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THEORY OF LINEAR ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATIONS AND APPLICATIONS

Abstract. The boundary value problem is considered for the abstract functional differential equation $Lx = f$, where $L : D \to B$ is a linear operator, B is a Banach space, and D is isomorphic to the direct product $B \times R^n$. The Green operator is constructed, continuous dependence on parameters is studied. The obtained results are applied to ordinary, impulse and singular equations.

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Key words and phrases. Linear operator, Green operator, Noether operator, boundary value problem.

რეზიუმე. ნაშრომში განხილულია სასაზღვრო ამოცანა $Lx = f$ აბსტრაქტული ფუნქციონალურ-დიფერენციალური განტოლებისათვის, სადაც $L : D \to B$ წრფივი ოპერატორია, B -ბანახის სივრცეა, ხოლო D იზომორფულია $B \times R^n$ პირდაპირი ნამრავლისა. აგებულია გრინის ოპერატორი, შესწავლილია ამონახსნის პარამეტრზე უწყვეტად დამოკიდებულება. მიღებული შედეგები გამოყენებულია ჩვეულებრივი, იმპულსური და სინგულარული განტოლებებისათვის.

PREFACE

The equation $\mathcal{L}x = f$ with a linear operator $\mathcal{L}$ acting from the space $\mathbf{D}^n$ of absolutely continuous functions $x : [a, b] \to \mathbf{R}^n$ into the space $\mathbf{L}^n$ of summable functions $f : [a, b] \to \mathbf{R}^n$ has been thoroughly studied in the works of the Perm Seminar. The results of these investigations are systematized in the book [1]. Such a generalization of the ordinary differential equation

$$(\mathcal{L}x)(t) \stackrel{def}{=} \dot{x}(t) + P(t)x(t) = f(t), \quad t \in [a, b], \tag{0.1}$$

covers many classes of equations containing the derivative of the unknown function, for instance, integro-differential equations and equations with deviated argument as well as their hybrids and so on. The theory of the equation $\mathcal{L}x = f$ with the linear operator $\mathcal{L} : \mathbf{D}^n \to \mathbf{L}^n$ is based upon the isomorphism between the space $\mathbf{D}^n$ and the direct product $\mathbf{L}^n \times \mathbf{R}^n$. The isomorphism may be defined by

$$x(t) = \int\limits_a^t z(s)ds + \beta, \quad x \in \mathbf{D}^n, \quad \{z, \beta\} \in \mathbf{L}^n \times \mathbf{R}^n.$$

As it turned out, the replacement of the space $\mathbf{L}^n$ by an arbitrary Banach space $\mathbf{B}$ does not violate validity of the fundamental theorems. Thus, a further generalization arises in the form of the theory of *abstract functional differential equation*

$$\mathcal{L}x = f \tag{0.2}$$

with a linear operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$, where $\mathbf{B}$ is a Banach space and $\mathbf{D}$ is isomorphic to the direct product $\mathbf{B} \times \mathbf{R}^n$ ($\mathbf{D} \simeq \mathbf{B} \times \mathbf{R}^n$).

The space $\mathbf{W}^n$ of the functions $x : [a, b] \to \mathbf{R}^1$ with absolutely continuous derivative $x^{(n-1)}$ is isomorphic to the direct product $\mathbf{L}^1 \times \mathbf{R}^n$. The isomorphism may be defined on the base of the representation

$$x(t) = \int\limits_a^t \frac{(t-s)^{n-1}}{(n-1)!} x^{(n)}(s)ds + \sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} x^{(k)}(a)$$

of the element $x \in \mathbf{W}^n$. Thus, the equation of the n-th order

$$(\mathcal{L}x)(t) \stackrel{def}{=} x^{(n)}(t) + \sum_{k=0}^{n-1} p_k(t) x^{(k)}[h_k(t)] = f(t), \quad t \in [a, b],$$

$$x^{(k)}(\xi) = 0, \quad \text{if} \quad \xi \notin [a, b], \quad k = 0, 1, \ldots, n-1,$$

as well as its generalization of the form

$$(\mathcal{L}x)(t) \stackrel{def}{=} x^{(n)}(t) + q(t)x^{(n)}[g(t)] +$$

$$+\sum_{k=0}^{n-1}\int_a^b x^{(k)}(s)d_s r_k(t,s) = f(t), \quad t \in [a,b],$$
$$x^{(n)}(\xi) = 0, \quad \text{if} \quad \xi \notin [a,b],$$

are the equations (0.2) in the space $\mathbf{W}^n$.

