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VARIATION FORMULAS OF A SOLUTION AND INITIAL DATA OPTIMIZATION PROBLEMS FOR QUASI-LINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS INITIAL CONDITION

Dedicated to the 125th birthday anniversary of Professor A. Razmadze

Abstract. For the quasi-linear neutral functional differential equation the continuous dependence of a solution of the Cauchy problem on the initial data and on the nonlinear term in the right-hand side of that equation is investigated, where the perturbation nonlinear term in the right-hand side and initial data are small in the integral and standard sense, respectively. Variation formulas of a solution are derived, in which the effect of perturbations of the initial moment and the delay function, and also that of the discontinuous initial condition are detected. For initial data optimization problems the necessary conditions of optimality are obtained. The existence theorem for optimal initial data is proved.

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Key words and phrases. Quasi-linear neutral functional differential equation, continuous dependence of solution, variation formula of solution, effect of initial moment perturbation, effect of a discontinuous initial condition, effect of delay function perturbation, initial data optimization problem, necessary conditions of optimality, existence of optimal initial data.

რეზიუმე. კვაზიწრფივი ნეიტრალური ფუნქციონალურ-დიფერენციალური განტოლებებისათვის გამოკვლეულია კოშის ამოცანის ამონახსნის უწყვეტობა საწყისი მონაცემების და განტოლების მარჯვენა მხარის არაწრფივი შესაკრების შეშფოთებების მიმართ, სადაც მარჯვენა მხარის არაწრფივი შესაკრების და საწყისი მონაცემების შეშფოთებები, შესაბამისად, მცირეა ინტეგრალური და სტანდარტული აზრით. დამტკიცებულია ამონახსნის ვარიაციის ფორმულები, რომლებშიც გამოვლენილია საწყისი მომენტის და დაგვიანების ფუნქციის შეშფოთებების ეფექტები, წყვეტილი საწყისი პირობის ეფექტი. საწყისი მონაცემების ოპტიმიზაციის ამოცანებისთვის მიღებულია ოპტიმალურობის აუცილებელი პირობები. დამტკიცებულია ოპტიმალური საწყისი მონაცემების არსებობის თეორემა.

Introduction

Neutral functional differential equation (briefly-neutral equation) is a mathematical model of such dynamical system whose behavior depends on the prehistory of the state of the system and on its velocity (derivative of trajectory) at a given moment of time. Such mathematical models arise in different areas of natural sciences as electrodynamics, economics, etc. (see e.g. [1, 2, 4–6, 12, 13, 16]). To illustrate this, we consider a simple model of economic growth. Let $N(t)$ be a quantity of a product produced at the moment t which is expressed in money units. The fundamental principle of the economic growth has the form

$$N(t) = C(t) + I(t), \tag{0.1}$$

where $C(t)$ is the so-called an apply function and $I(t)$ is a quantity of induced investment. We consider the case where the functions $C(t)$ and $I(t)$ are of the form

$$C(t) = \alpha N(t), \;\; \alpha \in (0,1), \tag{0.2}$$

and

$$I(t) = \alpha_1 N(t-\theta) + \alpha_2 \dot N(t) + \alpha_3 \dot N(t-\theta) + \alpha_0 \ddot N(t) + \alpha_4 \ddot N(t-\theta), \;\; \theta > 0. \tag{0.3}$$

From formulas (0.1)–(0.3) we get the equation

$$\ddot N(t) = \frac{1-\alpha}{\alpha_0}\, N(t) - \frac{\alpha_1}{\alpha_0}\, N(t-\theta) - \frac{\alpha_2}{\alpha_0}\, \dot N(t) - \frac{\alpha_3}{\alpha_0}\, \dot N(t-\theta) - \frac{\alpha_4}{\alpha_0}\, \ddot N(t-\theta)$$

which is equivalent to the following neutral equation:

$$\begin{cases} \dot x^1(t) = x^2(t), \\ \dot x^2(t) = \dfrac{1-\alpha}{\alpha_0}\, x^1(t) - \dfrac{\alpha_1}{\alpha_0}\, x^1(t-\theta) - \dfrac{\alpha_2}{\alpha_0}\, x^2(t) - \\ \qquad - \dfrac{\alpha_3}{\alpha_0}\, x^2(t-\theta) - \dfrac{\alpha_4}{\alpha_0}\, \dot x^2(t-\theta), \end{cases}$$

here $x^1(t) = N(t)$.

Many works are devoted to the investigation of neutral equations, including [1–7, 12–14, 17, 19, 25, 28].

We note that the Cauchy problem for the nonlinear with respect to the prehistory of velocity neutral equations is, in general, ill-posed when perturbation of the right-hand side of equation is small in the integral sense. Indeed, on the interval $[0, 2]$ we consider the system

$$\begin{cases} \dot x^1(t) = 0, \\ \dot x^2(t) = \big[x^1(t-1)\big]^2 \end{cases} \tag{0.4}$$

with the initial condition

$$\dot x^1(t) = 0, \;\; t \in [-1, 0), \quad x^1(0) = x^2(0) = 0. \tag{0.5}$$

The solution of the system (0.4) is

$$x^1_0(t) = x^2_0(t) \equiv 0.$$

We now consider the perturbed system

$$\begin{cases} \dot{x}_k^1(t) = p_k(t), \\ \dot{x}_k^2(t) = \left[x_k^1(t-1)\right]^2 \end{cases}$$

with the initial condition (0.5). Here,

$$p_k(t) = \begin{cases} \varsigma_k(t), & t \in [0,1], \\ 0, & t \in (1,2]. \end{cases}$$

The function $\varsigma_k(t)$ is defined as follows: for the given $k = 2, 3, \dots$, we divide the interval $[0,1]$ into the subintervals l_i, $i = 1, \dots, k$, of the length $1/k$; then we define $\varsigma_k(t) = 1$, $t \in l_1$, $\varsigma_k(t) = -1$, $t \in l_2$ and so on. It is easy to see that

$$\lim_{k\to\infty} \max_{s_1, s_2 \in [0,1]} \left| \int_{s_1}^{s_2} \varsigma_k(t)\, dt \right| = 0.$$

Taking into consideration the initial condition (0.5) and the structure of the function $\varsigma_k(t)$, we get

$$x_k^1(t) = \int_0^t \varsigma_k(s)\, ds \ \text{ for } \ t \in [0,1], \quad x_k^1(t) = x_k^1(1) \ \text{ for } \ t \in (1,2]$$

and

$$x_k^2(t) = \int_0^t \left[\dot{x}_k^1(s-1)\right]^2 ds = 0 \ \text{ for } \ t \in [0,1],$$

$$x_k^2(t) = \int_1^t \left[\dot{x}_k^1(s-1)\right]^2 ds = \int_1^t \varsigma_k^2(s-1)\, ds =$$

$$= \int_1^t 1\, ds = t - 1 \ \text{ for } \ t \in (1,2].$$

It is clear that

$$\lim_{k\to\infty} \max_{t\in[0,2]} \left|x_k^1(t) - x_0^1(t)\right| = 0, \quad \lim_{k\to\infty} \max_{t\in[0,2]} \left|x_k^2(t) - x_0^2(t)\right| \neq 0.$$

Thus, the Cauchy problem (0.4)–(0.5) is ill-posed.

The present work consists of two parts, naturally interconnected in their meaning.

Part I concerns the following quasi-linear neutral equation:

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f\big(t, x(t), x(\tau(t))\big) \tag{0.6}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \ \ \dot{x}(t) = v(t), \ \ t < t_0, \ \ x(t_0) = x_0. \tag{0.7}$$

We note that the symbol $\dot{x}(t)$ for $t < t_0$ is not connected with the derivative of the function $\varphi(t)$. The condition (0.7) is called the discontinuous initial condition, since, in general, $x(t_0) \neq \varphi(t_0)$.

In the same part we study the continuous dependence of a solution of the problem (0.6)–(0.7) on the initial data and on the nonlinear term in the right-hand side of the equation (0.6). Here, under initial data we mean the collection of an initial moment, delay function appearing in the phase coordinates, initial vector and initial functions. Moreover, we derive variation formulas of a solution.

In Part II we consider the control neutral equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f\big(t, x(t), x\big(\tau(t), u(t)\big)\big)$$

with the initial condition (0.7). Here under initial data we understand the collection of the initial moment t_0, delay function $\tau(t)$, initial vector x_0, initial functions $\varphi(t)$ and $v(t)$, and the control function $u(t)$. In the same part, the continuous dependence of a solution and variation formulas are used in proving both the necessary optimality conditions for the initial data optimization problem and the existence of optimal initial data.

In Section 1 we prove the theorem on the continuous dependence of a solution in the case where the perturbation of f is small in the integral sense and initial data are small in the standard sense. Analogous theorems without perturbation of a delay function are given [17, 28] for quasi-linear neutral equations. Theorems on the continuous dependence of a solution of the Cauchy and boundary value problems for various classes of ordinary differential equations and delay functional differential equations when perturbations of the right-hand side are small in the integral sense are given in [10, 11, 15, 18, 20, 21, 23, 26].

In Section 2 we prove derive variation formulas which show the effect of perturbations of the initial moment and the delay function appearing in the phase coordinates and also that of the discontinuous initial condition. Variation formulas for various classes of neutral equations without perturbation of delay can be found in [16, 24]. The variation formula of a solution plays the basic role in proving the necessary conditions of optimality [11, 15] and in sensitivity analysis of mathematical models [1,2,22]. Moreover, the variation formula allows one to obtain an approximate solution of the perturbed equation.

In Section 3 we consider initial data optimization problem with a general functional and under the boundary conditions. The necessary conditions are obtained for: the initial moment in the form of inequalities and equalities, the initial vector in the form of equality, and the initial functions and control function in the form of linearized integral maximum principle.

Finally, in Section 4 the existence theorem for an optimal initial data is proved.

1. Continuous Dependence of a Solution

1.1. **Formulation of main results.** Let $I=[a,b]$ be a finite interval and $\mathbb{R}^n$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$, where T is the sign of transposition. Suppose that $O\subset\mathbb{R}^n$ is an open set and let E_f be the set of functions $f:I\times O^2\to\mathbb{R}^n$ satisfying the following conditions: for each fixed $(x_1,x_2)\in O^2$ the function $f(\,\cdot\,,x_1,x_2):I\to\mathbb{R}^n$ is measurable; for each $f\in E_f$ and compact set $K\subset O$ there exist the functions $m_{f,K}(t),L_{f,K}(t)\in L(I,\mathbb{R}_+)$, where $\mathbb{R}_+=[0,\infty)$, such that for almost all $t\in I$

$$|f(t,x_1,x_2)|\le m_{f,K}(t),\ \ \forall\,(x_1,x_2)\in K^2,$$
$$\big|f(t,x_1,x_2)-f(t,y_1,y_2)\big|\le$$
$$\le L_{f,K}(t)\sum_{i=1}^{2}|x_i-y_i|,\ \ \forall\,(x_1,x_2)\in K^2,\ \ \forall\,(y_1,y_2)\in K^2.$$

We introduce the topology in E_f by the following basis of neighborhoods of zero:

$$\Big\{V_{K,\delta}:\ K\subset O \text{ is a compact set and } \delta>0 \text{ is an arbitrary number}\Big\},$$

where

$$V_{K,\delta}=\big\{\delta f\in E_f:\ \Delta(\delta f;K)\le\delta\big\},$$
$$\Delta(\delta f;K)=\sup\bigg\{\bigg|\int\limits_{t'}^{t''}\delta f(t,x_1,x_2)\,dt\bigg|:\ t',t''\in I,\ x_i\in K,\ i=1,2\bigg\}.$$

Let D be the set of continuously differentiable scalar functions (delay functions) $\tau(t)$, $t\in\mathbb{R}$, satisfying the conditions

$$\tau(t)<t,\ \ \dot\tau(t)>0,\ \ t\in\mathbb{R};\quad \inf\big\{\tau(a):\ \tau\in D\big\}:=\widehat\tau>-\infty,$$
$$\sup\big\{\tau^{-1}(b):\ \tau\in D\big\}:=\widehat\gamma<+\infty,$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$.

Let E_φ be the space of bounded piecewise-continuous functions $\varphi(t)\in\mathbb{R}^n$, $t\in I_1=[\widehat\tau,b]$, with finitely many discontinuities, equipped with the norm $\|\varphi\|_{I_1}=\sup\{|\varphi(t)|:\ t\in I_1\}$. By $\Phi_1=\{\varphi\in E_\varphi:\ \operatorname{cl}\varphi(I_1)\subset O\}$ we denote the set of initial functions of trajectories, where $\varphi(I_1)=\{\varphi(t):\ t\in I_1\}$; by E_v we denote the set of bounded measurable functions $v:I_1\to\mathbb{R}^n$, $v(t)$ is called the initial function of trajectory derivative.

By μ we denote the collection of initial data $(t_0,\tau,x_0,\varphi,v)\in[a,b)\times D\times O\times\Phi_1\times E_v$ and the function $f\in E_f$.

To each element $\mu=(t_0,\tau,x_0,\varphi,v,f)\in\Lambda=[a,b)\times D\times O\times\Phi_1\times E_v\times E_f$ we assign the quasi-linear neutral equation

$$\dot x(t)=A(t)\dot x(\sigma(t))+f\big(t,x(t),x(\tau(t))\big) \tag{1.1}$$

with the initial condition

$$x(t)=\varphi(t),\ \ \dot{x}(t)=v(t),\ \ t\in[\widehat{\tau},t_0),\ \ x(t_0)=x_0. \tag{1.2}$$

Here $A(t)$ is a given continuous $n\times n$ matrix function and $\sigma\in D$ is a fixed delay function in the phase velocity. We note that the symbol $\dot{x}(t)$ for $t<t_0$ is not connected with a derivative of the function $\varphi(t)$. The condition (1.2) is called the discontinuous initial condition, since $x(t_0)\neq\varphi(t_0)$, in general.

Definition 1.1. Let $\mu=(t_0,\tau,x_0,\varphi,v,f)\in\Lambda$. A function $x(t)=x(t;\mu)\in O$, $t\in[\widehat{\tau},t_1]$, $t_1\in(t_0,b]$, is called a solution of the equation (1.1) with the initial condition (1.2) or a solution corresponding to the element μ and defined on the interval $[\widehat{\tau},t_1]$ if it satisfies the condition (1.2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies the equation (1.1) almost everywhere (a.e.) on $[t_0,t_1]$.

To formulate the main results, we introduce the following sets:

$$W(K;\alpha)=\Big\{\delta f\in E_f:\ \exists\, m_{\delta f,K}(t),L_{\delta f,K}(t)\in L(I,\mathbb{R}_+),\\ \int\limits_I\big[m_{\delta f,K}(t)+L_{\delta f,K}(t)\big]\,dt\le\alpha\Big\},$$

where $K\subset O$ is a compact set and $\alpha>0$ is a fixed number independent of δf;

$$\begin{aligned}
B(t_{00};\delta)&=\{t_0\in I:|t_0-t_{00}|<\delta\},\\
B_1(x_{00};\delta)&=\{x_0\in O:|x_0-x_{00}|<\delta\},\\
V(\tau_0;\delta)&=\{\tau\in D:||\tau-\tau_0||_{I_2}<\delta\},\\
V_1(\varphi_0;\delta)&=\{\varphi\in\Phi_1:||\varphi-\varphi_0||_{I_1}<\delta\},\\
V_2(v_0;\delta)&=\{v\in E_v:||v-v_0||_{I_1}<\delta\},
\end{aligned}$$

where $t_{00}\in[a,b)$ and $x_{00}\in O$ are the fixed points; $\tau_0\in D$, $\varphi_0\in\Phi_1$, $v_0\in E_v$ are the fixed functions, $\delta>0$ is the fixed number, $I_2=[a,\widehat{\gamma}]$.

Theorem 1.1. *Let $x_0(t)$ be a solution corresponding to $\mu_0=(t_{00},\tau_0,x_{00},\varphi_0,v_0,f_0)\in\Lambda$, $t_{10}<b$, and defined on $[\widehat{\tau},t_{10}]$. Let $K_1\subset O$ be a compact set containing a certain neighborhood of the set $\operatorname{cl}\varphi_0(I_1)\cup x_0([t_{00},t_{10}])$. Then the following assertions hold:*

1.1. *there exist numbers $\delta_i>0$, $i=0,1$ such that to each element*

$$\begin{aligned}
\mu=(t_0,\tau,x_0,\varphi,v,f_0+\delta f)&\in V(\mu_0;K_1,\delta_0,\alpha)=\\
=B(t_{00};\delta_0)\times V(\tau_0;\delta_0)&\times B_1(x_{00};\delta_0)\times V_1(\varphi_0;\delta_0)\times V_2(v_0;\delta_0)\times\\
&\times\big[f_0+W(K_1;\alpha)\cap V_{K_1,\delta_0}\big]
\end{aligned}$$

there corresponds the solution $x(t;\mu)$ defined on the interval $[\widehat{\tau},t_{10}+\delta_1]\subset I_1$ and satisfying the condition $x(t;\mu)\in K_1$;

1.2. *for an arbitrary* $\varepsilon > 0$ *there exists a number* $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0]$ *such that the following inequality holds for any* $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$*:*

$$\big|x(t;\mu) - x(t;\mu_0)\big| \le \varepsilon, \ \ \forall\, t \in [\widehat{t}, t_{10} + \delta_1], \ \ \widehat{t} = \max\{t_{00}, t_0\};$$

1.3. *for an arbitrary* $\varepsilon > 0$ *there exists a number* $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0]$ *such that the following inequality holds for any* $\mu \in V(\mu_0; K_1, \delta_3, \alpha)$*:*

$$\int\limits_{\widehat{\tau}}^{t_{10}+\delta_1} |x(t;\mu) - x(t;\mu_0)|\, dt \le \varepsilon.$$

Due to the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$.

In the space $E_\mu - \mu_0$, where $E_\mu = \mathbb{R} \times D \times \mathbb{R}^n \times E_\varphi \times E_v \times E_f$, we introduce the set of variations:

$$\Im = \Big\{ \delta\mu = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi, \delta v, \delta f) \in E_\mu - \mu_0 : \ |\delta t_0| \le \beta, \ \|\delta\tau\|_{I_2} \le \beta,$$

$$|\delta x_0| \le \beta, \ \|\delta\varphi\|_{I_1} \le \beta, \ \|\delta v\|_{I_1} \le \beta, \ \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i,$$

$$|\lambda_i| \le \beta, \ i = 1, \dots, k \Big\},$$

where $\beta > 0$ is a fixed number and $\delta f_i \in E_f - f_0$, $i = i = 1, \dots, k$, are fixed functions.

Theorem 1.2. *Let* $x_0(t)$ *be a solution corresponding to* $\mu_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, v_0, f_0) \in \Lambda$ *and defined on* $[\widehat{\tau}, t_{10}]$, $t_{i0} \in (a, b)$, $i = 0, 1$. *Let* $K_1 \subset O$ *be a compact set containing a certain neighborhood of the set* $\operatorname{cl} \varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$*. Then the following conditions hold:*

1.4. *there exist the numbers* $\varepsilon_1 > 0$, $\delta_1 > 0$ *such that for an arbitrary* $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times \Im$ *we have* $\mu_0 + \varepsilon\delta\mu \in \Lambda$ *and the solution* $x(t; \mu_0 + \varepsilon\delta\mu)$ *defined on the interval* $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ *corresponds to that element. Moreover,* $x(t; \mu_0 + \varepsilon\delta\mu) \in K_1$*;*

1.5. $\lim\limits_{\varepsilon \to 0} \sup \Big\{ |x(t; \mu_0 + \varepsilon\delta\mu) - x(t;\mu_0)| : \ t \in [\widehat{t}, t_{10} + \delta_1] \Big\} = 0,$

$$\lim_{\varepsilon \to 0} \int\limits_{\widehat{\tau}}^{t_{10}+\delta_1} \big|x(t; \mu_0 + \varepsilon\delta\mu) - x(t;\mu_0)\big|\, dt = 0$$

uniformly for $\delta\mu \in \Im$*, where* $\widehat{t} = \max\{t_0, t_0 + \varepsilon\delta t_0\}$*.*

Theorem 1.2 is the corollary of Theorem 1.1.

Let E_u be the space of bounded measurable functions $u(t) \in \mathbb{R}^r$, $t \in I$. Let $U_0 \subset \mathbb{R}^r$ be an open set and $\Omega = \{u \in E_u : \ \operatorname{cl} u(I) \subset U_0\}$. Let Φ_{11} be the set of bounded measurable functions $\varphi(t) \in O$, $t \in I_1$, with $\operatorname{cl} \varphi(I_1) \subset O$.

To each element $w = (t_0, \tau, x_0, \varphi, v, u) \in \Lambda_1 = [a, b) \times D \times O \times \Phi_{11} \times E_v \times \Omega$ we assign the controlled neutral equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f\big(t, x(t), x(\tau(t)), u(t)\big) \tag{1.3}$$

with the initial condition (1.2). Here, the function $f(t, x_1, x_2, u)$ is defined on $I \times O^2 \times U_0$ and satisfies the following conditions: for each fixed $(x_1, x_2, u) \in O^2 \times U_0$, the function $f(\,\cdot\,, x_1, x_2, u) : I \to \mathbb{R}^n$ is measurable; for each compact sets $K \subset O$ and $U \subset U_0$ there exist the functions $m_{K,U}(t), L_{K,U}(t) \in L(I, R_+)$ such that for almost all $t \in I$,

$$\big|f(t, x_1, x_2, u)\big| \leq m_{K,U}(t), \;\; \forall\, (x_1, x_2, u) \in K^2 \times U,$$

$$\big|f(t, x_1, x_2, u_1) - f(t, y_1, y_2, u_2)\big| \leq L_{f,K}(t)\Big[\sum_{i=1}^{2} |x_i - y_i| + |u_1 - u_2|\Big],$$

$$\forall\, (x_1, x_2) \in K^2, \;\; \forall\, (y_1, y_2) \in K^2, \;\; (u_1, u_2) \in U^2.$$

Definition 1.2. Let $w = (t_0, \tau, x_0, \varphi, v, u) \in \Lambda_1$. A function $x(t) = x(t; w) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is called a solution of the equation (1.3) with the initial condition (1.2), or a solution corresponding to the element w and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies the condition (1.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (1.3) a. e. on $[t_0, t_1]$.

Theorem 1.3. *Let $x_0(t)$ be a solution corresponding to $w_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, v_0, u_0) \in \Lambda_1$ and defined on $[\widehat{\tau}, t_{10}]$, $t_{10} < b$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $\operatorname{cl} \varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following conditions hold:*

1.6. *there exist the numbers $\delta_i > 0, i = 0, 1$ such that to each element*

$$w = (t_0, \tau, x_0, \varphi, v, u) \in \widehat{V}(w_0; \delta_0) =$$
$$= B(t_{00}; \delta_0) \times V(\tau_0; \delta_0) \times B_1(x_{00}; \delta_0) \times V_1(\varphi_0; \delta_0) \times V_2(v_0; \delta_0) \times V_3(u_0; \delta_0)$$

there corresponds the solution $x(t; w)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$ and satisfying the condition $x(t; w) \in K_1$, where $V_3(u_0; \delta_0) = \{u \in \Omega : \; \|u - u_0\|_I < \delta_0\}$;

1.7. *for an arbitrary $\varepsilon > 0$, there exists the number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $w \in \widehat{V}(w_0; \delta_2)$:*

$$\big|x(t; w) - x(t; w_0)\big| \leq \varepsilon, \;\; \forall\, t \in [\widehat{t}, t_{10} + \delta_1], \quad \widehat{t} = \max\{t_0, t_{00}\};$$

1.8. *for an arbitrary $\varepsilon > 0$, there exists the number $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $w \in \widehat{V}(w_0; \delta_3)$:*

$$\int_{\widehat{\tau}}^{t_{10}+\delta_1} \big|x(t; w) - x(t; w_0)\big|\, dt \leq \varepsilon.$$

In the space $E_w - w_0$, where $E_w = \mathbb{R}\times D\times\mathbb{R}^n\times\Phi_{11}\times E_v\times E_u$, we introduce the set of variations

$$\Im_1=\Big\{\delta w=(\delta t_0,\delta\tau,\delta x_0,\delta\varphi,\delta v,\delta u)\in E_w-w_0:\ |\delta t_0|\le\beta,\ \|\delta\tau\|_{I_2}\le\beta,\\ |\delta x_0|\le\beta,\ \|\delta\varphi\|_{I_1}\le\beta,\ \|\delta v\|_{I_1}\le\beta,\ \|\delta u\|_I\le\beta\Big\}.$$

Theorem 1.4. *Let $x_0(t)$ be a solution corresponding to $w_0=(t_{00},\tau_0,x_{00},\varphi_0,v_0,u_0)\in\Lambda_1$ and defined on $[\widehat{\tau},t_{10}]$, $t_{i0}\in(a,b)$, $i=0,1$. Let $K_1\subset O$ be a compact set containing a certain neighborhood of the set $\operatorname{cl}\varphi_0(I_1)\cup x_0([t_{00},t_{10}])$. Then the following conditions hold:*

1.9. *there exist numbers $\varepsilon_1>0$, $\delta_1>0$ such that for an arbitrary $(\varepsilon,\delta w)\in[0,\varepsilon_1]\times\Im_1$ we have $w_0+\varepsilon\delta w\in\Lambda_1$, and the solution $x(t;w_0+\varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau},t_{10}+\delta_1]\subset I_1$ corresponds to that element. Moreover, $x(t;w_0+\varepsilon\delta w)\in K_1$;*

1.10. $\lim\limits_{\varepsilon\to0}\sup\Big\{|x(t;w_0+\varepsilon\delta w)-x(t;w_0)|:\ t\in[\widehat{t},t_{10}+\delta_1]\Big\}=0,$

$$\lim_{\varepsilon\to0}\int\limits_{\widehat{\tau}}^{t_{10}+\delta_1}\big|x(t;w_0+\varepsilon\delta w)-x(t;w_0)\big|\,dt=0$$

uniformly for $\delta w\in\Im_1$.

Theorem 1.4 is the corollary of Theorem 1.3.

1.2. **Preliminaries.** Consider the linear neutral equation

$$\dot x(t)=A(t)\dot x(\sigma(t))+B(t)x(t)+C(t)x(\tau(t))+g(t),\ \ t\in[t_0,b],\qquad(1.4)$$

with the initial condition

$$x(t)=\varphi(t),\ \ \dot x(t)=v(t),\ \ t\in[\widehat{\tau},t_0),\ \ x(t_0)=x_0,\qquad(1.5)$$

where $B(t)$, $C(t)$ and $g(t)$ are the integrable on I matrix- and vector-functions.

Theorem 1.5 (Cauchy formula)**.** *The solution of the problem* (1.4)–(1.5) *can be represented on the interval $[t_0,b]$ in the following form:*

$$x(t)=\Psi(t_0;t)x_0+\int\limits_{\sigma(t_0)}^{t_0}Y(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi)v(\xi)\,d\xi+\\ +\int\limits_{\tau(t_0)}^{t_0}Y(\gamma(\xi);t)C(\gamma(\xi))\dot\gamma(\xi)\varphi(\xi)\,d\xi+\int\limits_{t_0}^{t}Y(\xi;t)g(\xi)\,d\xi,\qquad(1.6)$$

where $\nu(t)=\sigma^{-1}(t)$, $\gamma(t)=\tau^{-1}(t)$; $\Psi(\xi;t)$ and $Y(\xi;t)$ are the matrix-functions satisfying the system

$$\begin{cases}\Psi_\xi(\xi;t)=-Y(\xi;t)B(\xi)-Y(\gamma(\xi);t)C(\gamma(\xi))\dot\gamma(\xi),\\ Y(\xi;t)=\Psi(\xi;t)+Y(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi)\end{cases}\qquad(1.7)$$

on (a,t) for any fixed $t \in (a,b]$ and the condition

$$\Psi(\xi;t) = Y(\xi;t) = \begin{cases} H, & \xi = t, \\ \Theta, & \xi > t. \end{cases} \tag{1.8}$$

Here, H is the identity matrix and Θ is the zero matrix.

This theorem is proved in a standard way [3, 9, 15]. The existence of a unique solution of the system (1.7) with the initial condition (1.8) can be easily proved by using the step method from right to left.

