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TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS OF STATICS OF THE THEORY OF ELASTIC MIXTURES

In the paper, two-dimensional boundary value problems of statics of elastic mixtures are investigated. Using the potential method and the theory of singular integral equations, existence and uniqueness theorems are proved. Parallelly, Fredholm type equations are obtained for all the considered problems. By the aid of these equations, explicit solutions are constructed for the half-plane, the disk and the exterior to a disk.

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Introduction

Since the early sixties, the theory of elastic mixtures has become very popular in mechanics and engineering sciences. A lot of important results have been obtained concerning mathematical problems of three-dimensional models (see [1] and references cited therein). As to the corresponding two-dimensional problems, they are not deeply investigated so far. The present paper deals with the two-dimensional version of the above theory. Using the potential method and the theory of integral equations, basic boundary value problems are studied and uniqueness and existence theorems are proved. Applying the theoretical results obtained, explicit solutions (in quadratures) are constructed for some particular domains with concrete geometry.

1. Basic Equations and Boundary Value Problems

Let the third component of the partial displacements $u' = (u_1', u_2', u_3')$ and $u'' = (u_1'', u_2'', u_3'')$ vanish and u_1', u_2', u_2'', u_2'' be functions only of the variables x_1 and x_2. Then we have plane deformations of elastic mixture, and the basic equations read as [1]

$$\begin{aligned} &a_1\Delta u' + b_1 \operatorname{grad}\theta' + c\Delta u'' + d\operatorname{grad}\theta'' = -\rho_1 F' \equiv \psi', \\ &c\Delta u' + d\operatorname{grad}\theta' + a_2\Delta u'' + b_2\operatorname{grad}\theta'' = -\rho_2 F'' \equiv \psi'', \end{aligned} \tag{1.1}$$

where $\Delta = \partial_1^2 + \partial_2^2$ is the Laplace operator, ρ_1 and ρ_2 are partial densities, F' and F'' are mass forces, $u' = (u_1', u_2')$ and $u'' = (u_1'', u_2'')$ are partial displacements depending on the variables x_1 and x_2, $\partial_k = \frac{\partial}{\partial x_k}$;

$$\theta' = \sum_{k=1}^{2} \partial_k u_k', \quad \theta'' = \sum_{k=1}^{2} \partial_k u_k'', \quad k = 1, 2, \tag{1.2}$$

$$\begin{aligned} &a_1 = \mu_1 - \lambda_5, \ \ b_1 = \mu_1 + \lambda_1 + \lambda_5 - \rho^{-1}\alpha_2\rho_2, \ \ a_2 = \mu_2 - \lambda_5, \\ &c = \mu_3 + \lambda_5, \ \ b_2 = \mu_2 + \lambda_2 + \lambda_5 + \rho^{-1}\alpha_2\rho_1, \\ &d = \mu_3 + \lambda_3 - \lambda_5 - \alpha_2\rho_1 \equiv \mu_3 + \lambda_4 - \lambda_5 + \rho^{-1}\alpha_2\rho_2, \\ &\rho = \rho_1 + \rho_2, \quad \alpha_2 = \lambda_3 - \lambda_4, \end{aligned} \tag{1.3}$$

where μ_1, μ_2, μ_3, λ_1, λ_2, λ_3, λ_4, λ_5 are constants which characterise mechanical properties of the elastic mixture in question and satisfy certain conditions (inequalities).

If $\psi' = \psi'' = 0$, then the system (1.1) becomes homogeneous, and we get

$$\begin{aligned} &a_1\Delta u' + b_1 \operatorname{grad}\theta' + c\Delta u'' + d\operatorname{grad}\theta'' = 0, \\ &c\Delta u' + d\operatorname{grad}\theta' + a_2\Delta u'' + b_2\operatorname{grad}\theta'' = 0. \end{aligned} \tag{1.4}$$

The equations (1.1) imply

$$\begin{aligned} &\partial_1\sigma_{11}' + \partial_2\sigma_{21}' = \psi_1', \quad \partial_1\sigma_{12}' + \partial_2\sigma_{22}' = \psi_2', \\ &\partial_1\sigma_{11}'' + \partial_2\sigma_{21}'' = \psi_1'', \quad \partial_1\sigma_{12}'' + \partial_2\sigma_{22}'' = \psi_2'', \end{aligned} \tag{1.5}$$

where

$$
\begin{aligned}
\sigma'_{11} &= L_1 + \partial_2 M_2, \quad \sigma'_{21} = -L_2 - \partial_1 M_2, \\
\sigma'_{12} &= L_2 - \partial_2 M_1, \quad \sigma'_{22} = L_1 + \partial_1 M_1, \\
\sigma''_{11} &= L_3 + \partial_2 M_4, \quad \sigma''_{21} = -L_4 - \partial_1 M_4, \\
\sigma''_{12} &= L_4 - \partial_2 M_3, \quad \sigma''_{22} = L_3 + \partial_1 M_3,
\end{aligned}
\tag{1.6}
$$

$$
\begin{aligned}
&L_1 = (a_1 + b_1)\theta' + (c + d)\theta'', \quad L_2 = a_1\omega' + c\omega'', \\
&L_3 = (c + d)\theta' + (a_2 + b_2)\theta'', \quad L_4 = c\omega' + a_2\omega'', \\
&M_k = (\varkappa_1 - 2\mu_1)u'_k + (\varkappa_3 - 2\mu_3)u''_k, \\
&M_{k+2} = (\varkappa_3 - 2\mu_3)u'_k + (\varkappa_2 - 2\mu_2)u''_k, \quad k = 1, 2,
\end{aligned}
\tag{1.7}
$$

$$
\omega' = \partial_1 u'_2 - \partial_2 u'_1, \ \omega'' = \partial_1 u''_2 - \partial_2 u''_1. \tag{1.8}
$$

The functions σ'_{11}, σ'_{21}, σ'_{12}, σ'_{22}, σ''_{11}, σ''_{21}, σ''_{12}, σ''_{22} are the components of the generalized stress tensor. The generalized stress vector $\overset{\varkappa}{T} u$ is defined as follows

$$
\begin{aligned}
(\overset{\varkappa}{T} u)_1 &= \sigma'_{11} n_1 + \sigma'_{21} n_2, \quad (\overset{\varkappa}{T} u)_2 = \sigma'_{12} n_1 + \sigma'_{22} n_2, \\
(\overset{\varkappa}{T} u)_3 &= \sigma''_{11} n_1 + \sigma''_{21} n_2, \quad (\overset{\varkappa}{T} u)_4 = \sigma''_{12} n_1 + \sigma''_{22} n_2,
\end{aligned}
\tag{1.9}
$$

where $n = (n_1, n_2)$ is an arbitrary unit vector, $u = (u', u'') = (u_1, u_2, u_3, u_4)$, $u_1 = u'_1$, $u_2 = u'_2$, $u_3 = u''_1$, $u_4 = u''_2$.

If $\varkappa_1 = \varkappa_2 = \varkappa_3 = 0$, then we set $\overset{\circ}{T} u \equiv Tu$; here Tu is the physical stress vector with the components

$$
\begin{aligned}
(Tu)_1 &= \tau'_{11} n_1 + \tau'_{21} n_2, \quad (Tu)_2 = \tau'_{12} n_1 + \tau'_{22} n_2, \\
(Tu)_3 &= \tau''_{11} n_1 + \tau''_{21} n_2, \quad (Tu)_4 = \tau''_{12} n_1 + \tau''_{22} n_2,
\end{aligned}
\tag{1.10}
$$

where τ'_{11}, τ'_{21}, τ'_{12}, τ'_{22}, τ''_{11}, τ''_{21}, τ''_{12}, τ''_{22} are the components of the physical stress tensor; their exprssions can be obtained from (1.6) and (1.7) by substitution $\varkappa_i = 0$, $i = 1, 2, 3$.

We have introduced the parameters $\varkappa_1$, $\varkappa_2$ and $\varkappa_3$ which are not involved in the basic equations (1.5). In what follows, we will see that the generalized stress vector (1.9) will be very useful and efficient. We note that similar generalized stress vector in the classical elasticity theory was introduced in [2,3]. It can be easily checked that

$$
\overset{\varkappa}{T} u = Tu + \varkappa \frac{\partial u}{\partial s(x)}, \tag{1.11}
$$

where

$$
\frac{\partial}{\partial s(x)} = n_1 \partial_2 - n_2 \partial_1, \tag{1.12}
$$

$$\varkappa = \begin{Vmatrix} 0 & \varkappa_1 & 0 & \varkappa_3 \\ -\varkappa_1 & 0 & -\varkappa_3 & 0 \\ 0 & \varkappa_3 & 0 & \varkappa_2 \\ -\varkappa_3 & 0 & -\varkappa_2 & 0 \end{Vmatrix}. \tag{1.13}$$

Let D^+ be a bounded two-dimensional domain (surrounded by the curve S) and let D^- be the complement of $\overline{D}^+ = D^+ \cup S$. We assume that $S \in C^{k+\beta}$, $k = 1, 2$, $0 < \beta \leq 1$ [4].

A vector $u = (u', u'') = (u_1, ...u_4)$ is said to be regular in D^+ $[D^-]$ if $u_k \in C^2(D^+) \cap C^1(\overline{D}^+)[u_k \in C^2(D^-) \cap C^1(\overline{D}^-]$ and the second order derivatives of u_k are summable in D^+ $[D^-]$; in the case of the domain D^-, we assume, in addition, the following conditions at infinity

$$u(x) = O(1), \quad \rho^2 \partial_k u = O(1), \quad k = 1, 2, \tag{1.14}$$

to be fulfilled with $\rho^2 = x_1^2 + x_2^2$.

The basic boundary value problems ($BVPs$) are formulated as follows [1].

Find a regular solution to the equation (1.1) in D^+ $[D^-]$ satisfying one of the following boundary conditions.

1. Problem $(\mathrm{I})^{\pm}_{\psi,f}$:

$$\{u(t)\}^{\pm} = f(t), \quad t \in S; \tag{1.15}$$

2. Problem $(\mathrm{II})^{\pm}_{\psi,f}$:

$$\{Tu(t)\}^{\pm} = f(t), \quad t \in S; \tag{1.16}$$

3. Problem $(\mathrm{III})^{\pm}_{\psi,f}$:

$$\begin{aligned} &\{u_j(t) - u_{j+2}(t)\}^{\pm} = f_j(t), \\ &\left\{[Tu(t)]_j + [Tu(t)]_{j+2}\right\}^{\pm} = f_{j+2}(t), \end{aligned} \quad t \in S, \quad j = 1, 2; \tag{1.17}$$

4. Problem $(\mathrm{IV})^{\pm}_{\psi,f}$: Let $S = \overline{S}_1 \cup \overline{S}_2$, $S_1 \cap S_2 = \varnothing$, and condition (1.15) is given on S_1, while either condition (1.16) or conditions (1.17) are given on S_2.

Here $\psi = (\psi', \psi'') = (\psi_1, \psi_2, \psi_3, \psi_4)$ and $f = (f_1, f_2, f_3, f_4)$ are known continuous vectors on $D^{\pm}$ and S, respectively. Throughout this paper $n(x)$ denotes the exterior to D^+ unit normal vector at the point $x \in S$.

Note that in the above formulations of BVPs, we can replace the physical stress vector by the generalized stress vector.

2. The Basic Fundamental Matrix

In this section, we will construct the basic fundamental matrix of the equation (1.4).

Upon taking the divergence operation, from (1.1) we get

$$(a_1 + b_1)\Delta\theta' + (c + d)\Delta\theta'' = \operatorname{div}\psi',$$
$$(c + d)\Delta\theta' + (a_2 + b_2)\Delta\theta'' = \operatorname{div}\psi''.$$

Whence

$$\begin{aligned}\Delta\theta' &= \frac{a_2 + b_2}{d_1}\operatorname{div}\psi' - \frac{c + d}{d_1}\operatorname{div}\psi'',\\ \Delta\theta'' &= -\frac{c + d}{d_1}\operatorname{div}\psi' + \frac{a_1 + b_1}{d_1}\operatorname{div}\psi'',\end{aligned} \tag{2.1}$$

where

$$d_1 = (a_1 + b_1)(a_2 + b_2) - (c + d)^2. \tag{2.2}$$

Later we will prove that $d_1 > 0$.

Further, upon taking the operator Δ and taking into account (2.1), we have

$$\begin{aligned}a_1\Delta\Delta u' + c\Delta\Delta u'' = \Delta\psi' &+ \frac{d(c + d) - b_1(a_2 + b_2)}{d_1}\operatorname{grad}\operatorname{div}\psi' +\\ &+ \frac{b_1(c + d) - d(a_1 + b_1)}{d_1}\operatorname{grad}\operatorname{div}\psi'',\\ c\Delta\Delta u' + a_2\Delta\Delta u'' = \Delta\psi'' &+ \frac{b_2(c + d) - d(a_2 + b_2)}{d_1}\operatorname{grad}\operatorname{div}\psi' +\\ &+ \frac{d(c + d) - b_2(a_1 + b_1)}{d_1}\operatorname{grad}\operatorname{div}\psi''.\end{aligned}$$

From the latter equation it follows that

$$\Delta\Delta u' = e_1\Delta\psi' + e_2\Delta\psi'' + e_4\operatorname{grad}\operatorname{div}\psi' + e_5\operatorname{grad}\operatorname{div}\psi'',$$
$$\Delta\Delta u'' = e_2\Delta\psi' + e_3\Delta\psi'' + e_5\operatorname{grad}\operatorname{div}\psi' + e_6\operatorname{grad}\operatorname{div}\psi'',$$

where

$$
\begin{aligned}
e_1 &= \frac{a_2}{d_2}, \quad e_2 = -\frac{c}{d_2}, \quad e_3 = \frac{a_1}{d_2}, \quad d_2 = a_1 a_2 - c^2,\\
e_4 &= \frac{(da_2 - cb_2)(c+d) + (cd - b_1 a_2)(a_2 + b_2)}{d_1 d_2},\\
e_5 &= \frac{(b_1 a_2 - cd)(c+d) + (cb_2 - da_2)(a_1 + b_1)}{d_1 d_2} =\\
&= \frac{(a_1 b_2 - cd)(c+d) + (cb_1 - da_1)(a_2 + b_2)}{d_1 d_2},\\
e_6 &= \frac{(da_1 - cb_1)(c+d) + (cd - a_1 b_2)(a_1 + b_1)}{d_1 d_2}.
\end{aligned}
\tag{2.3}
$$

It also will be shown that $d_2 > 0$. (2.3) implies

$$
e_1 + e_4 = \frac{a_2 + b_2}{d_1}, \quad e_2 + e_5 = -\frac{c+d}{d_1}, \quad e_3 + e_6 = \frac{a_1 + b_1}{d_1}. \tag{2.4}
$$

We look for u' and u'' in the form

$$
\begin{aligned}
u' &= e_1 \Delta\Phi' + e_2 \Delta\Phi'' + e_4 \operatorname{grad}\operatorname{div}\Phi' + e_5 \operatorname{grad}\operatorname{div}\Phi'',\\
u'' &= e_2 \Delta\Phi' + e_3 \Delta\Phi'' + e_5 \operatorname{grad}\operatorname{div}\Phi' + e_6 \operatorname{grad}\operatorname{div}\Phi'',
\end{aligned}
\tag{2.5}
$$

where Φ' and Φ'' are arbitrary vectors.

Substitution of (2.5) into (1.1) and (1.4) leads to

$$
\Delta\Delta\Phi' = \psi', \quad \Delta\Delta\Phi'' = \psi''
$$

and

$$
\Delta\Delta\Phi' = 0, \quad \Delta\Delta\Phi'' = 0,
$$

respectively.