The space $\mathbf{DS}^n(m)$ of the functions $x : [a,b] \to \mathbf{R}^n$ permitting finite discontinuity at the fixed points $t_1, \ldots, t_m \in (a,b)$ and absolutely continuous on the intervals $[a,t_1)$, $[t_1,t_2), \ldots,$ $[t_m,b]$ is isomorphic to $\mathbf{L}^n \times \mathbf{R}^{n(m+1)}$. The isomorphism is defined by

$$x(t) = \int_a^t z(s)ds + \beta^0 + \sum_{i=1}^m \chi_{[t_i,b]}(t)\beta^i,$$
$$z \in \mathbf{L}^n, \quad \{\beta^0, \beta^1, \ldots, \beta^m\} \in \mathbf{R}^{n(m+1)},$$

where χ_e denotes the characteristic function of the set e.

Any space $\mathbf{D}$ isomorphic to $\mathbf{B} \times \mathbf{R}^n$ forms its own proper class of equations. Some examples of nontraditional spaces isomorphic to the product $\mathbf{B} \times \mathbf{R}^n$ are provided in [2]. It is shown there in particular that the space of functions $x : [a,b] \to \mathbf{R}^1$ which have "quasi-derivatives" up to the n-th order inclusively is isomorphic to $\mathbf{L}^1 \times \mathbf{R}^n$. Thus, the linear equation with quasi-derivatives is one of the form (0.2).

The theory of abstract functional differential equations considers wide classes of equations from a single point of view. During the last ten-year period, this theory found various applications in studying old and new problems due to possibility of choosing proper space $\mathbf{D} \simeq \mathbf{B} \times \mathbf{R}^n$ for each class of problems. A successful choice of the space permits in virtue of the general theory direct using of standard schemes and theorems of analysis in such cases, where we have been forced before to invent special devices and put severe restrictions connected with application of these devices.

It is relevant to emphasize the principal difference between the generalization of the ordinary differential equation in the form of the abstract functional differential equation and the "ordinary differential equation in Banach spaces". The equation (0.1) is defined by the operator $\mathcal{L} : \mathbf{D}^n \to \mathbf{L}^n$ belonging to the class of the so called "local operators" [3, 4]. An operator $\mathcal{L}$ in a functional space is called local, if the value of the image $f(t) = (\mathcal{L}x)(t)$ in a neighborhood of each point t depends only on the value of the preimage $x(\cdot)$ in the neighborhood of the same point t. The generalization in the theory of ordinary differential equations in Banach spaces consists in replacement of the finite dimensional space $\mathbf{R}^n$ of the values $x(t)$ of the unknown function x by an arbitrary Banach space. In this connection, the property of $\mathcal{L}$ to be a local operator keeps. In the theory of abstract functional differential equation the generalization consists in replacement of the space $\mathbf{L}^n$ by an arbitrary Banach space $\mathbf{B}$ and in replacement of the local operator $\mathcal{L} : \mathbf{D}^n \to \mathbf{L}^n$ by an arbitrary linear operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$.

The main notation.

