_2 y(k) = y(k+1) - y(k) \quad \text{for} \quad y \in E(\widetilde{N}_m, R^n) \quad (k = 0, \dots, m-1),$$

respectively.

By $M(N_m \times R_+, R^n_+)$ we denote the set of all vector-functions $\omega : N_m \times R_+ \to R^n_+$ such that $\omega(k, \cdot)$ is a continuous nondecreasing vector-function, and $\omega(k, 0) = 0$ for $k \in N_m$.

Let $\widetilde{C}([a, b], R^{n\times m})$ be the set of all matrix-functions $X : [a, b] \to R^{n\times m}$ with absolutely continuous components.

For system of ordinary differential equations

$$\frac{dx_i(t)}{dt} = f_i(t, x_1(t), \dots, x_n(t)) \quad (i = 1, \dots, n) \tag{4.1}$$

consider the multipoint boundary value problem

$$x_i(t_i) = \varphi_i(x_1, \dots, x_n) \quad (i = 1, \dots, n), \tag{4.2}$$

where $t_1, \dots, t_n \in [a, b]$, $f = (f_i)_{i=1}^n \in K([a, b]\times R^n, R^n)$ and $\varphi_i : BV_S([a, b], R^n) \to R$ $(i = 1, \dots, n)$ are continuous functionals, in general, nonlinear.

The system (4.1) is a special case of the system (1.1), when $A^{(2)}(t) \equiv 0$ and $A^{(1)}(t)$ is the diagonal matrix-function whose every diagonal element equals $t_0 t$.

It is clear that a vector-function $x^0 \in BV([a,b],R^n)$ is a solution of the system (4.1) in the sense described in Section 1 if and only if $x^0 \in \widetilde{C}([a,b],R^n)$ and satisfies the system (4.1) for almost every $t \in [a,b]$.

Along with the problem (4.1), (4.2) consider the difference scheme

$$\Delta_j y_i(k) = \frac{1}{m} g_{jim}\big(k, y_1(k), \dots, y_n(k)\big) \quad (k+j-1 \in N_m,\ i \in N_{jm};\ j=1,2), \tag{4.1$_m$}$$

$$y_i(k_{im}) = \psi_{im}(y_1,\dots,y_n) \quad (i=1,\dots,n) \tag{4.2$_m$}$$

$(m = 2,3,\dots)$, where $k_{im} \in \widetilde{N}_m$ $(i=1,\dots,n)$, $N_{1m} \cup N_{2m} = N_n$, $N_{1m} \cap N_{2m} = \varnothing$, $g_{jim}(k,\cdot) : R^n \to R$ $(k \in \widetilde{N}_m,\ i \in N_{jm};\ j=1,2)$ and $\psi_{im} : E(\widetilde{N}_m, R^n) \to R$ $(i=1,\dots,n)$ are continuous functionals, in general, nonlinear.

Let $A_m = (a_{ilm})_{i,l=1}^n : [a,b] \to R^{n\times n}$ $(m=2,3,\dots)$, $f_m = (f_{im})_{i=1}^n : [a,b]\times R^n \to R^n$ $(m=2,3,\dots)$, $t_{im} \in [a,b]$ $(i=1,\dots,n;\ m=2,3,\dots)$ and $\varphi_{im} : BV_S([a,b],R^n) \to R$ $(i=1,\dots,n;\ m=2,3,\dots)$ be the matrix-functions, the vector-functions, the points and the functionals, respectively, defined by the equalities

$$a_{ilm}(t) = 0 \quad \text{for} \quad t \in [a,b] \quad (i \neq l;\ i,l=1,\dots,n),$$
$$a_{iim}(t) = k \quad \text{for} \quad t \in I_{jkm} \cap [a,b] \quad (k=0,\dots,m;\ i=1,\dots,n;\ j=1,2), \tag{4.3}$$

$$f_{im}(t,x) = \begin{cases} 0 & \text{for} \quad t \in I_{10m}\cap[a,b],\ x \in R^n \ (i \in N_{1m}),\\ 0 & \text{for} \quad t \in I_{2mm}\cap[a,b],\ x \in R^n \ (i \in N_{2m}),\\ \frac{1}{m} g_{jim}(k,x) & \text{for} \quad t \in I_{jkm}\cap[a,b],\ x \in R^n \end{cases} \tag{4.4}$$

$$(k+j-1 \in N_m;\ i \in N_{jm};\ j=1,2),$$
$$t_{im} = \tau_{k_{im}m} \quad (i=1,\dots,n), \tag{4.5}$$
$$\varphi_{im}(x_1,\dots,x_n) = \psi_{im}(y_1,\dots,y_n) \quad \text{for} \quad (x_l)_{l=1}^n \in BV([a,b],R^n) \quad (i=1,\dots,n), \tag{4.6}$$

where $I_{1km} = \big[\tau_{km} - \frac{\tau_m}{2},\ \tau_{km} + \frac{\tau_m}{2}\big[$ $(k=0,\dots,m)$, $I_{2km} = \big]\tau_{km} - \frac{\tau_m}{2},\ \tau_{km} + \frac{\tau_m}{2}\big]$ $(k=0,\dots,m)$, $\tau_{km} = a + k\tau_m$ $(k=0,\dots,m)$, $\tau_m = \dfrac{b-a}{m}$ and

$$y_i(k) = x_i(\tau_{km}) \quad \text{for} \quad (x_l)_{l=1}^n \in BV([a,b],R^n) \quad (k=0,\dots,m;\ i=1,\dots,n). \tag{4.7}$$

The following lemma is evident.

Let $m \in \{2,3,\dots\}$ be fixed and let a vector-function $(y_i)_{i=1}^n \in E(\widetilde{N}_m, R^n)$ be a solution of the problem (4.1_m), (4.2_m). Then the vector-function $x = (x_i)_{i=1}^n : [a,b] \to R^n$ defined by the equalities

$$x_i(t) = y_i(k) \quad \textit{for} \quad t \in I_{jkm} \cap [a,b] \quad (k+j-1 \in N_m;\ i \in N_{jm};\ j=1,2),$$

will be a solution of the problem (1.1_m), (1.2_m), where the matrix-function $A_m = (a_{ilm})_{i,l=1}^n : [a,b] \to R^{n\times n}$, the vector-function $f_m = (f_{im})_{i=1}^n : [a,b] \times R^n \to R^n$, the points t_{im} $(i=1,\dots,n)$ and the functionals $\varphi_{im} : BV_S([a,b],R^n) \to R$ $(i=1,\dots,n)$ are defined by the equalities (4.3)–(4.6), respectively. On the contrary, if a vector-function $x = (x_i)_{i=1}^n$ is a solution of the problem (1.1_m), (1.2_m), then the vector-function $(y_i)_{i=1}^n$ defined by (4.7) will be a solution of the problem (4.1), (4.2), where

$$g_{jim}(k,x) = m f_{im}(\tau_{km}, x) \quad \textit{for} \quad k+j-1 \in N_m,\ x \in R^n$$
$$(i \in N_{jm};\ j=1,2).$$

Will give the following definition from [2].

We will say that the pair $\big((q_{il})_{i,l=1}^n;\ (\varphi_{0i})_{i=1}^n\big)$ consisting of a matrix-function $(q_{il})_{i,l=1}^n \in L\,\big([a,b],R^{n\times n}\big)$ and of a positively homogeneous operator $(\varphi_{0i})_{i=1}^n : BV_S\big([a,b],R_+^n\big) \to R_+^n$ belongs to the set $\widetilde{U}(t_1,\dots,t_n)$ if

$$q_{il}(t) \ge 0 \quad \text{for} \quad t \in [a,b] \quad (i \ne l;\ i,l=1,\dots,n)$$

and the problem

$$x_i'(t)\,\mathrm{sign}(t-t_i) \le \sum_{l=1}^n q_{il}(t)x_l(t) \quad (i=1,\dots,n),$$
$$x_i(t_i) \le \varphi_{0i}\big(|x_1|,\dots,|x_n|\big) \quad (i=1,\dots,n)$$

has no nontrivial nonnegative absolutely continuous solution.

It is evident that if $(c_{il})_{i,l=1}^n \in \widetilde{C}\big([a,b],\,R^{n\times n}\big)$, then the condition (1.12) is fulfilled if and only if

$$\big((q_{il})_{i,l=1}^n;(\varphi_{0i})_{i=1}^n\big) \in \widetilde{U}(t_1,\dots,t_n), \tag{4.8}$$

where

$$q_{il}(t) \equiv c_{il}'(t) \quad (i,l=1,\dots,n).$$

Let $x^0 \in \widetilde{C}\big([a,b],R^n\big)$ and $r \in]0,+\infty[$ be arbitrary. By $y_m(x^0;r)$ we denote the set of all solutions $y = (y_i)_{i=1}^n$ of the problem (4.1_m), (4.2_m) such that

$$\max\big\{\|y(k) - x^0(\tau_{km})\| : k \in \widetilde{N}_m\big\} < r$$

for every $m \in \{2,3,\dots\}$.