Let us introduce $\Phi = (\Phi', \Phi'')$ and $\psi = (\psi', \psi'')$. Then previous equations yield

$$
\Delta\Delta\Phi = \psi \tag{2.6}
$$

and

$$
\Delta\Delta\Phi = 0. \tag{2.7}
$$

Let

$$
\Phi = E \operatorname{Re}\psi_0, \tag{2.8}
$$

where E is the 4×4 unit matrix, while

$$
\psi_0 = \frac{\sigma\overline{\sigma}}{4}(\ln\sigma - 1), \tag{2.9}
$$

$$
\sigma = z - \zeta, \quad \overline{\sigma} = \overline{z} - \overline{\zeta}, \quad z = x_1 + ix_2, \quad \zeta = y_1 + iy_2. \tag{2.10}
$$

Direct calculations give

$$\frac{\partial^2\psi_0}{\partial x_1^2}=\frac{1}{2}\ln\sigma+\frac{\overline{\sigma}}{4\sigma},\quad \frac{\partial^2\psi_0}{\partial x_1^2}=\frac{1}{2}\ln\sigma-\frac{\overline{\sigma}}{4\sigma},$$
$$\frac{\partial^2\psi_0}{\partial x_1\partial x_2}=i\frac{\overline{\sigma}}{4\sigma},\quad \triangle\psi_0=\ln\sigma. \tag{2.11}$$

Substituting (2.8) into (2.5), we obtain the basic fundamental matrix of the equation (1.4)

$$\Phi(x-y)=\operatorname{Re}\Gamma(x-y), \tag{2.12}$$

where

$$\Gamma(x-y)=m\ln\sigma+\frac{1}{4}n\frac{\overline{\sigma}}{\sigma} \tag{2.13}$$

$$m=\begin{Vmatrix} m_1 & 0 & m_2 & 0\\ 0 & m_1 & 0 & m_2\\ m_2 & 0 & m_3 & 0\\ 0 & m_2 & 0 & m_3\end{Vmatrix},\quad n=\begin{Vmatrix} e_4 & ie_4 & e_5 & ie_5\\ ie_4 & -e_4 & ie_5 & -e_5\\ e_5 & ie_5 & e_6 & ie_6\\ ie_5 & -e_5 & ie_6 & -e_6\end{Vmatrix}, \tag{2.14}$$

$$m_1=e_1+\frac{e_4}{2},\quad m_2=e_2+\frac{e_5}{2},\quad m_3=e_3+\frac{e_6}{2}. \tag{2.15}$$

It is evident that $\Phi(x-y)$ is a symmetric matrix. It easily follows from (2.12) and (2.13) that all elements of Φ are single-valued functions on the whole plane and they have a logarithmic singularity at most. It can be shown that columns of the matrices $\Gamma(x-y)$ and $\Phi(x-y)$ are solutions to the equation (1.4) with respect to x for any $x\neq y$.

Let us rewrite (2.12) as

$$\Phi(x-y)=\operatorname{Re}\begin{Vmatrix} L^{(1)} & L^{(2)}\\ L^{(3)} & L^{(4)}\end{Vmatrix}\psi_0,\quad L^{(i)}=\|L_{kj}^{(i)}\|_{2\times2},\quad i=\overline{1,4}, \tag{2.16}$$

$$L_{kj}^{(1)}=e_1\delta_{kj}\triangle+e_4\partial_k\partial_j,\quad L_{kj}^{(2)}=e_2\delta_{kj}\triangle+e_5\partial_k\partial_j,$$
$$L_{kj}^{(3)}=e_2\delta_{kj}\triangle+e_5\partial_k\partial_j,\quad L_{kj}^{(4)}=e_3\delta_{kj}\triangle+e_6\partial_k\partial_j. \tag{2.17}$$

We also rewrite (1.1) in the matrix form

$$Cu=\psi, \tag{2.18}$$

where

$$C=\begin{Vmatrix} C^{(1)} & C^{(2)}\\ C^{(3)} & C^{(4)}\end{Vmatrix},\quad C^{(i)}=\|C_{kj}^{(i)}\|_{2\times2},\quad i=\overline{1,4}, \tag{2.19}$$

$$C_{kj}^{(1)}=a_1\delta_{kj}\triangle+b_1\partial_k\partial_j,\quad C_{kj}^{(2)}=c\delta_{kj}\triangle+d\partial_k\partial_j,$$
$$C_{kj}^{(3)}=c\delta_{kj}\triangle+d\partial_k\partial_j,\quad C_{kj}^{(4)}=a_2\delta_{kj}\triangle+b_2\partial_k\partial_j. \tag{2.20}$$

We put

$$u_0(x) = \frac{1}{2\pi} \int_D \Phi(x-y)\psi(y)\, dy_1\, dy_2. \tag{2.21}$$

Then, due to the equation

$$\begin{Vmatrix} C^{(1)} & C^{(2)} \\ C^{(3)} & C^{(4)} \end{Vmatrix} \begin{Vmatrix} L^{(1)} & L^{(2)} \\ L^{(3)} & L^{(4)} \end{Vmatrix} = E \triangle \triangle,$$

we get

$$\begin{aligned} Cu_0(x) &= \frac{1}{2\pi} \triangle \triangle \int_D \operatorname{Re} \psi_0 \psi(y) dy_1 dy_2 = \\ &= \frac{1}{2\pi} \int_D \ln r\psi\, dy_1\, dy_2 = \psi(x), \quad x \in D. \end{aligned} \tag{2.22}$$

Thus we have proved that $u_0(x)$ is a particular solution to equation (1.1). In (2.21), D denotes either D^+ or D^-, ψ is a continuous vector in D along with its first order derivatives. When $D = D^-$, then the vector ψ has to satisfy the following decay condition at infinity

$$\psi(y) = O(R^{-1-\alpha}), \quad \alpha > 0, \quad R = \sqrt{y_1^2 + y_2^2}. \tag{2.23}$$

3. Singular Matrices of Solutions

Using the basic fundamental matrix, we will construct the so-called singular matrices of solutions and study their properties.

For simplicity, we will introduce the special generalized stress operators. Let the elements of the matrix (1.11) be defined as follows

$$\varkappa_1 = 2\mu_1, \quad \varkappa_2 = 2\mu_2, \quad \varkappa_3 = 2\mu_3. \tag{3.1}$$

Denote by L the generalized operator $\overset{\varkappa}{T}$ with $\varkappa$ defined by (3.1) (the corresponding matrix is denoted by $\varkappa_{_L}$). Then by (1.6),

$$\begin{aligned} &(Lu)_1 = L_1 n_1 - L_2 n_2, \quad (Lu)_2 = L_2 n_1 + L_1 n_2, \\ &(Lu)_3 = L_3 n_1 - L_1 n_2, \quad (Lu)_4 = L_4 n_1 + L_3 n_2, \end{aligned} \tag{3.2}$$

where L_1, L_2, L_3, L_4 are defined by (1.7).

It follows from (1.11) that

$$\overset{\varkappa}{T} u = Lu + (\varkappa - \varkappa_{_L}) \frac{\partial u}{\partial s(x)}\,. \tag{3.3}$$

First let us construct $L\Phi$, i.e., $L\Gamma$ (see (2.12)). Denote by $\Gamma^{(k)}$ the k-th column of the matrix Γ given by (2.13). $\theta_k', \theta_k'', \omega_k'$ and ω_k'' denote expressions given by (1.2) and (1.8) for the vector $\Gamma^{(k)}$, $k=\overline{1,4}$. Simple manipulations lead to

$$\begin{aligned}
&\theta_1' = (e_1+e_4)\frac{\partial}{\partial x_1}\ln\sigma = -(e_1+e_4)i\frac{\partial\ln\sigma}{\partial x_2}, \quad \theta_1'' = (e_2+e_5)\frac{\partial\ln\sigma}{\partial x_1},\\
&\omega_1' = -ie_1\frac{\partial}{\partial x_1}\ln\sigma, \quad \omega_1'' = -ie_2\frac{\partial}{\partial x_1}\ln\sigma,\\
&\theta_2' = -(e_1+e_4)\frac{\partial}{\partial x_2}\ln\sigma, \quad \theta_2'' = (e_2+e_5)\frac{\partial\ln\sigma}{\partial x_2},\\
&\omega_2' = e_1\frac{\partial}{\partial x_1}\ln\sigma, \quad \omega_2'' = e_2\frac{\partial}{\partial x_1}\ln\sigma,\\
&\theta_3' = (e_2+e_5)i\frac{\partial\ln\sigma}{\partial x_2}, \quad \theta_3'' = -(e_3+e_6)i\frac{\partial\ln\sigma}{\partial x_2},\\
&\omega_3' = -ie_2\frac{\partial\ln\sigma}{\partial x_1}, \quad \omega_3'' = -ie_3\frac{\partial\ln\sigma}{\partial x_1},\\
&\theta_4' = (e_2+e_5)i\frac{\partial\ln\sigma}{\partial x_2}, \quad \theta_4'' = (e_3+e_6)\frac{\partial\ln\sigma}{\partial x_2},\\
&\omega_4' = e_2\frac{\partial\ln\sigma}{\partial x_1}, \quad \omega_4'' = e_3\frac{\partial\ln\sigma}{\partial x_1}.
\end{aligned}$$

From these formulas together with (2.4), (1.7) and (3.2), it follows

$$L_x\Phi(x-y) = \operatorname{Im}\frac{\partial}{\partial s(x)}(E+iE_1)\ln\sigma, \tag{3.4}$$

where

$$E_1 = \begin{Vmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -1 \end{Vmatrix}. \tag{3.5}$$

Applying (3.4) and (3.3), we get

$$\overset{\varkappa}{T}_x\ \Phi(x-y) = \frac{\partial}{\partial s(x)}\operatorname{Im}\big[(E+iE_1)\ln\sigma + i(\varkappa-\varkappa_{_L})\Gamma(x-y)\big]. \tag{3.6}$$

If $\varkappa_1=\varkappa_2=\varkappa_3=0$, then $\varkappa=0$ (see (1.13)), and (3.6) implies

$$T_x\Phi(x-y) = \frac{\partial}{\partial s(x)}\operatorname{Im}\Big[(E+iA)\ln\sigma + \frac{B}{2}\frac{\overline{\sigma}}{\sigma}\Big], \tag{3.7}$$

where

$$A = \begin{Vmatrix} 0 & 1-A_1 & 0 & -A_2 \\ -1+A_1 & 0 & A_2 & 0 \\ 0 & -A_3 & 0 & 1-A_4 \\ A_3 & 0 & -1+A_4 & 0 \end{Vmatrix}, \qquad B = \begin{Vmatrix} B_1 & iB_1 & B_2 & iB_2 \\ iB_1 & -B_1 & iB_2 & -B_2 \\ B_3 & iB_3 & B_4 & iB_4 \\ iB_3 & -B_3 & iB_4 & -B_4 \end{Vmatrix}, \tag{3.8}$$

$$\begin{aligned} A_1 &= 2(\mu_1 m_1 + \mu_3 m_2), \quad A_2 = 2(\mu_1 m_2 + \mu_3 m_3), \\ A_3 &= 2(\mu_3 m_1 + \mu_2 m_2), \quad A_4 = 2(\mu_3 m_2 + \mu_2 m_3), \end{aligned} \tag{3.9}$$

$$\begin{aligned} B_1 &= \mu_1 e_4 + \mu_3 e_5, \quad B_2 = \mu_2 e_5 + \mu_3 e_6. \\ B_3 &= \mu_2 e_5 + \mu_3 e_4, \quad B_4 = \mu_2 e_6 + \mu_3 e_5. \end{aligned} \tag{3.10}$$

It is obvious that $T_x \Phi(x-y)$ is a singular kernel (in the sense of Cauchy) on Liapunov ($C^{1+\alpha}$) curves since the matrix A is not identical zero.

Replacing x by y and vice versa in matrix (3.6), we arrive to

$$\big[\overset{\varkappa}{T}_y \, \Phi(y-x)\big]' = \frac{\partial}{\partial s(y)} \operatorname{Im} \big[i\Gamma(y-x)(\varkappa_L - \varkappa) + (E - iE_1)\ln\sigma\big]. \tag{3.11}$$

where $(\;)'$ denotes transposition.

It is easy to check that the columns of the matrix (3.11) are solutions of the equation (1.4) with respect to the variable x for any $x \neq y$. It is also evident that the elements of the matrix (3.11) are singular kernels in the sense of Cauchy since $m(\varkappa_L - \varkappa) - E_1 \neq 0$. Let us note that if $m(\varkappa_L - \varkappa) = E_1$, then $[\overset{\varkappa}{T}_y \; \Phi(y-x)]'$ is a weakly singular kernel. The previous equation yields

$$\varkappa = \varkappa_L - m^{-1} E_1, \tag{3.12}$$

where

$$m^{-1} = \frac{1}{\Delta_0} \begin{Vmatrix} m_3 & 0 & -m_2 & 0 \\ 0 & m_3 & 0 & -m_2 \\ -m_2 & 0 & m_1 & 0 \\ 0 & -m_2 & 0 & m_1 \end{Vmatrix}, \quad \Delta_0 = m_1 m_3 - m_2^2. \tag{3.13}$$

From (3.12) and (3.13) it follows

$$\varkappa_1 = 2\mu_1 - \frac{m_3}{\Delta_0}, \quad \varkappa_2 = 2\mu_2 - \frac{m_1}{\Delta_0}, \quad \varkappa_3 = 2\mu_3 + \frac{m_2}{\Delta_0}. \tag{3.14}$$

Denote by N the stress operator $\overset{\varkappa}{T}$ with $\varkappa$ given by (3.12). Then we have

$$[N_y \Phi(y-x)]' = \frac{\partial}{\partial s(y)} \operatorname{Im} \Big(E \ln\sigma - \frac{\varepsilon}{2} \frac{\overline{\sigma}}{\sigma}\Big), \tag{3.15}$$

where

$$\varepsilon = \begin{Vmatrix} \varepsilon_1 & i\varepsilon_1 & \varepsilon_3 & i\varepsilon_3 \\ i\varepsilon_1 & -\varepsilon_1 & i\varepsilon_3 & -\varepsilon_3 \\ \varepsilon_2 & i\varepsilon_2 & \varepsilon_4 & i\varepsilon_4 \\ i\varepsilon_2 & -\varepsilon_2 & i\varepsilon_4 & -\varepsilon_4 \end{Vmatrix} \tag{3.16}$$

$$\begin{aligned} 2\Delta_0\varepsilon_1 &= e_5m_2 - e_4m_3, \quad 2\Delta_0\varepsilon_3 = e_4m_2 - e_5m_1, \\ 2\Delta_0\varepsilon_2 &= e_6m_2 - e_5m_3, \quad 2\Delta_0\varepsilon_4 = e_5m_2 - e_6m_1, \end{aligned} \tag{3.17}$$

Δ_0 is defined by (3.13).

Taking into account expressions for m_j $(j = \overline{1,3})$ and e_j $(j = \overline{4,6})$ (see (2.15) and (2.3)), we have for the coefficients ε_j $(j = \overline{1,4})$

$$\begin{gathered} \delta_0\varepsilon_1 = b_1(2a_2 + b_2) - d(2c + d), \quad \delta_0\varepsilon_3 = 2(da_2 - cb_2), \\ \delta_0\varepsilon_2 = 2(da_1 - cb_1), \quad \delta_0\varepsilon_4 = b_2(2a_1 + b_1) - d(2c + d), \\ \delta_0 = (2a_1 + b_1)(2a_2 + b_2) - (2c + d)^2 \equiv 4\Delta_0 d_1 d_2. \end{gathered} \tag{3.18}$$

Later on, we will show that $\Delta_0 > 0$, i.e., $\delta_0 > 0$.

It follows from (3.15)

$$N_x\Phi(x - y) = \frac{\partial}{\partial s(x)} \operatorname{Im}\left(E\ln\sigma - \frac{\varepsilon'}{2}\frac{\overline{\sigma}}{\sigma}\right) \equiv m^{-1}\frac{\partial}{\partial s(x)}\operatorname{Im}\Gamma(x - y). \tag{3.19}$$

Quite similarly we have

$$N_x \operatorname{Im}\Gamma(x - y) = -m^{-1}\frac{\partial\Phi(x - y)}{\partial s(x)}. \tag{3.20}$$

Due to the equation $\Phi(x - y) = \operatorname{Re}\Gamma(x - y)$, we get from (3.19) and (3.20)

$$N_x\Gamma(x - y) = -im^{-1}\frac{\partial\Gamma(x - y)}{\partial s(x)}, \tag{3.21}$$

Now (3.19) implies

$$T_x\Phi(x - y) = \operatorname{Im}(m^{-1} - i\varkappa_N)\frac{\partial\Gamma(x - y)}{\partial s(x)}, \tag{3.22}$$

where $\varkappa_N$ is defined by (1.13) with $\varkappa_1$, $\varkappa_2$ and $\varkappa_3$ given by (3.14).