$\mathbf{R}^n$	space of n-dimensional real vectors with the norm $\lvert\cdot\rvert$.
$\lVert \cdot \rVert_{\mathbf{X}}$	norm of an element of the normed space $\mathbf{X}$.
$\lVert A \rVert_{\mathbf{X}\to\mathbf{Y}}$	norm of an operator $A : \mathbf{X} \to \mathbf{Y}$. Usually the symbol "$X \to Y$" is ommited.
A^*	operator adjoint to the operator A.
$R(A)$	range of values of the operator A.
$D(A)$	domain of definition of the operator A.
$\dim M$	dimension of the linear set M.
$\ker A$	null-set (the kernel) of the operator A.
$\operatorname{ind} A$	index of the operator $A : \operatorname{ind} A = \dim\ker A - \dim\ker A^*$.
$[A_1, A_2]$	linear operator acting from the space X into the product $Y_1 \times Y_2$ by $[A_1, A_2]x = \{A_1x, A_2x\}$, $x \in X$, $A_1x \in Y_1$, $A_2x \in Y_2$.
$\{A_1, A_2\}$	linear operator acting from the product of the spaces $X_1 \times X_2$ into Y by $\{A_1, A_2\}\{x_1, x_2\} = A_1x_1 + A_2x_2$, $x_1 \in X_1$, $x_2 \in X_2$.
I	identity operator.
E	identity matrix or E_n under the necessity to emphasize the dimension of the identity $n \times n$-matrix.
$< \varphi, x >, \varphi x$	value of the functional φ on the element x.
S_r	composition operator defined by $$(S_rx)(t) = \begin{cases} x[r(t)], & if \quad r(t) \in [a,b], \\ 0, & if \quad r(t) \notin [a,b]. \end{cases}$$
$\chi_e(\cdot)$	characteristic function of the set e : $$\chi_e = \begin{cases} 1, & if \quad t \in e, \\ 0, & if \quad t \notin e. \end{cases}$$
δ_{ij}	Kronecker symbol: $\delta_{ij} = \begin{cases} 1, & if \quad i = j, \\ 0, & if \quad i \neq j. \end{cases}$

CHAPTER I
LINEAR ABCTRACT FUNCTIONAL DIFFERENTIAL EQUATIONS

§ 1. Preliminary Knowledge from the Theory of Linear Equations in Banach Spaces

The main assertions of the theory of linear abstract functional-differential equations are based on the theorems about linear equations in Banach spaces. We give here without proofs certain results of the book [5] which we will need below. We formulate some of these assertions not in the most general form, but in the form satisfying our aims. The enumeration of the theorems in brackets means that the assertion either coincides with the corresponding result of the book [5] or is only an extraction from this result.

We will use the following notation.

$\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ are Banach spaces; A, B are linear operators; $D(A)$ is a domain of definition of A; $R(A)$ is a range of values of A; A^* is an operator adjoined to A. The set of solutions of the equation $Ax = 0$ is said to be *a null space* or *a kernel* of A and is denoted by $\ker A$. The dimension of a linear set M is denoted by $\dim M$.

Let A be acting from $\mathbf{X}$ into $\mathbf{Y}$. The equation

$$Ax = y \tag{1.1}$$

(the operator A) is said to be *normal solvable*, if the set $R(A)$ is closed; (1.1) the operator A is said to be a *Noether equation*, if it is a normal solvable one and besides $\dim\ker A < \infty$ and $\dim\ker A^* < \infty$. The number $\operatorname{ind} A = \dim\ker A - \dim\ker A^*$ is said to be an *index of the operator* A (the equation (1.1)). If A is a Noether operator and $\operatorname{ind} A = 0$, the equation (1.1) (the operator A) is said to be a *Fredholm* one. The equation $A^*\varphi = g$ is said to be an *equation, adjoined* to (1.1).

Theorem 1.1 (Theorem 3.2). *An operator A is normal solvable if and only if the equation* (1.1) *is solvable for such and only such right hand side y which is orthogonal to all the solutions of the homogenous adjoined equation $A^*\varphi = 0$.*

Theorem 1.2 (Theorem 16.4). *The property of being Noether operator is stable in respect to completely continuous perturbations. By such perturbations, the index of the operator does not change.*

Theorem 1.3 (Theorem 12.2). *Let A be acting from $\mathbf{X}$ into $\mathbf{Y}$ and $D(B)$ be dense in $\mathbf{Y}$. If A and B are Noether operators, BA is also a Noether one and* $\operatorname{ind}(BA) = \operatorname{ind} A + \operatorname{ind} B$.