Theorem 1.6. *Let q be the minimal natural number for which the inequality*

$$\sigma^{q+1}(b) = \sigma^q(\sigma(b)) < a$$

holds. Then for each fixed instant $t \in (t_0,b]$, the matrix function $Y(\xi;t)$ on the interval $[t_0,t]$ can be represented in the form

$$Y(\xi;t) = \Psi(\xi;t) + \sum_{i=1}^{q} \Psi(\nu^i(\xi);t) \prod_{m=i}^{1} A(\nu^m(\xi)) \frac{d}{d\xi} \nu^m(\xi). \tag{1.9}$$

Proof. It is easy to see that as a result of a multiple substitution of the corresponding expression for the matrix functions $Y(\xi;t)$, using the second equation of the system (1.7), we obtain

$$Y(\xi;t) = \Psi(\xi;t) + \Big[\Psi(\nu(\xi);t) + Y(\nu^2(\xi);t)A(\nu^2(\xi))\dot\nu(\nu(\xi))\Big]A(\nu(\xi))\dot\nu(\xi) =$$

$$= \Psi(\xi;t) + \Psi(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi) + Y(\nu^2(\xi);t)A(\nu^2(\xi))A(\nu(\xi))\,\frac{d}{d\xi}\,\nu^2(\xi) =$$

$$= \Psi(\xi;t) + \Psi(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi) +$$

$$+ \Big[\Psi(\nu^2(\xi);t) + Y(\nu^3(\xi);t)A(\nu^3(\xi))\dot\nu(\nu^2(\xi))\Big]A(\nu^2(\xi))A(\nu(\xi))\,\frac{d}{d\xi}\,\nu^2(\xi) =$$

$$= \Psi(\xi;t) + \Psi(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi) + \Psi(\nu^2(\xi);t)A(\nu^2(\xi))A(\nu(\xi))\,\frac{d}{d\xi}\nu^2(\xi) +$$

$$+ Y(\nu^3(\xi);t)A(\nu^3(\xi))A(\nu^2(\xi))A(\nu(\xi))\,\frac{d}{d\xi}\,\nu^3(\xi).$$

Continuing this process and taking into account (1.8), we obtain (1.9). □

Theorem 1.7. *The solution $x(t)$ of the equation*

$$\dot x(t) = A(t)\dot x(\sigma(t)) + g(t), \quad t \in [t_0,b]$$

with the initial condition

$$\dot x(t) = v(t), \quad t \in [\widehat\tau, t_0), \quad x(t_0) = x_0,$$

on the interval $[t_0,b]$ can be represented in the form

$$x(t) = x_0 + \int_{\sigma(t_0)}^{t_0} Y(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi)v(\xi)\,d\xi + \int_{t_0}^{t} Y(\xi;t)g(\xi)\,d\xi, \tag{1.10}$$

where

$$Y(\xi;t)=\alpha(\xi;t)H+\sum_{i=1}^{q}\alpha(\nu^i(\xi);t)\prod_{m=i}^{1}A(\nu^m(\xi))\frac{d}{d\xi}\nu^m(\xi), \tag{1.11}$$

$$\alpha(\xi;t)=\begin{cases}1, & \xi<t,\\ 0, & \xi>t.\end{cases}$$

Proof. In the above-considered case, $B(t)=C(t)=\Theta$, therefore the first equation of the system (1.7) is of the form

$$\Psi_\xi(\xi;t)=0,\ \ \xi\in[t_0,t].$$

Hence, taking into account (1.8), we have $\Psi(\xi;t)=\alpha(\xi;t)H$. From (1.6) and (1.9), we obtain (1.10) and (1.11), respectively. □

Theorem 1.8. *Let the function $g:I\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ satisfy the following conditions: for each fixed $(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n$, the function $g(\,\cdot\,,x_1,x_2):I\to\mathbb{R}^n$ is measurable; there exist the functions $m(t),L(t)\in L(I,\mathbb{R}_+)$ such that for almost all $t\in I$,*

$$\big|g(t,x_1,x_2)\big|\le m(t),\ \ \forall\,(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n,$$

$$\big|g(t,x_1,x_2)-g(t,y_1,y_2)\big|\le$$

$$\le L(t)\sum_{i=1}^{2}|x_i-y_i|,\ \ \forall\,(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n,\ \ \forall\,(y_1,y_2)\in\mathbb{R}^n\times\mathbb{R}^n.$$

Then the equation

$$\dot x(t)=A(t)\dot x(\sigma(t))+g\big(t,x(t),x(\tau(t))\big) \tag{1.12}$$

with the initial condition

$$x(t)=\varphi(t),\ \ \dot x(t)=v(t),\ \ t\in[\widehat\tau,t_0),\ \ x(t_0)=x_0. \tag{1.13}$$

has the unique solution $x(t)\in\mathbb{R}^n$ defined on the interval $[\widehat\tau,b]$ (see Definition 1.1).

Proof. The existence of a global solution will be proved by the step method with respect to the function $\nu(t)$. We divide the interval $[t_0,b]$ into the subintervals $[\xi_i,\xi_{i+1}]$, $i=0,\dots,l$, where $\xi_0=t_0,\xi_i=\nu^i(t_0)$, $i=1,\dots,l$, $\xi_{l+1}=b$, $\nu^1(t_0)=\nu(t_0)$, $\nu^2(t_0)=\nu(\nu(t_0)),\dots$.

It is clear that on the interval $[\xi_0,\xi_1]$ we have the delay differential equation

$$\dot x(t)=g\big(t,x(t),x(\tau(t))\big)+A(t)v(\sigma(t)) \tag{1.14}$$

with the initial condition

$$x(t)=\varphi(t),\ \ t\in[\widehat\tau,\xi_0),\ \ x(\xi_0)=x_0. \tag{1.15}$$

The problem (1.14)–(1.15) has the unique solution $z_1(t)$ defined on the interval $[\widehat\tau,\xi_1]$, i.e. the function $z_1(t)$ satisfies the condition (1.13) and on the interval $[\xi_0,\xi_1)]$ is absolutely continuous and satisfies the equation (1.12) a.e. on $[\xi_0,\xi_1]$. Thus, $x(t)=z_1(t)$ is the solution of the problem (1.12)–(1.13) defined on the interval $[\widehat{\tau},\xi_1]$.

Further, on the interval $[\xi_1,\xi_2]$ we have the equation

$$\dot{x}(t)=g\big(t,x(t),x(\tau(t))\big)+A(t)\dot{z}(\sigma(t)) \tag{1.16}$$

with the initial condition

$$x(t)=z_1(t),\ \ t\in[\widehat{\tau},\xi_1]. \tag{1.17}$$

Here,

$$\dot{z}(t)=\begin{cases} v(t), & t\in[\widehat{\tau},\xi_0),\\ \dot{z}_1(t), & t\in[\xi_0,\xi_1]. \end{cases}$$

The problem (1.16)–(1.17) has the unique solution $z_2(t)$ defined on the interval $[\widehat{\tau},\xi_2]$. Thus, the function $x(t)=z_2(t)$ is the solution of the problem (1.12)–(1.13) defined on the interval $[\widehat{\tau},\xi_2]$.

Continuing this process, we can construct a solution of the problem (1.12)–(1.13) defined on the interval $[\widehat{\tau},b]$. □

Theorem 1.9. *Let $x(t)$, $t\in[\widehat{\tau},b]$, be a solution of the problem* (1.12)–(1.13), *then it is a solution of the integral equation*

$$x(t)=x_0+\int\limits_{\sigma(t_0)}^{t_0} Y(\nu(\xi);t)A(\nu(\xi))\dot{\nu}(\xi)v(\xi)\,d\xi+ \\ +\int\limits_{t_0}^{t} Y(\xi;t)g(t,x(\xi),x(\tau(\xi)))\,d\xi,\ \ t\in[t_0,b], \tag{1.18}$$

with the initial condition

$$x(t)=\varphi(t),\ \ t\in[\widehat{\tau},t_0), \tag{1.19}$$

where $Y(\xi;t)$ has the form (1.11).

This theorem is a simple corollary of Theorem 1.5.

Theorem 1.10. *If the integral equation* (1.18) *with the initial condition* (1.19) *has a solution, then it is unique.*

Proof. Let $x_1(t)$ and $x_2(t)$ be two solutions of the problem (1.18)–(1.19). We have

$$\big|x_1(t)-x_2(t)\big|\le \\ \le\|Y\|\int\limits_{t_0}^{t} L(\xi)\Big\{\big|x_1(\xi)-x_2(\xi)\big|+\big|x_1(\tau(\xi))-x_2(\tau(\xi))\big|\Big\}\,d\xi\le \\ \le\|Y\|\bigg\{\int\limits_{t_0}^{t}\big[L(\xi)+L(\gamma(\xi))\dot{\gamma}(\xi)\big]\,\big|x_1(\xi)-x_2(\xi)\big|\,d\xi\bigg\},$$

where

$$\|Y\| = \sup\big\{|Y(\xi;t)| :\ (\xi,t)\in I\times I\big\}.$$

By virtue of Gronwall's inequality, we have $x_1(t)=x_2(t)$, $t\in[t_0,b]$. □

Theorem 1.11. *The solution of the problem* (1.18)–(1.19) *is the solution of the problem* (1.12)–(1.13)*.*

This theorem is a simple corollary of Theorems 1.7–1.9.

Theorem 1.12 ([24])**.** *Let $x(t)\in K_1$, $t\in I_1$, be a piecewise-continuous function, where $K_1\subset O$ is a compact set, and let a sequence $\delta f_i\in W(K_1;\alpha)$, $i=1,2,\dots$, satisfy the condition*

$$\lim_{i\to\infty}\Delta(\delta f_i;K_1)=0.$$

Then

$$\lim_{i\to\infty}\sup\bigg\{\bigg|\int\limits_{s_1}^{s_2} Y(\xi;t)\delta f_i\big(\xi,x(\xi),x(\tau(\xi))\big)\,d\xi\bigg| :\ s_1,s_2\in I\bigg\}=0$$

uniformly in $t\in I$.

Theorem 1.13 ([24])**.** *The matrix functions $\Psi(\xi;t)$ and $Y(\xi;t)$ have the following properties:*

1.11. *$\Psi(\xi;t)$ is continuous on the set $\Pi=\{(\xi,t):\ a\le\xi\le t\le b\}$;*

1.12. *for any fixed $t\in(a,b)$, the function $Y(\xi;t)$, $\xi\in[a,t]$, has first order discontinuity at the points of the set*

$$I(t_0;t)=\Big\{\sigma^i(t)=\sigma(\sigma^{i-1}(t))\in[a,t],\ i=1,2,\dots,\sigma^0(t)=t\Big\};$$

1.13. *$\lim\limits_{\theta\to\xi-}Y(\theta;t)=Y(\xi-;t)$, $\lim\limits_{\theta\to\xi+}Y(\theta;t)=Y(\xi+;t)$ uniformly with respect to $(\xi,t)\in\Pi$;*

1.14. *Let $\xi_i\in(a,b)$, $i=0,1$, $\xi_0<\xi_1$ and $\xi_0\neq I(\xi_0;\xi_1)$. Then there exist numbers δ_i, $i=0,1$, such that the function $Y(\xi;t)$ is continuous on the set $[\xi_0-\delta_0,\xi_0+\delta_0]\times[\xi_1-\delta_1,\xi_1-\delta_1]\subset\Pi$.*

1.3. **Proof of Theorem 1.1.** *On the continuous dependence of a solution for a class of neutral equation.* To each element $\mu=(t_0,\tau,x_0,\varphi,v,f)\in\Lambda$ we assign the functional differential equation

$$\dot y(t)=A(t)h(t_0,v,\dot y)(\sigma(t))+f(t_0,\tau,\varphi,y)(t) \tag{1.20}$$

with the initial condition

$$y(t_0)=x_0, \tag{1.21}$$

where $f(t_0,\tau,\varphi,y)(t)=f(t,y(t),h(t_0,\varphi,y)(\tau(t)))$ and $h(\,\cdot\,)$ is the operator given by the formula

$$h(t_0,\varphi,y)(t)=\begin{cases}\varphi(t) & \text{for } t\in[\widehat\tau,t_0),\\ y(t) & \text{for } t\in[t_0,b].\end{cases} \tag{1.22}$$

Definition 1.3. An absolutely continuous function $y(t) = y(t;\mu) \in O$, $t \in [r_1, r_2] \subset I$, is called a solution of the equation (1.20) with the initial condition (1.21), or a solution corresponding to the element $\mu \in \Lambda$ and defined on $[r_1, r_2]$ if $t_0 \in [r_1, r_2]$, $y(t_0) = x_0$ and satisfies the equation (1.20) a.e. on the interval $[r_1, r_2]$.

Remark 1.1. Let $y(t;\mu), t \in [r_1, r_2]$ be the solution of the problem (1.20)–(1.21). Then the function

$$x(t;\mu) = h\big(t_0, \varphi, y(\,\cdot\,;\mu)\big)(t), \;\; t \in [\widehat{\tau}, r_2]$$

is the solution of the equation (1.1) with the initial condition (1.2).

Theorem 1.14. *Let $y_0(t)$ be a solution corresponding to $\mu_0 \in \Lambda$ defined on $[r_1, r_2] \subset (a, b)$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \operatorname{cl} \varphi_0(I_1) \cup y_0([r_1, r_2])$. Then the following conditions hold:*

1.15. *there exist numbers $\delta_i > 0, i = 0, 1$ such that a solution $y(t;\mu)$ defined on $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ corresponds to each element*

$$\mu = (t_0, \tau, x_0, \varphi, v, f_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha).$$

Moreover,

$$\varphi(t) \in K_1, t \in I_1; \;\; y(t;\mu) \in K_1, \;\; t \in [r_1 - \delta_1, r_2 + \delta_1],$$

for arbitrary $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$;

1.16. *for an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0]$ such that the following inequality holds for any $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$:*

$$\big|y(t;\mu) - y(t;\mu_0)\big| \le \varepsilon, \;\; \forall\, t \in [r_1 - \delta_1, r_2 + \delta_1]. \tag{1.23}$$

Proof. Let $\varepsilon_0 > 0$ be so small that a closed ε_0-neighborhood of the set K_0:

$$K(\varepsilon_0) = \Big\{x \in \mathbb{R}^n : \; \exists\, \widehat{x} \in K_0 \; |x - \widehat{x}| \le \varepsilon_0\Big\}$$

lies in $intK_1$. There exist a compact set Q: $K_0^2(\varepsilon_0) \subset Q \subset K_1^2$ and a continuously differentiable function $\chi : \mathbb{R}^{2n} \to [0, 1]$ such that

$$\chi(x_1, x_2) = \begin{cases} 1 & \text{for } \; (x_1, x_2) \in Q, \\ 0 & \text{for } \; (x_1, x_2) \notin K_1^2 \end{cases} \tag{1.24}$$

(see Assertion 3.2 in [11, p. 60]).

To each element $\mu \in \Lambda$, we assign the functional differential equation

$$\dot{z}(t) = A(t)h(t_0, v, \dot{z})(\sigma(t)) + g(t_0, \tau, \varphi, z)(t) \tag{1.25}$$

with the initial condition

$$z(t_0) = x_0, \tag{1.26}$$

where $g(t_0, \tau, \varphi, z)(t) = g(t, z(t), h(t_0, \varphi, z)(\tau(t)))$ and $g = \chi f$. The function $g(t, x_1, x_2)$ satisfies the conditions

$$|g(t, x_1, x_2)| \le m_{f,K_1}(t), \;\; \forall\, x_i \in \mathbb{R}^n, \;\; i = 1, 2, \tag{1.27}$$

for $\forall\, x_i', x_i'' \in \mathbb{R}^n$, $i=1,2$, and for almost all $t \in I$

$$\big|g(t,x_1',x_2') - g(t,x_1'',x_2'')\big| \le L_f(t)\sum_{i=1}^{2}|x_i' - x_i''|, \tag{1.28}$$

where

$$\begin{gathered} L_f(t) = L_{f,K_1}(t) + \alpha_1 m_{f,K_1}(t), \\ \alpha_1 = \sup\bigg\{\sum_{i=1}^{2}|\chi_{x_i}(x_1,x_2)| :\ x_i \in \mathbb{R}^n,\ i=1,2\bigg\} \end{gathered} \tag{1.29}$$

(see [15]).

By the definition of the operator $h(\,\cdot\,)$, the equation (1.25) for $t \in [a,t_0]$ can be considered as the ordinary differential equation

$$\dot z_1(t) = A(t)v(\sigma(t)) + g\big(t, z_1(t), \varphi(\tau(t))\big) \tag{1.30}$$

with the initial condition

$$z_1(t_0) = x_0, \tag{1.31}$$

and for $t \in [t_0,b]$, it can be considered as the neutral equation

$$\dot z_2(t) = A(t)\dot z_2(\sigma(t)) + g\big(t, z_2(t), z_2(\tau(t))\big) \tag{1.32}$$

with the initial condition

$$z_2(t) = \varphi(t),\ \ \dot z_2(t) = v(t),\ \ t \in [\widehat\tau, t_0),\ \ z_2(t_0) = x_0. \tag{1.33}$$

Obviously, if $z_1(t)$, $t \in [a,t_0]$, is a solution of problem (1.30)–(1.31) and $z_2(t), t \in [t_0,b]$, is a solution of problem (1.32)–(1.33), then the function

$$z(t) = \begin{cases} z_1(t), & t \in [a,t_0), \\ z_2(t), & t \in [t_0,b] \end{cases}$$

is a solution of the equation (1.25) with the initial condition (1.26) defined on the interval I.

We rewrite the equation (1.30) with the initial condition (1.31) in the integral form

$$z_1(t) = x_0 + \int_{t_0}^{t}\Big[A(\xi)v(\sigma(\xi)) + g\big(\xi, z_1(\xi), \varphi(\tau(\xi))\big)\Big]\,d\xi,\ \ t \in [a,t_0], \tag{1.34}$$

and the equation (1.32) with the initial condition (1.33) we write in the equivalent form

$$\begin{aligned} z_2(t) = x_0 + \int_{t_0}^{\nu(t_0)} & Y(\xi;t)A(\xi)v(\sigma(\xi))\,d\xi + \\ & + \int_{t_0}^{t} Y(\xi;t)g(\xi, z_2(\xi), z_2(\tau(\xi)))\,d\xi,\ \ t \in [t_0,b], \end{aligned} \tag{1.35}$$

where

$$z_2(t) = \varphi(t), \quad t \in [\widehat{\tau}, t_0)$$

(see Theorem 1.9 and (1.11)).

Introduce the following notation:

$$Y_0(\xi; t, t_0) = \begin{cases} H, & t \in [a, t_0), \\ Y(\xi; t), & t \in [t_0, b], \end{cases} \tag{1.36}$$

$$Y(\xi; t, t_0) = \begin{cases} H, & t \in [a, t_0), \\ Y(\xi; t), & t_0 \le t \le \min\{\nu(t_0), b\}, \\ \Theta, & \min\{\nu(t_0), b\} < t \le b. \end{cases} \tag{1.37}$$

Using this notation and taking into account (1.34) and (1.35), we can rewrite the equation (1.25) in the form of the equivalent integral equation

$$\begin{aligned} z(t) = x_0 + \int_{t_0}^{t} Y(\xi; t, t_0) A(\xi) v(\sigma(\xi))\, d\xi + \\ + \int_{t_0}^{t} Y_0(\xi; t, t_0) g(t_0, \tau, \varphi, z)(\xi)\, d\xi, \quad t \in I. \end{aligned} \tag{1.38}$$

A solution of the equation (1.38) depends on the parameter

$$\mu \in \Lambda_0 = I \times D \times O \times \Phi_1 \times E_v \times \big(f_0 + W(K_1; \alpha)\big) \subset E_\mu$$

The topology in Λ_0 is induced by the topology of the vector space E_μ. Denote by $C(I, \mathbb{R}^n)$ the space of continuous functions $y : I \to \mathbb{R}^n$ with the distance $d(y_1, y_2) = \|y_1 - y_2\|_I$.

On the complete metric space $C(I, \mathbb{R}^n)$, we define a family of mappings

$$F(\,\cdot\,; \mu) : C(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n) \tag{1.39}$$

depending on the parameter μ by the formula

$$\zeta(t) = \zeta(t; z, \mu) =$$

$$= x_0 + \int_{t_0}^{t} Y(\xi; t, t_0) A(\xi) v(\sigma(\xi))\, d\xi + \int_{t_0}^{t} Y_0(\xi; t, t_0) g(t_0, \tau, \varphi, z)(\xi)\, d\xi.$$

Clearly, every fixed point $z(t; \mu), t \in I$, of the mapping (1.39) is a solution of the equation (1.25) with the initial condition (1.26).

Define the kth iteration $F^k(z;\mu)$ by

$$\zeta_k(t) = x_0 + \int\limits_{t_0}^{t} Y(\xi;t,t_0)A(\xi)v(\sigma(\xi))\,d\xi + \\ + \int\limits_{t_0}^{t} Y_0(\xi;t,t_0)g(t_0,\tau,\varphi,\zeta_{k-1})(\xi)\,d\xi, \quad k=1,2,\dots,$$

$$\zeta_0(t) = z(t).$$

Let us now prove that for a sufficiently large k, the family of mappings $F^k(z;\mu)$ is uniformly contractive. Towards this end, we estimate the difference

$$\big|\zeta_k'(t) - \zeta_k''(t)\big| = \big|\zeta_k(t;z',\mu) - \zeta_k(t;z'',\mu)\big| \leq \\ \leq \int\limits_{a}^{t} \big|Y_0(\xi;t,t_0)\big|\,\Big|g(t_0,\tau,\varphi,\zeta_{k-1}')(\xi) - g(t_0,\tau,\varphi,\zeta_{k-1}'')(\xi)\Big|\,d\xi \leq \\ \leq \int\limits_{a}^{t} L_f(\xi)\Big[\big|\zeta_{k-1}'(\xi) - \zeta_{k-1}''(\xi)\big| + \\ + \big|h(t_0,\varphi,\zeta_{k-1}')(\tau(\xi)) - h(t_0,\varphi,\zeta_{k-1}'')(\tau(\xi))\big|\Big]\,d\xi, \quad k=1,2,\dots, \tag{1.40}$$

(see (1.28)), where the function $L_f(\xi)$ is of the form (1.29). Here, it is assumed that $\zeta_0'(\xi) = z'(\xi)$ and $\zeta_0''(\xi) = z''(\xi)$.

It follows from the definition of the operator $h(\,\cdot\,)$ that

$$h(t_0,\varphi,\zeta_{k-1}')(\tau(\xi)) - h(t_0,\varphi,\zeta_{k-1}'')(\tau(\xi)) = h(t_0,0,\zeta_{k-1}' - \zeta_{k-1}'')(\tau(\xi)).$$

Therefore, for $\xi \in [a,\gamma(t_0)]$, we have

$$h(t_0,0,\zeta_{k-1}' - \zeta_{k-1}'')(\tau(\xi)) = 0. \tag{1.41}$$

Let $\gamma(t_0) < b$; then for $\xi \in [\gamma(t_0),b]$, we obtain

$$\big|h(t_0,0,\zeta_{k-1}' - \zeta_{k-1}'')(\tau(\xi))\big| = \big|\zeta_{k-1}'(\tau(\xi)) - \zeta_{k-1}'')(\tau(\xi))\big|, \\ \sup\Big\{\big|\zeta_{k-1}'(\tau(t)) - \zeta_{k-1}''(\tau(t))\big| :\ t \in [\gamma(t_0),\xi]\Big\} \leq \\ \leq \sup\Big\{\big|\zeta_{k-1}'(t) - \zeta_{k-1}''(t)\big| :\ t \in [a,\xi]\Big\}. \tag{1.42}$$

If $\gamma(t_0) > b$, then (1.41) holds on the whole interval I. The relation (1.40), together with (1.41) and (1.42), imply that

$$\big|\zeta_k'(t) - \zeta_k''(t)\big| \leq \sup\Big\{\big|\zeta_k'(\xi) - \zeta_k''(\xi)\big| :\ \xi \in [a,t]\Big\} \leq \\ \leq 2\|Y_0\| \int\limits_{a}^{t} L_f(\xi_1)\sup\Big\{\big|\zeta_{k-1}'(\xi) - \zeta_{k-1}''(\xi)\big| :\ \xi \in [a,\xi_1]\Big\}\,d\xi_1, \quad k=1,2,\dots.$$

Therefore,

$$\begin{aligned}&\big|\zeta_k'(t)-\zeta_k''(t)\big|\le\\&\le 2^2\|Y_0\|^2\int\limits_a^t L_f(\xi_1)\,d\xi_1\int\limits_a^{\xi_1}L_f(\xi_2)\sup\Big\{\big|\zeta_{k-2}'(\xi)-\zeta_{k-2}''(\xi)\big|:\ \xi\in[a,\xi_2]\Big\}\,d\xi_2.\end{aligned}$$

By continuing this procedure, we obtain

$$\big|\zeta_k'(t)-\zeta_k''(t)\big|\le\big(2\|Y_0\|\big)^k\alpha_k(t)\|z'-z''\|_I,$$

where

$$\alpha_k(t)=\int\limits_a^t L_f(\xi_1)\,d\xi_1\int\limits_a^{\xi_1}L_f(\xi_2)\,d\xi_2\cdots\int\limits_a^{\xi_{k-1}}L_f(\xi_k)\,d\xi_k.$$

By the induction, one can readily show that

$$\alpha_k(t)=\frac{1}{k!}\bigg(\int\limits_a^t L_f(\xi)\,d\xi\bigg)^k.$$

Thus,

$$\begin{aligned}&d\big(F^k(z';\mu),F^k(z'';\mu)\big)=\\&\qquad=\|\zeta_k'-\zeta_k''\|_I\le\big(2\|Y_0\|\big)^k\alpha_k(b)\|z'-z''\|_I=\widehat{\alpha}_k\|z'-z''\|_I.\end{aligned}$$

Let us prove the existence of the number $\alpha_2>0$ such that

$$\int\limits_I L_f(t)\,dt\le\alpha_2,\ \ \forall\, f\in f_0+W(K_1;\alpha).$$

Indeed, let $(x_1,x_2)\in K_1^2$ and let $f\in f_0+W(K_1;\alpha)$, then

$$\big|f(t,x_1,x_2)\big|\le m_{f_0,K_1}(t)+m_{\delta f,K_1}(t):=m_{f,K_1}(t),\ \ t\in I.$$

Further, let $x_i',x_i{}''\in K_1$, $i=1,2$ then

$$\begin{aligned}&\big|f(t,x_1',x_2')-f(t,x_1'',x_2'')\big|\le\\&\le\big|f_0(t,x_1',x_2')-f_0(t,x_1'',x_2'')\big|+\big|\delta f(t,x_1',x_2')-\delta f(t,x_1'',x_2'')\big|\le\\&\le\big(L_{f_0,K_1}(t)+L_{\delta f,K_1}(t)\big)\sum_{i=1}^2|x_i'-x_i''|=L_{f,K_1}(t)\sum_{i=1}^2|x_i'-x_i''|,\end{aligned}$$

where $L_{f,K_1}(t) = L_{f_0,K_1}(t) + L_{\delta f,K_1}(t)$. By (1.29),

$$\int\limits_I L_f(t)\,dt = \int\limits_I \big(L_{f,K_1}(t) + \alpha_1 m_{f,K_1}(t)\big)\,dt =$$
$$= \int\limits_I \Big[L_{f_0,K_1}(t) + L_{\delta f,K_1}(t) + \alpha_1\big(m_{f_0,K_1}(t) + m_{\delta f,K_1}(t)\big)\Big]\,dt \le$$
$$\le \alpha(\alpha_1+1) + \int\limits_I \big[L_{f_0,K_1}(t) + \alpha_1 m_{f_0,K_1}(t)\big]\,dt = \alpha_2.$$

Taking into account this estimate, we obtain $\widehat{\alpha}_k \le (2\|Y_0\|\alpha_2)^k/k!$. Consequently, there exists a positive integer k_1 such that $\widehat{\alpha}_{k_1} < 1$. Therefore, the k_1st iteration of the family (1.39) is contracting. By the fixed point theorem for contraction mappings (see [11, p. 90], [27, p. 110]), the mapping (1.39) has a unique fixed point for each μ. Hence it follows that the equation (1.25) with the initial condition (1.26) has a unique solution $z(t;\mu)$, $t \in I$.