In turn, (3.22) yields

$$[T_y\Phi(x - y)]' = \operatorname{Im}\frac{\partial\Gamma(y - x)}{\partial s(x)}\,(m^{-1} + i\varkappa_N). \tag{3.23}$$

Analogously we have

$$\big[\overset{\varkappa}{T}_y\,\Phi(y - x)\big]' = \operatorname{Im}\frac{\partial\Gamma(y - x)}{\partial s(x)}\,[m^{-1} + i(\varkappa_N - \varkappa)]. \tag{3.24}$$

In what follows, we will see that the operator N plays an essential role in the study of the first boundary value problem (it enables us to reduce the BVP to a Fredholm equation of the second kind with a weakly singular kernel).

4. Matrix $M(x-y)$

In this section, we will construct the special fundamental matrix which reduces the second BVP to a Fredholm integral equation of the second kind. We denote the matrix by $M(x-y)$ and look for it as

$$M(y-x) = \mathrm{Re}(\Gamma - E_0 \ln \sigma X)Y, \tag{4.1}$$

where Γ is given by (2.13),

$$E_0 = iE + E_1, \tag{4.2}$$

E is again the unit matrix and E_1 is given by (3.5); the real matrices X and Y will be defined later on.

Each column of $M(x-y)$ is a solution to equation (1.4) with respect to the variable x provided $x \neq y$.

Upon acting the operation T_x on the matrix $M(x-y)$ and applying the equation (3.7), we get

$$T_x M(x-y) = \frac{\partial}{\partial s(x)} \mathrm{Im}\left[(E+iA)\ln\sigma + \frac{B}{2}\frac{\overline{\sigma}}{\sigma} + i\varkappa_L E_0 \ln \sigma X\right]Y. \tag{4.3}$$

We will try now to determine matrices X and Y in such a way that, on one hand, the coefficients of singular terms in (4.3) would vanish (i.e., the expression (4.3) would involve only weakly singular terms) and, on the other hand, the coefficient of the term $\frac{\partial\theta}{\partial s(x)}$ would be converted into the unit matrix. These requirements lead to the equations

$$A + \varkappa_L E_1 X = 0, \ (E - \varkappa_L \cdot X)Y = E. \tag{4.4}$$

Taking into account expressions for $\varkappa_L$ and E_1, we get from the first equation

$$A - 2\begin{Vmatrix} \mu_1 & 0 & \mu_3 & 0 \\ 0 & \mu_1 & 0 & \mu_3 \\ \mu_3 & 0 & \mu_2 & 0 \\ 0 & \mu_3 & 0 & \mu_2 \end{Vmatrix} X = 0,$$

whence

$$X = \frac{1}{2\Delta_1}\begin{Vmatrix} \mu_2 & 0 & -\mu_3 & 0 \\ 0 & \mu_2 & 0 & -\mu_3 \\ -\mu_3 & 0 & \mu_1 & 0 \\ 0 & -\mu_3 & 0 & \mu_1 \end{Vmatrix} A,$$

where

$$\Delta_1 = \mu_1\mu_2 - \mu_3^2. \tag{4.5}$$

Later on, it will be shown that $\Delta_1 > 0$.

Further, (3.8) along with the equations

$$\begin{aligned}\mu_2(1-A_1)+\mu_3 A_3 &= \mu_2 - 2\Delta_1 m_1,\\ \mu_2 A_2+\mu_3(1-A_4) &= \mu_3 + 2\Delta_1 m_2,\\ \mu_3(1-A_1)+\mu_1 A_3 &= \mu_3 + 2\Delta_1 m_2,\\ \mu_3 A_2+\mu_1(1-A_4) &= \mu_1 - 2\Delta_1 m_3,\end{aligned}$$

yields

$$X=\frac{1}{2\Delta_1}\begin{Vmatrix} 0 & \mu_2-2\Delta_1 m_1 & 0 & -\mu_3-2\Delta_1 m_2\\ -\mu_2+2\Delta_1 m_1 & 0 & \mu_3+2\Delta_1 m_2 & 0\\ 0 & -\mu_3-2\Delta_1 m_2 & 0 & \mu_1-2\Delta_1 m_3\\ \mu_3+2\Delta_1 m_2 & 0 & -\mu_1+2\Delta_1 m_3 & 0\end{Vmatrix}. \quad (4.6)$$

Let us note that

$$\varkappa_L X = -\begin{Vmatrix} 1-A_1 & 0 & -A_2 & 0\\ 0 & 1-A_1 & 0 & -A_2\\ -A_3 & 0 & 1-A_4 & 0\\ 0 & -A_3 & 0 & 1-A_4\end{Vmatrix}.$$

Then the second equation of (4.4) implies

$$\begin{Vmatrix} 2-A_1 & 0 & -A_2 & 0\\ 0 & 2-A_1 & 0 & -A_2\\ -A_3 & 0 & 2-A_4 & 0\\ 0 & -A_3 & 0 & 2-A_4\end{Vmatrix} Y = E,$$

whence finally we have

$$Y=\frac{1}{\Delta_2}\begin{Vmatrix} 2-A_4 & 0 & A_2 & 0\\ 0 & 2-A_4 & 0 & A_2\\ A_3 & 0 & 2-A_1 & 0\\ 0 & A_3 & 0 & 2-A_1\end{Vmatrix}, \quad (4.7)$$

where

$$\Delta_2 = (2-A_1)(2-A_4)-A_2A_3. \quad (4.8)$$

Thus we have determined matrices X and Y uniquely. Substituting them into (4.3), we get

$$T_x M(x-y) = \frac{\partial}{\partial s(x)}\,\mathrm{Im}\left(E\ln\sigma + \frac{H}{2\Delta_2}\frac{\overline{\sigma}}{\sigma}\right), \quad (4.9)$$

where

$$H=\begin{Vmatrix} H_1 & iH_1 & H_2 & iH_2\\ iH_1 & -H_1 & iH_2 & -H_2\\ H_3 & iH_3 & H_4 & iH_4\\ iH_3 & -H_3 & iH_4 & -H_4\end{Vmatrix}, \quad (4.10)$$

$$H_1 = B_1(2-A_4) + B_2A_3, \quad H_2 = B_1A_2 + B_2(2-A_1),$$
$$H_3 = B_3(2-A_4) + B_4A_3, \quad H_4 = B_3A_2 + B_4(2-A_1); \tag{4.11}$$

constants A_j and B_j ($j=\overline{1,4}$) are given by (3.9) and (3.10).

Throughout the paper, X and Y denote matrices determined by (4.6) and (4.7), respectively. The matrix $M(x-y)$ (see (4.1)) is a multifunction, since matrices X and Y are not zero-matrices. In what follows, we will show how to get rid of the multivalence of the matrix $M(x-y)$.

5. Generalized Green Formulas

Let u and v be four-dimensional vectors in D^+. The equations (1.1) can be written as follows

$$(Cu)_1 = \partial_1\sigma'_{11} + \partial_2\sigma'_{12}, \quad (Cu)_2 = \partial_1\sigma'_{12} + \partial_2\sigma'_{22},$$
$$(Cu)_3 = \partial_1\sigma''_{11} + \partial_2\sigma''_{21}, \quad (Cu)_4 = \partial_1\sigma''_{12} + \partial_2\sigma''_{22}, \tag{5.1}$$

where the σ'_{11} , ... ,σ''_{22} are the components of the generalized stress tensor given by (1.6), (1.7) and (1.8). We note that the derivatives in (5.1) are taken with respect to the coordinates of the point $y=(y_1,y_2)$ (u and v are functions of y and $\partial_k = \partial/\partial y_k$, $k=1,2$).

From (5.1) and (1.1) it follows that

$$(Cu)_k = \psi'_k(y), \quad (Cu)_{k+2} = \psi''_k(y), \quad k=1,2, \tag{5.2}$$

Multiplicating the k-th equation of (5.1) by v_k, integrating over D^+ and summing the results, we arrive to

$$\int\limits_{D^+} vCudy_1dy_2 = \int\limits_S v \overset{\varkappa}{T} udS - \int\limits_{D^+} \overset{\varkappa}{T}(u,v)dy_1dy_2, \tag{5.3}$$

where

$$\overset{\varkappa}{T}(u,v) = \sigma'_{11}\partial_1v'_1 + \sigma'_{21}\partial_2v'_1 + \sigma'_{12}\partial_1v'_2 + \sigma'_{22}\partial_2v'_2 +$$
$$+ \sigma''_{11}\partial_1v''_1 + \sigma''_{21}\partial_2v''_1 + \sigma''_{12}\partial_1v''_2 + \sigma''_{22}\partial_2v''_2. \tag{5.4}$$

Here we have used notation

$$v_1 = v'_1, \quad v_2 = v'_2, \quad v_3 = v''_1, \quad v_4 = v''_2. \tag{5.5}$$

To give a more symmetric form to the expression (5.4), we set

$$\theta' = 2\xi_1, \quad \theta'' = 2\xi_2, \quad \partial_1u'_1 - \partial_2u'_2 = 2\xi_3,$$
$$\partial_1u''_1 - \partial_2u''_2 = 2\xi_4, \quad \partial_1u'_2 + \partial_2u'_1 = 2\xi_5, \tag{5.6}$$
$$\partial_1u''_2 + \partial_2u''_1 = 2\xi_6, \quad \omega' = 2\xi_7, \quad \omega'' = 2\xi_8,$$

$$\begin{aligned}
&\partial_1 v_1' + \partial_2 v_2' = 2\eta_1, \quad \partial_1 v_1'' + \partial_2 v_2'' = 2\eta_2,\\
&\partial_1 v_1' - \partial_2 v_2' = 2\eta_3, \quad \partial_1 v_1'' - \partial_2 v_2'' = 2\eta_4,\\
&\partial_1 v_2' + \partial_2 v_1' = 2\eta_5, \quad \partial_1 v_2'' + \partial_2 v_1'' = 2\eta_6,\\
&\partial_1 v_2' - \partial_2 v_1' = 2\eta_7, \quad \partial_1 v_2'' - \partial_2 v_1'' = 2\eta_8.
\end{aligned} \tag{5.7}$$

Now (5.6) and (5.7) yield

$$\begin{aligned}
\partial_1 u_1' &= \xi_1 + \xi_3, \quad \partial_2 u_2' = \xi_1 - \xi_3,\\
\partial_1 u_1'' &= \xi_2 + \xi_4, \quad \partial_2 u_2'' = \xi_2 - \xi_4,\\
\partial_1 u_2' &= \xi_5 + \xi_7, \quad \partial_2 u_1' = \xi_5 - \xi_7,\\
\partial_1 u_2'' &= \xi_6 + \xi_8, \quad \partial_2 u_1'' = \xi_6 - \xi_8,
\end{aligned} \tag{5.8}$$

$$\begin{aligned}
\partial_1 v_1' &= \eta_1 + \eta_3, \quad \partial_2 v_2' = \eta_1 - \eta_3,\\
\partial_1 v_1'' &= \eta_2 + \eta_4, \quad \partial_2 v_2'' = \eta_2 - \eta_4,\\
\partial_1 v_2' &= \eta_5 + \eta_7, \quad \partial_2 v_1' = \eta_5 - \eta_7,\\
\partial_1 v_2'' &= \eta_6 + \eta_8, \quad \partial_2 v_1'' = \eta_6 - \eta_8.
\end{aligned} \tag{5.9}$$

Substitution of (5.8) and (5.9) into (5.4) leads to

$$\begin{aligned}
\overset{\varkappa}{T}(u,v) = {} & 2[2(b_1-\lambda_5)+\varkappa_1]\xi_1\eta_1 + 2[2(d+\lambda_5)+\varkappa_3](\xi_1\eta_2+\xi_2\eta_1) + \\
&+ 2[2(b_2-\lambda_5)+\varkappa_2]\xi_2\eta_2 + 2(2\mu_1-\varkappa_1)(\xi_3\eta_3+\xi_5\eta_5) + \\
&+ 2(2\mu_3-\varkappa_3)(\xi_3\eta_4+\xi_4\eta_3+\xi_5\eta_6+\xi_6\eta_5) + \\
&+ 2(2\mu_2-\varkappa_2)(\xi_4\eta_4+\xi_6\eta_6) + 2(-2\lambda_5+\varkappa_1)\xi_7\eta_7 + \\
&+ 2(2\lambda_5+\varkappa_3)(\xi_7\eta_8+\xi_8\eta_7) + 2(-2\lambda_5+\varkappa_2)\xi_8\eta_8.
\end{aligned} \tag{5.10}$$

Note that $\overset{\varkappa}{T}(u,v)$ is a symmetric function with respect to ξ_k and η_k $(k=\overline{1,8})$, i.e.,

$$\overset{\varkappa}{T}(u,v) = \overset{\varkappa}{T}(v,u) \tag{5.11}$$

Clearly we have (cf. (5.3))

$$\int\limits_{D^+} uCv\,dy_1dy_2 = \int\limits_{S} u\overset{\varkappa}{T}v\,ds - \int\limits_{D^+}\overset{\varkappa}{T}(v,u)\,dy_1dy_2 \tag{5.12}$$

Now (5.3) and (5.12) along with (5.11) imply

$$\int\limits_{D^+}(uCv - vCu)\,dy_1dy_2 = \int\limits_{S}(u\overset{\varkappa}{T}v - v\overset{\varkappa}{T}u)\,ds. \tag{5.13}$$

Let u and v be complex vectors and, in addition, $v = \overline{u}$. Then $\overset{\varkappa}{T}(u, \overline{u}) = \overset{\varkappa}{T}(\overline{u}, u)$ and

$$\int\limits_{D^+} (uC\overline{u} - \overline{u}Cu)dy_1dy_2 = \int\limits_S (u\overset{\varkappa}{T}\overline{u} - \overline{u}\overset{\varkappa}{T}u)ds. \tag{5.14}$$

Let now u be a solution to (1.4) and $v = u$. Then from (5.3) it follows that

$$\int\limits_{D^+} \overset{\varkappa}{T}(u, u)dy_1dy_2 = \int\limits_S u\overset{\varkappa}{T}uds, \tag{5.15}$$

where

$$\begin{aligned}\overset{\varkappa}{T}(u, u) = {} & 2[2(b_1 - \lambda_5) + \varkappa_1]\xi_1^2 + 4[2(d + \lambda_5) + \varkappa_3]\xi_1\xi_2 + \\ & + 2[2(b_2 - \lambda_5) + \varkappa_2]\xi_2^2 + 2(2\mu_1 - \varkappa_1)(\xi_3^2 + \xi_5^2) + \\ & + 4(2\mu_3 - \varkappa_3)(\xi_3\xi_4 + \xi_5\xi_6) + \\ & + 2(2\mu_2 - \varkappa_2)(\xi_4^2 + \xi_6^2) + 2(-2\lambda_5 + \varkappa_1)\xi_7^2 + \\ & + 4(2\lambda_5 + \varkappa_3)\xi_7\xi_8 + (-2\lambda_5 + \varkappa_2)\xi_8^2.\end{aligned} \tag{5.16}$$

It is evident that $\overset{\varkappa}{T}(u, u)$ is a quadratic form in variables $\xi_1, \dots, \xi_8$. The necessary and sufficient conditions for $\overset{\varkappa}{T}(u, u)$ to be positive definite read

$$\begin{gathered}2(b_1 - \lambda_5) + \varkappa_1 > 0, \\ [2(b_1 - \lambda_5) + \varkappa_1][2(b_2 - \lambda_5) + \varkappa_2] - [2(d + \lambda_5) + \varkappa_3]^2 > 0, \\ 2\mu_1 - \varkappa_1 > 0, \quad (2\mu_1 - \varkappa_1)(2\mu_2 - \varkappa_2) - (2\mu_3 - \varkappa_3)^2 > 0, \\ -2\lambda_5 + \varkappa_1 > 0, \quad (-2\lambda_5 + \varkappa_1)(-2\lambda_5 + \varkappa_2) - (2\lambda_5 + \varkappa_3)^2 > 0.\end{gathered} \tag{5.17}$$

If $\varkappa = 0$ (i.e., $\varkappa_1 = \varkappa_2 = \varkappa_3 = 0$), then (5.16) represents the doubled specific potential energy of elastic mixture at the point y

$$\begin{aligned}T(u, u) = {} & 4(b_1 - \lambda_5)\xi_1^2 + 8(d + \lambda_5)\xi_1\xi_2 + 4(b_2 - \lambda_5)\xi_2^2 + \\ & + 4\mu_1(\xi_3^2 + \xi_5^2) + 8\mu_3(\xi_3\xi_4 + \xi_5\xi_6) + \\ & + 4\mu_2(\xi_4^2 + \xi_6^2) - 4\lambda_5(\xi_7 - \xi_8)^2.\end{aligned} \tag{5.18}$$

Conditions (5.17) in that case read

$$\begin{gathered}b_1 - \lambda_5 > 0, \quad (b_1 - \lambda_5)(b_2 - \lambda_5) - (d + \lambda_5)^2 > 0, \\ \mu_1 > 0, \quad \mu_1\mu_2 - \mu_3^2 > 0, \quad -\lambda_5 > 0.\end{gathered} \tag{5.19}$$

In what follows, these conditions are supposed to be fulfilled since from the physical considerations it is obvious that the potential energy is a positive function.