The difference scheme (4.1_m), (4.2_m) $(m = 2,3,\dots)$ is said to be converging to the vector-function $x^0 \in \widetilde{C}([a,b],R^n)$ in the radius r if for every $\varepsilon \in]0,r[$ there exists a natural number m_0 such that

$$Y_m(x^0;r) \neq \varnothing$$

and

$$Y_m(x^0;r) \subset Y_m(x^0;\varepsilon)$$

for every $m \geq m_0$.

The difference scheme (4.1_m), (4.2_m) $(m = 2,3,\dots)$ is said to be converging to the solution of the problem (4.1), (4.2) if the latter problem has the unique solution x^0 and this difference scheme converges to x^0 in every radius $r > 0$.

Let $h(x) = \big(h_i(x)\big)_{i=1}^n$, $h_i(x) = x_i(t_i) - \varphi_i(x)$ $(i = 1,\dots,n)$ and $h_m(x) = \big(h_{im}(x)\big)_{i=1}^n$, $h_{im}(x) = x_i(\tau_{k_{im}m}) - \psi_{im}(y_1,\dots,y_n)$ $(i = 1,\dots,n;\ m = 2,3,\dots)$ for $x = (x_l)_{l=1}^n \in BV([a,b],R^n)$, where $y_l : \widetilde{N}_m \to R$ $(l = 1,\dots,n)$ are defined by (4.7). Let $g_m = (g_{im})_{i=1}^n : \widetilde{N}_m \times R^n \to R^n$ $(m = 2,3,\dots)$ be a vector-function such that

$$g_{im}(k,x) = g_{jim}(k,x) \quad \text{for} \quad k \in \widetilde{N}_m,\ x \in R^n \quad (i \in N_{jm};\ j = 1,2).$$

By $W(f,h)$ we denote the set of all sequences $(g_m,h_m)_{m=2}^{+\infty}$ such that

a)

$$\lim_{m\to+\infty} \max\left\{\left|\frac{1}{m}\sum_{l=\sigma+1}^{k} g_{jim}(l,x) - \int\limits_{\tau_{\sigma m}}^{\tau_{km}} f_i(\tau,x)d\tau\right| : \sigma < k;\right.$$
$$\left.\sigma + j - 1,\ k + j - 1 \in N_m\right\} = 0 \quad \text{for every} \quad x \in R^n \quad (i \in N_{jm};\ j = 1,2);$$

b) $\lim\limits_{m\to+\infty} h_{im}(x) = h_i(x)$ uniformly on $U_n(0;r)$ $(i = 1,\dots,n)$ for every $r > 0$;

c) there exist sequences $(\omega_{jim})_{i=1}^n \in M(N_m \times R_+, R_+^n)$ $(m = 2,3,\dots;\ j = 1,2)$ such that

$$\sup\left\{\frac{1}{m}\sum_{k=2-j}^{m-j+1} \omega_{jim}(k,r) : m = 2,3,\dots\right\} < +\infty \quad \text{for} \quad r > 0 \quad (j = 1,2),$$

$$\lim_{s\to0+} \sup\left\{\frac{1}{m}\sum_{k=2-j}^{m-j+1} \omega_{jim}(k,s) : m = 2,3,\dots\right\} = 0 \quad (j = 1,2)$$

and

$$\big|g_{jim}(k,x) - g_{jim}(k,y)\big| \leq \omega_{jim}\left(k, \|x-y\|\right) \quad \text{for} \quad k + j - 1 \in N_m$$
$$\text{and} \quad x,y \in R^n \quad (j = 1,2;\ i = 1,\dots,n;\ m = 2,3,\dots).$$

Theorems 4.1, 4.2 and 4.2′ and Corollaries 4.1–4.4, 4.1′ and 4.3′ below follow immediately from Theorems 1.1, 1.2 and from Corollaries 1.1–1.4 if we use Lemma 4.1.

Let the inequalities

$$f_i(t, x_1, \dots, x_n) \operatorname{sign}\big[(t - t_i)x_i\big] \le \sum_{l=1}^{n} p_{il}(t)|x_l| + q_i\big(t, \|x\|\big),$$
$$(i = 1, \dots, n) \tag{4.9}$$

and (1.10) *be fulfilled on* $[a,b] \times R^n$ *and* $BV\big([a,b], R^n\big)$, *respectively and let,*

$$\int_s^t p_{il}(\tau)d\tau \le c_{il}(t) - c_{il}(s) \quad \textit{for} \quad a \le s \le t \le b \quad (i, l = 1, \dots, n), \tag{4.10}$$

where $p_{il} \in L\big([a,b], R\big)$ $(i, l = 1, \dots, n)$, $q_i \in K\big([a,b] \times R_+, R_+\big)$ $(i = 1, \dots, n)$ *are functions nondecreasing in the second variable,* $\gamma \in C(R_+, R_+)$,

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b q_i(t, \rho)dt = 0 \quad (i = 1, \dots, n), \quad \lim_{\rho \to +\infty} \frac{\gamma(\rho)}{\rho} = 0 \tag{4.11}$$

and the matrix-function $(c_{il})_{i,l=1}^{n} \in BV\big([a,b], R^{n \times n}\big)$ *is such that the condition* (1.12) *holds. Moreover,let*

$$(g_m, h_m)_{m=2}^{+\infty} \in W(f, h). \tag{4.12}$$

If the problem (4.1), (4.2) *has no more than one solution, then the difference scheme* (4.1_m), (4.2_m) $(m = 2, 3, \dots)$ *converges to the solution of the problem* (4.1), (4.2).

4.1′. *Let the inequalities* (4.9) *and* (1.10) *be fulfilled on* $[a,b] \times R^n$ *and* $BV\big([a,b], R^n\big)$, *respectively, and let the conditions* (4.11), (4.12) *and*

$$p_{il}(t) \le q_{il}(t) \quad \textit{for} \quad t \in [a,b] \quad (i, l = 1, \dots, n) \tag{4.13}$$

hold, where $p_{il} \in L\big([a,b], R\big)$ $(i, l = 1, \dots, n)$, $q_i \in K\big([a,b] \times R_+, R_+\big)$ $(i = 1, \dots, n)$ *are functions nondecreasing in the second variable,* $\gamma \in C(R_+, R_+)$, q_{il} *and* φ_{0i} $(i, l = 1, \dots, n)$ *satisfy the condition* (4.8). *If the problem* (4.1), (4.2) *has no more than one solution, then the difference scheme* (4.1_m) (4.2_m) $(m = 2, 3, \dots)$ *converges to the solution of the problem* (4.1), (4.2).

Let the inequalities (4.9) *and* (1.16) *be fulfilled on* $[a,b] \times R^n$ *and* $BV\big([a,b], R^n\big)$, *respectively, and let the conditions* (1.3), (1.15), (4.10)–(4.12) *hold, where* $p_{il} \in L\big([a,b], R\big)$ $(i, l = 1, \dots, n)$, $q_i \in K\big([a,b] \times R_+, R_+\big)$ $(i = 1, \dots, n)$ *are functions nondecreasing in the second variable,* $l_{\sigma ik} \in$

R_+ $(\sigma = 0,1,2;\ i,k = 1,\dots,n)$, $\gamma \in C(R_+,R_+)$, $c_{il}(i \neq l;\ i,l = 1,\dots,n)$ *are functions nondecreasing on* $[a,b]$ *and continuous at the point* t_i, $c_{ii} \in BV\big([a,b],R\big)$ $(i = 1,\dots,n)$; $\alpha_l(l = 1,\dots,n)$ *are functions nondecreasing on* $[a,b]$ *and having not more than a finite number of discontinuity points;* $h_{il} \in L^\mu\big([a,b],R_+;\ \alpha_l\big)$ $(i,l = 1,\dots,n)$, $1 \le \mu \le +\infty$, $\frac{1}{\mu}+\frac{2}{\nu} = 1$. *Moreover, let the module of every characteristic value of the* $3n \times 3n$- *matrix* $\mathcal{H} = \big(\mathcal{H}_{j+1,\sigma+1}\big)_{j,\sigma=0}^{2}$ *appearing in Corollary* 1.1 *or of the* $n\times n$-*matrix appearing in Remark* 1.2 *be less than* 1. *Then the conclusion of Theorem* 4.1 *is true.*

If $\alpha_l(t) \equiv t$ $(l = 1,\dots,n)$, then Corollary 4.1 has the following form.