Let us prove that the mapping $F^k(z(\,\cdot\,;\mu_0);\,\cdot\,) : \Lambda_0 \to C(I,\mathbb{R}^n)$ is continuous at the point $\mu = \mu_0$ for an arbitrary $k = 1,2,\dots$. To his end, it suffices to show that if the sequence $\mu_i = (t_{0i},\tau_i,x_{0i},\varphi_i,v_i,f_i) \in \Lambda_0$, $i = 1,2,\dots$, where $f_i = f_0 + \delta f_i$, converges to $\mu_0 = (t_{00},\tau_0,x_{00},\varphi_0,v_0,f_0)$, i.e. if

$$\lim_{i\to\infty}\Big(|t_{0i}-t_{00}| + \|\tau_i-\tau_0\|_{I_2} +$$
$$+ |x_{0i}-x_{00}| + \|\varphi_i-\varphi_0\|_{1_1} + \|v_i-v_0\|_{1_1} + \Delta(\delta f_i;K_1)\Big) = 0,$$

then

$$\lim_{i\to\infty} F^k\big(z(\,\cdot\,;\mu_0);\mu_i\big) = F^k\big(z(\,\cdot\,;\mu_0);\mu_0\big) = z(\,\cdot\,;\mu_0). \tag{1.43}$$

We prove the relation (1.43) by induction. Let $k = 1$, then we have

$$\big|\zeta_1^i(t) - z_0(t)\big| \le |x_{0i}-x_{00}| +$$
$$+ \bigg|\int\limits_{t_{0i}}^{t} Y(\xi;t,t_{0i})A(\xi)v_i(\sigma(\xi))\,d\xi - \int\limits_{t_{00}}^{t} Y(\xi;t,t_{00})A(\xi)v_0(\sigma(\xi))\,d\xi\bigg| +$$
$$+ \bigg|\int\limits_{t_{0i}}^{t} Y_0(\xi;t,t_{0i})g_i(t_{0i},\tau_i,\varphi_i,z_0)(\xi)\,d\xi -$$
$$- \int\limits_{t_{00}}^{t} Y_0(\xi;t,t_{00})g_0(t_{00},\tau_0,\varphi_0,z_0)(\xi)\,d\xi\bigg| =$$
$$= |x_{0i}-x_{00}| + a_i(t) + b_i(t), \tag{1.44}$$

where

$$\zeta_1^i(t) = \zeta_1(t; z_0, \mu_i), \quad z_0(t) = z(t; \mu_0),$$
$$g_i = \chi f_i = g_0 + \delta g_i, \quad g_0 = \chi f_0, \quad \delta g_i = \chi \delta f_i;$$

$$a_i(t) = \Big| \int_{t_{0i}}^{t} Y(\xi; t, t_{0i}) A(\xi) v_i(\sigma(\xi))\, d\xi - \int_{t_{00}}^{t} Y(\xi; t, t_{00}) A(\xi) v_0(\sigma(\xi))\, d\xi \Big|;$$

$$b_i(t) = \Big| \int_{t_{0i}}^{t} Y_0(\xi; t, t_{0i}) g_i(t_{0i}, \tau_{1i}, \varphi_i, z_0)(\xi)\, d\xi -$$
$$- \int_{t_{00}}^{t} Y_0(\xi; t, t_{00}) g_0(t_{00}, \tau_{10}, \varphi_0, z_0)(\xi)\, d\xi \Big|.$$

First of all, let us estimate $a_i(t)$. We have

$$a_i(t) \le \Big| \int_{t_{0i}}^{t_{00}} Y(\xi; t, t_{00}) A(\xi) v_0(\sigma(\xi))\, d\xi \Big| +$$
$$+ \int_I \Big| Y(\xi; t, t_{0i}) A(\xi) v_i(\sigma(\xi)) - Y(\xi; t, t_{00}) A(\xi) v_0(\sigma(\xi)) \Big|\, d\xi =$$
$$= a_{i1}(t) + a_{i2}(t). \tag{1.45}$$

Obviously,

$$\lim_{i\to\infty} a_{i1}(t) = 0 \ \text{ uniformly in } \ t \in I. \tag{1.46}$$

Furthermore,

$$a_{i2}(t) \le \int_I \big| Y(\xi; t, t_{0i}) - Y(\xi; t, t_{00}) \big| \, \big| A(\xi) v_i(\sigma(\xi)) \big|\, d\xi +$$
$$+ \int_I \big| Y(\xi; t, t_0) A(\xi) \big| \, \big| v_i(\sigma(\xi)) - v_0(\sigma(\xi)) \big|\, d\xi \le$$
$$\le \|A\| \, \|v_i\|_{I_1} a_{i3}(t) + \|YA\| \, \|v_i - v_0\|_{I_1}, \tag{1.47}$$

where

$$a_{i3}(t) = \int_I \big| Y(\xi; t, t_{0i}) - Y(\xi; t, t_{00}) \big|\, d\xi.$$

Let $t_{0i} < t_{00}$, and let a number i_0 be so large that $\nu(t_{0i}) > t_{00}$ for $i \ge i_0$. Then taking into account (1.37), we have

$$a_{i3}(t) = \int_{t_{0i}}^{t_{00}} |Y(\xi; t) - H|\, d\xi + \int_{\nu(t_{0i})}^{\nu(t_{00})} |Y(\xi; t)|\, d\xi \le$$
$$\le \|Y - H\| (t_{00} - t_{0i}) + \|Y\| \big(\nu(t_{00}) - \nu(t_{0i})\big),$$

therefore,

$$\lim_{i\to\infty} a_{i3}(t)=0 \ \text{ uniformly in } \ I. \tag{1.48}$$

Let $t_{0i}>t_{00}$. Choose a number i_0 so large that $\nu(t_{00})>t_{0i}$ for $i\geq i_0$. Then

$$a_{i3}(t)=\int\limits_{t_{00}}^{t_{0i}} |H-Y(\xi;t)|\,d\xi+\int\limits_{\nu(t_{00})}^{\nu(t_{0i})} |Y(\xi;t)|\,d\xi.$$

This implies (1.48). Taking into account (1.46)–(1.48), we obtain from (1.45) that

$$\lim_{i\to\infty} a_{i}(t)=0 \ \text{ uniformly in } \ I. \tag{1.49}$$

Now, let us estimate the summand $b_i(t)$. We have

$$\begin{aligned} b_i(t)\leq \bigg|\int\limits_{t_{0i}}^{t_{00}} Y_0(\xi;t,t_{0i})g_0(t_{00},\tau_0,\varphi_0,z_0)(\xi)\,d\xi\bigg|+ \\ +\bigg|\int\limits_{t_{0i}}^{t} \Big[Y_0(\xi;t,t_{0i})g_i(t_{0i},\tau_i,\varphi_i,z_0)(\xi)- \\ -Y_0(\xi;t,t_{00})g_0(t_{00},\tau_0,\varphi_0,z_0)(\xi)\Big]\,d\xi=b_{i1}(t)+b_{i2}(t). \end{aligned} \tag{1.50}$$

Obviously,

$$\lim_{i\to\infty} b_{i1}(t)=0 \ \text{ uniformly in } \ I. \tag{1.51}$$

Furthermore,

$$\begin{aligned} b_{i2}(t)=\bigg|\int\limits_{t_{0i}}^{t} Y_0(\xi;t,t_{0i})\Big[g_i(t_{0i},\tau_i,\varphi_i,z_0)(\xi)-g_0(t_{0i},\tau_i,\varphi_i,z_0)(\xi)\Big]\,d\xi+ \\ +\int\limits_{t_{0i}}^{t} Y_0(\xi;t,t_{0i})\Big[g_0(t_{0i},\tau_i,\varphi_i,z_0)(\xi)-g_0(t_{00},\tau_0,\varphi_0,z_0)(\xi)\Big]\,d\xi+ \\ +\int\limits_{t_{0i}}^{t} \Big[Y_0(\xi;t,t_{0i})-Y_0(\xi;t,t_{00})\Big]g_0(t_{00},\tau_0,\varphi_0,z_0)(\xi)\,d\xi\bigg|\leq \\ \leq\sum_{j=1}^{3} b_{i2}^{j}(t), \end{aligned} \tag{1.52}$$

where

$$b^1_{i2}(t)=\Bigg|\int\limits_{t_{0i}}^{t} Y_0(\xi;t,t_{0i})\delta g_i(t_{0i},\tau_i,\varphi_i,z_0)(\xi)\,d\xi\Bigg|,$$

$$b^2_{i2}(t)=\Bigg|\int\limits_{t_{0i}}^{t} Y_0(\xi;t,t_{0i})\Big[g_0(t_{0i},\tau_i,\varphi_i,z_0)(\xi)-g_0(t_{00},\tau_0,\varphi_0,z_0)(\xi)\Big]\,d\xi\Bigg|,$$

$$b^3_{i2}(t)=\Bigg|\int\limits_{t_{0i}}^{t} \Big[Y_0(\xi;t,t_{0i})-Y_0(\xi;t,t_{00})\Big]g_0(t_{00},\tau_0,\varphi_0,z_0)(\xi)\,d\xi\Bigg|.$$

Now, let us estimate the expressions $b^1_{i2}(t)$. We have

$$\begin{aligned}
b^1_{i2}(t)=\Bigg|&\int\limits_{t_{0i}}^{t} Y_0(\xi;t,t_{0i})\Big[\delta g_i(t_{0i},\tau_i,\varphi_i,z_0)(\xi)-\delta g_i(t_{0i},\tau_i,\varphi_0,z_0)(\xi)\Big]\,d\xi+\\
&+\int\limits_{t_{0i}}^{t} Y_0(\xi;t,t_{0i})\delta g_i(t_{0i},\tau_i,\varphi_0,z_0)(\xi)\,d\xi\Bigg|\le\\
&\le\|Y_0\|\int\limits_I L_{\delta g_i,K_1}(\xi)\Big|h(t_{0i},\varphi_i,z_0)(\tau_i(\xi))-h(t_{0i},\varphi_0,z_0)(\tau_i(\xi))\Big|\,d\xi+\\
&+\max_{t',t''\in I}\Bigg|\int\limits_{t'}^{t''} Y_0(\xi;t,t_{0i})\delta g_i(t_{0i},\tau_i,\varphi_0,z_0)(\xi)\,d\xi\Bigg|=\\
&=b^4_{i2}+b^5_{i2}(t). \qquad (1.53)
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
b^4_{i2}&\le\|Y_0\|\int\limits_I L_{\delta g_i,K_1}(\xi)\big|\varphi_i(\tau_i(\xi))-\varphi_0(\tau_i(\xi))\big|\,d\xi\le\\
&\le\|\varphi_i-\varphi_0\|_{I_1}\int\limits_I L_{\delta g_i,K_1}(\xi)\,d\xi.
\end{aligned}$$

The sequence

$$\int\limits_I L_{\delta g_i,K_1}(\xi)\,d\xi,\quad i=1,2,\dots,$$

is bounded, therefore

$$\lim_{i\to\infty} b^4_{i2}=0.$$

Furthermore,

$$b_{i2}^5(t) \le \max_{t',t''\in I} \left| \int_{t'}^{t''} \delta g_i\big(\xi, z_0(\xi), \varphi_0(\tau_i(\xi))\big)\, d\xi \right| +$$

$$+ \max_{t',t''\in I} \left| \int_{t'}^{t''} Y(\xi;t)\delta g_i(t_{0i},\tau_i,\varphi_0,z_0)(\xi)\, d\xi \right| = b_{i2}^6 + b_{i2}^7(t).$$

The function $\varphi_0(\xi)$, $\xi \in I_1$, is piecewise-continuous with a finite number of discontinuity points of the first kind, i.e. there exist subintervals (θ_q,θ_{q+1}), $q = 1,\dots,m$, where the function $\varphi_0(t)$ is continuous, with

$$\theta_1 = \widehat{\tau},\ \ \theta_{m+1} = b,\ \ I_1 = \bigcup_{q=1}^{m-1} [\theta_q,\theta_{q+1}) \cup [\theta_m,\theta_{m+1}].$$

We define on the interval I_1 the continuous functions $z_i(t)$, $i = 1,\dots,m+1$, as follows:

$$z_1(t) = \varphi_{01}(t),\dots,z_m(t) = \varphi_{0m}(t),$$

$$z_{m+1}(t) = \begin{cases} z_0(a), & t \in [\widehat{\tau},a), \\ z_0(t), & t \in I, \end{cases}$$

where

$$\varphi_{0q}(t) = \begin{cases} \varphi_0(\theta_q+), & t \in [\widehat{\tau},\theta_q], \\ \varphi_0(t), & t \in (\theta_q,\theta_{q+1}), \\ \varphi_0(\theta_{q+1}-), & t \in [\theta_{q+1},b] \end{cases} \quad q = 1,\dots,m.$$

One can readily see that b_{i2}^6 satisfies the estimation

$$b_{i2}^6 \le \sum_{m_1=1}^{m} \max_{t',t''\in I} \left| \int_{t'}^{t''} \delta g_i\big(t, z_0(t), z_{m_1}(\tau_i(t))\big)\, dt \right| \le$$

$$\le \sum_{m_1=1}^{m} \max_{t',t''\in I} \left| \int_{t'}^{t''} \delta g_i\big(t, z_0(t), z_{m_1}(\tau_0(t))\big)\, dt \right| +$$

$$+ \sum_{m_1=1}^{m} \max_{t',t''\in I} \left| \int_{t'}^{t''} \Big|\delta g_i\big(t, z_0(t), z_{m_1}(\tau_i(t))\big) - \delta g_i\big(t, z_0(t), z_{m_1}(\tau_0(t))\big)\Big|\, dt \right| \le$$

$$\le \sum_{m_1=1}^{m} \max_{t',t''\in I} \left| \int_{t'}^{t''} \delta g_i\big(t, z_0(t), z_{m_1}(\tau_0(t))\big)\, dt \right| +$$

$$+ \sum_{m_1=1}^{m} \int_{I} L_{\delta f_i,K_1}(t) \big|z_{m_1}(\tau_i(t)) - z_{m_1}(\tau_0(t))\big|\, dt \le$$

$$\leq \sum_{m_1=1}^{m} \max_{t',t''\in I} \left| \int_{t'}^{t''} \delta g_i\big(t, z_0(t), z_{m_1}(\tau_0(t))\big)\, dt \right| +$$

$$+ \sum_{m_1=1}^{m} \max_{t\in I} \left| z_{m_1}(\tau_i(t)) - z_{m_1}(\tau_0(t)) \right| \int_I L_{\delta g_i, K_1}(t)\, dt. \quad (1.54)$$

Obviously,

$$\Delta(\delta g_i; K_1) = \Delta(\chi \delta f_i; K_1) \leq \Delta(\delta f_i; K_1)$$

(see (1.24)). Since $\Delta(\delta f_i; K_1) \to 0$ as $i \to \infty$, we have

$$\lim_{i\to\infty} \Delta(\delta g_i, K_1) = 0.$$

This allows us to use Theorem 1.12, which in turn, implies that

$$\lim_{i\to\infty} \max_{t',t''\in I} \left| \int_{t'}^{t''} \delta g_i\big(t, z_0(t), z_{m_1}(\tau_0(t))\big)\, dt \right| = 0, \quad \forall\, m_1 = 1, \dots, m.$$

Moreover, it is clear that

$$\lim_{i\to\infty} \max_{t\in I} \left| z_{m_1}(\tau_i(t)) - z_{m_1}(\tau_0(t)) \right| = 0.$$

The right-hand side of inequality (1.54) consists of finitely many summands, and therefore

$$\lim_{i\to\infty} b_{i2}^6 = 0.$$

For $b_{12}^7(t)$, in the analogous manner, we get

$$b_{12}^7(t) \leq \sum_{m_1=1}^{m+1} \max_{t',t''\in I} \left| \int_{t'}^{t''} Y(\xi; t) \delta g_i\big(\xi, z_0(\xi), z_{m_1}(\tau_0(\xi))\big)\, d\xi \right| +$$

$$+ \|Y\| \sum_{m_1=1}^{m+1} \max_{t\in I} \left| z_{m_1}(\tau_i(t)) - z_{m_1}(\tau_0(t)) \right| \int_I L_{\delta g_i, K_1}(t)\, dt,$$

from which we have

$$\lim_{i\to\infty} b_{i2}^7(t) = 0 \ \text{ uniformly in } \ I$$

(see Theorem 1.12).

Thus,

$$\lim_{i\to\infty} b_{i2}^5(t) = 0 \ \text{ uniformly in } \ I.$$

Consequently,

$$\lim_{i\to\infty} b_{i2}^1(t) = 0 \ \text{ uniformly in } \ I. \quad (1.55)$$

Next,

$$b_{i2}^2(t) \leq \|Y\| \int_I L_{f_0}(t) \Big| h(t_{0i}, \varphi_i, z_0)(\tau_i(t)) - h(t_{00}, \varphi_0, z_0)(\tau_0(t)) \Big|\, dt \leq$$

$$\leq \|Y\| \bigg\{ \int_I L_{f_0}(t) \Big| h(t_{0i}, \varphi_i, z_0)(\tau_i(t)) - h(t_{0i}, \varphi_0, z_0)(\tau_i(t)) \Big| \, dt +$$

$$+ \int_I L_{f_0}(t) \Big| h(t_{0i}, \varphi_0, z_0)(\tau_i(t)) - h(t_{00}, \varphi_0, z_0)(\tau_0(t)) \Big| \, dt \bigg\} \leq$$

$$\leq \|Y\| \bigg\{ \int_I L_{f_0}(t) \Big| h(t_{0i}, \varphi_i - \varphi_0)(\tau_i(t)) \Big| \, dt +$$

$$+ \int_I L_{f_0}(t) \Big| h(t_{0i}, \varphi_0, z_0)(\tau_i(t)) - h(t_{00}, \varphi_0, z_0)(\tau_i(t)) \Big| +$$

$$+ \int_I L_{f_0}(t) \Big| h(t_{00}, \varphi_0, z_0)(\tau_i(t)) - h(t_{00}, \varphi_0, z_0)(\tau_0(t)) \Big| \bigg\} \times$$

$$\times \|Y\| \bigg\{ \|\varphi_i - \varphi_0\|_{I_1} \int_I L_{f_0}(t) \, dt + b_{i21}^2 + b_{i22}^2 \bigg\}$$

(see (1.22) and (1.36)). Introduce the notation

$$\rho_{0i} = \min \big\{ \gamma_i(t_{00}), \gamma_i(t_{0i}) \big\}, \quad \theta_{0i} = \max \big\{ \gamma_i(t_{00}), \gamma_i(t_{0i}) \big\}.$$

We prove that

$$\lim_{i \to \infty} \gamma_i(t_{00}) = \lim_{i \to \infty} \gamma_i(t_{0i}) = \gamma_0(t_{00}).$$

The sequences $\{\gamma_i(t_{00})\}$ and $\gamma_i(t_{0i})$ are bounded. Without less of generality, we assume that

$$\lim_{i \to \infty} \gamma_i(t_{00}) = \gamma_0, \quad \lim_{i \to \infty} \gamma_i(t_{0i}) = \gamma_1.$$

We have

$$t_{00} = \tau_i(\gamma_i(t_{00})) = \tau_i(\gamma_i(t_{00})) - \tau_0(\gamma_i(t_{00})) + \tau_0(\gamma_i(t_{00})).$$

Clearly,

$$\lim_{i \to \infty} \Big| \tau_i(\gamma_i(t_{00})) - \tau_0(\gamma_i(t_{00})) \Big| \leq \lim_{i \to \infty} \|\tau_i - \tau_0\|_{I_2} = 0.$$

Passing to the limit, we obtain $t_{00} = \tau_0(\gamma_0)$. The equation $\tau_0(t) = t_{00}$ has a unique solution $\gamma_0(t_{00})$, i.e. $\gamma_0 = \gamma_0(t_{00})$.

Further,

$$t_{0i} = \tau_i(\gamma_i(t_{0i})) = \tau_i(\gamma_i(t_{0i})) - \tau_0(\gamma_i(t_{0i})) + \tau_0(\gamma_i(t_{0i})).$$

Hence we obtain $t_{00} = \tau_0(\gamma_1)$, i.e. $\gamma_1 = \gamma_0(t_{00})$.

Thus,

$$\lim_{i \to \infty} (\rho_{0i} - \theta_{0i}) = 0.$$

Consequently,

$$b_{i21}^2 = \int_{\rho_{0i}}^{\theta_{0i}} L_{f_0}(t) \Big| h(t_{0i}, \varphi_0, z_0)(\tau_i(t)) - h(t_{00}, \varphi_0, z_0)(\tau_i(t)) \Big| \, dt \longrightarrow 0.$$

Introduce the notation

$$\rho_{1i} = \min\big\{\gamma_i(t_{00}), \gamma_0(t_{00})\big\}, \quad \theta_{1i} = \max\big\{\gamma_i(t_{00}), \gamma_0(t_{00})\big\}.$$

For b_{i22}^2, we have

$$b_{i22}^2 = \int\limits_{\rho_{1i}}^{\theta_{1i}} L_{f_0}(t)\Big|h(t_{00}, \varphi_0, z_0)(\tau_i(t)) - h(t_{00}, \varphi_0, z_0)(\tau_0(t))\Big|\,dt.$$

Analogously, it can be proved that

$$\lim_{i\to\infty}(\rho_{1i} - \theta_{1i}) = 0.$$

Thus, $b_{i22}^2 \to 0$. Consequently,

$$b_{i2}^2(t) \to 0. \tag{1.56}$$

Finally, we have

$$b_{12}^3(t) \le \Bigg|\int\limits_{t_{0i}}^{t_{00}} |Y(\xi;t) - H| m_{f_0,K_1}(\xi)\,d\xi\Bigg| \le \|Y - H\|\Bigg|\int\limits_{t_{0i}}^{t_{00}} m_{f_0,K_1}(\xi)\,d\xi\Bigg|$$

i.e.

$$\lim_{i\to\infty} b_{i2}^3(t) = 0 \ \text{ uniformly in } \ I.$$

Therefore,

$$\lim_{i\to\infty} |\zeta_1^i(t) - z_0(t)| = 0 \ \text{ uniformly in } \ I$$

(see (1.44), (1.45), (1.49)–(1.52), (1.55), (1.56)). Assume that the relation (1.43) holds for a certain $k > 1$. Let us prove its fulfilment for $k+1$. Elementary transformations yield

$$\big|\zeta_{k+1}^i(t) - z_0(t)\big| \le |x_{0i} - x_{00}| + a_i(t) + b_{ik}(t), \tag{1.57}$$

where

$$b_{ik}(t) = \Bigg|\int\limits_{t_{0i}}^{t} Y_0(\xi;t,t_{0i}) g_i(t_{0i}, \tau_i, \varphi_i, \zeta_k^i)(\xi)\,d\xi - \\ - \int\limits_{t_{00}}^{t} Y_0(\xi;t,t_{00}) g_i(t_{00}, \tau_0, \varphi_0, z_0)(\xi)\,d\xi\Bigg|$$

(see (1.44)). The quantity $a_i(t)$ has been estimated above, it remains to estimate $b_{ik}(t)$. We have

$$b_{ik}(t) \le \|Y_0\| \int\limits_I \Big|g_i(t_{0i}, \tau_i, \varphi_i, \zeta_k^i)(\xi) - g_i(t_{0i}, \tau_i, \varphi_i, z_0)(\xi)\Big|\,d\xi + \\ + \Bigg|\int\limits_{t_{0i}}^{t} Y_0(\xi;t,t_{0i}) g_i(t_{0i}, \tau_i, \varphi_i, z_0)(\xi)\,d\xi -$$

$$-\int\limits_{t_{0i}}^{t} Y_0(\xi;t,t_{00})g_i(t_{00},\tau_0,\varphi_0,z_0)(\xi)\,d\xi\Bigg| = b_{ik}^1(t)+b_i(t).$$

The function $b_i(t)$ has been estimated above. It is not difficult to see that the following inequality holds for $b_{ik}(t)$:

$$b_{ik}(t) \le 2\|Y_0\|\,\|\zeta_k^i - z_0\| \int\limits_I L_{f_i}(t)\,dt.$$

By the assumptions,

$$\lim_{i\to\infty} \|\zeta_k^i - z_0\| = 0.$$

Therefore,

$$\lim_{i\to\infty} b_{ik}(t) = 0 \ \text{ uniformly in } \ I.$$

Thus, we obtain from (1.57) that

$$\lim_{i\to\infty} \|\zeta_{k+1}^i - z_0\| = 0.$$

We have proved (1.43) for every $k = 1,2,\dots$. Let the number $\delta_1 > 0$ be so small that $[r_1-\delta_1, r_2+\delta_1] \subset I$ and $|z(t;\mu_0) - z(r_1;\mu_0)| \le \varepsilon_0/2$ for $t \in [r_1-\delta_1, r_1]$ and $|z(t;\mu_0) - z(r_2;\mu_0)| \le \varepsilon_0/2$ for $t \in [r_2, r_2+\delta_1]$.

From the uniqueness of the solution $z(t;\mu_0)$, we can conclude that $z(t;\mu_0) = y_0(t)$ for $t \in [r_1, r_2]$. Taking into account the above inequalities, we have

$$\Big(z(t;\mu_0), h\big(t_{00},\varphi_0, z(\,\cdot\,;\mu_0)(\tau_0(t))\big)\Big) \in K^2(\varepsilon_0/2) \subset Q, \ \ t\in[r_1-\delta_1, r_2+\delta_1].$$

Hence,

$$\chi\Big(z(t;\mu_0), h\big(t_{00},\varphi_0, z(\,\cdot\,;\mu_0)(\tau_0(t))\big)\Big) = 1, \ \ t\in[r_1-\delta_1, r_2+\delta_1],$$

and the function $z(t;\mu_0)$ satisfies the equation (1.20) and the condition (1.21).

Therefore,

$$y(t;\mu_0) = z(t;\mu_0), \ \ t\in[r_1-\delta_1, r_2+\delta_1].$$

According to the fixed point theorem, for $\varepsilon_0/2$ there exists a number $\delta_0 \in (0,\varepsilon_0)$ such that a solution $z(t;\mu)$ satisfying the condition

$$\big|z(t;\mu) - z(t;\mu_0)\big| \le \frac{\varepsilon_0}{2}\,, \ \ t\in I,$$

corresponds to each element $\mu \in V(\mu_0;K_1,\delta_0,\alpha)$.

Therefore, for $t\in[r_1-\delta_1, r_2+\delta_1]$

$$z(t;\mu) \in K(\varepsilon_0), \ \ \forall\,\mu\in V(\mu_0;K_1,\delta_0,\alpha).$$

Taking into account the fact that $\varphi(t) \in K(\varepsilon_0)$, we can see that for $t \in [r_1-\delta_1, r_2+\delta_1]$, this implies

$$\chi\Big(z(t;\mu), h\big(t_0,\varphi, z(\,\cdot\,;\mu)(\tau(t))\big)\Big) = 1, \ \ \forall\,\mu\in V(\mu_0;K_1,\delta_0,\alpha).$$

Hence the function $z(t;\mu)$ satisfies the equation (1.20) and the condition (1.21), i.e.

$$y(t;\mu)=z(t;\mu)\in\operatorname{int}K_1,\ \ t\in[r_1-\delta_1,r_2+\delta_1],\ \ \mu\in V(\mu_0;K_1,\delta_0,\alpha). \quad (1.58)$$

The first part of Theorem 1.14 is proved. By the fixed point theorem, for an arbitrary $\varepsilon>0$, there exists a number $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0)$ such that for each $\mu\in V(\mu_0;K_1,\delta_2,\alpha)$,

$$\big|z(t;\mu)-z(t;\mu_0)\big|\le\varepsilon,\ \ t\in I,$$

whence using (1.58), we obtain (1.23). □

Proof of Theorem 1.1. In Theorem 1.14, let $r_1=t_{00}$ and $r_2=t_{00}$. Obviously, the solution $x_0(t)$ satisfies on the interval $[t_{00},t_{10}]$ the following equation:

$$\dot y(t)=A(t)h(t_{00},v_0,\dot y)(\sigma(t))+f_0(t_{00},\tau_0,\varphi_0,y)(t).$$

Therefore, in Theorem 1.14, as the solution $y_0(t)$ we can take the function $x_0(t),t\in[t_{00},t_{10}]$.

By Theorem 1.14, there exist numbers $\delta_i>0,i=0,1$, and for an arbitrary $\varepsilon>0$, there exists a number $\delta_2=\delta_2(\varepsilon)\in(0,\delta_0]$ such that the solution $y(t;\mu)$, $t\in[t_{00}-\delta_1,t_{10}+\delta_1]$, corresponds to each $\mu\in V(\mu_0;K_1,\delta_0,\alpha)$. Moreover, the following conditions hold:

$$\begin{cases}\varphi(t)\in K_1, & t\in I_1;\ \ y(t;\mu)\in K_1,\\ \big|y(t;\mu)-y(t;\mu_0)\big|\le\varepsilon, & t\in[t_{00}-\delta_1,t_{10}+\delta_1],\\ & \mu\in V(\mu_0;K_1,\delta_2,\alpha).\end{cases} \quad (1.59)$$

For an arbitrary $\mu\in V(\mu_0;K_1,\delta_0,\alpha)$, the function

$$x(t;\mu)=\begin{cases}\varphi(t), & t\in[\widehat\tau,t_0),\\ y(t;\mu), & t\in[t_0,t_1+\delta_1].\end{cases}$$

is the solution corresponding to μ. Moreover, if $t\in[\widehat t,t_{10}+\delta_1]$, then $x(t;\mu_0)=y(t;\mu_0)$ and $x(t;\mu)=y(t;\mu)$. Taking into account (1.59), we can see that this implies 1.1 and 1.2. It is easy to notice that for an arbitrary $\mu\in V(\mu_0;K_1,\delta_0,\alpha)$, we have

$$\int\limits_{\widehat\tau}^{t_{10}+\delta_1}\big|x(t;\mu)-x(t;\mu_0)\big|\,dt=\int\limits_{\widehat\tau}^{\bar t}\big|\varphi(t)-\varphi_0(t)\big|\,dt+$$

$$+\int\limits_{\bar t}^{\widehat t}\big|x(t;\mu)-x(t;\mu_0)\big|\,dt+\int\limits_{\widehat t}^{t_{10}+\delta_1}|x(t;\mu)-x(t;\mu_0)|\,dt\le$$

$$\le\|\varphi-\varphi_0\|_{I_1}(b-\widehat\tau)+N|t_0-t_{00}|+\max_{t\in[\widehat t,t_{10}+\delta_1]}\big|x(t;\mu)-x(t;\mu_0\big|(b-\widehat\tau),$$

where

$$\bar t=\min\{t_0,t_{00}\},\quad N=\sup\big\{|x'-x''|:\ x',x''\in K_1\big\}.$$

By 1.1 and 1.2, this inequality implies 1.3. □

1.4. **Proof of Theorem 1.3.** To each element $w \in \Lambda_1$ we correspond the equation

$$\dot y(t) = A(t)h(t_0, v, \dot y)(\sigma(t)) + f(t_0, \tau, \varphi, y, u)(t)$$

with the initial condition (1.21).