Let us consider one more particular case where $\varkappa$ is given by (3.12) and (3.14). Then $\overset{\varkappa}{T} \equiv N$, and we have

$$\begin{aligned} N(u,u) &= 2\Big[2(a_1+b_1) - \frac{m_3}{\Delta_0}\Big]\xi_1^2 + 4\Big[2(c+d) + \frac{m_2}{\Delta_0}\Big]\xi_1\xi_2 + \\ &+ 2\Big[2(a_2+b_2) - \frac{m_1}{\Delta_0}\Big]\xi_2^2 + \frac{2m_3}{\Delta_0}(\xi_5^2+\xi_5^3) - \frac{4m_2}{\Delta_0}(\xi_3\xi_4+\xi_5\xi_6) + \\ &+ \frac{2m_1}{\Delta_0}(\xi_4^2+\xi_6^2) + 2\Big(2a_1 - \frac{m_3}{\Delta_0}\Big)\xi_7^2 + \\ &+ 4\Big(2c + \frac{m_2}{\Delta_0}\Big)\xi_7\xi_8 + 2(2a_2 - \frac{m_1}{\Delta_0})\xi_8^2 \end{aligned} \tag{5.20}$$

due to (5.16).

Inequalities (5.17) now read as

$$\begin{gathered} 2(a_1+b_1) - \frac{m_3}{\Delta_0} > 0, \\ \Big[2(a_1+b_1) - \frac{m_3}{\Delta_0}\Big]\Big[2(a_2+b_2) - \frac{m_1}{\Delta_0}\Big] - \Big[2(c+d) + \frac{m_2}{\Delta_0}\Big]^2 > 0, \\ \frac{m_3}{\Delta_0} > 0, \quad \frac{1}{\Delta_0} > 0, \quad 2a_1 - \frac{m_3}{\Delta_0} > 0, \\ \Big(2a_1 - \frac{m_3}{\Delta_0}\Big)\Big(2a_2 - \frac{m_1}{\Delta_0}\Big) - \Big(2c + \frac{m_2}{\Delta_0}\Big)^2 > 0. \end{gathered} \tag{5.21}$$

Let us first show that (5.19) implies (5.21). We begin with the proof of the inequalities $d_1 > 0$ and $d_2 > 0$ (see (2.2) and (2.3)).

We have

$$\begin{aligned} d_2 &= a_1a_2 - c^2 = (\mu_1 - \lambda_5)(\mu_2 - \lambda_5) - (\mu_3 + \lambda_5)^2 = \\ &= \mu_1\mu_2 - \mu_3^2 - \lambda_5\big[(\sqrt{\mu_1} - \sqrt{\mu_2})^2 + 2(\sqrt{\mu_1\mu_2} + \mu_3)\big]. \end{aligned}$$

Since $\mu_1\mu_2 - \mu_3^2 > 0$, we get $-\sqrt{\mu_1\mu_2} < \mu_3 < \sqrt{\mu_1\mu_2}$ and $\sqrt{\mu_1\mu_2} + \mu_3 > 0$. Note that the inequality $-\lambda_5 > 0$ implies $d_2 > 0$. Quite similarly we have

$$\begin{aligned} d_1 &= (b_1 - \lambda_5 + \mu_1)(b_2 - \lambda_5 + \mu_2) - (d + \lambda_5 + \mu_3)^2 = \\ &= (b_1 - \lambda_5)(b_2 - \lambda_5) - (d + \lambda_5)^2 + \mu_1\mu_2 - \mu_3^2 + \\ &+ \big(\sqrt{\mu_2(b_1 - \lambda_5)} - \sqrt{\mu_1(b_2 - \lambda_5)}\big)^2 + \\ &+ 2\big[\sqrt{\mu_1\mu_2(b_1 - \lambda_5)(b_2 - \lambda_5)} - \mu_3(d + \lambda_5)\big], \end{aligned}$$

whence, applying again (5.19), we get the inequality $d_1 > 0$. Due to (2.15), we find

$$\begin{gathered} m_1 = \frac{1}{2}\Big(\frac{a_2}{d_2} + \frac{a_2 + b_2}{d_1}\Big), \quad m_2 = -\frac{1}{2}\Big(\frac{c}{d_2} + \frac{c + d}{d_1}\Big), \\ m_3 = \frac{1}{2}\Big(\frac{a_1}{d_2} + \frac{a_1 + b_1}{d_1}\Big). \end{gathered}$$

It is obvious that $d_1 > 0$ and $d_2 > 0$ yield $a_1 > 0$, $a_2 > 0$, $a_1 + b_1 > 0$, $a_2 + b_2 > 0$ and, consequently, $m_1 > 0$ and $m_3 > 0$.

Bearing in mind the equation $m_1 m_3 - m_2^2 = \Delta_0$, we have

$$\begin{aligned} 4\Delta_0 d_1 d_2 = \delta_0 = (2a_1 + b_1)(2a_2 + b_2) - 2c(c+d)^2 = \\ = d_2 + d_1 + a_1(a_2 + b_2) + a_2(a_1 + b_1) - 2c(c+d). \end{aligned}$$

We can easily prove that

$$a_1(a_2 + b_2) + a_2(a_1 + b_1) - 2c(c+d) > 0$$

from which $\Delta_0 > 0$ follows immediately.

By direct evaluation, we can verify that

$$\begin{aligned} &\Big(2a_1 - \frac{m_3}{\Delta_0}\Big)\Big(2a_2 - \frac{m_1}{\Delta_0}\Big) - \Big(2c + \frac{m_2}{\Delta_0}\Big)^2 = \frac{d_2}{d_1\Delta_0} > 0, \\ &\Big[2(a_1 + b_1) - \frac{m_3}{\Delta_0}\Big]\Big[2(a_2 + b_2) - \frac{m_1}{\Delta_0}\Big] - \Big[2(c+d) + \frac{m_2}{\Delta_0}\Big]^2 = \\ &\qquad = \frac{d_1}{d_2\Delta_0} > 0, \\ &2a_1 - \frac{m_3}{\Delta_0} = \frac{1}{2\Delta_0(a_2 + b_2)d_1 d_2}\big\{a_1(a_2 + b_2)d_2 + c^2 d_1 + \\ &\qquad + [a_1(a_2 + b_2) - c(c+d)]^2\big\} > 0, \\ &2(a_1 + b_1) - \frac{m_3}{\Delta_0} = \frac{1}{2\Delta_0 a_2 d_1 d_2}\big\{a_2(a_1 + b_1)d_1 + (c+d)^2 d_2 + \\ &\qquad + [a_2(a_1 + b_1) - c(c+d)]^2\big\} > 0. \end{aligned}$$

Thus all inequalities in (5.21) hold.

Formulas (5.13) and (5.15) can be generalized to unbounded domains of the type D^- if the conditions

$$\begin{gathered} \lim_{R\to\infty} \int\limits_{S(0,R)} u \overset{\varkappa}{T} v dS = 0, \quad \lim_{R\to\infty} \int\limits_{S(0,R)} v \overset{\varkappa}{T} u dS = 0, \\ \lim_{R\to\infty} \int\limits_{S(0,R)} u \overset{\varkappa}{T} u dS = 0 \end{gathered} \tag{5.22}$$

are fulfilled, where $S(0,R)$ is the circle centered at the origin and with the radius R; we assume that $(0,0) \in D^+$ and $S(0,R)$ envelopes the domain $\overline{D}^+$. Clearly, the conditions (5.22) hold if u and v meet conditions (1.14). As a result, we have the following formulas for the unbounded domain D^-

$$\int\limits_{D^-} (uCv - vCu) dy_1 dy_2 = \int\limits_{S} (v \overset{\varkappa}{T} u - u \overset{\varkappa}{T} v) dS, \tag{5.23}$$

$$\int\limits_{D^-} \overset{\varkappa}{T}(u,u)dy_1dy_2 = -\int\limits_S u \overset{\varkappa}{T} udS. \tag{5.24}$$

We note that (5.13) and (5.15) remain also valid for such D^+ which is a bounded, multiconnected domain surrounded by contours $S_1, \dots, S_m, S_{m+1}$ (we assume that S_{m+1} envelopes all other contours); $S = \bigcup\limits_{k=1}^{m+1} S_k$ is the boundary of D^+. The positive direction on S_k is the one which leaves the domain D^+ left-hand side.

6. General Representation of Solution

We will start with the following assertion.

Let $S \in C^{1+\beta}$, $0 < \beta \leq 1$, and let u be a regular solution of the equation (1.1) *in D^+. Then*

$$u(x) = \frac{1}{2\pi}\int\limits_S \big\{\big[\overset{\varkappa}{T}_y \,\Phi(y-x)\big]'(u)^+ - \Phi(y-x)(\overset{\varkappa}{T} u)^+\big\}dS +$$
$$+\frac{1}{2\pi}\int\limits_{D^+} \Phi(y-x)\psi(y)\,dy_1\,dy_2, \;\; x \in D^+, \tag{6.1}$$

where $\Phi(x-y)$ is the basic fundamental matrix and $[\overset{\varkappa}{T}_y \,\Phi(y-x)]'$ is given by (3.24).

Proof. Let $S(x,\varepsilon)$ be a circle centered at the point $x \in D^+$ and with the radius $\varepsilon > 0$, and let the corresponding closed disk $\overline{K(x,\varepsilon)} \subset D^+$. Denote $D_\varepsilon = D^+ \setminus \overline{K(x,\varepsilon)}$. Obviously $v(y) = \Phi^{(j)}(y-x)$ (j-th column of the matrix $\Phi(y-x)$) is a regular solution to (1.4) in D_ε. Now the equations

$$C_y\Phi^{(j)}(y-x) = 0, \quad Cu = \psi(y)$$

together with (5.13) give

$$-\int\limits_{D_\varepsilon} \Phi^{(j)}(y-x)\psi(y)\,dy_1\,dy_2 =$$
$$= \int\limits_S \big[(u)^+ \overset{\varkappa}{T}_y \,\Phi^{(j)}(y-x) - \Phi^{(j)}(y-x)(\overset{\varkappa}{T} u)^+\big]ds +$$
$$+ \int\limits_{S(x;\varepsilon)} \big[u(y)\,\overset{\varkappa}{T}_y \,\Phi^{(j)}(y-x) - \Phi^{(j)}(y-x)\,\overset{\varkappa}{T} u\big]dS. \tag{6.2}$$

We need to calculate the following integrals

$$J_1(x) = \int\limits_S \frac{\partial \ln\sigma}{\partial s(y)}\,dS, \tag{6.3}$$

$$J_2(x) = \int\limits_S \frac{\partial}{\partial s(y)} \frac{\overline{\sigma}}{\sigma}\, dS. \tag{6.4}$$

Applying the equation

$$0 = \int\limits_{D^+} \Big(\frac{\partial^2 u}{\partial y_1 \partial y_2} - \frac{\partial^2 u}{\partial y_2 \partial y_1} \Big)\, dy_1\, dy_2 = \int\limits_S \frac{\partial u}{\partial s}\, dS,$$

we get

$$\int\limits_S \frac{\partial \ln \sigma}{\partial s(y)}\, dS + \int\limits_{S(x;\varepsilon)} \frac{\partial \ln \sigma}{\partial s(y)}\, dS = 0.$$

Clearly, if $y \in S(x, \varepsilon)$, we have

$$y_1 - x_1 = \varepsilon \cos\varphi, \quad y_2 - x_2 = \varepsilon \sin\varphi, \quad dS = \varepsilon d\varphi,$$
$$n_1(y) = -\cos\varphi, \quad n_2(y) = -\sin\varphi.$$

Therefore

$$\frac{\partial \ln \sigma}{\partial s(y)}\, dS = -i d\varphi,$$
$$\frac{\partial}{\partial s(y)} \frac{\overline{\sigma}}{\sigma}\, dS = -2i \exp(-2i\varphi)\, d\varphi.$$

From the above results, it follows

$$\int\limits_{S(x;\varepsilon)} \frac{\partial \ln \sigma}{\partial s(y)}\, dS = -2\pi i, \quad \int\limits_S \frac{\partial}{\partial s(y)} \frac{\overline{\sigma}}{\sigma}\, dS = 0. \tag{6.5}$$

Thus

$$J_1(x) = 2\pi i, \quad J_2(x) = 0, \quad x \in D_\varepsilon. \tag{6.6}$$

By (6.5), it can be easily proved that

$$\lim_{\varepsilon \to 0} \int\limits_{S(x,\varepsilon)} u(y)\, \overset{\varkappa}{T}_y\, \Phi^{(j)}(y - x) dS = -2\pi u_j(x),$$
$$\lim_{\varepsilon \to 0} \int\limits_{S(x;\varepsilon)} \Phi^{(j)}\, \overset{\varkappa}{T}\, u\, dS = 0.$$

Now from (6.2) we get

$$-\int\limits_{D^+} \Phi^{(j)}(y - x)\psi(y)\, dy_1\, dy_2 =$$
$$= \int\limits_S \big[(u)^+\, \overset{\varkappa}{T}_y\, \Phi^{(j)}(y - x) - \Phi^{(j)}(y - x)(\overset{\varkappa}{T}\, u)^+\big] dS -$$
$$-2\pi u_j(x), \quad x \in D^+,$$

which completes the proof. ■

If $\psi = 0$, then (6.1) reads as

$$u(x)=\frac{1}{2\pi}\int\limits_S \{[T_y\Phi(y-x)]'(u)^+ - \Phi(y-x)(\overset{\varkappa}{T}u)^+\}dS, \quad x\in D^+. \tag{6.7}$$

Quite similarly we establish that for any $x \in D^-$,

$$0=\frac{1}{\pi}\int\limits_S \{[T_y\Phi(y-x)]'(u)^+ - \Phi(y-x)(\overset{\varkappa}{T}u)^+\}dS, \quad x\in D^-. \tag{6.8}$$

The representations (6.7) and (6.8) hold for an arbitrary $\varkappa$. Let $\varkappa = \varkappa_N$. We apply the identity

$$Nu = m^{-1}\frac{\partial v}{\partial s}, \quad Nv = -m^{-1}\frac{\partial u}{\partial s}. \tag{6.9}$$

These relations have been obtained for an arbitrary matrix. In this connection, if $u = \operatorname{Re} W$, then $v = \operatorname{Im} W$, i.e., $W = u + iv$.