4.1$'$. *Let the inequalities* (4.9) *and*

$$\big|\varphi_i(x_1,\dots,x_n)\big| \le \sum_{k=1}^{n} l_{ik}\|x_k\|_{L^\nu} + \gamma\Big(\sum_{k=1}^{n}\|x_k\|_S\Big)$$
$$(i = 1,\dots,n)^3$$

be fulfilled on $[a,b]\times R^n$ *and* $BV\big([a,b],R^n\big)$, *respectively, and let the conditions* (4.11) *and* (4.12) *hold, where* $p_{il} \in L^\mu\big([a,b];R_+\big)$ $(i,l = 1,\dots,n)$, $l_{ik} \in R_+$ $(i,k = 1,\dots,n)$, $1 \le \mu \le +\infty$, $\frac{1}{\mu}+\frac{2}{\nu} = 1$, $q_i \in K\big([a,b]\times R_+,R_+\big)$ $(i = 1,\dots,n)$ *are the functions nondecreasing with respect to the second variable,* $\gamma \in C(R_+,R_+)$. *Moreover, let the module of every characteristic value of the matrix*

$$\left((b-a)^{\frac{1}{\nu}} l_{ik} + \left[\frac{2(b-a)}{\pi}\right]^{\frac{2}{\nu}}\|p_{ik}\|_{L^\nu}\right)_{i,k=1}^{n} \tag{4.14}$$

be less than 1. *Then the conclusion of Theorem* 4.1 *is true.*

Let the inequalities

$$f_i(t,x_1,\dots,x_n)\,\mathrm{sign}\,\big[(t-t_i)x_i\big] \le \sum_{l=1}^{n}\eta_{il}|x_l| + q_i\big(t,\|x\|\big),$$
$$(i = 1,\dots,n) \tag{4.15}$$

and (1.17) *be fulfilled on* $[a,b]\times R^n$ *and* $BV\big([a,b],R^n\big)$, *respectively, and let the conditions* (4.11), (4.12) *and*

$$c_i \exp\big((-1)^j(\tau_i - t_i)\eta_{ii}\big) < 1 \quad \textit{if} \quad (-1)^j(\tau_i - t_i) > 0$$
$$(j = 1,2;\ i = 1,\dots,n), \tag{4.16}$$

hold, where $\eta_{il} \in R_+$ $(i \neq l;\ i,l = 1,\dots,n)$, $\eta_{ii} < 0$ $(i = 1,\dots,n)$, $q_i \in K\big([a,b]\times R_+,R_+\big)$ *are functions nondecreasing in the second variable,* $\gamma \in C(R_+,R_+)$, $c_i \in R_+$ $(i = 1,\dots,n)$, $\tau_i \in [a,b]$ *and* $\tau_i \neq t_i$

[3] Here $\|x_k\|_{L^\nu} = \Big(\int_a^b |x_k(t)|^\nu\,dt\Big)^{\frac{1}{\nu}}$ $(k = 1,\dots,n)$.

$(i=1,\dots,n)$. Moreover, let the real part of every characteristic value of the matrix $(\xi_{il})_{i,l=1}^n$ be negative, where

$$\xi_{il}=\eta_{il}\big[\delta_{il}+(1-\delta_{il})h_{ij}\big] \quad \textit{for} \quad (-1)^j(\tau_i-t_i)>0$$
$$(j=1,2;\; i,l=1,\dots,n),$$

$$h_{ij}=\begin{cases} 1 \quad \textit{for} \quad c_i\le 1,\\ 1+(c_i-1)\Big(1-c_i\exp\big((-1)^j(\tau_i-t_i)\eta_{ii}\big)\Big)^{-1} \quad \textit{for} \quad c_i>1 \end{cases}$$

$$(j=1,2;\; i=1,\dots,n).$$

Then the conclusion of Theorem 4.1. is true.

Let the inequalities

$$\big[f_i(t,x_1,\dots,x_n)-f_i(t,y_1,\dots,y_n)\big]\operatorname{sign}\big[(t-t_i)(x_i-y_i)\big]\le$$
$$\le\sum_{l=1}^n p_{il}(t)|x_l-y_l| \quad (i=1,\dots,n) \tag{4.17}$$

and (1.23) be fulfilled on $[a,b]\times R^n$ and $BV\big([a,b],R^n\big)$, respectively, and let the conditions (1.12), (4.10) and (4.12) hold, where $p_{il}\in L\big([a,b],R\big)$ $(i,l=1,\dots,n)$. Then the difference scheme (4.1_m), (4.2_m) $(m=2,3,\dots)$ converges to the solution of the problem (4.1), (4.2).

4.2$'$. *Let the inequalities (4.17) and (1.23) be fulfilled on $[a,b]\times R^n$ and $BV\big([a,b],R^n\big)$, respectively, and let the conditions (4.12) and (4.13) hold, where $p_{il}\in L\big([a,b],R\big)$ $(i,l=1,\dots,n)$, q_{il} and φ_{0i} $(i,l=1,\dots,n)$ satisfy the condition (4.8). Then the difference scheme (4.1_m), (4.2_m) $(m=2,3,\dots)$ converges to the solution of the problem (4.1), (4.2).*

Let the inequalities (4.17) and (1.24) be fulfilled on $[a,b]\times R^n$ and $BV\big([a,b],R^n\big)$, respectively, and let the conditions (1.3), (1.15), (4.10) and (4.12) hold, where $p_{il}\in L\big([a,b],R\big)$ $(i,l=1,\dots,n)$, $l_{\sigma ik}\in R_+$ $(\sigma=0,1,2;\; i,k=1,\dots,n)$, c_{il} $(i\ne l;\; i,l=1,\dots,n)$ are functions nondecreasing on $[a,b]$ and continuous at the point $t_i, c_{ii}\in BV\big([a,b],R\big)$ $(i=1,\dots,n)$, α_l $(l=1,\dots,n)$ are functions nondecreasing on $[a,b]$ and having not more than a finite number of discontinuity points, $h_{il}\in L^\mu\big([a,b],R_+;\alpha_l\big)$ $(i,l=1,\dots,n)$, $1\le\mu\le+\infty$, $\frac{1}{\mu}+\frac{2}{\nu}=1$. Moreover, let the module of every characteristic value of the $3n\times 3n$ -matrix $\mathcal{H}=(\mathcal{H}_{j+1,\sigma+1})_{j,\sigma=0}^2$ appearing in Corollary 1.1 or of the $n\times n$-matrix appearing in Remark 1.2 be less than 1. Then the conclusion of Theorem 4.2 is true.

4.3′ *Let the inequalities* (4.17) *and*

$$\big|\varphi_i(x_1,\dots,x_n)-\varphi_i(y_1,\dots,y_n)\big| \le \sum_{k=1}^{n} l_{ik}\|x_k-y_k\|_{L^\nu} \quad (i=1,\dots,n)$$

be fulfilled on $[a,b]\times R^n$ *and* $BV\big([a,b],R^n\big)$, *respectively, and let the condition* (4.12) *hold, where* $p_{il}\in L^\mu([a,b];R_+)$ $(i,l=1,\dots,n)$, $l_{ik}\in R_+$ $(i,k=1,\dots,n)$, $1\le\mu\le+\infty$, $\frac{1}{\mu}+\frac{2}{\nu}=1$. *Moreover, let the module of every characteristic value of the matrix* (4.14) *be less than* 1. *Then the conclusion of Theorem* 4.2 *is true.*

Let the inequalities

$$\big[f_i(t,x_1,\dots,x_n)-f_i(t,y_1,\dots,y_n)\big]\operatorname{sign}\big[(t-t_i)(x_i-y_i)\big]\le$$
$$\le\sum_{l=1}^{n}\eta_{il}|x_l-y_l| \quad (i=1,\dots,n)$$

be fulfilled on $[a,b]\times R^n$, *and let the conditions* (4.12) *and* (4.16) *hold, where* $\eta_{il}\in R_+$ $(i\ne l;\ i,l=1,\dots,n)$, $\eta_{ii}<0$ $(i=1,\dots,n)$, $c_i\in R_+$ $(i=1,\dots,n)$, $\tau_i\in[a,b]$ *and* $\tau_i\ne t_i$ $(i=1,\dots,n)$. *Moreover, let the real part of every characteristic value of the matrix* $(\xi_{il})_{i,l=1}^n$ *appearing in Corollary* 4.2 *be negative. Then the difference scheme* (4.1_m), (4.2_m) $(m=2,3,\dots)$ *converges to the solution of the problem* (4.1), (1.25) *for every* $\lambda_i\in[-c_i,c_i]$ *and* $\gamma_i\in R$ $(i=1,\dots,n)$.

We give the following notation from [16].