Theorem 1.15. *Let $y_0(t)$ be a solution corresponding to $w_0 = (t_0, \tau_0, x_{00}, \varphi_0, v_0, u_0) \in \Lambda_1$ and defined on $[r_1, r_2] \subset (a, b)$. Let $K_2 \subset O$ be a compact set containing a certain neighborhood of the set $\operatorname{cl}\varphi_0(I_1) \cup y_0([r_1, r_2])$. Then the following conditions hold:*

1.17. *there exist numbers $\delta_i > 0$, $i = 0, 1$ such that to each element*

$$w = (t_0, \tau, x_0, \varphi, v, u) \in \widehat V(w_0; \delta_0)$$

there corresponds the solution $y(t; w)$ defined on the interval $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ and satisfying the condition $y(t; w) \in K_2$;

1.18. *for an arbitrary $\varepsilon > 0$ there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0]$ such that the following inequality holds for any $w \in \widehat V(w_0; \delta_0$*

$$\big|y(t; w) - y(t; w_0)\big| \le \varepsilon, \ \ \forall\, t \in [r_1 - \delta_1, r_2 + \delta_1].$$

Theorem 1.15 is proved analogously to Theorem 1.14.

Proof of Theorem 1.3. In Theorem 1.15, let $r_1 = t_{00}$ and $r_2 = t_{10}$. Obviously, the solution $x_0(t)$ satisfies on the interval $[t_{00}, t_{10}]$ the following equation:

$$\dot y(t) = A(t)h(t_{00}, v_0, \dot y)(\sigma(t)) + f(t_{00}, \tau_0, \varphi_0, y, u_0)(t).$$

Therefore, in Theorem 1.15, as the solution $y_0(t)$ we can take the function $x_0(t), t \in [t_{00}, t_{10}]$. Then the proof of the theorem completely coincides with that of Theorem 1.1; for this purpose, it suffices to replace the element μ by the element w and the set $V(\mu_0; K_1, \delta_0, \alpha)$ by the set $\widehat V(w_0; \delta_0)$ everywhere. □

2. Variation Formulas of a Solution

Let $D_1 = \{\tau \in D : \ \dot\tau(t) \ge e = const > 0, \ t \in \mathbb{R}\}$ and let $E_f^{(1)}$ be the set of functions $f : I \times O^2 \to \mathbb{R}^n$ satisfying the following conditions: the function $f(t, \cdot) : O^2 \to \mathbb{R}^n$ is continuously differentiable for almost all $t \in I$; the functions $f(t, x_1, x_2)$, $f_{x_1}(t, x_1, x_2)$ and $f_{x_2}(t, x_1, x_2)$ are measurable on I for any $(x_1, x_2) \in O^2$; for each $f \in E_f^{(1)}$ and compact set $K \subset O$, there exists a function $m_{f,K}(t) \in L(I, \mathbb{R}_+)$, such that

$$\big|f(t, x_1, x_2)\big| + \big|f_{x_1}(t, x_1, x_2)\big| + \big|f_{x_2}(t, x_1, x_2)\big| \le m_{f,K}(t)$$

for all $(x_1, x_2) \in K^2$ and almost all $t \in I$.

To each element

$$\mu = (t_0, \tau, x_0, \varphi, v, f) \in \Lambda_2 = [a, b) \times D_1 \times O \times \Phi_1 \times E_v \times E_f^{(1)}$$

we assign the neutral equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f\big(t, x(t), x(\tau(t))\big)$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \;\; \dot{x}(t) = v(t), \;\; t \in [\widehat{\tau}, t_0), \;\; x(t_0) = x_0.$$

Let $\mu_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, v_0, f_0) \in \Lambda_2$ be a given element and $x_0(t)$ be the solution corresponding to μ_0 and defined on $[\widehat{\tau}, t_{10}]$, with $a < t_{00} < t_{10} < b$.

In the space $E_\mu^{(1)} - \mu_0$, where $E_\mu^{(1)} = \mathbb{R} \times D_1 \times \mathbb{R}^n \times E_\varphi \times E_v \times E_f^{(1)}$, we introduce the set of variations:

$$\begin{aligned} \Im_2 = \Big\{ \delta\mu = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi, \delta v, \delta f) \in E_\mu^{(1)} - \mu_0 : \\ |\delta t_0| \le \beta, \; \|\delta\tau\|_{I_2} \le \beta, \; |\delta x_0| \le \beta, \; \delta\varphi = \sum_{i=1}^k \lambda_i \delta\varphi_i, \\ \|\delta v\|_{I_1} \le \beta, \; \delta f = \sum_{i=1}^k \lambda_i \delta f_i, \; |\lambda_i| \le \beta, \; i = 1, \dots, k \Big\}, \end{aligned}$$

where $\delta\varphi_i \in E_\varphi - \varphi_0$, $\delta f_i \in E_f^{(1)} - f_0$, $i = 1, \dots, k$, are fixed functions.

The inclusion $E_f^{(1)} \subset E_f$ holds (see [15, Lemma 2.1.2]), therefore, according to Theorem 1.2, there exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im_2$ the element $\mu_0 + \varepsilon\delta\mu \in \Lambda_2$, and there corresponds the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$.

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$. Therefore, the solution $x_0(t)$ is assumed to be defined on the interval $[\widehat{\tau}, t_{10} + \delta_1]$.

Let us define the increment of the solution $x_0(t) = x(t; \mu_0)$:

$$\Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), \;\; \forall\, (t, \varepsilon, \delta\mu) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times \Im_2.$$

Theorem 2.1. *Let the following conditions hold:*

2.1. $\gamma_0(t_{00}) < t_{10}$, *where* $\gamma_0(t)$ *is the inverse function to* $\tau_0(t)$*;*
2.2. *the functions* $v_0(\sigma(t))$ *and* $v_0(t)$ *are continuous at the point* t_{00}*; the function* $\varphi_0(t)$ *is absolutely continuous and the function* $\dot{\varphi}_0(t)$ *is bounded;*
2.3. *for each compact set* $K \subset O$ *there exists a number* $m_K > 0$ *such that*

$$|f_0(z)| \le m_K, \;\; \forall\, z = (t, x, y) \in I \times K^2;$$

2.4. *there exist the limits*

$$\lim_{z \to z_0} f_0(z) = f_0^-, \;\; z \in (a, t_{00}] \times O^2,$$

$$\lim_{(z_1, z_2) \to (z_{10}, z_{20})} \big[f_0(z_1) - f_0(z_2)\big] = f_{01}^-, \;\; z_i \in (t_{00}, \gamma_0(t_{00})] \times O^2, \;\; i = 1, 2,$$

where

$$z_0=(t_{00},x_{00},\varphi_0(\tau_0(t_{00}))),\quad z_{10}=\big(\gamma_0(t_{00}),x_0(\gamma_0(t_{00})),x_{00}\big),$$
$$z_{20}=\big(\gamma_0(t_{00}),x_0(\gamma_0(t_{00})),\varphi_0(t_{00})\big).$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ *and* $\delta_2\in(0,\delta_1)$ *such that for arbitrary*

$$(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times\Im_2^-,$$

where

$$\Im_2^-=\big\{\delta\mu\in\Im_2:\ \delta t_0\le 0,\ \delta\tau(\gamma_0(t_{00}))>0\big\}$$

we have

$$\Delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x(t;\delta\mu)+o(t;\varepsilon\delta\mu),\tag{2.1}$$

where

$$\begin{aligned}\delta x(t;\delta\mu)=\Big\{&Y(t_{00}-;t)\Big[v_0(t_{00})-A(t_{00})v_0(\sigma(t_{00}))-f_0^-\Big]-\\&-Y(\gamma_0(t_{00})-;t)f_{01}^-\dot\gamma_0(t_{00})\Big\}\delta t_0+\\&+Y(\gamma_0(t_{00})-;t)f_{01}^-\dot\gamma_0(t_{00})\delta\tau(\gamma_0(t_{00}))+\\&+\int_{t_{00}}^{t}Y(s;t)\delta f[s]\,ds+\beta(t;\delta\mu),\end{aligned}\tag{2.2}$$

and

$$\begin{aligned}\beta(t;\delta\mu)=\;&\Psi(t_{00};t)\big[\delta x_0-v_0(t_{00})\delta t_0\big]+\\&+\int_{t_{00}}^{\gamma_0(t_{00})}Y(s;t)f_{0x_2}[s]\dot\varphi_0(\tau_0(s))\delta\tau(s)\,ds+\\&+\int_{\gamma(t_{00})}^{t}Y(s;t)f_{0x_2}[s]\dot x_0(\tau_0(s))\delta\tau(s)\,ds+\\&+\int_{\tau_0(t_{00})}^{t_{00}}Y(\gamma_0(s);t)f_{0x_2}[\gamma_0(s)]\dot\gamma_0(s)\delta\varphi(s)\,ds+\\&+\int_{\sigma(t_{00})}^{t_{00}}Y(\nu(s);t)A(\nu(s))\dot\nu(s)\delta v(s)\,ds\end{aligned}\tag{2.3}$$

Here, $\Psi(s;t)$ *and* $Y(s;t)$ *are* $n\times n$ *matrix functions satisfying the system*

$$\begin{cases}\Psi_s(s;t)=-Y(s;t)f_{0x_1}[t]-Y(\gamma_0(s);t)f_{0x_2}[\gamma_0(s)]\dot{\gamma}_0(s),\\ Y(s;t)=\Psi(s;t)+Y(\nu(s);t)A(\nu(s))\dot{\nu}(s),\\ \qquad\qquad s\in[t_{00}-\delta_2,t],\ \ t\in[t_{00},t_{10}+\delta_2]\end{cases}$$

and the condition

$$\Psi(s;t)=Y(s;t)=\begin{cases}H, & s=t,\\ \Theta, & s>t;\end{cases}$$

H *is the identity matrix and* Θ *is the zero matrix,* $\nu(s)$ *is the inverse function to* $\sigma(s)$*,*

$$f_{0x_1}[s]=f_{0x_1}\big(s,x_0(s),x_0(\tau_0(s))\big),\ \ \delta f[s]=\delta f\big(s,x_0(s),x_0(\tau_0(s))\big).$$

Some Comments. The function $\delta x(t;\delta\mu)$ is called the variation of the solution $x_0(t)$, $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$, and the expression (2.2) is called the variation formula.

Theorem 2.1 corresponds to the case where the variation at the point t_{00} is performed on the left.

The expression

$$-Y(\gamma_0(t_{00})-;t)f_{01}^{-}\dot{\gamma}_0(t_{00})\delta t_0$$

is the effect of the discontinuous initial condition and perturbation of the initial moment t_{00}.

The expression

$$Y(\gamma_0(t_{00})-;t)f_{01}^{-}\dot{\gamma}_0(t_{00})\delta\tau(\gamma_0(t_{00}))+$$
$$+\int\limits_{t_{00}}^{\gamma_0(t_{00})}Y(s;t)f_{0x_2}[s]\dot{\varphi}_0(\tau_0(s))\delta\tau(s)\,ds+\int\limits_{\gamma_0(t_{00})}^{t}Y(s;t)f_{0x_2}[s]\dot{x}_0(\tau_0(s))\delta\tau(s)\,ds$$

is the effect of perturbation of the delay function $\tau_0(t)$ (see (2.2) and (2.3)).

The addend

$$Y(t_{00}-;t)\Big[v_0(t_{00})-A(t_{00})v_0(\sigma(t_{00}))-f_0^{-}\Big]\delta t_0+\Psi(t_{00};t)\big[\delta x_0-v_0(t_{00})\delta t_0\big]$$

is the effect of perturbations of the initial moment t_{00} and the initial vector x_{00}.

The expression

$$\int\limits_{\tau_0(t_{00})}^{t_{00}}Y(\gamma_0(s);t)f_{0x_2}[\gamma_0(s)]\dot{\gamma}_0(s)\delta\varphi(s)\,ds+$$
$$+\int\limits_{\sigma(t_{00})}^{t_{00}}Y(\nu(s);t)A(\nu(s))\dot{\nu}(s)v(s)\,ds+\int\limits_{t_{00}}^{t}Y(s;t)\delta f[s]\,ds$$

is the effect of perturbations of the initial functions $\varphi_0(t)$ and $v_0(s)$ and the function $f_0(t,x,y)$.

If $\varphi_0(t_{00})=x_{00}$, then $f_{01}^-=0$. If $\gamma_0(t_{00})=t_{10}$, then Theorem 2.1 is valid on the interval $[t_{10},t_{10}+\delta_2]$. If $\gamma_0(t_{00})>t_{10}$, then Theorem 2.1 is valid, with $\delta_2\in(0,\delta_1)$ such that $t_{10}+\delta_2<\gamma_0(t_{00})$; in this case $Y(\gamma_0(t_{00})-;t)=\Theta$.

Finally, we note that the variation formula allows us to obtain an approximate solution of the perturbed equation

$$\begin{aligned}\dot x(t)=A(t)\dot x(\sigma(t))+f_0\big(t,x(t),x(\tau_0(t)+\varepsilon\delta\tau(t))\big)+\\ +\varepsilon\delta f\big(t,x(t),x(\tau_0(t)+\varepsilon\delta\tau(t))\big)\end{aligned}$$

with the perturbed initial condition

$$\begin{gathered}x(t)=\varphi_0(t)+\varepsilon\delta\varphi(t),\ \ \dot x(t)=v_0(t)+\varepsilon\delta v(t),\ \ t\in[\widehat\tau,t_{00}+\varepsilon\delta t_0),\\ x(t_{00}+\varepsilon\delta t_0)=x_{00}+\varepsilon\delta x_0.\end{gathered}$$

In fact, for a sufficiently small $\varepsilon\in(0,\varepsilon_2)$ it follows from (2.1) that

$$x(t;\mu_0+\varepsilon\delta\mu)\approx x_0(t)+\varepsilon\delta x(t;\delta\mu).$$

The matrix function $Y(\xi;t)$ for any fixed $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$ has first order discontinuity at the points of the set

$$\big\{\sigma(t),\sigma^2(t),\dots,\sigma^i(t),\dots\big\},$$

where $\sigma^i(t)=\sigma(\sigma^{i-1}(t))$, $i=1,2,\dots$; $\sigma^0(t)=t,\sigma^1(t)=\sigma(t)$ (see Theorem 1.13).

Theorem 2.2. *Let the conditions* 2.1–2.3 *of Theorem* 2.1 *hold. Moreover, there exist the limits*

$$\begin{gathered}\lim_{z\to z_0}f_0(z)=f_0^+,\ \ z\in[t_{00},\gamma_0(t_{00}))\times O^2,\\ \lim_{(z_1,z_2)\to(z_{10},z_{20})}\big[f_0(z_1)-f_0(z_2)\big]=f_{01}^+,\ \ z_i\in[\gamma_0(t_{00}),b)\times O^2,\ \ i=1,2.\end{gathered}$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary

$$(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times\Im_2^+,$$

where

$$\Im_2^+=\big\{\delta\mu\in\Im_2:\ \delta t_0\ge 0,\ \delta\tau(\gamma_0(t_{00}))<0\big\},$$

formula (2.1) is valid, where

$$\begin{aligned}\delta x(t;\delta\mu)=\Big\{Y(t_{00}+;t)\Big[v_0(t_{00})-A(t_{00})v_0(\sigma(t_{00}))-f_0^+\Big]-\\ -Y(\gamma_0(t_{00})+;t)f_{01}^+\dot\gamma_0(t_{00})\Big\}\delta t_0+\\ +Y(\gamma_0(t_{00})+;t)f_{01}^+\dot\gamma_0(t_{00})\delta\tau(\gamma_0(t_{00}))+\\ +\int_{t_{00}}^{t}Y(s;t)\delta f[s]\,ds+\beta(t;\delta\mu).\end{aligned}$$

Theorem 2.2 corresponds to the case where the variation at the point t_{00} is performed on the right.

Theorem 2.3. *Let the assumptions of Theorems* 2.1 *and* 2.2 *be fulfilled. Moreover,*

$$f_0^- = f_0^+ := \widehat{f}_0, \ \ f_{01}^- = f_{01}^+ := \widehat{f}_{01}$$

and

$$t_{00}, \gamma_0(t_{00}) \notin \{\sigma(t_{10}), \sigma^2(t_{10}), \dots\}.$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *such that for arbitrary* $(t, \varepsilon, \delta\mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2] \times \Im_2$ *formula* (2.1) *holds, where*

$$\begin{aligned} \delta x(t;\delta\mu) = \Big\{ Y(t_{00};t)\Big[v_0(t_{00}) - A(t_{00})v_0(\sigma(t_{00})) - \widehat{f}_0\Big] - \\ - Y(\gamma_0(t_{00});t)\widehat{f}_{01}\dot{\gamma}_0(t_{00})\Big\}\delta t_0 + \\ + Y(\gamma_0(t_{00});t)f_{01}\dot{\gamma}_0(t_{00})\delta\tau(\gamma_0(t_{00})) + \\ + \int_{t_{00}}^{t} Y(s;t)\delta f[s]\,ds + \beta(t;\delta\mu). \end{aligned}$$

Theorem 2.3 corresponds to the case where the variation at the point t_{00} two-sided is performed. If the function $f_0(t,x,y)$ is continuous, then

$$\widehat{f}_0 = f_0\big(t_{00}, \varphi_0(t_{00}), \varphi_0(\tau_0(t_{00}))\big)$$

and

$$\widehat{f}_{01} = f_0\big(\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), x_{00}\big) - f_0\big(\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), \varphi_0(t_{00})\big).$$

Let the function $f(t, x_1, x_2, u)$ be defined on $I \times O^2 \times U_0$ and satisfy the conditions: for almost all $t \in I$ the function $f(t,x,y,u)$ is continuously differentiable with respect to $(x_1,x_2,u) \in O^2 \times U_0$; for any fixed $(x_1,x_2,u) \in O_2 \times U_0$ the functions $f(t,x_1,x_2,u)$, $f_{x_1}(t,x_1,x_2,u)$, $f_{x_2}(t,x_1,x_2,u)$, $f_u(t,x_1,x_2,u)$ are measurable, for any compacts $K \subset O$ and $U \subset U_0$ there exists $m_{K,U}(t) \in L(I, R_+)$ such that

$$\begin{aligned} \big|f(t,x_1,x_2,u)\big| + \big|f_{x_1}(t,x_1,x_2,u)\big| + \big|f_{x_2}(t,x_1,x_2,u)\big| + \big|f_u(t,x_1,x_2,u)\big| \le \\ \le m_f(t) \end{aligned}$$

for all $(x_1,x_2,u) \in K^2 \times U$ and almost all $t \in I$.

Let $w_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, v_0, u_0) \in \Lambda_1$ be the given element and $x_0(t)$ be the solution corresponding to w_0 and defined on $[\widehat{\tau}, t_{10}]$, with $a < t_{00} < t_{10} < b$.

In the space $E_w - w_0$ we introduce the set of variations

$$\Im_3 = \Big\{\delta w = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi, \delta v, \delta u) \in E_w - w_0 :$$
$$|\delta t_0| \le \beta,\ \|\delta\tau\|_{I_2} \le \beta,\ |\delta x_0| \le \beta,\ \delta\varphi = \sum_1^k \lambda_i \delta\varphi_i,$$
$$|\lambda_i| \le \beta,\ i = 1, \dots, k,\ \|\delta v\|_{I_1} \le \beta,\ \|\delta u\|_I \le \beta\Big\}.$$

There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta w) \in (0, \varepsilon_1) \times \Im_3$ the element $w_0 + \varepsilon\delta w \in \Lambda_1$ and there corresponds the solution $x(t; w_0 + \varepsilon\delta w)$ defined on the interval $[\widehat{\tau}, t_{10} + \delta_1] \subset I_1$.

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau}, t_{10} + \delta_1]$. Therefore, the solution $x_0(t)$ is assumed to be defined on the interval $[\widehat{\tau}, t_{10} + \delta_1]$.

Let us define the increment of the solution $x_0(t) = x(t; w_0)$:

$$\Delta x(t; \varepsilon\delta w) = x(t; w_0 + \varepsilon\delta w) - x_0(t), \quad \forall\, (t, \varepsilon, \delta w) \in [\widehat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1) \times \Im_3.$$

Theorem 2.4. *Let the following conditions hold:*

2.5. $\gamma_0(t_{00}) < t_{10}$, *where $\gamma_0(t)$ is the inverse function to $\tau_0(t)$;*
2.6. *the functions $v_0(\sigma(t))$ and $v_0(t)$ are continuous at the point t_{00}; the function $\varphi_0(t)$ is absolutely continuous and the function $\dot\varphi_0(t)$ is bounded;*
2.7. *for each compact sets $K \subset O$ and $U \subset U_0$ there exists a number $m_{K,U} > 0$ such that*
$$|f_0(z)| \le m_{K,U}, \quad \forall\, z = (t, x, y, u) \in I \times K^2 \times U;$$
2.8. *there exist the limits*
$$\lim_{z \to z_0} f_0(z) = f_0^-, \quad z \in (a, t_{00}] \times O^2,$$
$$\lim_{(z_1, z_2) \to (z_{10}, z_{20})} \big[f_0(z_1) - f_0(z_2)\big] = f_{01}^-, \quad z_i \in (t_{00}, \gamma_0(t_{00})] \times O^2, \quad i = 1, 2,$$
where
$$z_0 = \big(t_{00}, x_{00}, \varphi_0(\tau_0(t_{00}))\big), \quad z_{10} = \big(\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), x_{00}\big),$$
$$z_{20} = \big(\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), \varphi_0(t_{00})\big), \quad f_0(z) = f(z, u_0(t)).$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for arbitrary $(t, \varepsilon, \delta w) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times \Im_3^-$, where
$$\Im_3^- = \big\{\delta w \in \Im_3 :\ \delta t_0 \le 0,\ \delta\tau(\gamma_0(t_{00})) > 0\big\}$$
we have
$$\Delta x(t; \varepsilon\delta w) = \varepsilon\delta x(t; \delta w) + o(t; \varepsilon\delta w), \tag{2.4}$$

where

$$\delta x(t;\delta w)=\Big\{Y(t_{00}-;t)\Big[v_0(t_{00})-A(t_{00})v_0(\sigma(t_{00}))-f_0^-\Big]-$$
$$-Y(\gamma_0(t_{00})-;t)f_{01}^-\dot\gamma_0(t_{00})\Big\}\delta t_0+$$
$$+Y(\gamma_0(t_{00})-;t)f_{01}^-\dot\gamma_0(t_{00})\delta\tau(\gamma_0(t_{00}))+$$
$$+\int\limits_{t_{00}}^{t}Y(s;t)f_{0u}[s]\delta u(s)\,ds+\beta(t;\delta w),$$

and

$$\beta(t;\delta w)=\beta(t;\delta\mu).$$

Theorem 2.5. *Let the conditions* 2.5–2.7 *of Theorem* 2.4 *hold. Moreover, there exist the limits*

$$\lim_{z\to z_0}f_0(z)=f_0^+,\ \ z\in[t_{00},\gamma_0(t_{00}))\times O^2,$$
$$\lim_{(z_1,z_2)\to(z_{10},z_{20})}\big[f_0(z_1)-f_0(z_2)\big]=f_{01}^+,\ \ z_i\in[\gamma_0(t_{00}),b)\times O^2,\ \ i=1,2,$$

where $f_0(z)=f(z,u_0(t))$. *Then there exist numbers* $\varepsilon_2\in(0,\varepsilon_1)$ *and* $\delta_2\in(0,\delta_1)$ *such that for arbitrary*

$$(t,\varepsilon,\delta w)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times\Im_3^+,$$

where

$$\Im_3^+=\big\{\delta w\in\Im_3:\ \delta t_0\geq 0,\ \delta\tau(\gamma_0(t_{00}))<0\big\},$$

formula (2.4) *is valid, where*

$$\delta x(t;\delta w)=\Big\{Y(t_{00}+;t)\Big[v_0(t_{00})-A(t_{00})v_0(\sigma(t_{00}))-f_0^+\Big]-$$
$$-Y(\gamma_0(t_{00})+;t)f_{01}^+\dot\gamma_0(t_{00})\Big\}\delta t_0+$$
$$+Y(\gamma_0(t_{00})+;t)f_{01}^+\dot\gamma_0(t_{00})\delta\tau(\gamma_0(t_{00}))+$$
$$+\int\limits_{t_{00}}^{t}Y(s;t)f_{0u}[s]\,ds+\beta(t;\delta\mu).$$

Theorem 2.6. *Let the assumptions of Theorems* 2.4 *and* 2.5 *be fulfilled. Moreover,*

$$f_0^-=f_0^+:=\widehat f_0,\quad f_{01}^-=f_{01}^+:=\widehat f_{01}$$

and

$$t_{00},\gamma_0(t_{00})\notin\big\{\sigma(t_{10}),\sigma^2(t_{10}),\dots\big\}.$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ *and* $\delta_2\in(0,\delta_1)$ *such that for arbitrary*

$$(t,\varepsilon,\delta w)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2]\times\Im_3$$

formula (2.4) *holds, where*

$$\delta x(t;\delta w)=\Big\{Y(t_{00};t)\Big[v_0(t_{00})-A(t_{00})v_0(\sigma(t_{00}))-\widehat{f}_0\Big]- \\ -Y(\gamma_0(t_{00});t)\widehat{f}_{01}\dot{\gamma}_0(t_{00})\Big\}\delta t_0+ \\ +Y(\gamma_0(t_{00});t)f_{01}\dot{\gamma}_0(t_{00})\delta\tau(\gamma_0(t_{00}))+ \\ +\int\limits_{t_{00}}^{t}Y(s;t)f_{0u}[s]\delta u(s)\,ds+\beta(t;\delta w).$$

2.1. **Proof of Theorem 2.1.** First of all, we note that Lemma 2.1 formulated below is a consequence of Theorem 1.14.

Lemma 2.1. *Let $y_0(t)$ be a solution corresponding to $\mu_0\in\Lambda$ and defined on $[r_1,r_2]\subset(a,b)$. Let $K_1\subset O$ be a compact set containing a certain neighborhood of the set $\operatorname{cl}\varphi_0(I_1)\cup y_0([r_1,r_2])$. Then there exist numbers $\varepsilon_1>0$ and $\delta_1>0$ such that for an arbitrary $(t,\delta\mu)\in(0,\varepsilon_1)\times\Im_2$, we have $\mu_0+\varepsilon\delta\mu\in\Lambda$, and the solution $y(t;\mu_0+\varepsilon\delta\mu)$ defined on $[r_1-\delta_1,r_2+\delta_1]\subset I$ corresponds to this element. Moreover,*

$$\varphi(t)\in K_1,\ \ t\in I_1;\quad y(t;\mu_0+\varepsilon\delta\mu)\in K_1,\ \ t\in[r_1-\delta_1,r_2+\delta_1]; \\ \lim_{\varepsilon\to0}y(t;\mu_0+\varepsilon\delta\mu)=y(t;\mu_0),$$

uniformly in $(t,\delta\mu)\in[r_1-\delta_1,r_2+\delta_1]\times\Im_2$.

The solution $y(t;\mu_0)$ on the interval $[r_1-\delta_1,r_2+\delta_1]$ is a continuation of the solution $y_0(t)$. Therefore, in what follows, we can assume that the solution $y_0(t)$ is defined on the whole interval $[r_1-\delta_1,r_2+\delta_1]$.

Let us define the increment of the solution $y_0(t)=y(t;\mu_0)$:

$$\Delta y(t)=\Delta y(t;\varepsilon\delta\mu)=y(t;\mu_0+\varepsilon\delta\mu)-y_0(t), \\ \forall\,(t,\varepsilon,\delta\mu)\in[r_1-\delta_1,r_2+\delta_1]\times(0,\varepsilon_1)\times\Im_2.$$

Obviously,

$$\lim_{\varepsilon\to\infty}\Delta y(t;\varepsilon\delta\mu)=0, \tag{2.5}$$

uniformly in $(t,\delta\mu)\in[r_1-\delta_1,r_2+\delta_1]\times\Im_2$.