Taking into account (6.9) and single-valuedness of Φ and u, we get from (6.7) by integration by parts

$$\begin{aligned} u(x) &= \frac{1}{2\pi}\int\limits_S \Big\{[N_y\Phi(y-x)]'(u)^+ + \frac{\partial\Phi}{\partial s}m^{-1}(v)^+\Big\}dS = \\ &= \frac{1}{2\pi}\int\limits_S \operatorname{Im}\frac{\partial\Gamma(y-x)}{\partial s(y)}m^{-1}[(u)^+ + i(v)^+]dS. \end{aligned} \tag{6.10}$$

Similarly we can write

$$v(x) = \frac{-1}{2\pi}\int\limits_S \operatorname{Re}\frac{\partial\Gamma}{\partial s(y)}m^{-1}[(u)^+ + i(v)^+]dS. \tag{6.11}$$

Further, (6.10) and (6.11) yield

$$W(x) = \frac{-1}{2\pi i}\int\limits_S \frac{\partial\Gamma(y-x)}{\partial s(y)}m^{-1}(W)^+dS, \quad x\in D^+, \tag{6.12}$$

$$0 = \frac{1}{2\pi i}\int\limits_S \frac{\partial\Gamma(y-x)}{\partial s(y)}m^{-1}(W)^+dS, \quad x\in D^-. \tag{6.13}$$

By quite the same way we can derive similar formulas for D^-

$$W(x) = W(\infty) - \frac{1}{2\pi i}\int\limits_S \frac{\partial\Gamma(y-x)}{\partial s(y)}m^{-1}(W)^-dS, \; x\in D^-, \tag{6.14}$$

$$W(x) = W(\infty), \quad x\in D^+. \tag{6.15}$$

Equations (6.12), (6.13) and (6.14), (6.15) represent the generalized Cauchy integral formulas in the theory of elastic mixtures.

Let $\varkappa_1 = \varkappa_2 = \varkappa_3 = 0$ and $\psi = 0$. Then (6.1) reads

$$U(x) = \frac{1}{2\pi} \int\limits_S \big[T_y \Phi(y-x)\big]'(u)^+ - \Phi(y-x)(Tu)^+ dS, \ \ x \in D^+. \tag{6.16}$$

Let, in addition,

$$(u)^+ = \varphi^{(j)}(y) = \begin{pmatrix} \delta_{1j} \\ \delta_{2j} \\ \delta_{3j} \\ \delta_{4j} \end{pmatrix} + \delta_{5j} \begin{pmatrix} -y_2 \\ y_1 \\ -y_2 \\ y_1 \end{pmatrix}, \quad j = \overline{1,5}, \tag{6.17}$$

where δ_{kj} is Kronecker's symbol. Due to the equation $T_y \varphi^{(j)}(y) = 0$, we obtain

$$\varphi^{(j)}(x) = \frac{1}{2\pi} \int\limits_S \big[T_y \Phi(y-x)\big]' \psi^{(j)}(y) dS, \ \ x \in D^+. \tag{6.18}$$

Finally, let us note that the formula (6.12) has been derived for a regular vector W, but nevertheless, it remains to hold true for a continuous vector W in $\overline{D}^+$.

7. Uniqueness Theorems

Before going over to uniqueness theorems, let us prove

Let u be a regular vector in D and let

$$T(u,u) = 0 \tag{7.1}$$

with $T(u,u)$ given by (5.18).

Then

$$u = (u', u''), \quad u' = a' + b' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad u'' = a'' + b' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \tag{7.2}$$

where $a' = (a_1', a_2')$, $a'' = (a_1'', a_2'')$ and a_1', a_2', a_1'', a_2'', b' are arbitrary constants.

Proof. We have from (5.18)

$$\partial_k u_j' + \partial_j u_k' = 0, \quad \partial_k u_j'' + \partial_j u_k'' = 0, \quad k,j = 1,2, \tag{7.3}$$

$$\omega' = \omega'' \tag{7.4}$$

due to (5.19). In turn, (7.3) yields [5]

$$u' = a' + b' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad u'' = a'' + b'' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix},$$

where a_1', a_2', a_1'', a_2'', b' and b'' are arbitrary constants. Now the condition (7.4) completes the proof. ■

Now we can prove the following uniqueness results.

Let $S \in C^{1+\beta}, 0 < \beta \leq 1$. Then the homogeneous problems $(\mathrm{I})^{\pm}_{0,0}$, *have no nontrivial regular solutions.*

The general solution of the problem $(\mathrm{II})^{+}_{0,0}$ *is represented by the formula* (7.2), *while the general solution of the problem* $(\mathrm{III})^{+}_{0,0}$ *is*

$$u' = u'' = a' + b' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

The general solution of the problem $(\mathrm{II})^{-}_{0,0}$ $[(\mathrm{III})^{-}_{0,0}]$ *reads* $u' = a'$, $u'' = a''$ $(u' = u'' = a')$.

Proof. It follows from (5.15), (5.24) (with $\varkappa = 0$) and Lemma 7.1 since $(uTu)^{\pm} = 0$ under the conditions of the Theorem. ■

8. Generalized Potentials and Their Properties

Let us introduce the following definitions.

The vector

$$u(x) = \frac{1}{\pi} \int_S \Phi(x-y) g(y)\, dS, \tag{8.1}$$

where $\Phi(x-y)$ is given by (2.12) and g is a continuous vector, is called a single layer potential.

The vector

$$u(x) = \frac{1}{\pi} \int_S \big[N_y \Phi(y-x)\big]' g(y)\, dS, \tag{8.2}$$

where $[N_y\Phi(y-x)]'$ is given by (3.15) and g is a continuous vector, is called a double layer potential.

The vector

$$u(x) = \frac{1}{\pi} \int_S \big[T_y \Phi(y-x)\big]' g(y)\, dS, \tag{8.3}$$

where $[T_y\Phi(y-x)]'$ is given by (3.23) and g is a Hölder continuous vector, is called a double layer potential of the second kind.

The vector

$$u(x) = \frac{1}{\pi} \int_S M(x-y) g(y)\, dS, \tag{8.4}$$

where $M(x-y)$ is given by (4.1) and g is a continuous vector, is called a single layer potential of the second kind.

It is evident that all potentials introduced above are solutions to the equation (1.4) in $\mathbb{R}^2 \backslash S$. These potentials have certain continuity and jump properties when the point x either crosses the surface S or approaches some point $t = (t_1, t_2) \in S$ from $\Omega^{\pm}$. Those properties can be obtained very easily since the kernel-functions of the above potentials are quite similar to those of classical potentials of isotropic elastostatics [3].

Therefore we will only formulate final results.

A single layer potential defined by (8.1) *is continuous on the whole plane and*

$$\left[T_t u(t)\right]^{\pm} = \mp g(t) + \frac{1}{\pi} \int_S T_t \Phi(t-y) g(y)\, dS, \tag{8.5}$$

where the symbols $[\cdot]^{\pm}$ *denote limits on* S *from* $\Omega^{\pm}$*.*

Let u *be a single layer potential* (8.1)*. Then*

$$\left[N_t u(t)\right]^{\pm} = \mp g(t) + \frac{1}{\pi} \int_S N_t \Phi(t-y) g(y)\, dS \tag{8.6}$$

hold for an arbitrary $t \in S$*.*

Let u *be a double layer potential given by* (8.2)*. Then for any* $t \in S$*,*

$$\left[u(t)\right]^{\pm} = \pm g(t) + \frac{1}{\pi} \int_S \left[N_y \Phi(y-t)\right]' g(y)\, dS. \tag{8.7}$$

Let u *be a double layer potential of the second kind given by* (8.3)*. Then for any* $t \in S$*,*

$$\left[u(t)\right]^{\pm} = \pm g(t) + \frac{1}{\pi} \int_S \left[T_y \Phi(y-t)\right]' g(y)\, dS. \tag{8.8}$$

Let

$$u(x) = \frac{1}{\pi} \int_S \left[M(x-y) - M(y)\right] g(y)\, dS$$

and let

$$\int_S g(y)\, dS = 0. \tag{8.9}$$

Then u is continuous in $\overline{\Omega}^+$.

Let

$$u(x) = \frac{1}{\pi} \int_S \big[M(x-y) - M(x)\big] g(y)\, dS$$

and let (8.9) *be fulfilled. Then u is continuous in $\overline{\Omega}^-$.*

Let u be a single layer potential of the second kind. Then for any $t \in S$,

$$\big[T_t u(t)\big]^{\pm} = \mp g(t) + \frac{1}{\pi} \int_S T_t M(t-y) g(y)\, dS. \tag{8.10}$$

Let u be a single layer potential (8.1) *with the density g satisfying* (8.9) *and let u be a constant vector in Ω^+. Then u is the same constant in the whole plane.*

Proof. Let $u(x) = a$ in Ω^+, where $a = (a', a'')$ is a constant vector. Clearly $T_x u(x) = 0, x \in \Omega^+$. From Theorem 8.5, it follows that $(u)^+ = (u)^- = a$ and $(Tu)^- - (Tu)^+ = 2g$. Now $(Tu)^+ = 0$ implies

$$\int_S (u)^-(Tu)^- dS = 2a \int_S g\, dS = 0,$$

which together with (5.24) completes the proof. ■

Let a single layer potential of the second kind be a constant in D^+. In addition, if (8.9) *is fulfilled, then this potential is equal to the same constant in the whole plane.*

Proof. Let $u(x) = a, x \in D^+$, where $a = (a', a'')$ is a constant vector. Then $Nu = 0$ and $v(x) = b$ due to (6.9), where $b = (b', b'')$ is a constant vector.

Taking into account the equation $(Tv)^+ - (Tv)^- = 0$ we get $(Tv)^- = 0$. Further, the condition (8.9) implies that $v(x)$ is bounded at infinity and therefore $v(x) = b, x \in D^-$, due to (5.24) with $\varkappa = 0$. Now from (6.9) it follows that $u(x) = a, x \in D^-$. ■

Let u be a single layer potential. If u is a constant vector in D^+ and, in addition,

$$(\omega' + \omega'')_{x=0} = 0, \tag{8.11}$$

then the potential is constant on the whole plane.

Proof. We assume, as above, that $0 \in D^+$ and ω' and ω'' are calculated by formula (1.8) and correspond to the single layer potential (8.1). Since $u(x) \equiv a$, $x \in D^-$, we have $(Tu)^- = 0$. Now (8.5) yields

$$\int_S g\,dS = 0.$$

Further, note that

$$\int_S (u)^+(Tu)^+ dS = -2a \int_S g\,dS = 0.$$

Applying formula (5.15) with $\varkappa = 0$, we deduce $u = (u', u'')$, where

$$u' = a' + b' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad u'' = a'' + b' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad x \in D^+,$$

whence

$$\omega' + \omega'' = 2b'.$$

Finally, bearing in mind (8.11), we get $b' = 0$ and

$$u' = a', \quad u'' = a'',$$

which completes the proof. ■

If the single layer potential of the second kind u is constant in D^- and the equation

$$(\omega' + \omega'')_{x=0} = 0 \tag{8.12}$$

holds, then u is constant on the whole plane.

Proof. Let $u(x) = a, x \in D^-$. Then $(Tu)^- = 0$ and, due to (8.10),

$$\int_S g\,dS = 0.$$

On the other hand, we have $Nu = m^{-1} \frac{\partial v}{\partial S} = 0$ in Ω^-, whence

$$v(x) = c, \quad x \in D^-$$

follows.

We also have $(Tv)^- - (Tv)^+ = 0$, i.e., $(Tv)^+ = 0$. By making use of (5.15) (with $\varkappa = 0$) we arrive to

$$v' = a' + b' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad v'' = a'' + b' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix},$$

which together with (8.14) gives $b' = 0$. Now $v' = a'$ and $v'' = a''$ yield $u' = c'$, $u'' = c''$ in D^+. ■

9. Existence Theorems of Problems $(\mathrm{I})^{\pm}_{0,f}$ and $(\mathrm{II})^{\pm}_{0,F}$

${}^{+}_{0,f}$ ${}^{-}_{0,F}$ We look for solutions to the problems $(\mathrm{I})^{+}_{0,f}$ and $(\mathrm{II})^{-}_{0,F}$ in the form of a second kind double layer potential and a single layer potential, respectively. Then we arrive to the singular integral equations

$$g(t)+\frac{1}{\pi}\int\limits_{S}\big[T_y\Phi(y-t)\big]'g(y)\,dS=f(t), \tag{9.1}$$

$$h(t)+\frac{1}{\pi}\int\limits_{S}T_t\Phi(t-y)h(y)\,dS=F(t), \tag{9.2}$$

where g and h are unknown Hölder continuous vectors – densities of the potentials

$$u(x)\equiv u(x;g)=\frac{1}{\pi}\int\limits_{S}\big[T_y\Phi(y-x)\big]'g(y)\,dS, \tag{9.3}$$

$$V(x)\equiv V(x;h)=\frac{1}{\pi}\int\limits_{S}\Phi(x-y)h(y)\,dS. \tag{9.4}$$

The kernels of the singular integral equations (9.1) and (9.2) are given by (3.23) and (3.7), rexpectively. They are mutually adjoint kernels and therefore (9.1) and (9.2) are mutually adjoint singular integral equations. Now we show that they are of normal type, i.e., their indices are equal to zero.

We begin with the equation (9.2). Due to the general theory [6], the index is calculated by the formula

$$\varkappa=\frac{1}{2\pi}\left[\arg\frac{\det(E+iA)}{\det(E-iA)}\right]_S. \tag{9.5}$$

By the direct evaluation, we get

$$\begin{aligned}\det(E+iA)&=\det(E-iA)=\\&=4\Delta_0\Delta_1\big[(2-A_1)(2-A_4)-A_2A_3\big];\end{aligned} \tag{9.6}$$

here A_1, A_2, A_3, A_4, Δ_0, Δ_1 are given by (3.9), (3.13), (4.5).

The positive definiteness of the potential energy implies that $\Delta_0>0$, $\Delta_1>0$ and $(2-A_1)(2-A_4)-A_2A_3>0$. Therefore the index (9.5) is equal to zero. Thus the left-hand side of the equation (9.2) (and consequently of (9.1)) is a singular integral operator of normal type and we can apply Fredholm theorems to them.

Let us prove that the homogeneous version of the equation (9.2) has only the trivial solution. Indeed, let h_0 be some solution to it. Then for the single layer potential $V(x, h_0)$ we have: $[T_t V(t, h_0)]^- = 0$. We can also easily establish

$$\int\limits_S h_0 \, dS = 0, \tag{9.7}$$

which implies that the corresponding single layer potential vanishes at infinity. Further, from (5.24) with $\varkappa = 0$ and the condition $[T_t V(t, h_0)]^- = 0$ it follows that $V(x, h_0) = 0$, $x \in D^-$, whence $[V(t, h_0)]^- = [V(t, h_0)]^+ = 0$. Now (5.15) with $\varkappa = 0$ yields $V(x, h_0) = 0$, $x \in D^+$.

Thus $V(x, h_0)$ vanishes on the whole plane and therefore $h_0 = 0$. Due to the Fredholm alternative we conclude that the nonhomogeneous equation (9.2) is solvable for an arbitrary Hölder continuous vector $F(t)$. Clearly, the same is valid for the equation (9.1).

From the solvability of the equations (9.1) and (9.2) it follows that the solutions of problems $(\mathrm{I})^+_{0,f}$ and $(\mathrm{II})^-_{0,F}$ are representable as second kind double layer and single layer potentials, respectively (see (9.3) and (9.4)). From the general theory we conclude that if $S \in C^{2+\beta}$ and $f \in C^{1+\alpha}(S), 0 < \alpha < \beta \le 1$, then $g \in C^{1+\alpha}(S)$, where g solwes the equation (9.1). Therefore the double layer potential of the second kind with density g is a regular vector.

${}^{-}_{0,f}$ ${}^{+}_{0,F}$ We look for solutions to the problems $(\mathrm{I})^-_{0,f}$ and $(\mathrm{II})^+_{0,F}$ in the form of the second kind double layer potential (9.3) and the single layer potential (9.4), respectively. We obtain then the following equations

$$-\,g(t) + \frac{1}{\pi} \int\limits_S \big[T_y \Phi(y - t)\big]' g(y) \, dS = f(t), \tag{9.8}$$

$$-\,h(t) + \frac{1}{\pi} \int\limits_S T_t \Phi(t - y) h(y) dS = F(t), \tag{9.9}$$

where g and h are Hölder continuous unknown vectors.