Let I be a set obtained by rejection of not more than a finite number of points from $[a,b]$. Then by $C^*(I,R)$ we denote the set of all functions $u\in C(I,R)$ such that

$$|u(t)|\le v_u(t) \quad \text{for} \quad t\in I,$$

where $v_u: I\to R_+$ is a continuous, Riemann integrable in improper sense, function monotone in sufficiently small left and right neighborhood of every point of the set $[a,b]\backslash I$. Let $C^*(I\times R^n,R^n)$ be the set of all $f=(f_i)_{i=1}^n\in C$ $(I\times R^n,R^n)$ such that

$$\max\Big\{\big|f_i(\cdot,x)\big| : \|x\|\le r\Big\}\in C^*(I,R) \quad (i=1,\dots,n)$$

for every $r>0$.

Remark 4.1. Analogously, as in the proof of Lemma 1.1 from [16], we can show that if $(f_i)_{i=1}^n\in K\big([a,b]\times R^n,R^n\big)$ and

$$g_{jim}(k,x)=m\int\limits_{\tau_{k-1m}}^{\tau_{km}} f_i(\tau,x)d\tau \quad \text{for} \quad k+j-1\in N_m,\ x\in R^n$$
$$(i\in N_{jm};\ j=1,2;\ m=2,3,\dots)$$

or $(f_i)_{i=1}^n \in C^*(I \times R^n, R^n)$ and

$$g_{jim}(k,x) = \begin{cases} 0 \quad \text{for} \quad \tau_{km} \in T_m,\ x \in R^n, \\ f_i(\tau_{km}, x) \quad \text{for} \quad \tau_{km} \in I \backslash T_m,\ x \in R^n \end{cases}$$

$$(k+j-1 \in N_m,\ \ i \in N_{jm},\ j = 1,2;\ \ m = 2,3,\dots),$$

where I is a set, obtained by rejection of not more than a finite number of points from $[a,b]$,

$$T_m = \bigcup_{t \in [a,b] \backslash I}]t - \tau_m, t + \tau_m[,$$

then the sequences $(g_{jim})_{m=2}^{+\infty}$ $(i \in N_{jm},\ j = 1,2)$ satisfy the conditions (a) and (c) given in the definition of the set $W(f,h)$. In particular, it is evident that the Euler and the Runge–Kutta schemes [17] satisfy these conditions.

§ 5. On the Stability of Solutions of the Multipoint Boundary Value Problem for a System of Difference Equations

In this section we intend to consider the results connected with the multipoint boundary value problem for system of difference equations which have been given in Section 1.

Let $m_0 \geq 2$ be a fixed natural number. For system of difference equations

$$y_i(k) - y_i(k-1) = g_i\big(k, y_1(k), \dots, y_n(k), y_1(k-1), \dots, y_n(k-1)\big) \quad \text{for} \quad k \in N_{m_0} \quad (i = 1, \dots, n) \tag{5.1}$$

consider the multipoint boundary value problem

$$y_i(k_i) = \psi_i(y_1, \dots, y_n) \quad (i = 1, \dots, n), \tag{5.2}$$

where $k_i \in \widetilde{N}_{m_0}$ $(i = 1, \dots, n)$, $g_i(k, \cdot) : R^{2n} \to R$ $(k = 1, \dots, m_0;\ i = 1, \dots, n)$ and $\psi_i : E(\widetilde{N}_{m_0}, R^n) \to R$ $(i = 1, \dots, n)$ are continuous functionals, in general nonlinear.

Consider along with the problem (5.1), (5.2) a sequence of problems

$$y_i(k) - y_i(k-1) = g_{im}\big(k, y_1(k), \dots, y_n(k), y_1(k-1), \dots, y_n(k-1)\big) \quad \text{for} \quad k \in N_{m_0} \quad (i = 1, \dots, n), \tag{5.1$_m$}$$

$$y_i(k_{im}) = \psi_{im}(y_1, \dots, y_n) \quad (i = 1, \dots, n) \tag{5.2$_m$}$$

$(m = 1, 2, \dots)$, where $k_{im} \in \widetilde{N}_{m_0}$ $(i = 1, \dots, n)$, $g_{im}(k, \cdot) : R^{2n} \to R$ $(k = 1, \dots, m_0;\ i = 1, \dots, n)$ and $\psi_{im} : E(\widetilde{N}_{m_0}, R^n) \to R$ $(i = 1, \dots, n)$ are continuous functionals, in general nonlinear.

We will rewrite every of the problems (5.1), (5.2) and (5.1$_m$), (5.2$_m$) $(m = 1, 2, \dots)$ in the form of the problems (1.1), (1.2) and (1.1$_m$), (1.2$_m$) $(m = 1, 2, \dots)$, respectively.

Consider the problem (5.1), (5.2). Let $y = (y_i)_{i=1}^n \in E(\widetilde{N}_{m_0}, R^n)$ be a solution of this problem. Assume

$$y_i(-1) = 0 \quad (i = 1, \dots, n)$$

and

$$y_{n+i}(k)=y_i(k-1) \quad \text{for} \quad k\in \widetilde{N}_{m_0} \quad (i=1,\dots,n).$$

Then the problem (5.1), (5.2) is equivalent to the following $2n\times 2n$ difference problem

$$\begin{gathered}
y_i(k)-y_i(k-1)=g_i\big(k,y_1(k),\dots,y_{2n}(k)\big) \quad \text{for} \quad k\in N_{m_0}\\
(i=1,\dots,n),\\
y_{n+i}(k)-y_{n+i}(k-1)=y_i(k-1)-y_{n+i}(k-1) \quad \text{for} \quad k\in N_{m_0}\\
(i=1,\dots,n),\\
y_i(k_i)=\psi_i(y_1,\dots,y_n) \quad (i=1,\dots,n),\\
y_{n+i}(k_{n+i})=\psi_i(y_1,\dots,y_n) \quad (i=1,\dots,n),
\end{gathered}$$

where

$$k_{n+i}=k_i+1 \quad (i=1,\dots,n).$$

Therefore the vector-function $x=(x_i)_{i=1}^n\in BV\big([0,m_0],R^n\big)$,

$$\begin{gathered}
x_i(t)=y_i(k) \quad \text{for} \quad t\in I_{jkm_0}\cap[0,m_0]\\
(k+j-1\in N_{m_0};\, i\in N_{jm_0};\, j=1,2),
\end{gathered} \tag{5.3}$$

where $I_{1km_0}=\big[\tau_{km_0}-\frac{\tau_{m_0}}{2},\tau_{km_0}+\frac{\tau_{m_0}}{2}\big[$ $(k\in \widetilde{N}_{m_0})$, $I_{2km_0}=\big]\tau_{km_0}-\frac{\tau_{m_0}}{2},\tau_{km_0}+\frac{\tau_{m_0}}{2}\big]$ $(k\in\widetilde{N}_{m_0})$, $\tau_{km_0}=k\tau_{m_0}$, $\tau_{m_0}=\frac{b-a}{m_0}$, $N_{1m_0}=\{1,\dots,n\}$, $N_{2m_0}=\{n+1,\dots,2n\}$, is a solution of the $2n\times 2n$ problem (1.1), (1.2), with $a=0$, $b=m_0$, $A_0(t)=\big(a_{il}(t)\big)_{i,l=1}^{2n}$,

$$\begin{gathered}
a_{il}(t)\equiv 0 \quad (i\neq l;\, i,l=1,\dots,2n),\\
a_{ii}(t)=k \quad \text{for} \quad t\in I_{jkm_0}\cap[a,b]\\
(k\in\widetilde{N}_{m_0};\, i\in N_{jm_0};\, j=1,2),
\end{gathered} \tag{5.4}$$

$$f_0(t,x)=\big(f_{i0}(t,x)\big)_{i=1}^{2n},$$

$$f_{i0}(t,x)=\begin{cases}0 \quad \text{for} \quad t\in I_{10m_0}\cap[a,b], \quad x\in R^{2n},\\ g_i(k,x) \quad \text{for} \quad t\in I_{1km_0}\cap[a,b], \quad x\in R^{2n}, \quad k\in N_{m_0}\end{cases}$$
$$(i=1,\dots,n),$$

$$\begin{gathered}
f_{n+io}(t,x_1,\dots,x_{2n})=x_i-x_{n+i} \quad \text{for} \quad t\in[a,b], \quad (x_l)_{l=1}^{2n}\in R^{2n}\\
(i=1,\dots,n),
\end{gathered} \tag{5.5}$$

$$t_i=k_i \quad (i=1,\dots,2n), \tag{5.6}$$

$$\begin{gathered}
\varphi_i(x_1,\dots,x_{2n})=\varphi_{n+i}(x_1,\dots,x_{2n})=\psi_i(y_1,\dots,y_n)\\
\text{for} \quad (x_l)_{l=1}^{2n}\in BV\big([a,b],R^{2n}\big) \quad (i=1,\dots,n),
\end{gathered} \tag{5.7}$$