Lemma 2.2. *Let $\gamma_0(t_{00})<r_2$ and let the conditions of Theorem 2.1 be fulfilled. Then there exists a number $\varepsilon_2\in(0,\varepsilon_1)$ such that for any $(t,\delta\mu)\in(0,\varepsilon_2)\times\Im_2^-$ the inequality*

$$\max_{t\in[t_{00},r_2+\delta_1]}|\Delta y(t)|\le O(\varepsilon\delta\mu) \tag{2.6}$$

is valid. Moreover,

$$\Delta y(t_{00})=\varepsilon\Big\{\delta x_0-\big[A(t_{00})v_0(\sigma(t_{00}))+f_0^-\big]\delta t_0\Big\}+o(\varepsilon\delta\mu). \tag{2.7}$$

Proof. Let $\varepsilon_2 \in (0, \varepsilon_1)$ be so small that for any $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times \Im_2^-$ the following relations are fulfilled:

$$\tau(t) := \tau_0(t) + \varepsilon\delta\tau(t) < t_0 := t_{00} + \varepsilon\delta t_0, \;\; \forall\, t \in [t_0, t_{00}]. \tag{2.8}$$

The function $\Delta y(t)$ on the interval $[t_{00}, r_2 + \delta_1]$ satisfies the equation

$$\dot{\Delta} y(t) = A(t)h(t_{00}, \varepsilon\delta v, \dot{\Delta} y)(\sigma(t)) + \sum_{i=1}^{3} W_i(t; \varepsilon\delta\mu),$$

where

$$W_1(t; \varepsilon\delta\mu) = A(t)\Big[h(t_0, v, \dot{y}_0 + \dot{\Delta} y)(\sigma(t)) - h(t_{00}, v, \dot{y}_0 + \dot{\Delta} y)(\sigma(t))\Big],$$
$$W_2(t; \varepsilon\delta\mu) = f_0(t_0, \tau, \varphi, y_0 + \Delta y)(t) - f_0(t_{00}, \tau_0, \varphi_0, y_0)(t),$$
$$W_3(t; \varepsilon\delta\mu) = \varepsilon\delta f(t_0, \tau, \varphi, y_0 + \Delta y)(t),$$
$$v := v_0 + \varepsilon\delta v, \quad \varphi := \varphi_0 + \varepsilon\delta\varphi.$$

We now consider the linear nonhomogeneous neutral equation

$$\dot{z}(t) = A(t)\dot{z}(\sigma(t)) + \sum_{i=1}^{3} W_i(t; \varepsilon\delta\mu) \tag{2.9}$$

with the initial condition

$$\dot{z}(t) = \varepsilon\delta v(t), \;\; t \in [\widehat{\tau}, t_0), \quad z(t_0) = \Delta y(t_0).$$

Due to the uniqueness it is easily seen that $z(t) = \Delta y(t)$, $t \in [t_{00}, r_2 + \delta_1]$. According to Theorem 1.7, the solution of the equation (2.9) can be written in the form

$$\Delta y(t) = \Delta y(t_{00}) + \varepsilon \int\limits_{\sigma(t_{00})}^{t_{00}} Y(\nu(\xi); t)A(\nu(\xi))\dot{\nu}(\xi)\delta v(\xi)\, d\xi +$$
$$+ \sum_{i=1}^{3} \int\limits_{t_{00}}^{t} Y(\xi; t)W_i(\xi; \varepsilon\delta\mu)\, d\xi,$$

where $Y(\xi; t)$ has the form (1.11). Hence

$$|\Delta y(t)| \le |\Delta y(t_{00})| + \varepsilon\|Y\|\,\|A\|\alpha\big[\nu(t_{00}) - t_{00}\big] + \|Y\| \sum_{i=1}^{3} W_i(\varepsilon\delta\mu), \tag{2.10}$$

where

$$W_1(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{r_2+\delta_2}|W_1(t;\varepsilon\delta\mu)|\,dt,\quad W_2(t;t_{00},\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t}|W_2(\xi;\varepsilon\delta\mu)|\,d\xi,$$

$$W_3(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{r_2+\delta_2}|W_1(t;\varepsilon\delta\mu)|\,dt,\quad \|A\|=\sup\big\{|A(t)|:\ t\in I\big\},$$

$$\|Y\|=\sup\Big\{|Y(\xi;t)|:\ (\xi,t)\in[t_{00},r_2+\delta_1]\times[t_{00},r_2+\delta_1]\Big\}.$$

Let us prove equality (2.7). We have

$$\begin{aligned}\Delta y(t_{00})&=y(t_{00};\mu_0+\varepsilon\delta\mu)-x_{00}=\\&=x_{00}+\varepsilon\delta x_0+\int\limits_{t_0}^{t_{00}}A(t)[v_0(\sigma(t))+\varepsilon\delta v(\sigma(t))]\,dt+\\&\qquad+\int\limits_{t_0}^{t_{00}}f_0(t,y(t;\mu_0+\varepsilon\delta\mu),\varphi(\tau(t)))\,dt+\\&\qquad+\varepsilon\int\limits_{t_0}^{t_{00}}\delta f(t,y(t;\mu_0+\varepsilon\delta\mu),\varphi(\tau(t)))\,dt-x_{00}=\\&=\varepsilon\big[\delta x_0-A(t_{00})v_0(\sigma(t_{00}))\delta t_0\big]+o(\varepsilon\delta\mu)+\\&\qquad+\int\limits_{t_0}^{t_{00}}f_0(t,y_0(t)+\Delta y(t),\varphi(\tau(t)))\,dt+\\&\qquad+\varepsilon\sum_{i=1}^{k}\lambda_i\int\limits_{t_0}^{t_{00}}\delta f_i(t,y_0(t)+\Delta y(t),\varphi(\tau(t)))\,dt.\end{aligned}\tag{2.11}$$

It is clear that if $t\in[t_0,t_{00}]$, then

$$\lim_{\varepsilon\to0}(t,y_0(t)+\Delta y(t),\varphi(t))=z_0$$

(see (2.5)). Consequently,

$$\lim_{\varepsilon\to0}\sup_{t\in[t_0,t_{00}]}\Big|f_0\big(t,y_0(t)+\Delta y(t),\varphi(\tau(t))\big)-f_0^-\Big|=0.$$

This relation implies that

$$\int_{t_0}^{t_{00}} f_0\big(t, y_0(t) + \Delta y(t), \varphi(\tau(t))\big)\, dt =$$

$$= -\varepsilon f_0^- \delta t_0 + \int_{t_0}^{t_{00}} [f_0(t, y_0(t) + \Delta y(t), \varphi(\tau(t))) - f_0^-]\, dt =$$

$$= -\varepsilon f_0^- \delta t_0 + o(\varepsilon\delta\mu). \tag{2.12}$$

Further, we have

$$|\lambda_i| \int_{t_0}^{t_{00}} \Big|\delta f_i\big(t, y_0(t) + \Delta y(t), \varphi(\tau(t))\big)\Big|\, dt \le \alpha \int_{t_0}^{t_{00}} m_{\delta f_i, K_1}(t)\, dt. \tag{2.13}$$

From (2.11), by virtue of (2.12) and (2.13), we obtain (2.7).

Now, let us prove inequality (2.6). To this end, we have to estimate the expressions $W_1(\varepsilon\delta\mu), W_2(t; t_{00}, \varepsilon\delta\mu)$ and $W_3(\varepsilon\delta\mu)$. We have

$$W_1(\varepsilon\delta\mu) \le \|A\| \int_{\nu(t_0)}^{\nu(t_{00})} \Big|\dot y(\sigma(t); \mu_0 + \varepsilon\delta\mu) - v_0(\sigma(t)) - \varepsilon\delta v(\sigma(t))\Big|\, dt.$$

Using the step method, we can prove the boundedness of $|\dot y(t; \mu_0 + \varepsilon\delta\mu)\|, t \in [r_1 - \delta_1, r_2 + \delta_1]$ uniformly in $\delta\mu \in \Im_2^-$ i.e. there exist $M > 0$ such that

$$\Big|\dot y\big(\sigma(t); \mu_0 + \varepsilon\delta\mu\big) - v_0(\sigma(t)) - \varepsilon\delta v(\sigma(t))\Big| \le M,$$
$$t \in [\nu(t_0), \nu(t_{00})], \ \ \forall\, \delta\mu \in \Im_2.$$

Moreover,

$$\nu(t_{00}) - \nu(t_0) = \int_{t_0}^{t_{00}} \dot\nu(t)\, dt = O(\varepsilon\delta\mu).$$

Thus,

$$W_1(\varepsilon\delta\mu) = O(\varepsilon\delta\mu). \tag{2.14}$$

Let us estimate $W_2(t; t_{00}, \varepsilon\delta\mu)$. It is clear that

$$\gamma_0(t_0) - \gamma(t_0) = \int_{\tau_0(\gamma(t_0))}^{t_0} \dot\gamma_0(\xi)\, d\xi = \int_{t_0 - \varepsilon\delta\tau(\gamma(t_0))}^{t_0} \dot\gamma_0(\xi)\, d\xi > 0$$

and $\gamma(t_0) > t_{00}$ (see (2.8)). For $t \in [t_{00}, \gamma(t_0)]$, we have $\tau(t) < t_0$ and $\tau_0(t) < t_{00}$, therefore we get

$$W_2(t; t_{00}, \varepsilon\delta\mu) \leq \int\limits_{t_{00}}^{t} L_{f_0,K_1}(\xi)\Big[|\Delta y(\xi)| + |\varphi(\tau(\xi)) - \varphi_0(\tau_0(\xi))|\Big]\,d\xi \leq$$
$$\leq \int\limits_{t_{00}}^{t} L_{f_0,K_1}(\xi)\Delta y(\xi)\,d\xi + \Bigg|\int\limits_{\tau_0(\xi)}^{\tau(\xi)} |\dot{\varphi}_0(s)|\,ds\Bigg| + O(\varepsilon\delta\mu) =$$
$$= \int\limits_{t_{00}}^{t} L_{f_0,K_1}(\xi)\Delta y(\xi)\,d\xi + O(\varepsilon\delta\mu). \tag{2.15}$$

For $t \in [\gamma(t_0), \gamma_0(t_{00})]$, we have

$$W_2(t; t_{00}, \varepsilon\delta\mu) = W_2(\gamma(t_0); t_{00}, \varepsilon\delta\mu) + \int\limits_{\gamma(t_0)}^{t} W_2(\xi; \varepsilon\delta\mu)\,d\xi \leq$$
$$\leq O(\varepsilon\delta\mu) + \int\limits_{\gamma(t_0)}^{\gamma_0(t_{00})} W_2(\xi; \varepsilon\delta\mu)\,d\xi \leq O(\varepsilon\delta\mu) + 2m_{K_1}|\gamma_0(t_{00}) - \gamma(t_0)|.$$

Next,

$$\big|\gamma_0(t_{00}) - \gamma(t_0)\big| = \int\limits_{\tau_0(\gamma(t_0))}^{t_{00}} \dot{\gamma}_0(\xi)\,d\xi = \int\limits_{\tau_0(\gamma(t_0))+\varepsilon\delta\tau(\gamma(t_0))-\varepsilon\delta\tau(\gamma(t_0))}^{t_{00}} \dot{\gamma}_0(\xi)\,d\xi =$$
$$= \int\limits_{t_0-\varepsilon\delta\tau(\gamma(t_0))}^{t_{00}} \dot{\gamma}_0(\xi)\,d\xi = O(\varepsilon\delta\mu,)$$

Consequently,

$$W_2(t; t_{00}, \varepsilon\delta\mu) = O(\varepsilon\delta\mu), \;\; t \in [\gamma(t_0), \gamma_0(t_{00})]. \tag{2.16}$$

For $t \in (\gamma_0(t_{00}), r_1 + \delta_1]$, we have

$$W_2(t; t_0, \varepsilon\delta\mu) = W_2\big(\gamma_0(t_{00}); t_0, \varepsilon\delta\mu\big) + \int\limits_{\gamma_0(t_{00})}^{t} W_2(\xi; \varepsilon\delta\mu)\,d\xi \leq$$
$$\leq O(\varepsilon\delta\mu) + \Bigg|\int\limits_{\gamma_0(t_{00})}^{\gamma(t_{00})} W_2(\xi; \varepsilon\delta\mu)\,d\xi\Bigg| +$$

$$+\left|\int\limits_{\gamma(t_{00})}^{t}\chi(\xi)L_{f_0,K_1}(\xi)|\Delta y(\tau(\xi))|\,d\xi\right|+\left|\int\limits_{\gamma(t_{00})}^{t}\big|y_0(\tau(\xi))-y_0(\tau_0(\xi))\big|\,d\xi\right|\le$$

$$\le O(\varepsilon\delta\mu)+2m_{K_1}|\gamma_0(t_{00})-\gamma(t_{00})|+$$

$$+\int\limits_{t_{00}}^{t}\chi(\gamma(\xi))L_{f_0,K_1}(\gamma(\xi))\dot\gamma(\xi)|\Delta y(\xi)|\,d\xi+\int\limits_{t_{00}}^{r_1+\delta_1}\left|\int\limits_{\tau_0(\xi)}^{\tau(\xi)}|\dot y_0(s)|\,ds\right|d\xi=$$

$$=O(\varepsilon\delta\mu)+2m_{K_1}\Big[\big|\gamma(t_{00})-\gamma(t_0)\big|+\big|\gamma(t_{00})-\gamma_0(t_{00})\big|\Big]+$$

$$+\int\limits_{t_{00}}^{t}\chi(\gamma(\xi))L_{f_0,K_1}(\gamma(\xi))\dot\gamma(\xi)|\Delta y(\xi)|\,d\xi$$

where $\chi(\xi)$ is the characteristic function of I. Next,

$$\gamma(t_{00})-\gamma(t_0)=\int\limits_{t_0}^{t_{00}}\dot\gamma(\xi)\,d\xi\le\frac{1}{e}(t_{00}-t_0)=O(\varepsilon\delta\mu)$$

and

$$\big|\gamma(t_{00})-\gamma_0(t_{00})\big|=\left|\int\limits_{t_{00}}^{\tau_0(\gamma(t_{00}))}\dot\gamma_0(t)\,dt\right|=$$

$$=\left|\int\limits_{t_{00}}^{\tau(\gamma(t_{00}))-\varepsilon\delta(\gamma(t_{00}))}\dot\gamma_0(t)\,dt\right|=O(\varepsilon\delta\mu).$$

Thus,

$$W_2(t;t_0,\varepsilon\delta\mu)=O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{t}\chi(\gamma(\xi))L_{f_0,K_1}(\gamma(\xi))\dot\gamma(\xi)|\Delta y(\xi)|\,d\xi. \quad (2.17)$$

Finally, we note that

$$W_3(t;\varepsilon\delta\mu)=O(\varepsilon\delta\mu),\ \ t\in[t_{00},r_2+\delta_1] \quad (2.18)$$

(see (2.12)).

According to (2.7), (2.14)–(2.18), inequality (2.10) directly implies that

$$|\Delta y(t)|\le O(\varepsilon\delta\mu)+\int\limits_{t_{00}}^{t}\Big[L_{f_0,K_1}(\xi)+\chi(\gamma(\xi))L_{f_0,K_1}(\gamma(\xi))\dot\gamma(\xi)\Big]\Delta y(\xi)|\,d\xi$$

By virtue of Grounwall's lemma, we obtain

$$|\Delta y(t)| \le$$

$$\le O(\varepsilon\delta\mu)\exp\bigg\{\int\limits_{t_{00}}^{t} L_{f_0,K_1}(\xi)\,d\xi + \int\limits_{t_{00}}^{t} \chi(\gamma(\xi))L_{f_0,K_1}(\gamma(\xi))\dot\gamma(\xi)\,d\xi\bigg\} \le$$

$$\le \exp\bigg\{2\int\limits_{I} L_{f_0,K_1}(\xi)\,d\xi\bigg\}.$$

The following assertion can be proved by analogy with Lemma 2.2. □

Lemma 2.3. *Let $\gamma_0(t_{00}) < r_2$ and let the conditions of Theorem 2.2 be fulfilled. Then there exists the number $\varepsilon_2 \in (0,\varepsilon_1)$ such that for any $(t,\delta\mu) \in (0,\varepsilon_2)\times\Im_2^+$ the inequality*

$$\max_{t\in[t_0,r_2+\delta_1]} |\Delta y(t)| \le O(\varepsilon\delta\mu)$$

is valid. Moreover,

$$\Delta y(t_{00}) = \varepsilon\Big\{\delta x_0 - \big[A(t_{00})v_0(\sigma(t_{00})) + f_0^+\big]\delta t_0\Big\} + o(\varepsilon\delta\mu).$$

Proof of Theorem 2.1. Let $r_1 = t_{00}$ and $r_2 = t_{10}$ in Lemma 2.1. Then

$$x_0(t) = \begin{cases} \varphi_0(t), & t \in [\widehat\tau, t_{00}),\\ y_0(t), & t\in[t_{00},t_{10}], \end{cases}$$

and for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_1)\times\Im_2^-$

$$x(t;\mu_0+\varepsilon\delta\mu) = \begin{cases} \varphi(t) = \varphi_0(t) + \varepsilon\delta\varphi(t), & t\in[\widehat\tau,t_0),\\ y(t;\mu_0+\varepsilon\delta\mu), & t\in[t_0,t_{10}+\delta_1] \end{cases}$$

(see Remark 1.1). We note that $\delta\mu\in\Im_2^-$, i.e. $t_0 < t_{00}$, therefore

$$\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t), & t\in[\widehat\tau,t_0),\\ y(t;\mu_0+\varepsilon\delta\mu) - \varphi_0(t), & t\in[t_0,t_{00}),\\ \Delta y(t), & t\in[t_{00},t_{10}+\delta_1]. \end{cases}$$

$$\dot{\Delta} x(t) = \begin{cases} \varepsilon\delta v(t), & t\in[\widehat\tau,t_0),\\ \dot y(t;\mu_0+\varepsilon\delta\mu) - v_0(t), & t\in[t_0,t_{00}),\\ \dot{\Delta} y(t), & t\in[t_{00},t_{10}+\delta_1]. \end{cases}$$

By Lemma 2.2, we have

$$|\Delta x(t)| \le O(\varepsilon\delta\mu),\ \ \forall\,(t,\varepsilon,\delta\mu)\in[t_{00},t_{10}+\delta_1]\times(0,\varepsilon_1)\times\Im_2^-, \tag{2.19}$$

$$\Delta x(t_{00}) = \varepsilon\Big\{\delta x_0 - \big[A(t_{00})v_0(\sigma(t_{00})) + f_0^-\big]\delta t_0\Big\} + o(\varepsilon\delta\mu). \tag{2.20}$$

The function $\Delta x(t)$ satisfies the equation

$$\dot{\Delta}x(t) = A(t)\dot{\Delta}x(\sigma(t)) + \\ + f_{0x}[t]\Delta x(t) + f_{0y}[t]\Delta x(\tau_0(t)) + \varepsilon\delta f[t] + R_1[t] + R_2[t], \qquad (2.21)$$

where

$$R_1[t] = f_0\big(t, x_0(t) + \Delta x(t), x_0(\tau(t))\big) + \Delta x(\tau(t)) - \\ - f_0[t] - f_{0x_1}[t]\Delta x(t) - f_{0x_2}[t]\Delta x(\tau_0(t)),$$

$$R_2[t] = \varepsilon\Big[\delta f\Big(t, x_0(t) + \Delta x(t), x_0(\tau(t)) + \Delta x(\tau(t))\Big) - \delta f[t]\Big].$$

By using the Cauchy formula, one can represent the solution of the equation (2.21) in the form

$$\Delta x(t) = \Psi(t_{00}; t)\Delta x(t_{00}) + \\ + \varepsilon \int_{t_{00}}^{t} Y(\xi; t)\delta f[\xi]\, d\xi + \sum_{i=-1}^{2} R_i[t; t_{00}], t \in [t_{00}, t_{10} + \delta_1], \qquad (2.22)$$

where

$$R_{-1}[t; t_{00}] = \int_{\sigma(t_{00})}^{t_{00}} Y(\nu(\xi); t)A(\nu(\xi))\dot{\nu}(\xi)\dot{\Delta}x(\xi)\, d\xi,$$

$$R_0[t; t_{00}] = \int_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(\xi); t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\, d\xi,$$

$$R_i[t; t_{00}] = \int_{t_{00}}^{t} Y(\xi; t)R_i[\xi]\, d\xi, \quad i = 1, 2,$$

By Theorem 1.13, we get

$$\Phi(t_{00}; t)\Delta x(t_{00}) = \\ = \varepsilon\Phi(t_{00}; t)\Big\{\delta x_0 - \big[A(t_{00})v_0(\sigma(t_{00})) + f_0^-\big]\delta t_0\Big\} + o(t; \delta\mu) \qquad (2.23)$$

(see (2.20)).

Now, let us transform $R_{-1}[t; t_{00}]$. We have

$$R_{-1}[t; t_{00}] = \varepsilon \int_{\sigma(t_{00})}^{t_0} Y(\nu(\xi); t)A(\nu(\xi))\dot{\nu}(\xi)\delta v(\xi)\, d\xi + \\ + \int_{t_0}^{t_{00}} Y(\nu(\xi); t)A(\nu(\xi))\dot{\nu}(\xi)\dot{\Delta}x(\xi)\, d\xi =$$

$$
\begin{aligned}
&= \varepsilon \int\limits_{\sigma(t_{00})}^{t_{00}} Y(\nu(\xi);t)A(\nu(\xi))\dot{\nu}(\xi)\delta v(\xi)\,d\xi + o(t;\varepsilon\delta\mu)+\\
&\qquad + \int\limits_{t_0}^{t_{00}} Y(\nu(\xi);t)A(\nu(\xi))\dot{\nu}(\xi)\times\\
&\times \Big[A(\xi)\big(v_0(\sigma(\xi)) + \varepsilon\delta v(\sigma(\xi))\big) + f_0(t_0,\tau,\varphi,y_0+\Delta y)(\xi)+\\
&\qquad + \varepsilon\delta f(t_0,\tau,\varphi,y_0+\Delta y)(\xi) - v_0(\xi)\Big]\,d\xi =\\
&+ \varepsilon \int\limits_{\sigma(t_{00})}^{t_{00}} Y(\nu(\xi);t)A(\nu(\xi))\dot{\nu}(\xi)\delta v(\xi)\,d\xi - \varepsilon Y(\nu(t_{00})-;t)A(\nu(t_{00}))\times\\
&\times \dot{\nu}(t_{00})\big[A(t_{00})v_0(\sigma(t_{00})) + f_0^- - v_0(t_{00})\big]\delta t_0 + o(t;\varepsilon\delta\mu) =\\
&= \varepsilon \int\limits_{\sigma(t_{00})}^{t_{00}} Y(\nu(\xi);t)A(\nu(\xi))\dot{\nu}(\xi)\delta v(\xi)\,d\xi + \varepsilon\big[Y(t_{00}-;t) - \Phi(t_{00};t)\big]\times\\
&\qquad \times \big[v_0(t_{00}) - A(t_{00})v_0(\sigma(t_{00})) - f_0^-\big]\delta t_0 + o(\varepsilon\delta\mu) \qquad (2.24)
\end{aligned}
$$

(see (1.7)).

For $R_0[t;t_{00}]$, we have

$$
\begin{aligned}
R_0[t;t_{00}] &= \varepsilon \int\limits_{\tau_0(t_{00})}^{t_0} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\delta\varphi(\xi)\,d\xi+\\
&\quad + \int\limits_{t_0}^{t_{00}} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\,d\xi =\\
&= \varepsilon \int\limits_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\delta\varphi(\xi)\,d\xi + o(t;\varepsilon\delta\mu)+\\
&\quad + \int\limits_{t_0}^{t_{00}} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\,d\xi. \qquad (2.25)
\end{aligned}
$$

Let a number $\delta_2 \in (0,\delta_1)$ be so small that $\gamma_0(t_{00}) < t_{10} - \delta_2$. Since $\gamma_0(t_{00}) > \gamma(t_0)$, therefore for $t \in [t_{10}-\delta_2, t_{10}+\delta_2]$, we have

$$
R_1[t;t_{00}] = \sum_{i=1}^{3} \alpha_i[t],
$$

where

$$\alpha_1[t]=\int\limits_{t_{00}}^{\gamma(t_0)} r[\xi;t]\,d\xi,\quad \alpha_2[t]=\int\limits_{\gamma(t_0)}^{\gamma_0(t_{00})} r[\xi;t]\,d\xi,$$

$$\alpha_3[t]=\int\limits_{\gamma_0(t_{00})}^{t} r[\xi;t]\,d\xi,\quad r[\xi;t]=Y(\xi;t)R_1[\xi].$$

Introducing the notation,

$$f_0[\xi;s]=$$
$$=f_0\big(\xi,x_0(\xi)+s\Delta x(\xi),x_0(\tau_0(\xi))+s\big(x_0(\tau(\xi))-x_0(\tau_0(\xi))+\Delta x(\tau(\xi))\big)\big),$$
$$\theta[\xi;s]=f_{0x_1}[\xi;s]-f_{0x_1}[\xi],\quad \rho[\xi;s]=f_{0x_2}[\xi;s]-f_{0x_2}[\xi],$$

Then we have

$$R_1[\xi]=\int\limits_0^1 \frac{d}{ds}\,f_0[\xi;s]\,ds=$$
$$=\int\limits_0^1\Big\{f_{0x_1}[\xi;s]\Delta x(\xi)+f_{0x_2}[\xi;s]\Big(x_0(\tau(\xi))-x_0(\tau_0(\xi))+\Delta x(\tau(\xi))\Big)\Big\}\,ds-$$
$$-f_{0x_1}[\xi]\Delta x(\xi)-f_{0x_2}[\xi]\Delta x(\tau_0(\xi))=\bigg[\int\limits_0^1\theta[\xi;s]\,ds\bigg]\Delta x(\xi)+$$
$$+\bigg[\int\limits_0^1\rho[\xi;s]\,ds\bigg]\big(x_0(\tau(\xi))-x_0(\tau_0(\xi))+\Delta x(\tau(\xi))\big)+$$
$$+f_{0x_2}[\xi]\Big\{\big[x_0(\tau(\xi))-x_0(\tau_0(\xi))\big]+\big[\Delta x(\tau(\xi))-\Delta x(\tau_0(\xi))\big]\Big\}.$$

Taking into account the latter relation, we have

$$\alpha_1[t]=\sum_{i=1}^4\alpha_{1i}[t],$$

where

$$\alpha_{11}[t]=\int\limits_{t_{00}}^{\gamma(t_0)} Y(\xi;t)\theta_1[\xi]\Delta x(\xi)\,d\xi,\ \ \theta_1[\xi]=\int\limits_0^1\theta[\xi;s]\,ds,$$

$$\alpha_{12}[t]=\int\limits_{t_{00}}^{\gamma(t_0)} Y(\xi;t)\rho_1[\xi]\Big[x_0(\tau(\xi))-x_0(\tau_0(\xi))+\Delta x(\tau(\xi))\Big]\,d\xi,$$

$$\rho_1[\xi]=\int\limits_0^1\rho[\xi;s]\,ds,$$

$$\alpha_{13}[t]=\int\limits_{t_{00}}^{\gamma(t_0)}Y(\xi;t)f_{0x_2}[\xi]\big[\Delta x(\tau(\xi))-\Delta x(\tau_0(\xi))\big]\,d\xi,$$

$$\alpha_{14}[t]=\int\limits_{t_{00}}^{\gamma(t_0)}Y(\xi;t)f_{0x_2}[\xi]\big[x_0(\tau(\xi))-x_0(\tau_0(\xi))\big]\,d\xi.$$

Further,

$$\gamma_0(t_0)-\gamma(t_0)=\int\limits_{\tau_0(\gamma(t_0))}^{t_0}\dot\gamma_0(\xi)\,d\xi=\int\limits_{t_0-\varepsilon\delta\tau(\gamma(t_0))}^{t_0}\dot\gamma_0(\xi)\,d\xi>0.$$

Therefore. for $\xi\in(t_{00},\gamma(t_0))$, we have $\tau(\xi)<t_0$, $\tau_0(\xi)<t_0$. Thus,

$$x_0(\tau(\xi))-x_0(\tau_0(\xi))=\varphi_0(\tau(\xi))-\varphi_0(\tau_0(\xi))$$

and

$$\Delta x(\tau(\xi))-\Delta x(\tau_0(\xi))=\varepsilon\big[\delta\varphi(\tau(\xi))-\delta\varphi(\tau_0(\xi))\big].$$

The function $\varphi_0(t), t\in I_1$ is absolutely continuous, therefore for each fixed Lebesgue point $\tau_0(\xi)\in I_1$ we get

$$\varphi_0(\tau(\xi))-\varphi_0(\tau_0(\xi))=\int\limits_{\tau_0(\xi)}^{\tau(\xi)}\dot\varphi_0(s)\,ds=\varepsilon\dot\varphi_0(\tau_0(\xi))\delta\tau(\xi)+\gamma(\xi;\varepsilon\delta\mu),\tag{2.26}$$

where

$$\lim_{\varepsilon\to0}\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}=0\ \text{ uniformly for }\ \delta\mu\in\Im_2^-.\tag{2.27}$$

Thus, (2.26) is valid for almost all points of the interval $(t_{00},\gamma(t_0))$. From (2.26), taking into account the boundedness of the function $\dot\varphi_0(t)$, we have

$$\big|\varphi_0(\tau(\xi))-\varphi_0(\tau_0(\xi))\big|\le O(\varepsilon\delta\mu)\ \text{ and }\ \Big|\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big|\le const.\tag{2.28}$$

According to (2.19) and (2.26)–(2.28). for the expressions $\alpha_{1i}[t]$, $i=1,\dots,4$, we have

$$\big|\alpha_{11}[t]\big|\le\|Y\|O(\varepsilon\delta\mu)\theta_2(\varepsilon\delta\mu),\quad|\alpha_{12}[t]|\le\|Y\|O(\varepsilon\delta\mu)\rho_2(\varepsilon\delta\mu),$$

$$\big|\alpha_{13}[t]\big|\le o(\varepsilon\delta\mu),\quad\alpha_{14}[t]=\varepsilon\int\limits_{t_{00}}^{\rho_\varepsilon}Y(\xi;t)f_{0x_2}[\xi]\dot\varphi_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi,$$

where

$$\theta_2(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{b}\int\limits_0^1\Big|f_{0x_1}\big(\xi,x_0(\xi)+s\Delta x(\xi),\varphi_0(\tau_0(\xi))\big)+$$

$$+s\Big(\varphi_0(\tau(\xi))-\varphi_0(\tau_0(\xi))-\varepsilon\delta\varphi(\tau_0(\xi))\Big)-f_{0x_1}\big(\xi,x_0(\xi),\varphi_0(\tau_0(\xi))\big)\Big|\,ds\,d\xi,$$

$$\rho_2(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{b}\int\limits_{0}^{1}\Big|f_{0x_2}\big(\xi,x_0(\xi)+s\Delta x(\xi),\varphi_0(\tau_0(\xi))\big)+$$

$$+\,s\Big(\varphi_0(\tau(\xi))-\varphi_0(\tau_0(\xi))-\varepsilon\delta\varphi(\tau_0(\xi))\Big)-$$

$$-\,f_{0x_2}\big(\xi,x_0(\xi),\varphi_0(\tau_0(\xi))\big)\Big|\,ds\,d\xi,$$

$$\gamma_1(t;\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t}Y(\xi;t)f_{0x_2}[\xi]\gamma(\xi;\varepsilon\delta\mu)\,d\xi.$$

Obviously,

$$\Big|\frac{\gamma(t;\varepsilon\delta\mu)}{\varepsilon}\Big|\le\|Y\|\int\limits_{t_{00}}^{\gamma_0(t_{00})}\big|f_{0x_2}[\xi]\big|\,\Big|\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big|\,d\xi.$$

By the Lebesgue theorem on the passage under the integral sign, we have

$$\lim_{\varepsilon\to0}\theta(\varepsilon\delta\mu)=\lim_{\varepsilon\to0}\rho(\varepsilon\delta\mu)=\Big|\frac{\gamma_1(t;\varepsilon\delta\mu)}{\varepsilon}\Big|=0$$

uniformly for $(t,\delta\mu)\in[t_{00},\gamma_0(t_{00})]\times\Im_2^-$ (see (2.26)).