In quite the same way as in the previous subsection, it can be proved that (9.8) and (9.9) are mutually adjoint singular integral equations with index equal to zero (note that the corresponding determinants are the same as for equations (9.1) and (9.2)).

From (6.18) it follows

$$-\varphi^{(j)}(t) + \frac{1}{\pi} \int\limits_S \big[T_y \Phi(y - t)\big]' \varphi^{(j)}(y) \, dS = 0, \quad j = \overline{1, 5}, \tag{9.10}$$

where $\varphi^{(j)}$ are given by (6.17).

It can be easily proved that the homogeneous version of the equation (9.8) has a 5-dimensional null-space. Clearly the same is valid for the homogeneous version of the equation (9.9). Therefore the nonhomogeneous equations (9.8) and (9.9) are not solvable for arbitrary right-hand side f and F.

Let us consider the equation

$$
\begin{aligned}
&-h(t)+\frac{1}{\pi}\int\limits_S T_t\Phi(t-y)h(y)\,dS+\\
&+\frac{1}{2\pi}T_t\Phi(t)\cdot\int\limits_S h(y)\,dS+\frac{1}{4\pi}\,T_t\Psi(t)\cdot M=F(t),
\end{aligned}
\tag{9.11}
$$

where

$$
\begin{aligned}
\Psi(t)&=\begin{pmatrix}\frac{\mu_2-\mu_3}{2\Delta_1}\,\mathrm{grad}\,\theta\\ \frac{\mu_1-\mu_3}{2\Delta_2}\,\mathrm{grad}\,\theta\end{pmatrix},\quad \theta=\mathrm{arctg}\,\frac{t_2}{t_1},\\
T_t\Psi(t)&=-\begin{pmatrix}\frac{\partial}{\partial S(t)}\,\mathrm{grad}\ln\rho\\ \frac{\partial}{\partial S(t)}\,\mathrm{grad}\ln\rho\end{pmatrix},\quad \rho=\sqrt{t_1^2+t_2^2},
\end{aligned}
\tag{9.12}
$$

$$
\begin{aligned}
M=&\Big(\frac{\partial V_2'(x;h)}{\partial x_1}-\frac{\partial V_1'(x;h)}{\partial x_2}+\frac{\partial V_2''(x;h)}{\partial x_1}-\frac{\partial V_1''(x;h)}{\partial x_2}\Big)_{x=0}=\\
=&\frac{1}{\pi}\int\limits_S\Big[(e_1+e_2)\Big(-\frac{y_2}{R^2}\,h_1+\frac{y_1}{R^2}h_2\Big)+\\
&+(e_2+e_3)\Big(-\frac{y_2}{R^2}h_3+\frac{y_1}{R^2}h_4\Big)\Big]dS,\quad R=\sqrt{y_1^2+y_2^2}.
\end{aligned}
\tag{9.13}
$$

The constants e_1, e_2, e_3 are defined by (2.3), while $\Delta_1=\mu_1\mu_2-\mu_3^2>0$. From (9.11) by integration it follows

$$
\int\limits_S h(y)\,dS=\int\limits_S F(y)\,dS,
\tag{9.14}
$$

$$
M=\int\limits_S\big[y_1F_2(y)-y_2F_1(y)+y_1F_4(y)-y_2F_3(y)\big]dS.
\tag{9.15}
$$

Therefore if the right-hand side of (9.11) is orthogonal to all solutions of the adjoint homogeneous equation, then

$$
\int\limits_S F\,dS=0,
\tag{9.16}
$$

$$
\int\limits_S\big[y_1(F_2+F_4)-y_2(F_1+F_3)\big]dS=0.
\tag{9.17}
$$

In turn, if the conditions (9.16) and (9.17) hold, then (9.14) and (9.15) imply

$$
\int\limits_S h(y)\,dS=0,
\tag{9.18}
$$

$$M = 0. \tag{9.19}$$

Thus if (9.16) and (9.17) are fulfilled, then an arbitrary solution $h(y)$ of (9.11) solves at the same time the original equation (9.9).

Now we will prove that the equation (9.11) is always solvable.

To this end, let us consider the corresponding homogeneous equation (i.e., $F = 0$) and show that it has no non-trivial solutions.

Let h_0 be an arbitrary solution of the homogeneous equation under consideration. Since $F \equiv 0$, conditions (9.18) and (9.19) are fulfilled and the above homogeneous equation corresponds to the boundary condition

$$\big[T_t V_0(t)\big]^+ = 0, \tag{9.20}$$

where $V_0(x) = V(x, h_0)$ is defined by (9.4).

Further, (9.20) and the uniqueness theorem for the problem $(\mathrm{II})^+_{o,o}$ yield

$$V_0(x) = (V_0', V_0''),$$

where

$$V_0'(x) = a_0' + b_{10}' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad V_0''(x) = a_0'' + b_{10}' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \tag{9.21}$$

and a_0', a_0'' are arbitrary constant vectors while b_{10}' is an arbitrary scalar constant.

Taking into account the equation $M_0 = 0$ and (9.21), we get

$$V_0(x) = \begin{pmatrix} a_0' \\ a_0'' \end{pmatrix}, \quad x \in D^+. \tag{9.22}$$

Thus we have obtained that the single layer potential is constant in D^+ and (9.18) holds, in addition. Applying Theorem 8.12, we conclude

$$V_0(x) = \begin{pmatrix} a_0' \\ a_0'' \end{pmatrix}, \quad x \in D^-. \tag{9.23}$$

Since

$$\big[T_t V_0(t)\big]^- - \big[T_t V_0(t)\big]^+ = 2h_0(t),$$

we easily obtain that $h_0(t) = 0$.

Thus the homogeneous version of the equation (9.11) has only the trivial solution. Consequently the nonhomogeneous equation (9.11) has only one solution $h(t)$ for an arbitrary right-hand side F. If conditions (9.16) and (9.17) are fulfilled, the same $h(t)$ is a solution to (9.4) as well. Finally we note that the problem $(\mathrm{II})^+_{0,F}$ is solvable if the conditions (9.16) and (9.17) are satisfied. In this connection, the partial the displacements are defined to within the summands

$$a' + b_1' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad \text{and} \quad a'' + b_1' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix},$$

where a' and a'' are constant vectors while b_1' is a constant scalar. The stress vector is defined uniquely.

The adjoint equation to (9.11) reads

$$-g(t)+\frac{1}{\pi}\int\limits_S \big[T_y\Phi(y-t)\big]'g(y)\,dS+$$
$$+\frac{1}{2\pi}\int\limits_S \big[T_y\Phi(y)\big]'g(y)\,dS+\frac{1}{4\pi}X(t)\cdot L=f(t), \tag{9.24}$$

where

$$X(t)=\begin{pmatrix}(e_1+e_3)\,\mathrm{grad}\,\theta\\(e_2+e_3)\,\mathrm{grad}\,\theta\end{pmatrix},\quad \theta=\mathrm{arctg}\,\frac{t_2}{t_1}, \tag{9.25}$$

$$L=\Big(\frac{\partial u_2'}{\partial x_1}-\frac{\partial u_1'}{\partial x_2}+\frac{\partial u_2''}{\partial x_1}-\frac{\partial u_2''}{\partial x_2}\Big)_{x=0}=\frac{1}{\pi}\int\limits_S \big[T_y\Psi(y)\big]'g(y)\,dS, \tag{9.26}$$

here $u=(u',u'')$ is given by (9.3). The equation (9.24) corresponds to the exterior limit on S of the potential

$$u(x)=\frac{1}{\pi}\int\limits_S \big[T_y\Phi(y-x)\big]'g(y)\,dS+$$
$$+\frac{1}{2\pi}\int\limits_S \big[T_y\Phi(y)\big]'g(y)\,dS+\frac{1}{4\pi}X(x)\cdot L. \tag{9.27}$$

It is evident that the homogeneous version of the equation (9.24) has only the trivial solution since its adjoint possesses the same property. This results that (9.24) is solvable for an arbitrary right-hand side $f\in C^{1+\alpha}(S)$ and $g\in C^{1+\alpha}(S)$, povided $S\in C^{2+\beta}$, $0<\alpha<\beta\le 1$. Therefore the vector u defined by (9.27) is a regular solution of the problem $(\mathrm{I})^-_{0,f}$.

Thus we have studied the solvability of the problems $(\mathrm{I})^{\pm}_{0,f}$ and $(\mathrm{II})^{\pm}_{0,F}$ by reduction the original boundary value problems to corresponding singular equations.

10. An Alternative Approach to the Problem $(\mathrm{I})^{\pm}_{0,f}$

In this section, we will reduce the problems $(\mathrm{I})^{\pm}_{0,f}$ to second kind Fredholm equations (with weakly singular kernels).

First we consider the problem $(\mathrm{I})^{+}_{0,f}$ and look for its solution in the form of the double layer potential

$$u(x)=\frac{1}{\pi}\int\limits_S \big[N_y\Phi(y-x)\big]'g(y)\,dS, \tag{10.1}$$

where $[N_y\Phi(y-x)]'$ is given by (3.15) and the continuous vector g is an unknown density.

Due to Theorem 8.7, we get the equation on S

$$g(t)+\frac{1}{\pi}\int\limits_{S}\big[N_y\Phi(y-t)\big]'g(y)\,dS=f(t),\quad t\in S, \tag{10.2}$$

where $f(t)$ is a given vector.

Let us prove that (10.2) is solvable for an arbitrary continuous vector f.

The corresponding adjoint equation reads

$$h(t)+\frac{1}{\pi}\int\limits_{S}N_t\Phi(t-y)h(y)dS=0. \tag{10.3}$$

In what follows, we prove that the latter equation has only the zero solution. As usual, we denote by $h_0(t)$ an arbitrary solution of (10.3) and construct the single layer potential

$$V_0(x)=\frac{1}{\pi}\int\limits_{S}\Phi(x-y)h_0(y)\,dS.$$

It is obvious that

$$\big[N_tV_0(t)\big]^-=0,\qquad \int\limits_{S}h_0(t)\,dS=0.$$

Applying formula (5.25) with $\varkappa=\varkappa_N$ (in D^-), we get

$$V_0(x)=0,\quad x\in D^-.$$

Thus the potential $V_0(x)$ vanishes in D^- and in addition $\int\limits_S h_0(t)dS=0$. Since $[N_tV_0(t)]^+-[N_tV_0(t)]^-=2h_0(t)$, we conclude

$$\int\limits_{S}(V_0)^+\big[N_tV_0(t)\big]^+dS=0.$$

Now by (5.15) with $\varkappa=\varkappa_N$, we easily get $V_0(x)=0$, $x\in D^+$, whence $h_0(t)=0$ follows directly.

From the above results it follows that the equation (10.2) is solvable for an arbitrary continuous right-hand side f.

It can be easily proved that, if $S\in C^{1+\beta}$ and $f\in C^{1+\alpha}$, $0<\alpha<\beta\le 1$, then $g\in C^{1+\alpha}(S)$, and the corresponding potential (10.1) is a regular vector (note that the tangent derivative of the kernel of the equation (10.2) is a Hölder continuous function on S).

Let us now consider the problem $(\mathrm{I})^-_{0,f}$. We look for its solution as

$$u(x)=\frac{1}{\pi}\int\limits_{S}\big[N_y\Phi(y-x)\big]'g(y)\,dS+\frac{1}{2\pi}\int\limits_{S}\big[N_y\Phi(y)\big]'g(y)\,dS, \tag{10.4}$$

which reduces the boundary value problem to the second kind Fredholm equation on S with respect to g

$$-g(t)+\frac{1}{\pi}\int_S \big[N_y\Phi(y-t)\big]'g(y)\,dS+ \\ +\frac{1}{2\pi}\int_S \big[N_y\Phi(y)\big]'g(y)\,dS=f(t) \tag{10.5}$$

with f given on S.

We will show that (10.5) is uniquely solvable for an arbitrary f. To this end, we consider the corresponding adjoint homogeneous equation

$$-h(t)+\frac{1}{\pi}\int_S N_t\Phi(t-y)h(y)\,dS+\frac{1}{2\pi}N_t\Phi(t)\int_S h(y)dS=0. \tag{10.6}$$

Let h_0 be some solution to (10.6). From (10.6), by integration we obtain

$$\int_S h_0(y)\,dS=0. \tag{10.7}$$

But the equation (10.6) then corresponds to the boundary condition

$$\big[N_tV_0(t)\big]^+=0, \tag{10.8}$$

where

$$V_0(x)=\frac{1}{\pi}\int_S \Phi(x-y)h_0(y)\,dS. \tag{10.9}$$

Now (5.15) with $\varkappa=\varkappa_N$ implies

$$V_0(x)=c,\quad x\in D^+,$$

where c is a constant 4-dimensional vector.

The latter equation together with (10.7) and Theorem 8.12 yields $V_0(x)=a, x\in D^-$, where a is a constant vector.

Now again applying the equations $[N_tV_0(t)]^- - [N_tV_0(t)]^+ = 2h_0(t)$ and $[N_tV_0(t)]^+=0$, we conclude $h_0(t)=0$.

Thus (10.6) has no nontrivial solutions and therefore (10.5) is solvable for an arbitrary continuous right-hand side vector.

Note that, if $S\in C^{2+\beta}$ and $f\in C^{1+\alpha}(S), 0<\alpha<\beta\le 1$, then $g\in C^{1+\alpha}(S)$ and, clearly, the vector u defined by (10.4) is regular.

11. An Alternative Approach to the Problem $(\mathrm{II})^{\pm}_{0,F}$

As in the previous section, here we will study the problems $(\mathrm{II})^{\pm}_{0,F}$ by reduction to the second kind Fredholm integral equations.

First we consider the problem $(\mathrm{II})^{+}_{0,F}$. We look for the solution as

$$u(x)=\frac{1}{\pi}\int\limits_{S}\big[M(x-y)-M(-y)\big]g(y)\,dS,\quad x\in D^{+},\tag{11.1}$$

where $M(x-y)$ is given by (4.1) and g is a continuous unknown vector.

By Theorem 8.11, we get

$$-g(t)+\frac{1}{\pi}\int\limits_{S}T_tM(t-y)g(y)\,dS=F(t).\tag{11.2}$$

The adjoint (homogeneous) equation reads

$$-h(t)+\frac{1}{\pi}\int\limits_{S}\big[T_yM(y-t)\big]'h(y)\,dS=0.\tag{11.3}$$

It can be easily proved that the equation (11.3) has only 5 linearly independent solutions

$$h^{(j)}(t)=\begin{pmatrix}\delta_{ij}\\ \delta_{2j}\\ \delta_{3j}\\ \delta_{4j}\end{pmatrix}+\delta_{5j}\begin{pmatrix}-t_2\\ t_1\\ -t_2\\ t_1\end{pmatrix},\quad j=\overline{1,5}.\tag{11.4}$$

Therefore the equation (11.2) is not solvable for an arbitrary F.

Let us consider the following equation

$$\begin{aligned}&-g(t)+\frac{1}{\pi}\int\limits_{S}T_tM(t-y)g(y)\,dS+\\ &+\frac{1}{2\pi}T_tM(t)\int\limits_{S}g\,dS+\frac{1}{4\pi}T_t\Psi(t)M=F(t),\end{aligned}\tag{11.5}$$

where $T_t\Psi(t)$ is defined by (9.12), while

$$\begin{aligned}M&=\Big(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}+\frac{\partial u_4}{\partial x_1}-\frac{\partial u_3}{\partial x_2}\Big)_{x=0}=\\ &=\frac{1}{\pi\Delta_2}\int\limits_{S}\Big[-\frac{y_2}{R^2}(A_0h_1+B_0h_3)+\frac{y_1}{R^2}(A_0h_2+B_0h_4)\Big]dS,\end{aligned}\tag{11.6}$$

$$\begin{aligned}A_0&=(2-A_4)(e_1+e_2)+A_3(e_2+e_3),\\ B_0&=A_2(e_1+e_2)+(2-A_1)(e_2+e_3).\end{aligned}\tag{11.7}$$

Note that in (11.6) $u=(u_1,\dots,u_4)$ is given by (11.1).