Theorem 1.4 (Theorem 15.1). *Let BA be a Noether operator and $D(B) \subset R(A)$. Then B is a Noether operator.*

Theorem 1.5 (Theorem 2.4 and Lemma 8.1). *Let A be defined on $\mathbf{X}$ and acting into $\mathbf{Y}$. A is normal solvable and* $\dim\ker A^* = n$ *if and only if the space $\mathbf{Y}$ is representable in the form of a direct sum $\mathbf{Y} = R(A) \oplus M_n$, where M_n is a finite-dimensional subspace of the dimension n.*

Theorem 1.6 (Theorem 12.2). *Let $D(A) \subset \mathbf{X}$, M_n be a n-dimensional subspace of $\mathbf{X}$ and $D(A) \cap M_n = \{0\}$. If A is a Noether operator, then its linear extension $\widetilde{A}$ on $D(A) \oplus M_n$ is also a Noether operator. Besides* $\operatorname{ind}\widetilde{A} = \operatorname{ind} A + n$.

Theorem 1.7. *Let a Noether operator A be defined on $\mathbf{X}$ and acting into $\mathbf{Y}$, $D(B) = \mathbf{Y}$, $BA : \mathbf{X} \to \mathbf{Z}$ is a Noether operator. Then B is also a Noether operator.*

Proof. Thanks to Theorem 1.4, we are in need only of the proof of the case $R(A) \neq \mathbf{Y}$. From this Theorem 1.4 we obtain also that restriction $\bar{B}$ of B on $R(A)$ is a Noether operator.

Let $\dim\ker A^* = n$. Then we have from Theorem 1.5 that

$$\mathbf{Y} = R(A) \oplus M_n = D(\bar{B}) \oplus M_n,$$

where $\dim M_n = n$.

From Theorem 1.6 we see that B is a Noether operator as a linear extension of $\bar{B}$ on $\mathbf{Y}$. ∎

Let a linear operator A acting from a direct product $\mathbf{X}_1 \times \mathbf{X}_2$ into $\mathbf{Y}$ be defined by a pair of operators $A_1 : \mathbf{X}_1 \to \mathbf{Y}$ and $A_2 : \mathbf{X}_2 \to \mathbf{Y}$ in such a way, that

$$A\{x_1, x_2\} = A_1x_1 + A_2x_2, \quad x_1 \in \mathbf{X}_1, \quad x_2 \in \mathbf{X}_2,$$

where $A_1x_1 = A\{x_1, 0\}$, $A_2x_2 = A\{0, x_2\}$. We will denote such an operator by $A = \{A_1, A_2\}$.

Let a linear operator A acting from $\mathbf{X}$ into a direct product $\mathbf{Y}_1 \times \mathbf{Y}_2$ be defined by a pair of operators $A_1 : \mathbf{X} \to \mathbf{Y}_1$ and $A_2 : \mathbf{X} \to \mathbf{Y}_2$ in such a way, that $Ax = \{A_1x, A_2x\}$, $x \in \mathbf{X}$. We will denote such an operator by $A = [A_1, A_2]$.

The theory of linear abstract functional differential equation is using some operators defined on a product $\mathbf{B} \times \mathbf{R}^n$ or acting in such a product. We will formulate here certain assertions about such operators, reserving as far as possible the notations from [1].

A linear operator $\{\Lambda, \psi\}$ acting from a direct product $\mathbf{B} \times \mathbf{R}^n$ of the Banach spaces $\mathbf{B}$ and $\mathbf{R}^n$ into a Banach space $\mathbf{D}$ is defined by a pair of linear operators $\Lambda : \mathbf{B} \to \mathbf{D}$ and $Y : \mathbf{R}^n \to \mathbf{D}$ in such a way that

$$\{\Lambda, Y\}\{z, \beta\} = \Lambda z + Y\beta, \quad z \in \mathbf{B}, \quad \beta \in \mathbf{R}^n.$$

A linear operator $[\delta, r]$ acting from a space $\mathbf{D}$ into a direct product $\mathbf{B}\times\mathbf{R}^n$ is defined by a pair of linear operators $\delta : \mathbf{D} \to \mathbf{B}$ and $r : \mathbf{D} \to \mathbf{R}^n$ in such a way that

$$[\delta, r]x = \{\delta x, rx\}, \quad x \in \mathbf{D}.$$

If the norm in the space $\mathbf{B} \times \mathbf{R}^n$ is defined in an appropriate way, for instance by

$$\left\|\{z, \beta\}\right\|_{\mathbf{B}\times\mathbf{R}^n} = \|z\|_{\mathbf{B}} + |\beta|,$$

the space $\mathbf{B} \times \mathbf{R}^n$ will be a Banach one.