Thus,

$$\alpha_{1i}[t]=o(\varepsilon\delta\mu),\quad i=1,2,3; \tag{2.29}$$

$$\alpha_{14}[t]=\varepsilon\int\limits_{t_{00}}^{\gamma(t_0)}Y(\xi;t)f_{0x_2}[\xi]\dot\varphi_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi+o(t;\varepsilon\delta\mu).$$

It is clear that

$$\varepsilon\int\limits_{\gamma(t_0)}^{\gamma_0(t_{00})}Y(\xi;t)f_{0x_2}[\xi]\dot\varphi_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi=o(t;\varepsilon\delta\mu),$$

i.e.

$$\alpha_{14}[t]=\varepsilon\int\limits_{t_{00}}^{\gamma_0(t_{00})}Y(\xi;t)f_{0x_2}[\xi]\dot\varphi_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi+o(t;\varepsilon\delta\mu). \tag{2.30}$$

On the basis of (2.28) and (2.29), we obtain

$$\alpha_1[t]=\varepsilon\int\limits_{t_{00}}^{\gamma_0(t_{00})}Y(\xi;t)f_{0x_2}[\xi]\dot\varphi_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi+o(t;\varepsilon\delta\mu). \tag{2.31}$$

Let us now transform $\alpha_2[t]$. We have

$$\alpha_2[t]=\sum_{i=1}^{3}\alpha_{2i}(t;\varepsilon\delta\mu),$$

where

$$\alpha_{21}[t]=\int\limits_{\gamma(t_0)}^{\gamma_0(t_{00})}Y(\xi;t)\Big[f_0\Big(\xi,x_0(\xi)+\Delta x(\xi),x_0(\tau(\xi))+\Delta x(\tau(\xi))\Big)-f_0[\xi]\Big]\,d\xi,$$

$$\alpha_{22}[t]=-\int\limits_{\gamma(t_0)}^{\gamma_0(t_{00})}Y(\xi;t)f_{0x_1}[\xi]\Delta x(\xi)\,d\xi,$$

$$\alpha_{23}[t]=-\int\limits_{\gamma(t_0)}^{\gamma_0(t_{00})}Y(\xi;t)f_{0x_2}[\xi]\Delta x(\tau_0(\xi))\,d\xi.$$

If $\xi\in(\gamma(t_0),\gamma_0(t_{00}))$, then

$$|\Delta x(\xi)|\le O(\varepsilon\delta\mu),$$
$$x_0(\tau(\xi))+\Delta x(\tau(\xi))=y(\tau(\xi);\varepsilon\delta\mu)=y_0(\tau(\xi))+\Delta y(\tau(\xi);\varepsilon\delta\mu),$$
$$x_0(\tau_0(\xi))=\varphi_0(\tau_0(\xi)),$$

therefore,

$$\alpha_{22}[t]=o(t;\varepsilon\delta\mu),$$
$$\lim_{\varepsilon\to 0}\Big(\xi,x_0(\xi)+\Delta x(\xi),x_0(\tau(\xi))+\Delta x(\tau(\xi))\Big)=$$
$$=\lim_{\xi\to\gamma_0(t_{00})-}\big(\xi,x_0(\xi),y_0(\tau_0(\xi))\big)=z_{10},$$
$$\lim_{\varepsilon\to 0}\big(\xi,x_0(\xi),x_0(\tau_0(\xi))\big)=\lim_{\xi\to\gamma_0(t_{00})-}\big(\xi,x_0(\xi),\varphi_0(\tau_0(\xi))\big)=z_{20},$$

i.e.

$$\lim_{\varepsilon\to 0}\sup_{\xi\in[\gamma(t_0),\gamma_0(t_{00})]}\Big[f_0\Big(\xi,x_0(\xi)+\Delta x(\xi),x_0(\tau(\xi))+\Delta x(\tau(\xi))\Big)-$$
$$-f_0\big(\xi,x_0(\xi),x_0(\tau_0(\xi))\big)\Big]=f_{01}^-.$$

It is clear that

$$\gamma_0(t_{00})-\gamma(t_0)=\int\limits_{\tau_0(\gamma(t_0))}^{t_{00}}\dot\gamma_0(\xi)\,d\xi=$$
$$=\int\limits_{\tau(\gamma(t_0))-\varepsilon\delta\tau(\gamma(t_0))}^{t_{00}}\dot\gamma_0(\xi)\,d\xi=\int\limits_{t_0-\varepsilon\delta\tau(\gamma(t_0))}^{t_{00}}\dot\gamma_0(\xi)\,d\xi=O(\varepsilon\delta\mu)>0.$$

It is not difficult to see that

$$\alpha_{21}[t] = \int\limits_{\gamma(t_0)}^{\gamma_0(t_{00})} Y(\xi;t)f_{01}^{-}\,d\xi + o(t;\varepsilon\delta\mu) =$$

$$= \int\limits_{\tau_0(\gamma(t_0))}^{t_{00}} Y(\gamma_0(\xi);t)f_{01}^{-}\dot{\gamma}_0(\xi)\,d\xi + o(t;\varepsilon\delta\mu) =$$

$$= \int\limits_{t_{00}-\varepsilon(\delta\tau(\gamma_0(t_{00}))-\delta t_0)+o(\varepsilon\delta\mu)}^{t_{00}} Y(\gamma_0(\xi);t)f_{01}^{-}\dot{\gamma}_0(\xi)\,d\xi + o(t;\varepsilon\delta\mu) =$$

$$= \varepsilon Y(\gamma(t_{00})-;t)f_{01}^{-}\dot{\gamma}_0(t_{00})\big(\delta\tau(\gamma_0(t_{00})) - \delta t_0\big) + o(t;\varepsilon\delta\mu).$$

For $\xi \in [\gamma(t_0), \gamma_0(t_{00})]$, we have $\Delta x(\tau_0(\xi)) = \varepsilon\delta\varphi(\tau_0(\xi))$, therefore

$$\alpha_{23}[t] = -\varepsilon \int\limits_{\gamma(t_0)}^{\gamma_0(t_0)} Y(\xi;t)f_{0x_2}[\xi]\delta\varphi(\tau_0(\xi))\,d\xi -$$

$$- \int\limits_{\gamma_0(t_0)}^{\gamma_0(t_{00})} Y(\xi;t)f_{0x_2}[\xi]\Delta x(\tau_0(\xi))\,d\xi =$$

$$= -\int\limits_{t_0}^{t_{00}} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\,d\xi + o(t;\varepsilon\delta\mu).$$

Consequently,

$$\alpha_2[t] = \varepsilon Y(\gamma_0(t_{00})-;t)f_{01}^{-}\dot{\gamma}_0(t_{00})\big(\delta\tau(\gamma_0(t_{00})) - \delta t_0\big) =$$

$$- \int\limits_{t_0}^{t_{00}} Y(\gamma_0(\xi);t)f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\,d\xi + o(t;\varepsilon\delta\mu). \tag{2.32}$$

Transforming the expression $\alpha_3[t]$ for $t \in [t_{10} - \delta_2, t_{10} + \delta_2]$, we have

$$\alpha_3[t] = \sum_{i=1}^{4} \alpha_{3i}[t],$$

where

$$\alpha_{31}[t] = \int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t)\theta_1[\xi]\Delta x(\xi)d\xi,$$

$$\alpha_{32}[t] = \int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t)\rho_1[\xi]\Big[x_0(\tau(\xi)) - x_0(\tau_0(\xi)) + \Delta x(\tau(\xi))\Big]\,d\xi,$$

$$\alpha_{33}[t]=\int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t)f_{0x_2}[\xi]\big[\Delta x(\tau(\xi))-\Delta x(\tau_0(\xi))\big]\,d\xi,$$

$$\alpha_{34}[t]=\int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t)f_{0x_2}[\xi]\big[x_0(\tau(\xi))-x_0(\tau_0(\xi))\big]\,d\xi.$$

For each Lebesgue point $\tau_0(\xi)$ of the function $\dot{x}_0(t), t\in[t_{00},t_{10}+\delta_2]$, we get

$$x_0(\tau(\xi))-x_0(\tau_0(\xi))=\int\limits_{\tau_0(\xi)}^{\tau(\xi)}\dot{x}_0(\xi)\,d\xi=\varepsilon\dot{x}_0(\tau_0(\xi))\delta\tau(\xi)+\widehat{\gamma}(\xi;\varepsilon\delta\mu), \quad (2.33)$$

where

$$\lim_{\varepsilon\to 0}\frac{\widehat{\gamma}(\xi;\varepsilon\delta\mu)}{\varepsilon}=0 \ \text{ uniformly for } \ \delta\mu\in\Im_2^-. \quad (2.34)$$

From (2.32), taking into account the boundedness of the function $\dot{x}_0(t)$, we have

$$\big|x_0(\tau(\xi))-x_0(\tau_0(\xi))\big|\le O(\varepsilon\delta\mu) \ \text{ and } \ \Big|\frac{\gamma(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big|\le const. \quad (2.35)$$

Further,

$$\big|\Delta x(\tau(\xi))-\Delta x(\tau_0(\xi))\big|\le\int\limits_{\tau_0(\xi)}^{\tau(\xi)}|\dot{\Delta}(x(s))|\,ds\le$$

$$\le\int\limits_{\tau_0(\xi)}^{\tau(\xi)}|A(s)|\,\big|\dot{\Delta}x(\sigma(s))\big|\,ds\le$$

$$\le\int\limits_{\tau_0(\xi)}^{\tau(\xi)}L_{f_0,K_1}(s)\Big(|\Delta x(s)|+\big|x_0(\tau(s))-x_0(\tau_0(s))\big|+\big|\Delta x(\tau(s))\big|\Big)\,ds\le$$

$$\le\|A\|\int\limits_{\tau(\xi)}^{\tau(\xi)}\big|\dot{\Delta}x(\sigma(s))\big|\,ds+o(\xi;\varepsilon\delta\mu).$$

If $[\sigma(\tau_0(\xi)),\sigma(\tau(\xi))]\subset[t_0,\nu(t_0)]$, then

$$\dot{\Delta}x(\sigma(s))=\varepsilon\delta v(\sigma(s)).$$

Thus, in this case we have

$$\big|\Delta x(\tau(\xi))-\Delta x(\tau_0(\xi))\big|=o(\xi;\varepsilon\delta\mu).$$

If $[\sigma(\tau_0(\xi)),\sigma(\tau(\xi))]\subset[\nu(t_0),\nu(t_{00})]$, then

$$\big|\dot{\Delta}x(\sigma(s))\big|=\big|\dot{x}(\sigma(s);\mu_0+\varepsilon\delta\mu)-v_0(\sigma(s))\big|$$

and

$$\big|\Delta x(\tau(\xi)) - \Delta x(\tau_0(\xi))\big| = O(\xi;\varepsilon\delta\mu).$$

It is clear that if $\xi \in [\gamma_0(\nu(t_0)), \gamma(\nu(t_{00}))]$, then

$$\big[\sigma(\tau_0(\xi)), \sigma(\tau(\xi))\big] \subset \big[\nu(t_0), \nu(t_{00})\big]$$

with

$$\lim_{\varepsilon\to 0}\big[\gamma_0(\nu(t_0)) - \gamma(\nu(t_{00}))\big] = 0,$$

therefore

$$\int\limits_{\gamma_0(\nu(t_0))}^{\gamma(\nu(t_{00}))} Y(\xi;t) f_{0y}\big[\Delta x(\tau(\xi)) - \Delta x(\tau_0)(\xi)\big]\,d\xi = o(\varepsilon\delta\mu).$$

Continuing this process analogously for $a_{33}[t]$, we get

$$\alpha_{33}[t] = o(t;\varepsilon\delta\mu).$$

According to (2.32) and (2.34), for the above expressions we have

$$\big|\alpha_{31}[t]\big| \le \|Y\| O(\varepsilon\delta\mu)\theta_3(\varepsilon\delta\mu), \quad \big|\alpha_{32}[t]\big| \le \|Y\| O(\varepsilon\delta\mu)\rho_3(\varepsilon\delta\mu),$$

$$\alpha_{34}[t] = \widehat{\gamma}_1(t;\varepsilon\delta\mu) + \varepsilon \int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t) f_{0x_2}[\xi]\dot{x}_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi,$$

where

$$\theta_3(\varepsilon\delta\mu) = \int\limits_{\gamma_0(t_{00})}^{t_{10}+\delta_2} \Big| f_{0x_1}(\xi, x_0(\xi) + s\Delta x(\xi), x_0(\tau_0(\xi))) + \\ + s\Big(x_0(\tau(\xi)) - x_0(\tau_0(\xi)) + \Delta x(\tau(\xi))\Big) - \\ - f_{0x_1}\big(\xi, x_0(\xi), x_0(\xi)\big)\Big|\,d\xi,$$

$$\rho_3(\varepsilon\delta\mu) = \int\limits_{\gamma_0(t_{00})}^{t_{10}+\delta_2} \Big| f_{0x_2}(\xi, x_0(\xi) + s\Delta x(\xi), x_0(\tau_0(\xi))) + \\ + s\Big(x_0(\tau(\xi)) - x_0(\tau_0(\xi)) + \Delta x(\tau(\xi))\Big) - \\ - f_{0x_2}\big(\xi, x_0(\xi), x_0(\xi)\big)\Big|\,d\xi,$$

$$\widehat{\gamma}_1(t;\varepsilon\delta\mu) = \int\limits_{\gamma_0(t_{00})}^{t_{10}+\delta_2} Y(\xi;t) f_{0x_2}[\xi]\widehat{\gamma}(\xi;\varepsilon\delta\mu)\,d\xi.$$

Obviously,

$$\left|\frac{\widehat{\gamma}(t;\varepsilon\delta\mu)}{\varepsilon}\right| \le \|Y\| \int\limits_{\gamma_0(t_{00}}^{t_{10}+\delta_2} \left|f_{0x_2}[\xi]\right| \left|\frac{\widehat{\gamma}(\xi;\varepsilon\delta\mu)}{\varepsilon}\right| d\xi.$$

By the Lebesgue theorem on the passage under the integral sign, we have

$$\lim_{\varepsilon\to 0} \theta_3(\varepsilon\delta\mu) = \lim_{\varepsilon\to 0} \rho_{\varepsilon\delta\mu} = 0$$

and

$$\lim_{\varepsilon\to 0} \left|\frac{\widehat{\gamma}(\xi;\varepsilon\delta\mu)}{\varepsilon}\right| = 0$$

uniformly for $(t,\delta\mu) \in [\gamma_0(t_{00}), t_{10}+\delta_2]$ (see (2.33)).

Thus,

$$\alpha_{3i}[t] = o(t;\varepsilon\delta\mu), \quad i = 1,2,$$

$$\alpha_{34}[t] = \varepsilon \int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t) f_{0x_2}[\xi]\dot{x}_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi + o(t;\varepsilon\delta\mu).$$

Consequently,

$$\alpha_3[t] = \varepsilon \int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t) f_{0x_2}[\xi]\dot{x}_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi + o(t;\varepsilon\delta\mu). \tag{2.36}$$

On the basis of (2.31), (2.32) and (2.36),

$$\begin{aligned} R_1[t;t_{00}] = \varepsilon &\int\limits_{t_{00}}^{\gamma_0(t_{00})} Y(\xi;t) f_{0x_2}[\xi]\dot{\varphi}_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi + \\ &+ \varepsilon Y(\gamma_0(t_{00})-;t) f_{01}^{-}\dot{\gamma}_0(t_{00})\big(\delta\tau(\gamma_0(t_{00})) - \delta t_0\big) - \\ &- \int\limits_{t_0}^{t_{00}} Y(\gamma_0(\xi);t) f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\,d\xi \times \\ &\times \varepsilon \int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t) f_{0x_2}[\xi]\dot{x}_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi + o(t;\varepsilon\delta\mu). \end{aligned} \tag{2.37}$$

Finally, let us estimate $R_2[t;t_{00}]$. We have

$$\left|R_2[t;t_{00}]\right| \le \varepsilon\alpha\|Y\| \sum_{i=1}^{k} \beta_i(\varepsilon\delta\mu),$$

where

$$\beta_i(\varepsilon\delta\mu) =$$

$$= \int_{t_{00}}^{t_{10}+\delta_2} L_{\delta f_i,K_1}(\xi)\Big[\,\big|\Delta x(\xi)\big| + \big|x_0(\tau(\xi)) - x_(\tau_0(\xi))\big| + \big|\Delta x(\tau(\xi))\big|\,\Big]\,d\xi.$$

It is clear that

$$\beta_i(\varepsilon\delta\mu) \le$$

$$\le \int_{t_{00}}^{\gamma(t_0)} L_{\delta f_i,K_1}(\xi)\Big[O(\varepsilon\delta\mu) + \big|\varphi_0(\tau(\xi)) - \varphi_0(\tau_0(\xi))\big| + \varepsilon\big|\varphi(\tau(\xi))\big|\,\Big]\,d\xi +$$

$$+ \int_{\gamma(t_0)}^{\gamma_0(t_{00})} L_{\delta f_i,K_1}(\xi)\Big[O(\varepsilon\delta\mu) + \big|x_0(\tau(\xi)) - x_0(\tau_0(\xi))\big| + \big|\Delta x(\tau(\xi))\big|\,\Big]\,d\xi +$$

$$+ \int_{\gamma_0(t_{00})}^{t_{10}+\delta_2} L_{\delta f_i,K_1}(\xi)\Big[O(\varepsilon\delta\mu) + \big|x_0(\tau(\xi)) - x_0(\tau_0(\xi))\big| + O(\varepsilon\delta\mu)\Big]\,d\xi.$$

Obviously,

$$\lim_{\varepsilon\to 0}\beta(\varepsilon\delta\mu) = 0.$$

Thus,

$$R_2[t;t_{00}] = o(t;\varepsilon\delta\mu) \tag{2.38}$$

From (2.22), by virtue of (2.23)–(2.25), (2.37) and (2.38), we obtain (2.1), where $\delta x(t;\delta\mu)$ has the form (2.2). □

2.2. **Proof of Theorem 2.2.** Let $r_1 = t_{00}$ and $r_2 = t_{10}$ in Lemma 2.3. Then

$$x_0(t) = \begin{cases} \varphi_0(t), & t \in [\widehat{\tau}, t_{00}), \\ y_0(t), & t \in [t_{00}, t_{10}], \end{cases}$$

and for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_1) \times \Im_2^+$,

$$x(t;\mu_0+\varepsilon\delta\mu) = \begin{cases} \varphi(t) = \varphi_0(t) + \varepsilon\delta\varphi(t), & t \in [\widehat{\tau}, t_0), \\ y(t;\mu_0+\varepsilon\delta\mu), & t \in [t_0, t_{10}+\delta_1]. \end{cases}$$

We note that $\delta\mu \in \Im_2^+$, i.e. $t_0 > t_{00}$, therefore

$$\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t), & t \in [\widehat{\tau}, t_{00}), \\ \varphi(t) - x_0(t), & t \in [t_{00}, t_0), \\ \Delta y(t), & t \in [t_0, t_{10}+\delta_1], \end{cases}$$

$$\dot{\Delta} x(t) = \begin{cases} \varepsilon\delta v(t), & t \in [\widehat{\tau}, t_{00}), \\ v_0(t) + \varepsilon\delta v(t) - \dot{x}_0(t), & t \in [t_{00}, t_0), \\ \dot{\Delta} y(t), & t \in [t_0, t_{10}+\delta_1]. \end{cases}$$

By Lemma 2.3, we have

$$\big|\Delta x(t)\big| \le O(\varepsilon\delta\mu),\ \ \forall\,(t,\varepsilon,\delta\mu)\in[t_0,t_{10}+\delta_1]\times(0,\varepsilon_1)\times\Im_2^+, \tag{2.39}$$

$$\Delta x(t_0)=\varepsilon\Big\{\delta x_0-\big[A(t_{00})v_0(\sigma(t_{00}))+f_0^+\big]\delta t_0\Big\}+o(\varepsilon\delta\mu). \tag{2.40}$$

The function $\Delta x(t)$ satisfies the equation (2.21) on the interval $[t_0,t_{10}+\delta_1;]$ therefore, by using the Cauchy formula, we can represent it in the form

$$\Delta x(t)=\Psi(t_0;t)\Delta x(t_0)+\varepsilon\int\limits_{t_0}^{t}Y(\xi;t)\delta f[\xi]\,d\xi+\sum_{i=-1}^{2}R_i[t;t_0], \tag{2.41}$$
$$t\in[t_0,t_{10}+\delta_1].$$

Let $\delta_2\in(0,\delta_2)$ be so small that $\gamma_0(t_{00})<t_{10}-\delta_2$. The matrix function is continuous on $[t_{00},\gamma_0(t_{00})]\times[t_{10}-\delta_2,t_{10}+\delta_2]$ (see Theorem 1.13), therefore

$$\Phi(t_0;t)\Delta x(t_0)=$$
$$=\varepsilon\Phi(t_{00};t)\Big\{\delta x_0-\Big[A(t_{00})v_0(\sigma(t_{00}))+f_0^+\Big]\delta t_0\Big\}+o(t;\delta\mu) \tag{2.42}$$

(see (2.40)).

Let us now transform $R_{-1}[t;t_0]$. We have

$$R_{-1}[t;t_0]=\varepsilon\int\limits_{\sigma(t_0)}^{t_{00}}Y(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi)\delta v(\xi)\,d\xi+$$
$$+\int\limits_{t_{00}}^{t_0}Y(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi)\dot\Delta x(\xi)\,d\xi=$$
$$=\varepsilon\int\limits_{\sigma(t_{00})}^{t_{00}}Y(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi)\delta v(\xi)\,d\xi+o(t;\varepsilon\delta\mu)+$$
$$+\int\limits_{t_{00}}^{t_0}Y(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi)\times$$
$$\times\Big[A(\xi)\big(v_0(\sigma(\xi))+\varepsilon\delta v(\sigma(\xi))\big)+f_0\big(\xi,x_0(\xi),x_0(\tau_0(\xi))\big)\Big]\,d\xi=$$
$$=\varepsilon\int\limits_{\sigma(t_{00})}^{t_{00}}Y(\nu(\xi);t)A(\nu(\xi))\dot\nu(\xi)\delta v(\xi)\,d\xi+\varepsilon\big[Y(t_{00}+;t)-\Phi(t_{00};t)\big]\times$$
$$\times\Big[v_0(t_{00})-A(t_{00})v_0(\sigma(t_{00}))-f_0^+\Big]\delta t_0+o(\varepsilon\delta\mu). \tag{2.43}$$

For $R_0[t;t_0]$, we have

$$R_0[t;t_0] = \varepsilon \int\limits_{\tau_0(t_0)}^{t_{00}} Y(\gamma_0(\xi);t) f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\delta\varphi(\xi)\,d\xi + \\ + \int\limits_{t_{00}}^{t_0} Y(\gamma_0(\xi);t) f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\,d\xi = \\ = \varepsilon \int\limits_{\tau_0(t_{00})}^{t_{00}} Y(\gamma_0(\xi);t) f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\delta\varphi(\xi)\,d\xi + o(t;\varepsilon\delta\mu) + \\ + \int\limits_{t_{00}}^{t_0} Y(\gamma_0(\xi);t) f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\,d\xi. \tag{2.44}$$

In a similar way, with inessential changes one can prove

$$R_1[t;t_0] = \varepsilon \int\limits_{t_{00}}^{\gamma_0(t_{00})} Y(\xi;t) f_{0x_2}[\xi]\dot{\varphi}_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi + \\ + \varepsilon Y(\gamma_0(t_{00})-;t) f_{01}^{+}\dot{\gamma}_0(t_{00})\big(\delta\tau(\gamma_0(t_{00})) - \delta t_0\big) - \\ - \int\limits_{t_{00}}^{t_0} Y(\gamma_0(\xi);t) f_{0x_2}[\gamma_0(\xi)]\dot{\gamma}_0(\xi)\Delta x(\xi)\,d\xi \times \\ \times \varepsilon \int\limits_{\gamma_0(t_{00})}^{t} Y(\xi;t) f_{0x_2}[\xi]\dot{x}_0(\tau_0(\xi))\delta\tau(\xi)\,d\xi + o(t;\varepsilon\delta\mu) \tag{2.45}$$

and

$$R_2(t;t_0) = o(t;\varepsilon\delta\mu). \tag{2.46}$$

Obviously,

$$\varepsilon \int\limits_{t_0}^{t} Y(\xi;t)\delta f[\xi]\,d\xi = \varepsilon \int\limits_{t_{00}}^{t} Y(\xi;t)\delta f[\xi]\,d\xi + o(t;\varepsilon\delta\mu). \tag{2.47}$$

Bearing in mind (2.42)–(2.47), from (2.41), we obtain (2.1) and the variation formula.

In the conclusion we note that the Theorems 2.3–2.6 can be proved by the scheme using in the proof of Theorems 2.1 and 2.2.

3. Initial Data Optimization Problem

3.1. **The Necessary conditions of optimality.** Let $t_{01}, t_{02}, t_1 \in (a,b)$ be the given numbers with $t_{01} < t_{02} < t_1$ and let $X_0 \subset O$, $K_0 \subset O$, $K_1 \subset O$,

$U\subset U_0$ be compact and convex sets. Then

$$D_2=\{\tau\in D:\ e_2>\dot\tau(t)>e_1>0\},$$
$$\Phi_1=\{\varphi\in E_\varphi:\ \varphi(t)\in K_0,\ t\in I_1\},\quad \Phi_2=\{v\in E_v:\ v(t)\in K_1,\ t\in I_1\},$$
$$\Omega_1=\{u\in\Omega:\ u(t)\in U,\ t\in I\}.$$

Consider the initial data optimization problem

$$\dot x(t)=A(t)\dot x(\sigma(t))+f\big(t,x(t),x(\tau(t)),u(t)\big),\ \ t\in[t_0,t_1],$$
$$x(t)=\varphi(t),\ \ \dot x(t)=v(t),\ \ t\in[\widehat\tau,t_0),\ \ x(t_0)=x_0,$$
$$q^i\big(t_0,x_0,x(t_1)\big)=0,\ \ i=1,\dots,l,$$
$$q^0\big(t_0,x_0,x(t_1)\big)\longrightarrow\min,$$

where

$$w=(t_0,\tau,x_0,\varphi,v,u)\in W_1=[t_{01},t_{02})\times D_2\times X_0\times\Phi_1\times\Phi_2\times\Omega_1$$

and $x(t)=x(t;w)$; $q^i(t_0,x_0,x)$, $i=0,\dots,l$, are the continuously differentiable functions on the set $I\times O^2$.