From (11.5) it follows that

$$\int_S g\,dS = \int_S F\,dS, \tag{11.8}$$

$$M = \int_S \big[y_1(F_2 + F_4) - y_2(F_1 + F_3)\big]\,dS. \tag{11.9}$$

The conditions

$$\int_S F\,dS = 0, \tag{11.10}$$

$$\int_S \big[y_1(F_2 + F_4) - y_2(F_1 + F_3)\big]\,dS = 0 \tag{11.11}$$

are necessary for orthogonality of the right-hand side vector F and vector-functions $\varphi^{(j)}$, $j = \overline{1,6}$.

If equations (11.10) and (11.11) hold, then (11.8) and (11.9) imply

$$\int_S g\,dS = 0, \tag{11.12}$$

$$M = 0, \tag{11.13}$$

whence it follows that each solution g of the equation (11.5) with conditions (11.10) and (11.11) at the same time solves the equation (11.2).

Now we will show that (11.5) is solvable for an arbitrary right-hand side, i.e., we have to show that the corresponding homogeneous equation has no nontrivial solution. In fact, let g_0 be some solution to that homogeneous equation. It is evident that the conditions (11.12) and (11.13) are fulfilled, since $F \equiv 0$. But then the equation (11.5) coincides with (11.2) (with $F \equiv 0$); therefore we have

$$\big[T_t u_0(t)\big]^+ = 0, \tag{11.14}$$

where $u_0(x)$ is given by (11.1) with g_0 instead of g.

Applying (5.15) with $\varkappa = \varkappa_N$ and (11.14), we get

$$u_0(x) = (u_0'(x), u_0''(x)),$$

where

$$\begin{aligned} u_0'(x) &= a_0' + b_{10}' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \\ u_0''(x) &= a_0'' + b_{10}'' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \end{aligned} \tag{11.15}$$

a_0', a_0'' are arbitrary constant vectors, while b_{10}' is an arbitrary scalar constant.

Due to (11.15) and (11.13), we arrive to

$$u_0(x) = \begin{pmatrix} a_0' \\ a_0'' \end{pmatrix},$$

whence by the use of $u_0(0) = 0$, we get

$$u_0(x) = 0, \quad x \in D^+.$$

Thus we have obtained that the single layer potential of the second kind vanishes in D^+ and the condition $M_0 = 0$ holds, in addition (cf. (11.13)). Now by Theorem 8.13 $u_0(x) = c$, $x \in D^-$, where c is a constant vector. From the above results along with the equation $[T_t u_0(t)]^- - [T_t u_0(t)]^+ = 2g_0(t)$, we have $g_0(t) = 0$. Thus the homogeneous equation corresponding to (11.5) has only the trivial solution. As a result, we have that (11.10) and (11.11) are necessary and sufficient conditions for the nonhomogeneous equation (11.2) to be solvable.

Now we go over to the problem $(\mathrm{II})_{0,F}^-$. We look for the solution in the form

$$W(x) = \frac{1}{\pi} \int_S M(x-y) g(y)\, dS + \frac{1}{4\pi} \Phi(x) \varepsilon, \tag{11.16}$$

where

$$\Psi(x) = \begin{pmatrix} \frac{\mu_2 - \mu_3}{2\Delta_1} \operatorname{grad} \ln \rho \\ \frac{\mu_1 - \mu_3}{2\Delta_1} \operatorname{grad} \ln \rho \end{pmatrix}, \quad \rho = \sqrt{x_1^2 + x_2^2}, \quad \Delta_1 > 0, \tag{11.17}$$

$$\varepsilon = \Big(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} + \frac{\partial v_4}{\partial x_1} - \frac{\partial v_3}{\partial x} \Big)_{x=0}, \tag{11.18}$$

while the vector V is defined as follows: if $M(x-y) = \operatorname{Re} \widetilde{\Gamma}(x-y)$,

$$u(x) = \frac{1}{\pi} \int_S \operatorname{Re} \widetilde{\Gamma}(x-y) g(y)\, dS, \tag{11.19}$$

then

$$v(x) = \frac{1}{\pi} \int_S \operatorname{Im} \widetilde{\Gamma}(x-y) g(y)\, dS. \tag{11.20}$$

From the last equation and (4.1) we have

$$\widetilde{\Gamma}(x-y) = \big[\Gamma(x-y) - E_0 \ln \sigma X\big] Y,$$

where the matrices X and Y are given by (4.6) and (4.7). Obviously, $v(x)$ and $u(x)$ solve the homogeneous equation (1.4) for $x \notin S$.

Let us calculate $TW(x)$:

$$TW(x) = \frac{1}{\pi} \int_S T_x M(x-y) g(y)\, dS + \frac{1}{4\pi} T_x \Phi(x) \varepsilon, \tag{11.21}$$

where the matrix $T_x M(x-y)$ is given by (4.9), while

$$T_x \Phi(x) = \frac{\partial}{\partial s(x)} \begin{pmatrix} \operatorname{grad}\theta \\ \operatorname{grad}\theta \end{pmatrix}, \quad \theta = \operatorname{arctg} \frac{x_2}{x_1}. \tag{11.22}$$

Applying properties of the single layer potential of the second kind, we get from (11.21)

$$g(t) + \frac{1}{\pi} \int_S T_t M(t-y) g(y)\, dS + \frac{1}{4\pi} T_t \Phi(t)\varepsilon = F(t), \tag{11.23}$$

where F is a given vector.

Now we will prove that the homogeneous version of (11.23) has only the trivial solution. Indeed, let g_0 be some of its solution. Then we easily get

$$\int_S g_0\, dS = 0. \tag{11.24}$$

In turn, (11.24) along with the uniqueness theorem for the problem $(\mathrm{II})^-_{0,0}$, implies

$$W_0(x) = u_0(x) + \frac{1}{4\pi} \Phi(x)\varepsilon_0 = 0, \quad x \in D^-, \tag{11.25}$$

whence by (6.9) and (11.25) it follows

$$v_0(x) + \frac{1}{4\pi} \Psi(x)\varepsilon_0 = 0, \quad x \in D^-, \tag{11.26}$$

where

$$\Psi(x) = \begin{pmatrix} \frac{\mu_2-\mu_3}{2\Delta_1} \operatorname{grad}\theta \\ \frac{\mu_1-\mu_3}{2\Delta_1} \operatorname{grad}\theta \end{pmatrix},$$

θ is given by (11.22).

The equation (11.26) yields

$$Tv_0(x) - \frac{1}{4\pi} \begin{pmatrix} \frac{\partial}{\partial s(x)} \operatorname{grad}\ln\rho \\ \frac{\partial}{\partial s(x)} \operatorname{grad}\ln\rho \end{pmatrix} \varepsilon_0 = 0, \quad x \in D^-. \tag{11.27}$$

Using the equations $[Tv_0(t)]^+ = [Tv_0(t)]^- = Tv_0(t)$ and passing to limit in (11.27), we arrive to

$$Tv_0(t) - \frac{1}{4\pi} \begin{pmatrix} \frac{\partial}{\partial s(t)} \operatorname{grad}\ln\rho \\ \frac{\partial}{\partial s(t)} \operatorname{grad}\ln\rho \end{pmatrix} \varepsilon_0 = 0, \quad t \in S, \ \ \rho = \sqrt{t_1^2 + t_2^2}.$$

The last equation and

$$\int_S \big\{ t_1 \big[(Tv_0)_2 + (Tv_0)_4\big] - t_2 \big[(Tv_0)_1 + (Tv_0)_3\big] \big\} dS = 0$$

result

$$\varepsilon_0 = 0. \tag{11.28}$$

Then from (11.25)

$$u_0(x) = 0, \quad x \in D^-, \tag{11.29}$$

whence

$$0 = Nu_0(x) = m^{-1}\,\frac{\partial v_0(x)}{\partial s}.$$

Consequently

$$v_0(x) = C, \quad x \in D^-, \tag{11.30}$$

where c is a 4-dimensional constant vector.

Due to the above mentioned properties of the potential $v_0(x)$, we get

$$\big(Tv_0\big)^- = \big(Tv_0\big)^+ = 0. \tag{11.31}$$

Now applying (5.15) with $\varkappa = 0$, we obtain

$$v_0(x) = \begin{pmatrix} a' \\ a'' \end{pmatrix} + b' \begin{pmatrix} -x_2 \\ x_1 \\ -x_2 \\ x_1 \end{pmatrix}, \quad x \in D^+.$$

Taking into account (11.18) and (11.28), we conclude

$$\varepsilon_0 = 4b' = 0,$$

$$v_0(x) = \begin{pmatrix} a' \\ a'' \end{pmatrix}, \quad x \in D^+.$$

Therefore

$$u_0(x) = \begin{pmatrix} c' \\ c'' \end{pmatrix}, \quad x \in D^+.$$

We recall

$$\big[Tu_0(t)\big]^- - \big[Tu_0(t)\big]^+ = 2g_0(t),$$

which together with $[Tu_0(t)]^+ = 0$ leads to $g_0(t) = 0$.

Thus the homogeneous equation corresponding to (11.23) has no nontrivial solution and therefore the nonhomogeneous equation is solvable for an arbitrary right-hand side. Note that if the condition

$$\int_S F\,dS = 0$$

does not hold, then the single layer potential of the second kind with density g will not be bounded at infinity.

12. Solution of the Third Boundary Value Problem

In this section we will investigate the third boundary value problem formulated in Section 1. We reformulate the problem in question as follows:

$$\begin{gathered}\big[u_j(t)-u_{j+2}(t)\big]^{\pm}=f_j(t),\\ \int\limits_0^{s(t)}\big\{\big[Tu(t)\big]_j+\big[Tu(t)\big]_{j+2}\big\}^{\pm}dS=f_{j+2}(t)+c_j,\quad t\in S,\end{gathered}\tag{12.1}$$

where c_j, $j=1,2$ are constants.

We will consider only the interior problem. The exterior one can be treated quite similarly.

We look for the solution in the form

$$\begin{aligned}u(x)=\frac{1}{\pi}\int\limits_S \operatorname{Im}\frac{\partial}{\partial s(y)}\Big(E\ln\sigma-\frac{\varepsilon}{2}\frac{\overline{\sigma}}{\sigma}\Big)\times\\ \times\begin{pmatrix}g+\alpha_0 g+E_1\beta_0 h+i(E_1\gamma_o g+\delta_0 h)\\ \alpha_0 g+E_1\beta_0 h+i(E_1\gamma_0 g+\delta_0 h)\end{pmatrix}dS,\end{aligned}\tag{12.2}$$

where g and h are two-dimensional unknown (Hölder continuous) vectors,

$$E_1=\begin{Vmatrix}0, & 1\\ -1, & 0\end{Vmatrix},\tag{12.3}$$

α_0, β_0, γ_0, δ_0 are constants:

$$\begin{gathered}\alpha_0=\frac{m_2-m_3}{2(\alpha-\beta)\Delta_0}+\frac{(\mu_1+\mu_3)(2\beta-\alpha)}{2\beta(\alpha-\beta)},\quad \beta_0=\frac{2\beta-\alpha}{4\beta(\beta-\alpha)},\\ \gamma_0=\frac{m_2-m_3}{2(\alpha-\beta)\Delta_0}+\frac{(\mu_1+\mu_3)\alpha}{2\beta(\alpha-\beta)},\quad \delta_0=-\frac{\alpha}{4\beta(\beta-\alpha)}\end{gathered}\tag{12.4}$$

with

$$\begin{gathered}\alpha=\frac{m_1+m_3-2m_2}{\Delta_0},\quad \beta=\mu_1+\mu_2+2\mu_3,\\ \Delta_0=m_1m_3-m_3^2;\end{gathered}\tag{12.5}$$

all parameters involved in (12.2) are defined in Sections 1 and 2.

From (12.2) we get

$$\begin{aligned}(u)^+=\begin{pmatrix}g+\alpha_0 g+E_1\beta_0 h\\ \alpha_0 g+E_1\beta_0 h\end{pmatrix}+\frac{1}{\pi}\int\limits_S \operatorname{Im}\frac{\partial}{\partial s(y)}\Big(E\ln\sigma-\frac{\varepsilon}{2}\frac{\overline{\sigma}}{\sigma}\Big)\times\\ \times\begin{pmatrix}g+\alpha_0 g+E_1\beta_0 h+i(E_1\gamma_0 g+\delta_0 h)\\ \alpha_0 g+E_1\beta_0 h+i(E_1\gamma_0 g+\delta_0 h)\end{pmatrix}dS,\end{aligned}\tag{12.6}$$

$$\int\limits_0^{s(t)} (Tu)^+ dS = m^{-1} \begin{pmatrix} E_1\gamma_0 g + \delta_0 h \\ E_1\gamma_0 g + \delta_0 h \end{pmatrix} - \varkappa_N \begin{pmatrix} g + \alpha_0 g + E_1\beta_0 h \\ \alpha_0 g + E_1\beta_0 h \end{pmatrix} +$$
$$+ \frac{1}{\pi} \int\limits_S \operatorname{Re} \frac{\partial}{\partial s(y)} (-m^{-1} + i\varkappa_N)\Big(E \ln\sigma - \frac{\varepsilon}{2}\frac{\overline{\sigma}}{\sigma}\Big) \times$$
$$\times \begin{pmatrix} g + \alpha_0 g + E_1\beta_0 h + i(E_1\gamma_0 g + \delta_0 h) \\ \alpha_0 g + E_1\beta_0 h + i(E_1\gamma_0 g + \delta_0 h) \end{pmatrix} dS. \tag{12.7}$$

Further, (12.6) and (12.7) along with (12.1) and (12.4) yield

$$g + \int\limits_S (K_{11} g + K_{12} h) dS = f, \quad h + \int\limits_S (K_{21} g + K_{22} h) dS = F, \tag{12.8}$$

where K_{ij} are known 2×2 matrices with weakly singular elements, while

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad F = \begin{pmatrix} f_3 \\ f_4 \end{pmatrix}.$$

It can be proved that the system of Fredholm equations (12.8) is solvable in $C^{1+\alpha}(S)$ for arbitrary right-hand sides $f_j \in C^{1+\alpha}(S)$, $j = \overline{1,4}$, $S \in C^{2+\beta}$, $0 < \alpha < \beta \le 1$.

13. Explicit Solutions of Boundary Value Problems for Concrete Domains

In this section, we will explicitly (in quadratures) construct solutions to the above boundary value problems for a half-plane, circle and exterior to circle. We will essentially use the results obtained in the previous sections.

Let us consider the first boundary value problem for a half-plane.

Let D denote the upper half-plane $(x_2 > 0)$. Clearly the boundary of D is x_1 axis. Let us choose the exterior unit normal $n = (0, -1)$ and the unit tangent vector $\tau = (1, 0)$.

Let us look for the solution to the first boundary value problem in the form of a double layer potential

$$u(x) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \big[N_y \Phi(y - x)\big]'_{y_2=0} g(y_1)\, dy_1, \tag{13.1}$$

where the matrix $[N_y\Phi(y-x)]'$ is defined by (3.15).

Taking into account the properties of the double layer potential, we arrive to the integral equation

$$g(x_1) + \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \big[N_y \Phi(y - x)\big]'_{y_2=0, x_2=0} \cdot g(y_1)\, dy_1 = f(x_1).$$

It is easy to check that $[N_y\Phi(y-x)]'_{\substack{y_2=0,\\ x_2=0}}=0$, which results $g(x_1)=f(x_1)$.

Therefore we have the following formula for the solution of the original problem

$$u(x)=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{d}{dy_1}\,\mathrm{Im}\Big[E\ln(z-y_1)-\frac{\varepsilon}{2}\frac{\overline{z}-y_1}{z-y_1}\Big]g(y_1)\,dy_1, \qquad (13.2)$$

where $z=x_1+ix_2$.