where

$$y_i(k)=x_i(\tau_{km_0}) \quad (k=0,\dots,m_0;\, i=1,\dots,n). \tag{5.8}$$

Analogously we can see that if $y = (y_i)_{i=1}^n \in E(\widetilde{N}_{m_0}, R^n)$ is a solution of the problem (5.1_m), (5.2_m), then the vector-function $x = (x_i)_{i=1}^n \in BV\big([0,m_0], R^n\big)$ defined by (5.3) is a solution of the problem (1.1_m), (1.2_m), where $a = 0$, $b = m_0$,

$$A_m(t) \equiv A_0(t), \tag{5.9}$$

$A_0(t)$ is defined by (5.4);

$$f_m(t,x) = \big(f_{im}(t,x)\big)_{i=1}^{2n},$$
$$f_{im}(t,x) = \begin{cases} 0 \quad \text{for} \quad t \in I_{10m_0} \cap [a,b], \ \ x \in R^{2n}, \\ g_{im}(k,x) \quad \text{for} \quad t \in I_{1km_0} \cap [a,b], \ \ x \in R^{2n}, \ \ k \in N_{m_0} \end{cases}$$
$$(i = 1, \dots, n),$$
$$f_{n+im}(t, x_1, \dots, x_{2n}) = x_i - x_{n+i} \quad \text{for} \quad t \in [a,b], (x_l)_{l=1}^{2n} \in R^{2n} \tag{5.10}$$
$$(i = 1, \dots, n)$$
$$t_{im} = k_{im} \ (i = 1, \dots, 2n), \ k_{n+im} = k_{im} + 1 \quad (i = 1, \dots, n); \tag{5.11}$$
$$\varphi_{im}(x_1, \dots, x_{2n}) = \varphi_{n+im}(x_1, \dots, 2n) = \psi_{im}(y_1, \dots, y_n)$$
$$\text{for} \quad (x_l)_{l=1}^{2n} \in BV\big([a,b], R^{2n}\big) \quad (i = 1, \dots, n), \tag{5.12}$$

where $y_l : \widetilde{N}_{m_0} \to R$ $(l = 1, \dots, n)$ are defined by (5.8) for every natural m.

Therefore we have proved the following

Let m be a fixed natural number. Let $(y_i)_{i=1}^n \in E(\widetilde{N}_{m_0}, R^n)$ be a solution of the problem (5.1), (5.2) (*of the problem* (5.1_m), (5.2_m)). *Then the vector-function $(x_i)_{i=1}^n \in BV\big([a,b], R^n\big)$ defined by* (5.3) *will be a solution of the $2n \times 2n$ problem* (1.1), (1.2) (*of the $2n \times 2n$ problem* (1.1_m), (1.2_m))*, where $a = 0$, $b = m_0$, $A_0 = (a_{il0})_{i,l=1}^{2n}$, $f_0 = (f_{i0})_{i=1}^{2n}$, $t_1, \dots, t_{2n}$ and $\varphi_1, \dots, \varphi_{2n}$ $\big(A_m = (a_{ilm})_{i,l=1}^{2n}$, $f_m = (f_{im})_{i=1}^n$, $t_{1m}, \dots, t_{2nm}$ and $\varphi_{1m}, \dots, \varphi_{2nm}\big)$ are defined by* (5.4)–(5.7) (*by* (5.4), (5.9)–(5.12)), *respectively. On the contrary, if the vector-function $(x_i)_{i=1}^{2n} \in BV\big([a,b], R^{2n}\big)$ is a solution of the $2n \times 2n$ problem* (1.1), (1.2) (*of the $2n \times 2n$ problem* (1.1_m), (1.2_m)) *circumscribed above, then the vector-function $(y_i)_{i=1}^n \in E(\widetilde{N}_{m_0}, R^n)$ defined by* (5.8) *will be a solution of the problem* (5.1), (5.2) (*of the problem* (5.1_m), (5.2_m)), *where*

$$g_i(k,x) = f_{i0}(\tau_{km_0}, x) \quad \big(g_{im}(k,x) = f_{im}(\tau_{km_0}, x)\big)$$
$$\textit{for} \quad k \in N_{m_0}, \quad x \in R^{2n} \quad (i = 1, \dots, n).$$

Basing on the above facts, we can give the following

We will say that the pair $\big((q_{il})_{i,l=1}^{2n}; (\psi_{0i})_{i=1}^{2n}\big)$ consisting of a matrix-function $(q_{il})_{i,l=1}^{2n} : N_{m_0} \to R^{2n \times 2n}$ and a positive homogeneous operator $(\psi_{0i})_{i=1}^{2n} : E(\widetilde{N}_{m_0}, R_+^{2n}) \to R_+^{2n}$ belongs to the set $U_{m_0}(k_1,\dots,k_{2n})$ if the problem

$$\begin{gathered}\big(y_i(k)-y_i(k-1)\big)\operatorname{sign}\left(k-k_i-\frac{1}{2}\right) \le \sum_{l=1}^{n}\big(q_{il}(k)y_l(k)+\\ +q_{in+l}(k)y_{n+l}(k)\big)\quad (k=1,\dots,m_0;\ i=1,\dots,n),\\ \big(y_{n+i}(k+1)-y_{n+i}(k)\big)\operatorname{sign}\left(k-k_i-\frac{1}{2}\right) \le \sum_{l=1}^{n} q_{n+il}(k+1)y_l(k)\\ (k=0,\dots,m_0-1;\ i=1,\dots,n);\end{gathered} \tag{5.13}$$

$$y_i(k_i) \le \psi_{0i}\big(|y_1|,\dots,|y_{2n}|\big) \quad (i=1,\dots,2n) \tag{5.14}$$

has no nontrivial nonnegative solution.

Let $y^0=(y_i^0)_{i=1}^n \in E(\widetilde{N}_{m_0}, R^n)$ and $r \in]0,+\infty[$. By $Z_m(y^0;r)$ we denote the set of all solutions $y=(y_i)_{i=1}^n$ of the problem (5.1_m), (5.2_m) such that

$$\|y-y^0\|_{\widetilde{m}_0} < r$$

for every natural m. $D_{m_0n}(y^0;R)$ is the set of all $x\in R^n$ satisfying the inequality

$$\min\big\{\|x-y^0(k)\| : k\in \widetilde{N}_{m_0}\big\} < r.$$

$$U_{m_0n}(y^0;r) = \big\{y\in E(\widetilde{N}_{m_0}, R^n) : \|x-y^0\|_{\widetilde{m}_0} < r\big\}.$$

Let

$$g=(g_i)_{i=1}^n,\quad g_m=(g_{im})_{i=1}^n \quad (m=1,2,\dots)$$

and let h, h_m $(m=1,2,\dots)$ be the operator from $E(\widetilde{N}_{m_0}, R^{2n})$ into R^{2n} defined by the equalities

$$\begin{gathered}h(h_1,\dots,y_{2n}) = \big(h_i(y_1,\dots,y_{2n})\big)_{i=1}^{2n} \quad \text{for} \quad (y_l)_{l=1}^{2n}\in E(\widetilde{N}_{m_0}, R^{2n}),\\ h_i(y_1,\dots,y_{2n}) = y_i(k_i) - \psi_{i-(j-1)n}(y_1,\dots,y_{2n})\\ (i\in N_{jm_0};\ j=1,2);\\ h_m(y_1,\dots,y_{2n}) = \big(h_{im}(y_1,\dots,y_{2n})\big)_{i=1}^{2n}\\ \text{for} \quad (y_l)_{l=1}^{2n}\in E(\widetilde{N}_{m_0}, R^{2n}) \quad (m=1,2,\dots),\\ h_{im}(y_1,\dots,y_{2n}) = y_i(k_i) - \psi_{i-(j-1)n,m}(y_1,\dots,y_{2n})\\ (i\in N_{jm_0};\ j=1,2).\end{gathered}$$

By $W_r(g,h;y^0)$ we denote the set of all sequences $(g_m,h_m)_{m=1}^{+\infty}$ such that

a) $\lim\limits_{n\to+\infty} g_m(k,x,y) = g(k,x,y)$ for $k\in N_{m_0};\ x,y\in D_{m_0n}(y^0;r)$;

b) $\lim\limits_{m\to+\infty} h_m(x,y) = h(x,y)$ uniformly for $x,y\in U_{m_0n}(y^0;r)$;

c) there exists a sequence $\omega_m \in M\big(N_{m_0} \times R_+, R_+^n\big)$ $(m = 1, 2, \dots)$ [4] such that

$$\sup\big\{\|\omega_m(k,r)\| : m = 1, 2, \dots\big\} < +\infty \quad (k \in N_{m_0}),$$
$$\lim_{s\to 0+} \sup\big\{\|\omega_m(k,s)\| : m = 1, 2, \dots\big\} = 0 \quad (k \in N_{m_0}),$$
$$\big|g_m(k,x) - g_m(k,y)\big| \le \omega_m\big(k, \|x-y\|\big) \quad \text{on} \quad N_{m_0} \times D_{m_0 n}(y^0; r)$$
$$(m = 1, 2, \dots).$$

Let now y^0 be a solution of the problem (5.1), (5.2).