Definition 3.1. The initial data $w=(t_0,\tau,x_0,\varphi,v,u)\in W_1$ are said to be admissible, if the corresponding solution $x(t)=x(t;w)$ is defined on the interval $[\widehat\tau,t_1]$ and the conditions hold

$$q^i\big(t_0,x_0,x(t_1)\big)=0,\ \ i=1,\dots,l,$$

hold.

The set of admissible initial data will be denoted by W_{10}.

Definition 3.2. The initial data $w_0=(t_{00},\tau_0,x_{00},\varphi_0,v_0,u_0)\in W_{10}$ are said to be optimal, if for any $w=(t_0,\tau,x_0,\varphi,v,u)\in W_{10}$ we have

$$q^0\big(t_{00},x_{00},x_0(t_1)\big)\le q^0\big(t_0,x_0,x(t_1)\big),$$

where $x_0(t)=x(t;w_0),x(t)=x(t;w)$.

The initial data optimization problem consists in finding optimal initial data w_0.

Theorem 3.1. *Let $w_0\in W_{10}$ be optimal initial data and $t_{00}\in[t_{01},t_{02})$. Let the following conditions hold:*

(a) $\gamma_0(t_{00})<t_1$;
(b) *the functions $v_0(\sigma(t))$ and $v_0(t)$ are continuous at the point t_{00}; the function $\varphi_0(t)$ is absolutely continuous and the function $\dot\varphi_0(t)$ is bounded;*
(c) *for each compact sets $K\subset O$ and $U\subset U_0$ there exists a number $m_{K,U}>0$ such that*

$$|f_0(z)|\le m_{K,U},\ \ \forall\, z=(t,x_1,x_2,u)\in I\times K^2\times U;$$

(d) *there exist the limits*

$$\lim_{z\to z_0} f_0(z) = f_0^+, \;\; z\in[t_{00},t_{02})\times O^2,$$

$$\lim_{(z_1,z_2)\to(z_{10},z_{20})} \big[f_0(z_1)-f_0(z_2)\big] = f_{01}^+, \;\; z_i\in[\gamma_0(t_{00}),t_1)\times O^2, \;\; i=1,2,$$

where

$$z_0=\big(t_{00},x_{00},\varphi_0(\tau_0(t_{00}))\big), \quad z_{10}=\big(\gamma_0(t_{00}),x_0(\gamma_0(t_{00})),x_{00}\big),$$
$$z_{20}=\big(\gamma_0(t_{00}),x_0(\gamma_0(t_{00})),\varphi_0(t_{00})\big), \quad f_0(z)=f(z,u_0(t)).$$

Then there exist a vector $\pi=(\pi_0,\dots,\pi_l)\neq 0$, $\pi_0\le 0$, *and a solution* $(\chi(t),\psi(t))$ *of the system*

$$\begin{cases} \dot\chi(t)=-\psi(t)f_{0x_1}[t]-\psi(\gamma_0(t))f_{0x_2}[\gamma_0(t)]\dot\gamma_0(t), & \\ \psi(t)=\chi(t)+\psi(\nu(t))A(\nu(t))\dot\nu(t), & t\in[t_{00},t_1], \\ \chi(t)=\psi(t)=0, & t>t_1 \end{cases} \tag{3.1}$$

such that the conditions listed below hold:

3.1. *the condition for* $\chi(t)$ *and* $\psi(t)$

$$\chi(t_1)=\psi(t_1)=\pi Q_{0x},$$

where

$$Q=(q^0,\dots,q^l)^T, \quad Q_{0x}=Q_x\big(t_{00},x_{00},x_0(t_1)\big);$$

3.2. *the condition for the optimal initial moment* t_{00}

$$\pi Q_{0t_0}+\big(\psi(t_{00}+)-\chi(t_{00})\big)v_0(t_{00})-$$
$$-\psi(t_{00}+)\big(A(t_{00})v_0(\sigma(t_{00}))+f_0^+\big)-\psi(\gamma_0(t_{00}+))f_{01}^+\dot\gamma(t_{00})\le 0;$$

3.3. *the condition for the optimal initial vector* x_{00}

$$\big(\pi Q_{0x_0}+\psi(t_{00})\big)x_{00}\ge\big(\pi Q_{0x_0}+\psi(t_{00})\big)x_0, \;\; \forall\, x_0\in X_0;$$

3.4. *the condition for the optimal delay function* $\tau_0(t)$

$$\psi\big(\gamma_0(t_{00}+)\big)f_{01}^+t_{00}+\int\limits_{t_{00}}^{\gamma_0(t_{00})}\psi(t)f_{0x_2}[t]\dot\varphi_0(\tau_0(t))\tau_0(t)\,dt+$$
$$+\int\limits_{\gamma_0(t_{00})}^{t_1}\psi(t)f_{0x_2}[t]\dot x_0(\tau_0(t))\tau_0(t)\,dt\ge$$
$$\ge\psi(\gamma_0(t_{00}+))f_{01}^+\tau(\gamma_0(t_{00}))+\int\limits_{t_{00}}^{\gamma_0(t_{00})}\psi(t)f_{0x_2}[t]\dot\varphi_0(\tau_0(t))\tau(t)\,dt+$$
$$+\int\limits_{\gamma_0(t_{00})}^{t_1}\psi(t)f_{0x_2}[t]\dot x_0(\tau_0(t))\tau(t)\,dt, \;\; \forall\,\tau\in D_{21}=\big\{\tau\in D_2:\ \tau(\gamma_0(t_{00}))<t_{00}\big\};$$

3.5. *the condition for the optimal initial function* $\varphi_0(t)$

$$\int\limits_{\tau_0(t_{00})}^{t_{00}} \psi(\gamma_0(t)) f_{0x_2}[\gamma_0(t)]\dot{\gamma}_0(t)\varphi_0(t)\,dt \geq$$
$$\geq \int\limits_{\tau_0(t_{00})}^{t_{00}} \psi(\gamma_0(t)) f_{0x_2}[\gamma_0(t)]\dot{\gamma}_0(t)\varphi(t)\,dt, \ \ \forall\,\varphi\in\Phi_1;$$

3.6. *the condition for the optimal initial function* $v_0(t)$

$$\int\limits_{\sigma(t_{00})}^{t_{00}} \psi(\nu(t))A(\nu(t))\dot{\nu}(t)v_0(t)\,dt \geq$$
$$\geq \int\limits_{\sigma(t_{00})}^{t_{00}} \psi(\nu(t))A(\nu(t))\dot{\nu}(t)v(t)\,dt, \ \ \forall\,v\in\Phi_2;$$

3.7. *the condition for the optimal control function* $u_0(t)$

$$\int\limits_{t_0}^{t_1} \psi(t)f_{0u}[t]u_0(t)\,dt \geq \int\limits_{t_0}^{t_1} \psi(t)f_{0u}[t]u(t)\,dt, \ \ \forall\,u\in\Omega_1.$$

Here

$$f_{0x}[t]=f_x\big(t,x_0(t),x_0(\tau_0(t)),u_0(t)\big),$$

Theorem 3.2. *Let* $w_0\in W_{10}$ *be optimal initial data and* $t_{00}\in(t_{01},t_{02})$. *Let the conditions* (a), (b), (c) *hold. Moreover, there exist the limits*

$$\lim_{z\to z_0} f_0(z)=f_0^-, \ \ z\in(t_{01},t_{00}]\times O^2,$$
$$\lim_{(z_1,z_2)\to(z_{10},z_{20})}\big[f_0(z_1)-f_0(z_2)\big]=f_{01}^-, \ \ z_i\in(t_{00},\gamma_0(t_{00})]\times O^2, \ \ i=1,2,$$

Then there exist a vector $\pi=(\pi_0,\dots,\pi_l)\neq 0$, $\pi_0\leq 0$, *and a solution* $(\chi(t),\psi(t))$ *of the system* (3.1) *such that the conditions* 3.1, 3.3 *and* 3.5–3.7 *are fulfilled. Moreover,*

$$\pi Q_{0t_0}+\big(\psi(t_{00}-)-\chi(t_{00})\big)v_0(t_{00})-\psi(t_{00}-)\big(A(t_{00})v_0(\sigma(t_{00})+f_0^-\big)-$$
$$-\psi(\gamma_0(t_{00}-))f_{01}^-\dot{\gamma}(t_{00})\geq 0,$$

$$\psi(\gamma_0(t_{00}-))f_{01}^-t_{00}+\int\limits_{t_{00}}^{\gamma_0(t_{00})}\psi(t)f_{0x_2}[t]\dot{\varphi}_0(\tau_0(t))\tau_0(t)\,dt+$$
$$+\int\limits_{\gamma_0(t_{00})}^{t_1}\psi(t)f_{0x_2}[t]\dot{x}_0(\tau_0(t))\tau_0(t)\,dt\geq$$

$$\geq \psi(\gamma_0(t_{00}-))f_{01}^{-}\tau(\gamma_0(t_{00})) + \int\limits_{t_{00}}^{\gamma_0(t_{00})} \psi(t)f_{0x_2}[t]\dot{\varphi}_0(\tau_0(t))\tau(t)\,dt +$$

$$+ \int\limits_{\gamma_0(t_{00})}^{t_1} \psi(t)f_{0x_2}[t]\dot{x}_0(\tau_0(t))\tau(t)\,dt, \ \ \forall\,\tau\in D_{22}=\big\{\tau\in D_2:\ \tau(\gamma_0(t_{00}))>t_{00}\big\}.$$

Theorem 3.3. *Let $w_0 \in W_{10}$ be optimal initial data and $t_{00} \in (t_{01}, t_{02})$. Let the conditions of Theorems* 3.1 *and* 3.2 *hold. Moreover,*

$$f_0^{-} = f_0^{+} := \widehat{f}_0, \quad f_{01}^{-} = f_{01}^{+} := \widehat{f}_{01}$$

and

$$t_{00}, \gamma_0(t_{00}) \notin \big\{\sigma(t_1), \sigma^2(t_1), \dots\big\}.$$

Then there exist a vector $\pi = (\pi_0, \dots, \pi_l) \neq 0$, $\pi_0 \leq 0$, and a solution $(\chi(t), \psi(t))$ of the system (3.1) *such that the conditions* 3.1, 3.3 *and* 3.5–3.7 *are fulfilled, Moreover,*

$$\pi Q_{0t_0} + \big(\psi(t_{00}) - \chi(t_{00})\big)v_0(t_{00}) - \psi(t_{00})\big(A(t_{00})v_0(\sigma(t_{00})) + \widehat{f}_0\big) -$$
$$-\psi(\gamma_0(t_{00}))\widehat{f}_{01}\dot{\gamma}(t_{00}) = 0,$$

$$\psi(\gamma_0(t_{00}))\widehat{f}_{01}t_{00} + \int\limits_{t_{00}}^{\gamma_0(t_{00})} \psi(t)f_{0x_2}[t]\dot{\varphi}_0(\tau_0(t))\tau_0(t)\,dt +$$

$$+ \int\limits_{\gamma_0(t_{00})}^{t_1} \psi(t)f_{0x_2}[t]\dot{x}_0(\tau_0(t))\tau_0(t)\,dt \geq$$

$$\geq \psi(\gamma_0(t_{00}))\widehat{f}_{01}\tau(\gamma_0(t_{00})) + \int\limits_{t_{00}}^{\gamma_0(t_{00})} \psi(t)f_{0x_2}[t]\dot{\varphi}_0(\tau_0(t))\tau(t)\,dt +$$

$$+ \int\limits_{\gamma_0(t_{00})}^{t_1} \psi(t)f_{0x_2}[t]\dot{x}_0(\tau_0(t))\tau(t)\,dt, \ \ \forall\,\tau \in D_2.$$

3.2. **Proof of Theorem 3.1.** Denote by G_0 the set of such elements $w \in W_1^{+} = [t_{00}, t_{02}) \times D_{21} \times X_0 \times \Phi_1 \times \Phi_2 \times \Omega_1$ to which there corresponds the solution $x(t; w)$, $t \in [\widehat{\tau}, t_1]$. On the basis of Theorem 3.3, there exist $\widehat{V}(w_0; \delta_0)$ such that

$$\widehat{V}_0(w_0; \delta_0) = \widehat{V}(w_0; \delta_0) \cap W_1^{+} \subset G_0.$$

On the set $\widehat{V}_{01}(z_0; \delta_0) = [0, \delta_0) \times \widehat{V}_0(w_0; \delta_0)$, where $z_0 = (0, w_0))$, we define the mapping

$$P : \widehat{V}_{01}(z_0; \delta_0) \longrightarrow R_p^{1+l} \tag{3.2}$$

by the formula

$$
\begin{gathered}
P(z)=Q(t_0,x_0,x(t_1;w))+(s,0\dots,0)^T=\\
=\Big(q^0\big(t_1,x_0,x(t_1;w)\big)+s,q^1\big(t_1,x_0,x(t_1;w)\big),\dots,q^l\big(t_1,x_0,x(t_1;w)\big)\Big)^T,\\
z=(s,w).
\end{gathered}
$$

Lemma 3.1. *The mapping P is differentiable at the point $z_0=(0,w_0)$ and*

$$
\begin{gathered}
dP_{z_0}(\delta z)=\Big\{Q_{0t_0}+Q_{0x}\big[Y(t_{00}+;t_1)-\Psi(t_{00};t_1)\big]v_0(t_{00})-\\
-Q_{0x}Y(t_{00}+;t_1)\Big[A(t_{00})v_0(\sigma(t_{00}))+f_0^+\Big]-\\
-Q_{0x}Y(\gamma_0(t_{00})+;t_1)f_{01}^+\dot\gamma_0(t_{00})\Big\}\delta t_0+\Big\{Q_{0x_0}+Q_{0x}\Psi(t_{00};t_1)\Big\}\delta x_0+\\
+Q_{0x}\Big\{Y(\gamma_0(t_{00})+;t_1)f_{01}^+\dot\gamma_0(t_{00})\delta\tau(\gamma_0(t_{00}))+\\
+\int\limits_{t_{00}}^{\gamma_0(t_{00})}Y(t;t_1)f_{0x_2}[t]\dot\varphi_0(\tau(t))\delta\tau(t)\,dt+\int\limits_{\gamma_0(t_{00})}^{t_{00}}Y(t;t_1)f_{0x_2}[t]\dot x_0(\tau(t))\delta\tau(t)\,dt\Big\}+\\
+Q_{0x}\Big\{\int\limits_{\tau_0(t_{00})}^{t_{00}}Y(\gamma_0(t);t_1)f_{0x_2}[\gamma_0(t)]\dot\gamma_0(t)\delta\varphi(t)\,dt+\\
+\int\limits_{\sigma_0(t_{00})}^{t_{00}}Y(\nu(t);t_1)f_{0x_2}[\nu(t)]\dot\nu_0(t)\delta v(t)\,dt\Big\}+\\
+Q_{0x}\int\limits_{t_{00}}^{t_1}Y(t;t_1)f_{0u}[t]\delta u(t)\,dt+(\delta s,0,\dots,0). \qquad (3.3)
\end{gathered}
$$

Proof. Obviously, for arbitrary $(\varepsilon,\delta z)\in(0,\delta_0)\times[\widehat V_{01}(z_0;\delta_0)-z_0]$, we have

$$z_0+\varepsilon\delta z\in\widehat V_{01}(z_0;\delta_0).$$

Now we transform the difference

$$
\begin{gathered}
P(z_0+\varepsilon\delta z)-P(z_0)=\\
=Q\Big(t_{00}+\varepsilon\delta t_0,x_{00}+\varepsilon\delta x_0,x(t_1;w_0+\varepsilon\delta w)\Big)-Q_0+\varepsilon(\delta s,0,\dots,0)^T.
\end{gathered}
$$

It is easy to see that

$$Q(t_{00}+\varepsilon\delta t_{00},x_{00}+\varepsilon\delta x_0,x(t_1;w_0+\varepsilon\delta w))-Q_0$$
$$=\int_0^1 \frac{d}{d\xi}\,Q\Big(t_0+\varepsilon\xi\delta t_0,x_{00}+\varepsilon\xi\delta x_0,x_0(t_1)+\xi\big(x(t_1;w_0+\varepsilon\delta w)-x_0(t_1)\big)\Big)\,d\xi=$$
$$=\varepsilon\Big[Q_{0t_0}\delta t_0+Q_{0x_0}\delta x_0+Q_{0x}\delta x(t_1;\delta w)\Big]+\alpha(\varepsilon\delta w),$$

where

$$\alpha(\varepsilon\delta w)=\varepsilon\int_0^1\big[Q_{t_0}(\varepsilon;\xi)-Q_{0t_0}\big]\delta t_0\,d\xi+\varepsilon\int_0^1\big[Q_{x_0}(\varepsilon;\xi)-Q_{0x_0}\big]\delta x_0\,d\xi+$$
$$+\varepsilon\int_0^1\big[Q_x(\varepsilon;t)-Q_{0x}\big]\delta x(t_1;\delta w)\,d\xi+o(\varepsilon\delta w)\int_0^1 Q_{0x}(\varepsilon;\xi)\,d\xi,$$
$$Q_{t_0}(\varepsilon;\xi)=$$
$$=Q_{t_0}\Big(t_{00}+\varepsilon\xi\delta t_0,x_{00}+\varepsilon\xi\delta x_0,x_0(t_1)+\xi\big(x(t_1;w_0+\varepsilon\delta w)-x_0(t_1)\big)\Big).$$

Clearly, $\alpha(\varepsilon\delta w)=o(\varepsilon\delta w)$. Thus

$$P(z_0+\varepsilon\delta z)-P(z_0)=$$
$$=\varepsilon\Big[Q_{0t_0}\delta t_0+Q_{0x_0}\delta x_0+Q_{0x}\delta x(t_1;\delta w)+(\delta s,0,\dots,0)^\top\Big]+o(\varepsilon\delta w).$$

On the basis of Theorem 2.5, we have (3.3).

The set $\widehat{V}_{01}(z_0;\delta_0)$ is convex and the mapping (3.2) is continuous and differentiable. In a standard way we can prove the criticality of point z_0 with respect to the mapping (3.2), i.e. $P(z_0)\in\partial P(\widehat{V}_{01}(z_0;\delta_0))$ [10, 15]. These conditions guarantee fulfilment of the necessary condition of criticality [10, 15]. Thus, there exists the vector $\pi=(\pi_0,\dots,\pi_l)\neq 0$ such that the inequality

$$\pi dP_{z_0}(\delta z)\le 0,\quad \delta z\in \mathrm{Cone}\,\big(\widehat{V}_{01}(z_0;\delta_0)-z_0\big), \tag{3.4}$$

is valid, where $dP_{z_0}(\delta z)$ has the form (3.3).

Let us introduce the functions

$$\chi(t)=\pi Q_{0x}\Psi(t;t_1),\quad \psi(t)=\pi Q_{0x}Y(t;t_1). \tag{3.5}$$

It is clear that the functions $\chi(t)$ and $\psi(t)$ satisfy the system (3.1) and the conditions

$$\chi(t_1)=\psi(t_1)=\pi Q_{0x},\quad \chi(t_1)=\psi(t_1)=0,\quad t>t_1. \tag{3.6}$$

Taking into consideration (3.3))–(3.5) and (3.6), from (3.4) we have

$$\Big\{\pi Q_{0t_0} + \big[\psi(t_{00}+) - \chi(t_{00})\big] v_0(t_{00}) - \\ - \psi(t_{00}+)\big[A(t_{00})v_{00}(\sigma(t_{00})) + f_0^+\big] - \psi(\gamma_0(t_{00}+))f_{01}^+\dot{\gamma}_0(t_{00})\Big\}\delta t_0 + \\ + \big\{\pi Q_{0x_0} + \chi(t_{00})\big\}\delta x_0 + \psi(\gamma_0(t_{00})+)f_{01}^+\dot{\gamma}_0(t_{00})\delta\tau(\gamma_0(t_{00})) + \\ + \int\limits_{t_{00}}^{\gamma_0(t_{00})} \psi(t) f_{0x_2}[t]\dot{\varphi}_0(\tau(t))\delta\tau(t)\,dt + \int\limits_{\gamma_0(t_{00})}^{t_{00}} \psi(t) f_{0x_2}[t]\dot{x}_0(\tau(t))\delta\tau(t)\,dt + \\ + \int\limits_{\tau_0(t_{00})}^{t_{00}} \psi(\gamma_0(t)) f_{0x_2}[\gamma(t)]\dot{\gamma}_0(t)\delta\varphi(t)\,dt + \int\limits_{\sigma(t_{00})}^{t_{00}} \psi(\nu(t)) f_{0x_2}[\nu(t)]\dot{\nu}(t)\delta v(t)\,dt + \\ + \int\limits_{t_{00}}^{t_1} \psi(t) f_{0u}[t]\delta u(t)\,dt + \pi_0\delta s, \ \ \forall\,\delta z \in \mathrm{Cone}\,\big(\widehat{V}_{01}(z_0;\delta_0) - z_0\big). \tag{3.7}$$

The condition $\delta z \in Cone(\widehat{V}_{01}(z_0;\delta_0) - z_0)$ is equivalent to $\delta s \in [0,\infty)$, $\delta t_0 \in [0,\infty)$,

$$\delta x_0 \in \mathrm{Cone}\,\big(B(x_{00};\delta_0) \cap X_0 - x_{00}\big) \supset X_0 - x_{00}, \\ \delta\tau \in \mathrm{Cone}\,\big(V(\tau_0 : \delta_0) \cap D_{21} - \tau_0\big) \supset D_{21} - \tau_0, \\ \delta\varphi \in \mathrm{Cone}\,\Big(V_1(\varphi_0;\delta_0) \cap \Phi_1 - \varphi_0\Big) \supset \Phi_1 - \varphi_0, \\ \delta v \in \mathrm{Cone}\,\big(V_2(v_0;\delta_0) \cap \Phi_2 - v_0\big) \supset \Phi_2 - v_0, \\ \delta u \in \mathrm{Cone}\,\big(V_3(u_0;\delta_0) \cap \Omega_1 - u_0\big) \supset \Omega_1 - u_0.$$

Let $\delta t_0 = 0, \delta\tau = 0, \delta x_0 = 0, \delta\varphi = \delta v = 0, \delta u = 0$, then from (3.7) we have $\pi\delta s \le 0$, $\forall\,\delta s \in [0,\infty)$, thus $\pi_0 \le 0$.

Let $\delta s = 0, \delta\tau = 0, \delta x_0 = 0, \delta\varphi = \delta v = 0, \delta u = 0$, then we have

$$\Big\{\pi Q_{0t_0} + \big[\psi(t_{00}+) - \chi(t_{00})\big] v_0(t_{00}) - \psi(t_{00}+)\big[A(t_{00})v_{00}(\sigma(t_{00})) + f_0^+\big] - \\ - \psi(\gamma_0(t_{00}+))f_{01}^+\dot{\gamma}_0(t_{00})\Big\}\delta t_0 \le 0, \ \ \forall\,\delta t_0 \in [0,\infty).$$

From this we obtain the condition for t_{00}.

If $\delta s = 0$, $\delta t_0 = 0$, $\delta\tau = 0$, $\delta\varphi = \delta v = 0$, $\delta u = 0$, then we obtain the condition for x_{00}. Let $\delta s = 0$, $\delta t_0 = 0$, $\delta x_0 = 0$, $\delta\varphi = \delta v = 0$, $\delta u = 0$, then we have the condition for the optimal delay function $\tau_0(t)$ (see 3.4). Let $\delta s = 0$, $\delta t_0 = 0$, $\delta\tau = 0$, $\delta x_0 = 0$, $\delta v = 0$, $\delta u = 0$, then from (3.7) follows the condition for the initial function $\varphi_0(t)$. If $\delta s = 0$, $\delta t_0 = 0$, $\delta\tau = 0$, $\delta x_0 = 0$, $\delta\varphi = 0$, $\delta u = 0$, then we obtain the condition for $v_0(t)$. Finally, we consider the case, where $\delta s = 0$, $\delta t_0 = 0$, $\delta\tau = 0$, $\delta x_0 = 0$, $\delta\varphi = 0$, $\delta v = 0$, then we have the condition for the optimal control $u_0(t)$. Theorem 3.1 is proved completely. □

In the conclusion we note that the Theorems 3.2 and 3.3 are proved analogously by using the corresponding variation formulas.

4. The Existence Theorem of Optimal Initial Data

4.1. **Formulation of the main result.** Let $t_{01}, t_{02}, t_1 \in (a, b)$ be the given numbers with $t_{01} < t_{02} < t_1$ and let $X_0 \subset O$, $K_0 \subset O$, $U \subset U_0$ be compact sets. Then Φ_{11} is the set of measurable initial functions $\varphi(t) \in K_0$, $t \in I_1$, $\Omega_2 = \{u \in \Omega : \ u(t) \in U, \ t \in I\}$.

Consider the initial data optimization problem

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f\big(t, x(t), x(\tau(t)), u(t)\big), \ \ t \in [t_0, t_1],$$
$$x(t) = \varphi(t), \ \ \dot{x}(t) = v(t), \ \ t \in [\widehat{\tau}, t_0), \ \ x(t_0) = x_0,$$
$$q^i\big(t_0, x_0, x(t_1)\big) = 0, \ \ i = 1, \dots, l,$$
$$J(w) = q^0\big(t_0, x_0, x(t_1)\big) \longrightarrow \min,$$

where

$$w = (t_0, \tau, x_0, \varphi, v, u) \in W_2 = [t_{01}, t_{02}] \times D_2 \times X_0 \times \Phi_{12} \times \Phi_2 \times \Omega_2$$

and $x(t) = x(t; w)$. The set of admissible elements we denote by W_{20}.

Theorem 4.1. *There exists an optimal element w_0 if the following conditions hold:*

4.1. $W_{20} \neq \varnothing$;

4.2. *there exists a compact set $K_2 \subset O$ such that for an arbitrary $w \in W_{20}$,*

$$x(t; w) \in K_2, t \in [\widehat{\tau}, t_1];$$

4.3. *the sets*

$$P(t, x_1) = \big\{f(t, x_1, x_2, u) : \ (x_2, u) \in K_0 \times U\big\}, \ \ (t, x_1) \in I \times O$$

and

$$P_1(t, x_1, x_2) = \big\{f(t, x_1, x_2, u) : \ u \in U\big\}, \ \ (t, x_1, x_2) \in I \times O^2$$

are convex.

Remark 4.1. Let K_0 and U be convex sets, and

$$f(t, x_1, x_2, u) = B(t, x_1)x_2 + C(t, x_1)u.$$

Then the condition 4.3 of Theorem 4.1 holds.

4.2. **Auxiliary assertions.** To each element $w = (t_0, \tau, x_0, \varphi, v, u) \in W_2$ we correspond the functional differential equation

$$\dot{q}(t) = A(t)h(t_0, v, \dot{q})(\sigma(t)) + f\big(t, q(t), h(t_0, \varphi, q)(\tau(t)), u(t)\big) \qquad (4.1)$$

with the initial condition

$$q(t_0) = x_0. \qquad (4.2)$$

Let $K_i \subset O$, $i = 3, 4$ be compact sets and let K_4 contain a certain neighborhood of the set K_3.