Now let us consider the second boundary value problem. We look for the solution as a single layer potential of the second kind, which leads to the integral equation

$$-g(x_1)+\frac{1}{\pi}\int\limits_{-\infty}^{\infty}T_xM(x-y)\big|_{\substack{y_2=0,\\ x_2=0}}g(y_1)\,dy_1=F(x_1),$$

where $F(x_1)=(Tu)^+$. Here also we have $T_xM(x-y)|_{\substack{x=0,\\ y=0}}=0$, and, clearly, $g(x_1)=-F(x_1)$.

Finally, for the solution to the second boundary value problem, we have

$$u(x)=-\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\mathrm{Re}\big[\Gamma-E_0\ln(z-y_1)\big]f(y_1)\,dy_1.$$

The stress vector in this case has the form

$$Tu(x)=-\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{d}{dx_1}\Big[E\ln(z-y_1)+\frac{H}{2\Delta_2}\,\frac{\overline{z}-y_1}{z-y_1}\Big]f(y_1)\,dy_1. \quad (13.3)$$

In quite the same way, we can construct the solution to the third boundary value problem in D. The solution reads

$$\begin{aligned}u(x)=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{d}{dy_1}\,\mathrm{Im}\Big[E\ln(z-y_1)-\frac{\varepsilon}{2}\frac{\overline{z}-y_1}{z-y_1}\Big]\times\\ \times\begin{pmatrix}f+\alpha_0f+E_1\beta_0F+i(E_1\gamma_0f+\delta_0F)\\ \alpha_0f+E_1\beta_0F+i(E_1\gamma_0f+\delta_0F)\end{pmatrix}dy_1,\end{aligned} \qquad (13.4)$$

where

$$(u')^+-(u'')^+=f=\begin{pmatrix}f_1\\ f_2\end{pmatrix},\quad F=\begin{pmatrix}f_3\\ f_4\end{pmatrix},$$

and $f_1,\dots,f_4$ are given by (12.1).

Thus for the first, the second and the third boundary value problems we have obtained the Poisson type formulas.

We note that in the above formulas, we assume the following conditions to be fulfilled at infinity

$$f = c + \frac{a}{|y_1|^{1+\alpha}}, \quad F = d + \frac{b}{|y_1|^{1+\alpha}}, \tag{13.5}$$

where a, b, c and d are constant vectors and $\alpha > 0$.

Let us now consider the first BVP for a circle centered at the origin and radius R.

First let us note that

$$\frac{\partial}{\partial s(y)} \operatorname{Im}\Big(\ln\sigma - \frac{1}{2}\ln\zeta\Big) = 0, \quad \frac{\overline{\sigma}}{\sigma} + \frac{\overline{z}}{\zeta} = 0 \tag{13.6}$$

if both points belong to the circle.

Indeed, we have:

$$\begin{aligned} t_1 &= R\cos\psi, \ \ t_2 = R\sin\psi, \ \ y_1 = R\cos\varphi, \ \ y_2 = R\sin\varphi, \\ \theta &= \operatorname{arctg}\frac{y_2 - x_2}{y_1 - x_1} = \operatorname{arctg}\operatorname{tg}\Big(\frac{\pi}{2} + \frac{\varphi+\psi}{2}\Big) = \frac{\pi+\varphi+\psi}{2}, \\ &\frac{\partial}{\partial s(y)}\Big(\theta - \frac{1}{2}\varphi\Big) = \frac{1}{R}\frac{d}{d\varphi}\Big(\frac{\pi+\psi}{2}\Big) = 0, \\ \frac{\overline{\sigma}}{\sigma} + \frac{\overline{z}}{\zeta} &= e^{-i(\pi+\varphi+\psi)} + e^{-i(\varphi+\psi)} = (e^{-i\pi}+1)e^{-i(\varphi+\psi)} = 0. \end{aligned}$$

Further we look for the solution to the first BVP as

$$u(x) = \frac{1}{\pi}\int_S \operatorname{Im}\frac{\partial}{\partial s(y)}\Big[E\big(\ln\sigma - \frac{1}{2}\ln\zeta\big) - \frac{\varepsilon}{2}\Big(\frac{\overline{\sigma}}{\sigma} + \frac{\overline{z}}{\zeta}\Big)\Big]g(y)dS, \tag{13.7}$$

where g is an unknown vector, $\zeta = y_1 + iy_2 = Re^{i\varphi}$, $z = \rho e^{i\psi}$, $\rho = \sqrt{x_1^2 + x_2^2}$ (see also (3.16) and (3.17)).

It is obvious that the additional summands to the double layer potential (see (13.7)) do not cause difficulties, since they are solutions to the differential equation under consideration and represent vector-functions continuous up to the boundary of the disk. Passing to limit as $x \to t$, from (13.7) we get

$$g(t) + \frac{1}{\pi}\int_S \operatorname{Im}\frac{\partial}{\partial s(y)}\Big[E\big(\ln\sigma - \frac{1}{2}\ln\zeta\big) - \frac{\varepsilon}{2}\Big(\frac{\overline{\sigma}}{\sigma} + \frac{\overline{z}}{\zeta}\Big)\Big]g(y)\,dS = f(t).$$

The last equation together with (13.6) implies $g(t) = f(t)$. Now (13.7) yields (the Poisson type formula)

$$u(x) = \frac{1}{2\pi}\int_0^{2\pi}\Big[E\,\frac{R^2-\rho^2}{r^2} + \frac{\varepsilon}{2}(R^2-\rho^2)\frac{d}{d\varphi}\operatorname{Im}\frac{1}{\zeta(\zeta - z)}\Big]f(\varphi)\,d\varphi, \tag{13.8}$$

where

$$r^2 = \rho^2 - 2\rho R\cos(\varphi - \psi) + R^2, \quad \zeta = Re^{i\varphi}, \quad z = \rho e^{i\psi}.$$

Next we consider the second BVP for the same circle as above. We look for the solution as

$$u(x) = \frac{1}{\pi}\int\limits_S \mathrm{Re}(\widetilde{\Gamma} - E_0 \ln\sigma X)Y g(y)\, dS, \tag{13.9}$$

where g is an unknown vector,

$$\widetilde{\Gamma} = m\ln\sigma + \frac{n}{4}\Big(\frac{\overline{\sigma}}{\sigma} + \frac{\overline{z}}{\zeta}\Big), \tag{13.10}$$

other parameters involved in (13.9) and (13.10) are defined by (2.14),(2.15), (4.2), (4.6) and (4.7). The representation (13.9) and the boundary condition of the second BVP lead to the integral equation with respect to g:

$$-g(t) + \frac{1}{\pi}\int\limits_S \frac{\partial}{\partial s(t)}\,\mathrm{Im}\,\Big[E\ln\sigma + \frac{H}{2\Delta_2}\Big(\frac{\overline{\sigma}}{\sigma} + \frac{\overline{z}}{\zeta}\Big)\Big] g(y)\, dS = F(t).$$

By (13.6), we get

$$-g(\psi) + \frac{1}{2\pi}\int\limits_0^{2\pi} g(\varphi)\, d\varphi = F(\psi),$$

whence

$$g(\psi) = -F(\psi) + c \tag{13.11}$$

follows with an arbitrary constant vector c. Clearly the solution to the integral equation exists if the following conditions hold

$$\int\limits_S F(t)\varphi^{(j)}(t)\, dt = 0, \quad j = \overline{1,5},$$

where $\varphi^{(j)}(t)$ are determined by (6.17).

Substituting (13.11) into (13.9) yields

$$u(x) = \frac{1}{\pi}\int\limits_S \mathrm{Re}(E_0\ln\sigma X - \widetilde{\Gamma})Y F(y)\, dS.$$

The corresponding stress vector reads

$$Tu(x) = -\frac{1}{\pi}\int\limits_0^{2\pi} \frac{d}{d\psi}\,\mathrm{Im}\,\Big[E\ln\sigma + \frac{H}{2\Delta_2}\Big(\frac{\overline{\sigma}}{\sigma} + \frac{\overline{z}}{\zeta}\Big)\Big] F(\varphi)\, d\varphi. \tag{13.12}$$

The solution (Poisson type formula) to the third BVP can be obtained in the same way. It reads as

$$u(x) = \frac{1}{\pi}\int\limits_S \operatorname{Im}\frac{\partial}{\partial s(y)}\Big[E\ln\sigma - \frac{\varepsilon}{2}\Big(\frac{\overline{\sigma}}{\sigma}+\frac{\overline{z}}{\zeta}\Big)\Big] \times$$
$$\times\begin{pmatrix} f+\alpha_0 f + E_1\beta_0 F + i(E_1\gamma_0 f+\delta_0 F)\\ \alpha_0 f + E_1\beta_0 F + i(E_1\gamma_0 f + \delta_0 F)\end{pmatrix} dS, \tag{13.13}$$

where

$$f = (u')^+ - (u'')^+, \quad F = \begin{pmatrix} f_3\\ f_4\end{pmatrix}.$$

Finally we treat the BVPs for the exterior domain to the above circle. Let us first consider the first BVP. As above, we have

$$\frac{\partial}{\partial s(y)}\operatorname{Im}\Big(\ln\sigma - \frac{1}{2}\ln\zeta\Big) = 0, \quad \frac{\overline{\sigma}}{\sigma}+\frac{\overline{\zeta}}{z} = 0 \tag{13.14}$$

if the points are on the circle.

We look for the solution of the first BVP in the following form

$$u(x) = \frac{1}{\pi}\int\limits_S \operatorname{Im}\frac{\partial}{\partial s(y)}\Big[E\big(\ln\sigma - \frac{1}{2}\ln\zeta\big) - \frac{\varepsilon}{2}\Big(\frac{\overline{\sigma}}{\sigma}+\frac{\overline{\zeta}}{z}\Big)\Big]g(y)\,dS, \tag{13.15}$$

where g is the unknown continuous vector. Here the additional terms again facilitate the procedure of solution. Indeed, the above representation leads to the integral equation

$$-g(t) + \frac{1}{\pi}\int\limits_S \operatorname{Im}\frac{\partial}{\partial s(y)}\Big[E\big(\ln\sigma - \frac{1}{2}\ln\zeta\big) - \frac{\varepsilon}{2}\Big(\frac{\overline{\sigma}}{\sigma}+\frac{\overline{\zeta}}{z}\Big)\Big]g(y)\,dS = f(t),$$

whence $g(t) = -f(t)$ follows. Finaly we get the following Poisson type formula for the first BVP in the exterior to disk

$$u(x) = \frac{1}{2\pi}\int\limits_0^{2\pi}\Big[E\,\frac{\rho^2-R^2}{r^2} + \frac{\varepsilon}{2}(\rho^2-R^2)\frac{d}{d\varphi}\operatorname{Im}\frac{1}{z(z-\zeta)}\Big]f(\varphi)\,d\varphi. \tag{13.16}$$

The solution of the second BVP is respesented as

$$u(x) = \frac{1}{\pi}\int\limits_S \operatorname{Re}(\widetilde{\Gamma} - E_0\ln\sigma X)Y g(y)\,dS, \tag{13.17}$$

with the unknown density g and

$$\widetilde{\Gamma} = m\ln\sigma + \frac{n}{4}\Big(\frac{\overline{\sigma}}{\sigma}+\frac{\overline{\zeta}}{z}\Big).$$

The boundary condition and the representation formula (13.17) imply the following integral equation

$$g(t)+\frac{1}{\pi}\int\limits_S \frac{\partial}{\partial s(t)}\,\mathrm{Im}\,\Big[E\ln\sigma+\frac{H}{2\Delta_2}\Big(\frac{\overline{\sigma}}{\sigma}+\frac{\overline{\zeta}}{z}\Big)g(y)\,dS=F(t).$$

Now according to (13.14), we have

$$g(\psi)+\frac{1}{2\pi}\int\limits_0^{2\pi} g(\varphi)\,d\varphi=F(\psi),$$

whence

$$g(\psi)=F(\psi)-\frac{1}{4\pi}\int\limits_0^{2\pi} F(\varphi)\,d\varphi.$$

If the displacements are bounded at infinity, then we have

$$\int\limits_0^{2\pi} F\,d\varphi=0.$$

and, finally,

$$g(\psi)=F(\psi).$$

These results lead to the following formulas (see (13.17))

$$u(x)=\frac{1}{\pi}\int\limits_S \mathrm{Re}(\widetilde{\Gamma}-E_0\ln\sigma X)Y\cdot F(y)\,dS,$$

$$Tu(x)=\frac{1}{\pi}\int\limits_0^{2\pi}\frac{d}{d\psi}\,\mathrm{Im}\,\Big[E\ln\sigma+\frac{H}{2\Delta_2}\Big(\frac{\overline{\sigma}}{\sigma}+\frac{\overline{\zeta}}{z}\Big)\Big]F(\varphi)\,d\varphi.$$

Quite samillary we can solve the third BVP for the exterior of disk. The final expression for the solution reads

$$u(x)=\frac{1}{\pi}\int\limits_S \mathrm{Im}\,\frac{\partial}{\partial s(y)}\Big[-E\ln\sigma+\frac{\varepsilon}{2}\Big(\frac{\overline{\sigma}}{\sigma}+\frac{\overline{\zeta}}{z}\Big)\Big]\times$$
$$\times\begin{pmatrix} f+\alpha_0 f+E_1\beta_0 F+i(E_1\gamma_0 f+\delta_0 F)\\ \alpha_0 f+E_1\beta_0 F+i(E_1\gamma_0 f+\delta_0 F)\end{pmatrix}dS,$$

where

$$f=(u')^- -(u'')^-,\quad F=\begin{pmatrix} f_3\\ f_4\end{pmatrix}.$$

Other applications of the Fredholm integral equations, obtained in the present paper, will be treated in the forthcomming publications of the author.

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CONTENTS

ON THE STABILITY OF SOLUTIONS OF THE MULTIPOINT BOUNDARY VALUE PROBLEM FOR THE SYSTEM OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS 1

§1. Statement of the problem and formulation of main results 3
§2. Auxiliary propositions 16
§3. Proof of main results 31
§4. On the numerical solution of the multipoint boundary value problem for a system of ordinary differential equations ...38
§5. On the stability of solutions of the multipoint boundary value problem for a system of difference equations 46
References 56

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEMS OF STATICS OF THE THEORY OF ELASTIC MIXTURES 59

Introduction 61
1. Basic Equations and Boundary Value Problems 61
2. The Basic Fundamental Matrix 64
3. Singular Matrices of Solutions 67
4. Matrix $M(x-y)$ 71
5. Generalized Green Formulas 73
6. General Representation of Solution 78
7. Uniqueness Theorems 81
8. Generalized Potentials and Their Properties 82
9. Existence Theorems of Problems $(\mathrm{I})^{\pm}_{0,f}$ and $(\mathrm{II})^{\pm}_{0,F}$ 86
   1. Problems $(\mathrm{I})^{+}_{0,f}$ and $(\mathrm{II})^{-}_{0,F}$ 86
   2. Problems $(\mathrm{I})^{-}_{0,f}$ and $(\mathrm{II})^{+}_{0,F}$ 87
10. An Alternative Approach to the Problem $(\mathrm{I})^{\pm}_{0,f}$ 90
11. An Alternative Approach to the Problem $(\mathrm{II})^{\pm}_{0,F}$ 93
12. Solution of the Third Boundary Value Problem 98
13. Explicit Solutions of Boundary Value Problems for Concrete Domains 99
References 105

REPORTS OF THE TBILISI SEMINAR
ON QUALITATIVE THEORY OF
DIFFERENTIAL EQUATIONS

On Solvability of Functional Equations in the Space of Continuous Vector Functions 109

Functional Differential Equations with Pulses 113

On Asymptotic Behaviour of Solutions of Linear Functional Differential Equations 116

Criteria for Oscillation of Solutions of Two-Dimensional Differential Systems with Deviating Arguments 119

On the Question of Solvability of the Periodic Boundary Value Problem for a System of Linear Generalized Ordinary Differential Equations 121

On a Two-Point Boundary Value Problem for Second Order Functional Differential Equations 124

On Oscillatory Properties of the n-th Order System of Differential Equations with Deviating Arguments 127

On Asymptotic Behavior of Solutions of Second Order Linear Differential Equations 130