The problem (5.1) (5.2) is said to be $(y^0; r)$ -correct if for every $\varepsilon \in]0, r[$ and $(g_m, h_m)_{m=1}^{+\infty} \in W_r(g, h; y^0)$ there exists a natural m_0 such that

$$Z_m(y^0; r) \ne \varnothing$$

and

$$Z_m(y^0; r) \subset Z_m(y^0; \varepsilon)$$

for every $m \ge m_0$.

The problem (5.1), (5.2) is said to be correct if it has a unique solution y^0, and it is $(y^0; r)$-correct for every $r > 0$.

Theorems and Corollaries below follow immediately from those given in Section 1 if we use Lemma 5.1.

Let the conditions

$$g_i(k, x_1, \dots, x_{2n}) \operatorname{sign}\left[\left(k - k_i - \frac{1}{2}\right) x_i\right] \le$$
$$\le \sum_{l=1}^{2n} p_{il}(k)|x_l| + q_i\big(k, \|x\|\big) \quad (k = 1, \dots, m_0;\ i = 1, \dots, n), \tag{5.15}$$

$$(x_i - x_{n+i}) \operatorname{sign}\left[\big(k - k_i - \frac{1}{2}\big) x_{n+i}\right] \le p_{n+ii}(k)|x_i| +$$
$$+ p_{n+in+i}(k)|x_{n+i}| + q_{n+i}\big(k, \|x\|\big)$$
$$(k = 0, \dots, m_0 - 1;\ i = 1, \dots, n) \tag{5.16}$$

be fulfilled on R^{2n}, and let the inequalities

$$\big|\psi_i(y_1, \dots, y_n)\big| \le \psi_{0i}\big(|y_1|, \dots, |y_n|\big) + \gamma\left(\sum_{k=1}^{n} \|y_k\|_{\widetilde{m}_0}\right)$$
$$(i = 1, \dots, n), \tag{5.17}$$

be fulfilled on $E(\widetilde{N}_{m_0}, R^n)$, where $P_{il} \in E(\widetilde{N}_{m_0}, R)$ $(i = 1, \dots, n;\ l = 1, \dots, 2n)$, $P_{n+il} \in E(\widetilde{N}_{m_0}, R)$ $(i = 1, \dots, n;\ l = i, n+i)$, $q_i(k, \cdot)$ $(i =$

[4] We take here the n-vector-function, since the $(n+i)$-th $(i = 1, \dots, n)$ components of $\omega_m(k,s)$ $(k \in N_{m_0}; m = 1, 2, \dots)$ are assumed to be equal to s.

$1,\dots,2n;\ k=1,\dots,m_0)$ *and* $\gamma\in C(R_+,R_+)$ *are nondecreasing functions satisfying the conditions*

$$\begin{gathered}\lim_{\rho\to+\infty}\frac{q_i(k,\rho)}{\rho}=0\quad(k=1,\dots,m_0;\ i=1,\dots,2n),\\ \lim_{\rho\to+\infty}\frac{\gamma(\rho)}{\rho}=0,\end{gathered}\tag{5.18}$$

$\psi_{0i}:E(\widetilde{N}_{m_0},R_+^n)\to R_+$ $(i=1,\dots,n)$ *are positively homogeneous nondecreasing functionals. Moreover, let there exist a matrix-function* $(q_{il})_{i,l=1}^{2n}:N_{m_0}\to R_+^{2n}$ *such that*

$$\big((q_{il})_{i,l=1}^{2n},(\psi_{0i})_{i=1}^{2n}\big)\in U_{m_0}(k_1,\dots,k_{2n}),\tag{5.19}$$

and

$$\begin{gathered}p_{il}(k)\le q_{il}(k)\quad(k=1,\dots,m_0;\ i=1,\dots,n;\ l=1,\dots,2n),\\ p_{n+il}(k-1)\le q_{n+il}(k)\quad(k=1,\dots,m_0;\ i=1,\dots,n;\ l=i,n+i),\end{gathered}\tag{5.20}$$

where

$$\psi_{0i}(y_1,\dots,y_{2n})=\psi_{on+i}(y_1,\dots,y_{2n})=\psi_{0i}(y_1,\dots,y_n)\quad(i=1,\dots,n)$$

for $(y_l)_{l=1}^{2n}\in E(\widetilde{N}_{m_0},R_+^{2n})$, $k_{n+i}=k_i+1$ $(i=1,\dots,n)$. *If the problem* (5.1), (5.2) *has no more than one solution, then it is correct.*

Let the conditions (5.18), (5.20) *hold, and let the inequalities* (5.15), (5.16) *and*

$$\big|\psi_i(y_1,\dots,y_n)\big|\le\sum_{k=1}^n l_{ik}\|y_k\|_{\nu,m_0}+\gamma\Bigg(\sum_{k=1}^n\|y_k\|_{\widetilde{m}_0}\Bigg)\quad(i=1,\dots,n)^5$$

be fulfilled on R^{2n} *and* $E(\widetilde{N}_{m_0},R^n)$, *respectively, where* $p_{il},q_{il}\in E(\widetilde{N}_{m_0},R)$ $(i=1,\dots,n;\ l=1,\dots,2n)$, $p_{n+il},q_{n+il}\in E(\widetilde{N}_{m_0},R)$ $(i=1,\dots,n;\ l=i,n+i)$, $q_i(k,\cdot)$ $(i=1,\dots,2n;\ k=1,\dots,m_0)$ *and* $\gamma\in C(R_+,R_+)$ *are nondecreasing functions,* $l_{ik}\in R_+$ $(i,k=1,\dots,n)$, $1\le\mu\le+\infty$, $\frac{1}{\mu}+\frac{2}{\nu}=1$. *Moreover, let the module of every characteristic value of the* $2n\times 2n$*-matrix* $\mathcal{H}=(h_{ik})_{i,k=1}^{2n}$ *be less than* 1, *where*

$$\begin{gathered}h_{ik}=m_0^{\frac{1}{\nu}}l_{ik}+\left(\frac{1}{2}\sin^{-1}\frac{\pi}{4m_0+2}\right)^{\frac{2}{\nu}}\|q_{ik}\|_{\mu,m_0}\\ (i=1,\dots,n;\ k=1,\dots,2n),\\ h_{n+ik}=m_0^{\frac{1}{\nu}}l_{ik}+\left(\frac{1}{2}\sin^{-1}\frac{\pi}{4m_0+2}\right)^{\frac{2}{\nu}}\|q_{ik}\|_{\mu,\widetilde{m_0-1}}\end{gathered}$$

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[5]Here we assume that $\|y\|_{\nu,m_0}=\Big(\sum_{l=1}^{m_0}\|y(l)\|^\nu\Big)^{\frac{1}{\nu}}$ and $\|y\|_{\nu,\widetilde{m}_0}=\Big(\sum_{l=0}^{m_0}|y(l)|^\nu\Big)^{\frac{1}{\nu}}$ if $1\le\nu<+\infty$ and $\|y\|_{+\infty,m_0}=\|y\|_{m_0}$, $\|y\|_{+\infty,\widetilde{m}_0}=\|y\|_{\widetilde{m}_0}$.

$$(i = 1, \dots, n;\ k = i, n+i),$$
$$h_{n+ik} = 0 \quad (i = 1, \dots, n;\ k \neq i, k \neq n+i;\ k = 1, \dots, 2n).$$