Theorem 4.2. *Let $q_i(t) \in K_3, i = 1, 2, \dots,$ be a solution corresponding to the element $w_i = (t_{0i}, \tau_i, x_{0i}, \varphi_i, v_i, u_i) \in W_2$, $i = 1, 2, \dots,$ defined on the interval $[t_{0i}, t_1]$. Moreover,*

$$\lim_{i\to\infty} t_{0i} = t_{00}, \quad \lim_{i\to\infty} \|\tau_i - \tau_0\|_{I_2} = 0, \quad \lim_{i\to\infty} x_{0i} = x_{00}. \tag{4.3}$$

Then there exist numbers $\delta > 0$ and $M > 0$ such that for a sufficiently large i_0 the solution $\psi_i(t)$, $i \geq i_0$, corresponding to the element w_i, $i \geq i_0$, is defined on the interval $[t_{00} - \delta, t_1] \subset I$. Moreover,

$$\psi_i(t) \in K_4, \ \ \big|\dot{\psi}_i(t)\big| \leq M, \ \ t \in [t_{00} - \delta, t_1]$$

and

$$\psi_i(t) = q_i(t), \ \ t \in [t_{0i}, t_1] \subset [t_{00} - \delta, t_1].$$

Proof. Let $\varepsilon > 0$ be so small that a closed ε-neighborhood of the set K_3 : $K_3(\varepsilon) = \{x \in O : \ \exists \widehat{x} \in K_3, \ |x - \widehat{x}| \leq \varepsilon\}$ is contained $\operatorname{int} K_4$. There exist a compact set $Q \subset \mathbb{R}^n \times \mathbb{R}^n$ and a continuously differentiable function $\chi : \mathbb{R}^n \times \mathbb{R}^n \to [0, 1]$ such that

$$\chi(x_1, x_2) = \begin{cases} 1, & (x_1, x_2) \in Q, \\ 0, & (x_1, x_2) \notin K_4 \times [K_0 \cup K_4] \end{cases} \tag{4.4}$$

and

$$K_3(\varepsilon) \times [K_0 \cup K_3(\varepsilon)] \subset Q \subset K_4 \times [K_0 \cup K_4].$$

For each $i = 1, 2, \dots,$ the differential equation

$$\dot{\psi}(t) = A(t)h(t_{0i}, v_i, \dot{\psi})(\sigma(t)) + \phi\big(t, \psi(t), h(t_{0i}, \varphi_i, \psi)(\tau_i(t)), u_i(t)\big),$$

where

$$\phi(t, x_1, x_2, u) = \chi(x_1, x_2) f(t, x_1, x_2, u),$$

with the initial condition

$$\psi(t_{0i}) = x_{0i},$$

has the solution $\psi_i(t)$ defined on the interval I (see Theorem 1.15). Since

$$\big(q_i(t), h(t_{0i}, \varphi_i, q_i)(\tau_i(t))\big) \in K_3 \times [K_0 \cup K_3] \subset Q, t \in [t_{0i}, t_1],$$

therefore

$$\chi\big(q_i(t), h(t_{0i}, \varphi_i, q_i)(\tau_i(t))\big) = 1, \ \ t \in [t_{0i}, t_{1i}],$$

(see (4.6)), i.e.

$$\phi\big(t, q_i(t), h(t_{0i}, \varphi_i, q_i)(\tau_i(t)), u_i(t)\big) = \\ = f\big(t, q_i(t), h(t_{0i}, \varphi_i, q_i)(\tau_i(t)), u_i(t)\big), \ \ t \in [t_{0i}, t_1].$$

By the uniqueness,

$$\psi_i(t) = q_i(t), \ \ t \in [t_{0i}, t_1]. \tag{4.5}$$

There exists a number $M > 0$ such that

$$\big|\dot{\psi}_i(t)\big| \leq M, \ \ t \in I, \ \ i = 1, 2, \dots . \tag{4.6}$$

Indeed, first of all we note that

$$\Big|\phi\big(t,\psi_i(t),h(t_{0i},\varphi_i,\psi_i)(\tau_i(t)),u_i(t)\big)\Big| \leq$$
$$\leq \sup\Big\{\big|\phi(t,x_1,x_2,u)\big| : \ t\in I,\ x_1\in K_4,\ x_2\in K_4\cup K_0,\ u\in U\Big\}=N_1,$$
$$i=1,2,\dots .$$

It is not difficult to see that if $t\in[a,\nu(t_{0i}))$, then

$$\big|\dot\psi_i(t)\big| = \Big|A(t)v_i(\sigma_i(t))+\phi\big(t,\psi_i(t),h(t_{0i},\varphi_i,\psi_i)(\tau_i(t)),u_i(t)\big)\Big| \leq$$
$$\leq \|A\|N_2+N_1=M_1,$$

where

$$N_2=\sup\big\{|x| : \ x\in K_1\big\}.$$

Let $t\in[\sigma(t_{0i}),\sigma^2(t_{0i}))$, then

$$\big|\dot\psi_i(t)\big|\leq\|A\|\,\big|\dot\psi_i(\sigma(t))\big|+N_1\leq\|A\|M_1+N_1=M_2.$$

Continuing this process, we obtain (4.6). Further, there exists a number $\delta_0>0$ such that for an arbitrary $i=1,2,\dots$, $[t_{0i}-\delta_0,t_1]\subset I$, and the following conditions hold:

$$\big|\psi_i(t_{0i})-\psi_i(t)\big|\leq\int\limits_t^{t_{0i}}\Big[\,\big|A(s)h(t_{0i},v_i,\dot\psi_i)(\sigma(s))\big|+$$
$$+\Big|\phi\big(s,\psi_i(s),h(t_{0i},\varphi_i,\psi_i)(\tau_i(\xi)),u_i(s)\big)\Big|\Big]\,ds\leq\varepsilon,\ \ t\in[t_{0i}-\delta_0,t_{0i}],$$

This inequality, with regard for $\psi_i(t_{0i})\in K_3$ (see (4.5)), yields

$$\big(\psi_i(t),h(t_{0i},\varphi_i,\psi_i)(\tau_i(t))\big)\in K_3(\varepsilon)\times[K_0\cup K_3(\varepsilon)],\ \ t\in[t_{0i}-\delta_0,t_1],$$

i.e.

$$\chi\big(\psi_i(t),h(t_{0i},\varphi_i,\psi_i)(\tau_i(t))\big)=1,\ \ t\in[t_{0i}-\delta_0,t_1],\ \ i=1,2,\dots,$$

Thus, $\psi_i(t)$ satisfies the equation (4.1) and the conditions $\psi_i(t_{0i})=x_{0i}$, $\psi_i(t)\in K_4$, $t\in[t_{0i}-\delta_0,t_1]$, i.e. $\psi_i(t)$ is the solution corresponding to the element w_i and defined on the interval $[t_{0i}-\delta_0,t_1]\subset I$. Let $\delta\in(0,\delta_0)$, according to (4.3), for a sufficiently large i_0, we have

$$[t_{0i}-\delta_0,t_1]\supset[t_{00}-\delta,t_1]\supset[t_{0i},t_1],\ \ i\geq i_0.$$

Consequently, $\psi_i(t), i\geq i_0$ are the solutions defined on the interval $[t_{00}-\delta,t_1]$ and satisfy the conditions $\psi_i(t)\in K_4$,

$$\big|\dot\psi_i(t)\big|\leq M,\ \ t\in[t_{00}-\delta,t_1],$$
$$\psi_i(t)=q_i(t),\ \ t\in[t_{0i},t_1]. \qquad \square$$

Theorem 4.3 ([8]). *Let $p(t,u)\in\mathbb{R}^n$ be a continuous function on the set $I\times U$ and let*

$$P(t)=\big\{p(t,u):\ u\in U\big\}$$

be the convex set and

$$p_i\in L(I,\mathbb{R}^n),\ \ p_i(t)\in P(t)\ \ a.e.\ \ on\ I,\ \ i=1,2,\dots\,.$$

Moreover,

$$\lim_{i\to\infty}p_i(t)=p(t)\ \ weakly\ on\ \ I.$$

Then

$$p(t)\in P(t)\ \ a.e.\ \ on\ \ I$$

and there exists a measurable function $u(t)\in U$, $t\in I$ such that

$$p(t,u(t))=p(t)\ \ a.e.\ \ on\ \ I.$$

4.3. **Proof of Theorem 4.1.** Let $w_i=(t_{0i},\tau_i,x_{0i},\varphi_i,v_i,u_i)\in W_{20}$, $i=1,2,\dots$, be a minimizing sequence, i.e.

$$\lim_{i\to\infty}J(w_i)=\widehat{J}=\inf_{w\in W_{20}}J(w).$$

Without loss of generality, we assume that

$$\lim_{i\to\infty}t_{0i}=t_{00},\quad \lim_{i\to\infty}x_{0i}=x_{00}.$$

The set $D_2\subset C(I_2,\mathbb{R}^n)$ is compact and the set $\Phi_2\subset L(I_1,\mathbb{R}^n)$ is weakly compact (see Theorem 4.3), therefore we assume that

$$\lim_{i\to\infty}\tau_i(t)=\tau_0(t)\ \ \text{uniformly in}\ \ t\in I_2=[a,\widehat{\gamma}],$$

and

$$\lim_{i\to\infty}v_i(t)=v_0(t)\ \ \text{weakly in}\ \ t\in I_1,$$

the solution $x_i(t)=x(t;w_i)\in K_3$ is defined on the interval $[t_{0i},t_1]$. In a similar way (see proof of Theorem 4.2) we prove that $|\dot{x}_i(t)|\le N_3$, $t\in[t_{0i},t_1]$, $i=1,2,\dots$, $N_3>0$. By Theorem 4.2, there exists a number $\delta>0$ such that for a sufficiently large i_0 the solutions $x_i(t)$, $i\ge i_0$, are defined on the interval $[t_{00}-\delta,t_1]\subset I$. The sequence $x_i(t)$, $t\in[t_{00}-\delta,t_1]$, $i\ge i_0$, is uniformly bounded and equicontinuous. By the Arzèla–Ascoli lemma, from this sequence we can extract a subsequence which will again be denoted by $x_i(t)$, $i\ge i_0$, such that

$$\lim_{i\to\infty}x_i(t)=x_0(t)\ \ \text{uniformly in}\ \ [t_{00}-\delta,t_1].$$

Further, from the sequence $\dot{x}_i(t)$, $i\ge i_0$, we can extract a subsequence which will again be denoted by $\dot{x}_i(t)$, $i\ge i_0$, such that

$$\lim_{i\to\infty}\dot{x}_i(t)=\varrho(t)\ \ \text{weakly in}\ \ [t_{00}-\delta,t_1].$$

Obviously,

$$x_0(t) = \lim_{i\to\infty} x_i(t) =$$

$$= \lim_{i\to\infty} \left[x_i(t_{00}-\delta) + \int\limits_{t_{00}-\delta}^{t} \dot{x}_i(s)\,ds \right] = x_0(t_{00}-\delta) + \int\limits_{t_{00}-\delta}^{t} \varrho(s)\,ds.$$

Thus, $\dot{x}_0(t) = \varrho(t)$, i.e.

$$\lim_{i\to\infty} \dot{x}_i(t) = \dot{x}_0(t) \;\text{ weakly in }\; [t_{00}-\delta, t_1].$$

Further, we have

$$x_i(t) = x_{0i} +$$

$$+ \int\limits_{t_{0i}}^{t} \Big[A(s)h(t_{0i}, v_i, \dot{x}_i)(\sigma(s)) + f\big(s, x_i(s), h(t_{0i}, \varphi_i, x_i)(\tau_i(s)), u_i(s)\big) \Big]\,ds =$$

$$= x_{0i} + \vartheta_{1i}(t) + \vartheta_{2i} + \theta_{1i}(t) + \theta_{2i}, \;\; t \in [t_{00}, t_1], \;\; i \geq i_0,$$

where

$$\vartheta_{1i}(t) = \int\limits_{t_{00}}^{t} A(s)h(t_{0i}, v_i, \dot{x}_i)(\sigma(s))\,ds,$$

$$\theta_{1i}(t) = \int\limits_{t_{00}}^{t} f\big(s, x_i(s), h(t_{0i}, \varphi_i, x_i)(\tau_i(s)), u_i(s)\big)\,ds,$$

$$\vartheta_{2i} = \int\limits_{t_{0i}}^{t_{00}} A(s)h(t_{0i}, v_i, \dot{x}_i)(\sigma(s))\,ds,$$

$$\theta_{2i} = \int\limits_{t_{0i}}^{t_{00}} f\big(s, x_i(s), h(t_{0i}, \varphi_i, x_i)(\tau_i(s)), u_i(s)\big)\,ds.$$

Obviously, $\vartheta_{2i} \to 0$ and $\theta_{2i} \to 0$ as $i \to \infty$.

First of all, we transform the expression $\vartheta_{1i}(t)$ for $t \in [t_{00}, t_1]$. For this purpose, we consider two cases. Let $t \in [t_{00}, \nu(t_{00})]$, we have

$$\vartheta_{1i}(t) = \vartheta_{1i}^{(1)}(t) + \vartheta_{1i}^{(2)}(t),$$

where

$$\vartheta_{1i}^{(1)}(t) = \int\limits_{t_{00}}^{t} A(s)h(t_{00}, v_i, \dot{x}_i)(\sigma(s))\,ds,$$

$$\vartheta_{1i}^{(2)}(t) = \int\limits_{t_{00}}^{t} \vartheta_{1i}^{(3)}(s)\,ds,$$

$$\vartheta_{1i}^{(3)}(s) = A(s)\Big[h(t_{0i}, v_i, \dot{x}_i)(\sigma(s)) - h(t_{00}, v_i, \dot{x}_i)(\sigma(s))\Big].$$

It is clear that

$$\big|\vartheta_{1i}^{(2)}(t)\big| \le \int\limits_{t_{00}}^{t_1} \big|\vartheta_{1i}^{(3)}(s)\big|\,ds, \ \ t \in [t_{00}, t_1]. \tag{4.7}$$

Suppose that $\nu(t_{0i}) > t_{00}$ for $i \ge i_0$, then

$$\vartheta_{1i}^{(3)}(s) = 0, \ \ s \in [t_{00}, t_{0i}^{(1)}) \cup (t_{0i}^{(2)}, t_1],$$

where

$$t_{0i}^{(1)} = \min\big\{\nu(t_{0i}), \nu(t_{00})\big\}, \quad t_{0i}^{(2)} = \max\big\{\nu(t_{0i}), \nu(t_{00})\big\}.$$

Since

$$\lim_{i\to\infty}(t_{0i}^{(2)} - t_{0i}^{(1)}) = 0,$$

therefore

$$\lim_{i\to\infty}\vartheta_{1i}^{(2)}(t) = 0 \ \text{ uniformly in } \ t \in [t_{00}, t_1] \tag{4.8}$$

(see (4.7)).

For $\vartheta_{1i}^{(1)}(t)$, $t \in [t_{00}, \nu(t_{00})]$, we get

$$\vartheta_{1i}^{(1)}(t) = \int\limits_{\sigma(t_{00})}^{\sigma(t)} A(\nu(s))h(t_{00}, v_i, \dot{v}_i)(s)\dot{\nu}(s)\,ds = \\ = \int\limits_{\sigma(t_{00})}^{\sigma(t)} A(\nu(s))\dot{\nu}(s)v_i(s)\,ds.$$

Obviously,

$$\lim_{i\to\infty}\vartheta_{1i}(t) = \int\limits_{\sigma(t_{00})}^{\sigma(t)} A(\nu(s)\dot{\nu}(s)v_0(s)\,ds = \int\limits_{t_{00}}^{t} A(s)v_0(\sigma(s))\,ds, \\ t \in [t_{00}, \sigma(t_{00})] \tag{4.9}$$

(see (4.8)).

Let $t \in [\nu(t_{00}), t_1]$, then

$$\vartheta_{1i}^{(1)}(t) = \vartheta_{1i}^{(1)}(\nu(t_{00})) + \vartheta_{1i}^{(4)}(t),$$

where

$$\vartheta_{1i}^{(4)}(t) = \int\limits_{\nu(t_{00})}^{t} A(s)h(t_{0i}, v_i, \dot{x}_i)(\sigma(s))\,ds.$$

Further,

$$\vartheta_{1i}^{(4)}(t)=\int\limits_{\nu(t_{00})}^{t} A(s)h(t_{00},v_i,\dot{x}_i)(\sigma(s))\,ds=\int\limits_{t_{00}}^{\sigma(t)} A(\nu(s))\dot{\nu}(t)\dot{x}_i(s)\,ds.$$

Thus, for $t\in[\nu(t_{00}),t_1]$, we have

$$\lim_{i\to\infty}\vartheta_{1i}^{(1)}(t)=\int\limits_{t_{00}}^{\nu(t_{00})} A(t)v_0(\sigma(t))\,dt+\int\limits_{\nu(t_{00})}^{t} A(s)\dot{x}_0(\sigma(s))\,ds. \tag{4.10}$$

Now we transform the expression $\theta_{1i}(t)$ for $t\in[t_{00},t_1]$. We consider two cases again. Letting $t\in[t_{00},\gamma_0(t_{00})]$, we have

$$\begin{aligned}
&\theta_{1i}(t)=\theta_{1i}^{(1)}(t)+\theta_{1i}^{(2)}(t),\\
&\theta_{1i}^{(1)}(t)=\int\limits_{t_{00}}^{t} f\big(s,x_i(s),h(t_{00},\varphi_i,x_i)(\tau_i(s)),u_i(s)\big)\,ds,\\
&\theta_{1i}^{(2)}(t)=\int\limits_{t_{00}}^{t}\theta_{1i}^{(3)}(s)\,ds,\\
&\theta_{1i}^{(3)}(s)=f\big(s,x_i(s),h(t_{0i},\varphi_i,x_i)(\tau_i(s)),u_i(s)\big)-\\
&\qquad\quad -f\big(s,x_i(s),h(t_{00},\varphi_i,x_i)(\tau_i(s)),u_i(s)\big).
\end{aligned}$$

It is clear that

$$\big|\theta_{1i}^{(2)}(t)\big|\le\int\limits_{t_{00}}^{t_{10}}\big|\theta_{1i}^{(3)}(s)\big|\,ds,\quad t\in[t_{00},t_1].$$

Suppose that $\gamma_i(t_{0i})>t_{00}$ for $i\ge i_0$, then

$$\theta_{1i}^{(3)}(s)=0,\quad s\in[t_{00},t_{0i}^{(3)})\cup(t_{0i}^{(4)},t_1],$$

where

$$t_{1i}^{(3)}=\min\big\{\gamma_i(t_{0i}),\gamma_i(t_{00})\big\},\quad t_{1i}^{(4)}=\max\big\{\gamma_i(t_{0i}),\gamma_i(t_{00})\big\}.$$

Since

$$\lim_{i\to\infty}(t_{0i}^{(4)}-t_{0i}^{(3)})=0$$

therefore

$$\lim_{i\to\infty}\theta_{1i}^{(2)}(t)=0\ \text{ uniformly in }\ t\in[t_{00},t_{10}].$$

For $\theta_{1i}^{(1)}(t)$, $t\in[t_{00},\gamma_0(t_{00})]$, we have

$$\theta_{1i}^{(1)}(t)=\int\limits_{\tau_i(t_{00})}^{\tau_i(t)} f\big(\gamma_i(s),x_i(\gamma_i(s)),h(t_{00},\varphi_i,x_i)(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)\,ds= \\ =\theta_{1i}^{(4)}(t)+\theta_{1i}^{(5)}(t),\ \ i\geq i_0,$$

where

$$\theta_{1i}^{(4)}(t)=\int\limits_{\tau_0(t_{00})}^{\tau_0(t)} f\big(\gamma_0(s),x_0(\gamma_0(s)),\varphi_i(s),u_i(\gamma_i(s))\big)\dot\gamma_0(s)\,ds,$$

$$\theta_{1i}^{(5)}(t)=\int\limits_{\tau_i(t_{00})}^{\tau_i(t)} f\big(\gamma_i(s),x_i(\gamma_i(s)),h(t_{00},\varphi_i,x_i)(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)\,ds- \\ -\int\limits_{\tau_0(t_{00})}^{\tau_0(t)} f\big(\gamma_0(s),x_0(\gamma_0(s)),\varphi_i(s),u_i(\gamma_i(s))\big)\dot\gamma_0(s)\,ds.$$

For $t\in[t_{00},\gamma_0(t_{00})]$, we obtain

$$\theta_{1i}^{(5)}(t)=\int\limits_{\tau_i(t_{00})}^{\tau_0(t_{00})} f\big(\gamma_i(s),x_i(\gamma_i(s)),h(t_{00},\varphi_i,x_i)(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)\,ds+ \\ +\int\limits_{\tau_0(t_{00})}^{\tau_0(t)} \Big[f\big(\gamma_i(s),x_i(\gamma_i(s)),\varphi_i(s),u_i(\gamma_i(s))\big)- \\ -f\big(\gamma_0(s),x_0(\gamma_0(s)),\varphi_i(s),u_i(\gamma_i(s))\big)\Big]\dot\gamma_i(s)\,ds+ \\ +\int\limits_{\tau_0(t)}^{\tau_i(t)} f\big(\gamma_i(s),x_i(\gamma_i(s)),h(t_{00},\varphi_i,x_i)(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)\,ds.$$

Suppose that $\|\tau_i-\tau_0\|\leq\delta$ as $i\geq i_0$, then

$$\lim_{i\to\infty} f\big(\gamma_i(s),x_i(\gamma_i(s)),x_2,u\big)=f\big(\gamma_0(s),x_0(\gamma_0(s)),x_2,u\big)$$

uniformly in $(s,x_2,u)\in[\tau_0(t_{00}),t_{00}]\times K_0\times U$, we have

$$\lim_{i\to\infty}\theta_{1i}^{(5)}(t)=0\ \text{ uniformly in }\ t\in[t_{00},\gamma_0(t_{00})].$$

From the sequence

$$f_i(s)=f\big(\gamma_0(s),x_0(\gamma_0(s)),\varphi_i(s),u_i(\gamma_i(s))\big),\ \ i\geq i_0,\ \ t\in[\tau_0(t_{00}),t_{00}],$$

we extract a subsequence which will again be denoted by $f_i(s), i \geq i_0$, such that

$$\lim_{i\to\infty} f_i(s) = f_0(s) \ \text{ weakly in the space } \ L\big([\tau_0(t_{00}), t_{00}], \mathbb{R}^n\big).$$

It is not difficult to see that

$$f_i(s) \in P(\gamma_0(s), x_0(\gamma_0(s))), \ \ s \in [\tau_0(t_{00}), t_{00}].$$

By Theorem 4.3,

$$f_0(s) \in P(\gamma_0(s), x_0(\gamma_0(s))) \ \text{ a.e. } \ s \in [\tau_0(t_{00}), t_{00}]$$

and on the interval $[\tau_0(t_{00}), t_{00}]$ there exist measurable functions $\varphi_{01}(s) \in K_0, u_{01}(s) \in U$ such that

$$f_0(s) = f\big(\gamma_0(s), x_0(\gamma_0(s)), \varphi_{01}(s), u_{01}(s)\big) \ \text{ a.e. } \ s \in [\tau_0(t_{00}), t_{00}].$$

Thus,

$$\begin{aligned}
\lim_{i\to\infty} \theta_{1i}^{(1)} &= \lim_{i\to\infty} \theta_{1i}^{(4)}(t) = \int\limits_{\tau_0(t_{00})}^{\tau_0(t)} f_0(s)\dot{\gamma}_0(s)\,ds = \\
&= \int\limits_{\tau_0(t_{00})}^{\tau_0(t)} f\big(\gamma_0(s), x_0(\gamma_0(s)), \varphi_{01}(s), u_{01}(s)\big)\dot{\gamma}_0(s)\,ds = \\
&= \int\limits_{t_{00}}^{t} f\big(s, x_0(s), \varphi_{01}(\tau_0(s)), u_{01}(\tau_0(s))\big)\,ds, \ \ t \in [t_{00}, \gamma_0(t_{00})]. \qquad (4.11)
\end{aligned}$$

Let $t \in [\gamma_0(t_{00}), t_1]$, then

$$\theta_{1i}^{(1)}(t) = \theta_{1i}^{(1)}(\gamma_0(t_{00})) + \theta_{1i}^{(6)}(t),$$

where

$$\theta_{1i}^{(6)}(t) = \int\limits_{\gamma_0(t_{00})}^{t} f\big(s, x_i(s), h(t_{00}, \varphi_i, x_i)(\tau_i(s)), u_i(s)\big)\,ds.$$

It is clear that

$$\begin{aligned}
\theta_{1i}^{(6)}(t) &= \int\limits_{\tau_i(\gamma_0(t_{00}))}^{\tau_i(t)} f\big(\gamma_i(s), x_i(\gamma_i(s)), h(t_{00}, \varphi_i, x_i)(s), u_i(\gamma_i(s))\big)\dot{\gamma}_i(s)\,ds = \\
&= \theta_{1i}^{(7)}(t) + \theta_{1i}^{(8)}(t), \ \ i \geq i_0,
\end{aligned}$$

where

$$\theta^{(7)}_{1i}(t)=\int\limits_{t_{00}}^{\tau_0(t)} f\big(\gamma_0(t),x_0(\gamma_0(s)),x_0(s),u_i(\gamma_i(s))\big)\dot\gamma_0(s)\,ds,$$

$$\theta^{(8)}_{1i}(t)=\int\limits_{\tau_i(\gamma_0(t_{00}))}^{\tau_i(t)} f\big(\gamma_i(t),x_i(\gamma_i(s)),h(t_{00},\varphi_i,x_i)(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)\,ds-$$
$$-\int\limits_{t_{00}}^{\tau_0(t)} f\big(\gamma_0(s),x_0(\gamma_0(s)),x_0(s),u_i(\gamma_i(s))\big)\dot\gamma_0(s)\,ds.$$

For $t\in[\gamma_0(t_{00}),t_1]$, we have

$$\theta^{(8)}_{1i}(t)=\int\limits_{\tau_i(\gamma_0(t_{00}))}^{t_{00}} f\big(\gamma_i(s),x_i(\gamma_i(s)),h(t_{00},\varphi_i,x_i)(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)\,ds+$$
$$+\int\limits_{t_{00}}^{\tau_0(t)}\Big[f\big(\gamma_i(s),x_i(\gamma_i(s)),x_i(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)-$$
$$-f\big(\gamma_0(s),x_0(s+\gamma_0(s)),v_0(s),u_i(\gamma_i(s))\big)\dot\gamma_0(s)\Big]\,ds+$$
$$+\int\limits_{\tau_0(t)}^{\tau_i(t)} f\big(\gamma_i(s),x_i(\gamma_i(s)),h(t_{00},\varphi_i,x_i)(s),u_i(\gamma_i(s))\big)\dot\gamma_i(s)\,ds.$$

Thus,

$$\theta^{(8)}_{1i}(t)=0 \ \text{ uniformly in } \ t\in[\gamma_0(t_{00}),t_1].$$

From the sequence

$$f_i(s)=f\big(\gamma_0(s),x_0(\gamma_0(s)),x_0(s),u_i(\tau_i(s))\big),\ \ i\ge i_0,\ \ t\in[t_{00},\tau_0(t_1)],$$

we extract a subsequence which will again be denoted by $F_i(s)$, $i\ge i_0$, such that

$$\lim_{i\to\infty} f_i(s)=f_0(s) \ \text{ weakly in the space } \ L\big([t_{00},\tau_0(t_1)],\mathbb{R}^n\big).$$

It is not difficult to see that

$$f_i(s)\in P_1\big(\gamma_0(s),x_0(\gamma_0(s)),x_0(s)\big),\ \ s\in[t_{00},\tau_0(t_1)].$$

By Theorem 4.3,

$$f_0(s)\in P_1\big(\gamma_0(s),x_0(\gamma_0(s)),x_0(s)\big) \ \text{ a.e. } \ s\in[t_{00},\tau_0(t_1)]$$

and on the interval $[t_{00},\tau_0(t_1)]$ there exists a measurable function $u_{02}(s)\in U$ such that

$$f_0(s)=f\big(\gamma_0(s),x_0(\gamma_0(s)),x_0(s),u_{02}(s)\big) \ \text{ a.e. } \ s\in[t_{00},\tau_0(t_1)].$$

Thus, for $t \in [\gamma_0(t_{00}), t_1]$, we have

$$\lim_{i\to\infty}\theta_{1i}^{(1)}(t) = \lim_{i\to\infty}\theta_{1i}^{(1)}(\gamma_0(t_{00})) + \lim_{i\to\infty}\theta_{1i}^{(7)}(t) =$$

$$= \int\limits_{t_{00}}^{\gamma_0(t_{00})} f\big(s, x_0(s), x_0(\tau_0(s)), u_{02}(s)\big)\, ds +$$

$$+ \int\limits_{\gamma_0(t_{00})}^{t} f\big(s, x_0(s), x_0(\tau_0(s)), u_{02}(\tau_0(s))\big)\, ds, \quad t \in [\gamma_0(t_{00}), t_1]. \tag{4.12}$$

We introduce the following notation:

$$\varphi_0(s) = \begin{cases} \widehat{\varphi}, & s \in [\widehat{\tau}, \tau_0(t_{00})) \cup (t_{00}, t_{02}], \\ \varphi_{01}(s), & s \in [\tau_0(t_{00}), t_{00}], \end{cases}$$

$$u_0(s) = \begin{cases} \widehat{u}, & s \in [a, t_{00}) \cup (t_1, b], \\ u_{01}(\tau_0(s)), & s \in [t_{00}, \tau_0(t_{00})], \\ u_{02}(\tau_0(s)), & s \in (\gamma_0(t_{00}), t_1], \end{cases}$$

where $\widehat{\varphi} \in K_0$ and $\widehat{u} \in U$ are the fixed points

$$x_0(t) = \begin{cases} \varphi_0(t), & t \in [\widehat{\tau}, t_{00}), \\ v_0(t), & t \in [t_{00}, t_1]; \end{cases}$$

$$\dot{x}_0(t) = v_0(t), \quad t \in [\widehat{\tau}, t_{00}),$$

Clearly, $w_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, v_0, u_0) \in W_2$. Taking into account (4.9)–(4.12), we obtain

$$x_0(t) = x_{00} + \int\limits_{t_{00}}^{t} \Big[A(s)\dot{x}_0(\sigma_0(t)) + f\big(s, x_0(s), x_0(\tau_0(s)), u_0(s)\big)\Big]\, ds,$$

$$t \in [t_{00}, t_{10}],$$

and

$$0 = \lim_{i\to\infty} q^i\big(t_{0i}, x_{0i}, x_i(t_1)\big) = q^i\big(t_{00}, x_{00}, x_0(t_1)\big), \quad i = 1, \dots, l,$$

i.e. the element w_0 is admissible and $x_0(t) = x(t; w_0)$, $t \in [\widehat{\tau}, t_1]$.

Further, we have

$$\widehat{J} = \lim_{i\to\infty} q^0\big(t_{0i}, x_{0i}, x_i(t_1)\big) = q^0\big(t_{00}, x_{00}, x_0(t_1)\big) = J(w_0).$$

Thus, w_0 is an optimal element.

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