Then the conclusion of Theorem 5.1 *is true.*

Let the inequalities

$$g_i(k, x_1, \dots, x_{2n}) \operatorname{sign}\left[\left(k - k_i - \frac{1}{2}\right)x_i\right] \leq \sum_{l=1}^{2n} \eta_{il}|x_l| + q_i\big(k, \|x\|\big)$$
$$(k = 1, \dots, m_0;\ i = 1, \dots, n),$$
$$(x_i - x_{n+i}) \operatorname{sign}\left[\left(k - k_i - \frac{1}{2}\right)x_{n+i}\right] \leq \eta_{n+ii}|x_i| +$$
$$+\eta_{n+in+i}|x_{n+i}| + q_{n+i}\big(k, \|x\|\big)$$
$$(k = 0, \dots, m_0 - 1;\ i = 1, \dots, n)$$

be fulfilled on R^{2n}*, and let the inequalities*

$$\big|\psi_i(y_1, \dots, y_n)\big| \leq c_i\big|y_i(m_i)\big| + \gamma\left(\sum_{k=1}^{n} \|y_k\|_{\widetilde{m}_0}\right) \quad (i = 1, \dots, n)$$

be fulfilled on $E\big(N_{\widetilde{m}_0}, R^n\big)$*, where* $\eta_{il} \in R_+, \eta_{n+ii} \in R_+$ $(i \neq l;\ i = 1, \dots, n;$ $l = 1, \dots, 2n)$*,* $\eta_{ii} < 0$ $(i = 1, \dots, 2n)$*,* $q_i(k, \cdot)$ $(i = 1, \dots, 2n;\ k = 1, \dots, m_0)$ *and* $\gamma \in C(R_+, R_+)$ *are nondecreasing functions satisfying the conditions* (5.18)*,* $c_i \in R_+$ *and* $m_i \in \widetilde{N}_{m_0}$ *are such that*

$$c_i\zeta_{ij} < 1 \quad \textit{if} \quad (-1)^j(m_i - k_i) > 0 \quad (j = 1, 2;\ i = 1, \dots, n), \qquad (5.21)$$

$\zeta_{ij} = \exp\big((-1)^j(m_i - k_i)\eta_{ii}\big)$ $(j = 1, 2;\ i = 1, \dots, n)$*. Moreover, let the real part of every characteristic value of the matrix* $(\xi_{il})_{i,l=1}^{n}$ *be negative, where*

$$\xi_{il} = \eta_{il}\big[\delta_{il} + (1 - \delta_{il})h_{ij}\big] \quad \textit{for} \quad (-1)^j(m_i - k_i) > 0$$
$$(j = 1, 2;\ i, l = 1, \dots, n),^{6}$$
$$h_{ij} = \begin{cases} 1 \quad \textit{for} \quad c_i \leq 1, \\ 1 + (c_i - 1)(1 - c_i\zeta_{ij})^{-1} \quad \textit{for} \quad c_i > 1 \end{cases}$$
$$(j = 1, 2;\ i = 1, \dots, n).$$

Then the conclusion of Theorem 5.1 *is true.*

Let the conditions

$$\big[g_i(k, x_1, \dots, x_{2n}) - g_i(k, y_1, \dots, y_{2n})\big] \operatorname{sign}\Big[\big(k -$$
$$-k_i - \frac{1}{2}\big)(x_i - y_i)\Big] \leq$$

[6] Here δ_{il} is the Kroneker symbol.

$$\leq \sum_{l=1}^{2n} p_{il}(k)|x_l - y_l| \quad (k = 1, \dots, m_0;\ i = 1, \dots, n), \tag{5.22}$$

$$(x_i - y_i - x_{n+i} + y_{n+i}) \operatorname{sign}\left[\left(k - k_i - \frac{1}{2}\right)(x_{n+i} - y_{n+i})\right] \leq$$
$$\leq p_{n+ii}(k)|x_i - y_i| + p_{n+in+i}|x_{n+i} - y_{n+i}|$$
$$(k = 0, \dots, m_0 - 1;\ i = 1, \dots, n) \tag{5.23}$$

be fulfilled on R^{2n} *and let the inequalities*

$$\big|\psi_i(x_1, \dots, x_n) - \psi_i(y_1, \dots, y_n)\big| \leq \psi_{0i}\big(|x_1 - y_1|, \dots, |x_n - y_n|\big)$$
$$(i = 1, \dots, n)$$

be fulfilled on $E\big(\widetilde{N}_{m_0}, R^n\big)$, *where* $p_{il} \in E(\widetilde{N}_{m_0}, R)$ $(i = 1, \dots, n;\ l = 1, \dots, 2n), p_{n+il} \in E\big(\widetilde{N}_{m_0}, R\big)$ $(i=1, \dots, n;\ l=i, n+i)$, $\psi_{0i} : E(\widetilde{N}_{m_0}, R^n) \to R_+$ $(i = 1, \dots, n)$ *are positively homogeneous nondecreasing functionals. Moreover, let there exist a matrix-function* $(q_{il})_{i,l=1}^{2n} : N_{m_0} \to R_+^{2n}$ *such that the conditions* (5.19) *and* (5.20) *hold, where*

$$\psi_{0i}(y_1, \dots, y_{2n}) = \psi_{on+i}(y_1, \dots, y_{2n}) = \psi_{0i}(y_1, \dots, y_n)$$

for $(y_l)_{l=1}^{2n} \in E\big(\widetilde{N}_{m_0}, R_+^{2n}\big)$, $k_{n+i} = k_i$ $(i = 1, \dots, n)$. *Then the problem* (5.1), (5.2) *is correct.*

Let the condition (5.20) *hold, and let the inequalities* (5.22), (5.23) *and*

$$\big|\psi_i(x_1, \dots, x_n) - \psi_i(y_1, \dots, y_n)\big| \leq \sum_{k=1}^{n} l_{ik} \|y_k\|_{m_0, \nu}$$

be fulfilled on R^{2n} *and* $E\big(\widetilde{N}_{m_0}, R^n\big)$, *respectively, where* $p_{il}, q_{il} \in E(\widetilde{N}_{m_0}, R)$ $(i = 1, \dots, n;\ l = 1, \dots, 2n)$, $p_{n+il}, q_{n+il} \in E(\widetilde{N}_{m_0}, R)$ $(i = 1, \dots, n;\ l = i, n+i)$, $l_{ik} \in R_+$ $(i, k = 1, \dots, n)$, $1 \leq \mu \leq +\infty$, $\frac{1}{\mu} + \frac{2}{\nu} = 1$. *Moreover, let the module of every characteristic value of the* $2n \times 2n$*-matrix* $\mathcal{H} = (h_{ik})_{i,k=1}^{2n}$ *appearing in Corollary* 5.1 *be less than* 1. *Then the problem* (5.1), (5.2) *is correct.*

Let the inequalities

$$\big[g_i(k, x_1, \dots, x_{2n}) - g_i\big(k, y_1, \dots, y_{2n}\big)\big] \operatorname{sign}\left[\left(k - k_i - \frac{1}{2}\right)(x_i - y_i)\right] \leq$$
$$\leq \sum_{l=1}^{2n} \eta_{il}|x_l - y_l| \quad (k = 1, \dots, m_0;\ i = 1, \dots, n),$$

$$(x_i - y_i - x_{n+i} + y_{n+i}) \operatorname{sign}\left[\left(k - k_i - \frac{1}{2}\right)(x_{n+i} - y_{n+i})\right] \leq$$
$$\leq \eta_{n+ii}|x_i - y_i| + \eta_{n+in+i}|x_{n+i} - y_{n+i}|$$

$$(k = 0, \dots, m_0;\ i = 1, \dots, n)$$

be fulfilled on R^{2n}, *where* $\eta_{il} \in R_+$, $\eta_{n+ii} \in R_+$ $(i \neq l;\ i = 1, \dots, n;\ l = 1, \dots, 2n)$, $\eta_{ii} < 0$ $(i = 1, \dots, 2n)$. *Moreover, let* $c_i \in R_+$ *and* $m_i \in \widetilde{N}_{m_0}$, $m_i \neq k_i$ $(i = 1, \dots, n)$ *be such that the condition* (5.21) *holds and the real part of every characteristic value of the matrix* $(\xi_{il})_{i,l=1}^n$ *is negative, where* ζ_{ij} *and* ξ_{il} $(i, j, l = 1, \dots, n)$ *are the numbers appearing in Corollary* 5.2. *Then the problem* (5.1),

$$y_i(k_i) = \lambda_i y_i(m_i) + \gamma_i \quad (i = 1, \dots, n)$$

is correct for every $\lambda_i \in [c_i, c_i]$ *and* $\gamma_i \in R$ $(i = 1, \dots, n)$.

Finally, let us consider a method of construction of the solution of the problem (5.1), (5.2). We take an arbitrary vector-function $(y_{i0})_{i=1}^n \in E(\widetilde{N}_{m_0}, R^n)$ as the zero approximation of the solution of the problem (5.1), (5.2). If the $(m-1)$-th approximation is constructed, then as the m-th approximation we take a vector-function $(y_{im})_{i=1}^n \in E(\widetilde{N}_{m_0}, R)$ whose i-th component is the solution of the Cauchy problem

$$\begin{aligned}
y_{im}(k) - y_{im}(k-1) = g_i\big(&k, y_{1m-1}(k), \dots, y_{i-1m-1}(k), y_{im}(k),\\
&y_{i+1m-1}(k), \dots, y_{nm-1}(k), y_{1m-1}(k-1), \dots, y_{i-1m-1}(k-1),\\
&y_{im}(k-1), y_{i+1m-1}(k-1), \dots, y_{nm-1}(k-1)\big)\\
&\text{for} \quad k \in N_{m_0} \ (i = 1, \dots, n), \qquad (5.24)\\
y_{im}(k_i) = \psi_{i0}&(y_{1m-1}, \dots, y_{nm-1}) \quad (i = 1, \dots, n). \qquad (5.25)
\end{aligned}$$

Let the conditions of Theorem 5.2 *be fulfilled and*

$$\begin{aligned}
q_{ii}(k) < 1 \quad &\textit{for} \quad k = k_i + 1, \dots, m_0 \quad (i = 1, \dots, n),\\
q_{n+in+i}(k) < 1 \quad &\textit{for} \quad k = 1, \dots, k_i \quad (i = 1, \dots, n).
\end{aligned} \qquad (5.26)$$

Then for every $(y_{i0})_{i=1}^n \in E(\widetilde{N}_{m_0}, R^n)$ *there exists a unique sequence* $(y_{im})_{i=1}^n \in E(\widetilde{N}_{m_0}, R^n)$ $(m = 1, 2, \dots)$ *such that the vector-function* $(y_{im})_{i=1}^n$ *is a solution of the problem* (5.24), (5.25) *for every natural* m *and*

$$\sum_{i=1}^n \big|y_i(k) - y_{im}(k)\big| \le r_0 \delta^m \quad \textit{for} \quad k \in \widetilde{N}_{m_0} \quad (m = 1, 2, \dots), \qquad (5.27)$$

where $(y_i)_{i=1}^n$ *is the solution of the problem* (5.1), (5.2), *and* $r_0 > 0$ *and* $\delta \in]0, 1[$ *are numbers independent of* m.

Let the conditions of Corollary 5.3 *and the conditions* (5.26) *hold. Then the conclusion of Theorem* 5.3 *is true.*

Let the conditions of Corollary 5.4 *hold and*

$$-1 < \eta_{ii} < 0 \quad (i = 1, \dots, n).$$

Moreover, let $c_i \in R_+$ *and natural* $m_i (i = 1, \dots, n)$ *be the numbers appearing in Corollary* 5.4, $\lambda_i \in [-c_i, c_i]$ *and* $\gamma_i \in R$ $(i = 1, \dots, n)$. *Then for every* $(y_{i0})_{i=1}^n \in E(\widetilde{N}_{m_0}, R^n)$ *there exists a unique sequence* $(y_{im})_{i=1}^n \in E(\widetilde{N}_{m_0}, R^n)$ $(m = 1, 2, \dots)$ *such that the vector-function* $(y_{im})_{i=1}^n$ *is a solution of the system* (5.24), *satisfying the condition*

$$y_{im}(k_i) = \lambda_i y_{im-1}(m_i) + \gamma_i \quad (i = 1, \dots, n)$$

for every natural m, *and the estimates* (5.27) *hold, where* $(y_i)_{i=1}^n$ *is the solution of the problem* (5.1), (5.2), *and* $r_0 > 0$ *and* $\delta \in]0, 1[$ *are numbers independent of* m.

Remark 5.1. The process of construction of the solution of the problem (5.1), (5.2) under consideration is stable in the following sense: let the conditions of Theorem 5.3 (of Corollaries 5.5 and 5.6) be fulfilled. Then for every $(\overline{y}_{i0})_{i=1}^n \in E(\widetilde{N}_{m_0}, R^n)$, $(\overline{u}_{im})_{i=1}^n \in E(\widetilde{N}_{m_0}, R^n)$ $(m = 1, 2, \dots)$ and $(\overline{\gamma}_{im})_{i=1}^n \in R^n$ $(m = 1, 2, \dots)$ there exists a unique sequence of vector-functions $(\overline{y}_{im})_{i=1}^n \in E(\widetilde{N}_{m_0}, R^n)$ $(m = 1, 2, \dots)$ such that the vector-function $(\overline{y}_{im})_{i=1}^n$ is a solution of the Cauchy problem

$$\begin{gathered}
\overline{y}_{im}(k) - \overline{y}_{im}(k-1) = g_i\big(k, \overline{y}_{1m-1}(k), \dots, \overline{y}_{i-1m-1}(k), \overline{y}_{im}(k), \\
\overline{y}_{i+1m-1}(k), \dots, \overline{y}_{nm-1}(k), \overline{y}_{1m-1}(k-1), \dots, \overline{y}_{i-1m-1}(k-1), \\
\overline{y}_{im}(k-1), \overline{y}_{i+1m-1}(k-1), \dots, \overline{y}_{nm-1}(k-1)\big) + \\
+\overline{u}_{im}(k) - \overline{u}_{im}(k-1) \quad \text{for} \quad k \in \widetilde{N}_{m_0} \quad (i = 1, \dots, n), \\
\overline{y}_{im}(k_i) = \psi_{i0}(\overline{y}_{1m-1}, \dots, \overline{y}_{nm-1}) \quad (i = 1, \dots, n)
\end{gathered}$$

for every natural m. Let

$$\eta_m = \sum_{i=1}^n \left(\left|\overline{\gamma}_{im}\right| + \frac{1}{n} \sum_{k=1}^{m_0} \left|\overline{u}_{im}(k) - \overline{u}_{im}(k-1)\right| \right)$$

for every natural m. Then the condition

$$\lim_{m \to +\infty} \eta_m = 0$$

guarantees the condition

$$\lim_{m \to +\infty} \|y_{im} - y_{im}\|_{\widetilde{m}_0} = 0 \quad (i = 1, \dots, n).$$

In conclusion it should be noted that using the results given above, we can obtain sufficient conditions guaranteeing stability of the difference schemes, provided we define their stability properly.

References

1. M. T. Ashordia, On correctness of the Cauchy–Nicoletti boundary value problem for systems of nonlinear generalized ordinary differential equations. (Russian) *Differentsial'nye Uravnenija* (1995), No. 3, 382–392.

2. I. T. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) *Current problems in mathematics. Newest results, vol.* 30, (Russian), 3–103; *Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuzn. Inst. Nauchn. i Tekh. Inform., Moscow,* 1987.

3. D. G. Bitsadze and I. T. Kiguradze, On correctness of boundary value problems for systems of ordinary differential equations. (Russian) *Bull. Acad. Sci. Georgian SSR* (1983), No. 2, 241–244.

4. D. G. Bitsadze, On the problem of dependence on a parameter of solution of multipoint nonlinear boundary value problems. (Russian) *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy.* (1987), 42–55.

5. J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter. *Czechoslovak Math. J.* (1957), No. 3, 418–449.

6. T. H. Hildebrandt, On systems of linear differentio-Stieltjes-integral equations. *Illinois J. Math.* (1959), 352–373.

7. S. Schwabik, M. Tvrdy, and O. Vejvoda, Differential and integral equations: Boundary value problems and adjoints. *Academia, Praha,* 1979.

8. J. Groh, A nonlinear Volterra-Stieltjes integral equation and Gronwall inequality in one dimension. *Illinois J. Math.* (1980), No. 2, 244–263.

9. M. T. Ashordia, On the continuous dependence on a parameter of the solution of Cauchy problem of the system of generalized ordinary differential equations. (Russian). *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy.* (1988), 5–22.

10. M. T. Ashordia, On the correctness of nonlinear boundary value problems for systems of generalized ordinary differential equations. *Georgian Math. J.* (1996), No. 6, 501-524.

11. M. T. Ashordia, The conditions of existence and uniqueness of solutions of the multipoint boundary value problem for a system of generalized ordinary differential equations. *Georgian Math. J.* (*to appear*).

12. A. Lasota, A discrete boundary value problem.*Ann. Polon. Math.* (1968), No. 2, 183–190.

13. Sh. Gelashvili, On the Cauchy-Nicoletti problem for systems of nonlinear difference equations. (Russian).*Trudy. Tbilissi Univ. Mat. Mekh. Astvonom.* (1981), 96–113.

14. Sh. Gelashvili and I. Kiguradze, On multi-point boundary value problems for systems of functional differential and difference equations. *Memoirs on Differential Equations Math. Phys.* (1995), 3–115.

15. Sh. Gelashvili, On one boundary value problem for the systems of functional differential equations. *Arch. Math.* (*Brno*) (1984), No. 4,

157–168.

16. I. T. Kiguradze and G.S. Tabidze, On the numerical approximation of solutions of two-point boundary value problems for ordinary differential equations. (Russian). *Tbiliss. Gos. Umv. Inst. Prikl. Mat. Trudy* (1983), 69–127.

17. A. Samarski and A. Goolin, Numerical methods. *Nauka, Moscow,* 1989.

(Received 20.06.1995)

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