If a bounded operator $\{\Lambda, Y\} : \mathbf{B} \times \mathbf{R}^n \to \mathbf{D}$ is an inverse to a bounded operator $[\delta, r] : \mathbf{D} \to \mathbf{B} \times \mathbf{R}^n$, then

$$x = \Lambda\delta x + Yrx, \quad x \in \mathbf{D}, \tag{1.2}$$

$\delta(\Lambda z + Y\beta) = z$, $r(\Lambda z + Y\beta) = beta$, $\{z, \beta\} \in \mathbf{B} \times \mathbf{R}^n$.

Hence

$$\Lambda\delta + Yr = I, \quad \delta\Lambda = I, \quad \delta Y = 0, \quad r\Lambda = 0, \quad rY = I.$$

We will identify the finite-dimensional operator $Y : \mathbf{R}^n \to \mathbf{D}$ with such a vector $(y_1, \ldots, y_n)$, $y_i \in \mathbf{D}$, that

$$Y\beta = \sum_{i=1}^{n} y_i\beta^i, \quad \beta = \operatorname{col}\left\{\beta^1, \ldots, \beta^n\right\}.$$

We denote the components of a vector-functional r by $r^1, \ldots, r^n$.

If $l = \{l^1, \ldots, l^m\} : \mathbf{D} \to \mathbf{R}^m$ is a linear vector-functional and $X = (x_1, \ldots, x_n)$ is a vector with components $x_i \in \mathbf{D}$, then lX denotes the $m\times n$-matrix whose columns are the values of the vector-functional l on the components of $X : lX = (l^i x_j)$, $i = 1, \ldots, m$; $j = 1, \ldots, n$.

The operators Λ, Y, δ, r for the spaces suggested above in the Introduction have the following forms.

For the space $\mathbf{D}^n$,

$$(\Lambda z)(t) = \int_a^t z(s)ds, \quad Y = E, \quad \delta x = \dot{x}, \quad rx = x(a),$$

where E is the identical $n \times n$-matrix.

For the space $\mathbf{W}^n$,

$$(\Lambda z)(t) = \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} z(s)ds, \quad Y = \left(1, t-a, \ldots, \frac{(t-a)^{n-1}}{(n-1)!}\right),$$
$$\delta x = x^{(n)}, \quad rx = \left\{x(a), \dot{x}(a), \ldots, x^{(n-1)}(a)\right\}.$$

For the space $\mathbf{DS}^n(m)$,

$$(\Lambda z)(t) = \int_a^t z(s)ds, \quad Y = \Big(E, E \cdot \chi_{[t_1,b]}(t), \ldots, E \cdot \chi_{[t_m,b]}(t)\Big),$$

$$\delta x = \dot{x}, \quad rx = \big\{x(a), \Delta x(t_1), \ldots, \Delta x(t_m)\big\},$$

$$\Delta x(t_i) = x(t_i) - x(t_i - 0).$$

Theorem 1.8. *A linear bounded operator* $\{\Lambda, Y\} : \mathbf{B} \times \mathbf{R}^n \to \mathbf{D}$ *has the bounded inverse if and only if the following conditions are satisfied.*

a) *The operator* $\Lambda : \mathbf{B} \to \mathbf{D}$ *is a Noether one and* $\operatorname{ind} \Lambda = -n$.

b) $\dim \ker \Lambda = 0$.

c) *If* $\lambda^1, \ldots, \lambda^n$ *is a basis for* $\ker \Lambda^*$ *and* $\lambda = [\lambda^1, \ldots, \lambda^n]$, *then* $\det \lambda Y \neq 0$.

Proof. Sufficiency. From a) and b) it follows that $\dim \ker \Lambda^* = n$. By virtue of Theorem 1.5, $\mathbf{D} = \mathbf{R}(\Lambda) \oplus M_n$, where $\dim M_n = n$. It follows from c) that any nontrivial linear combination of the elements $y_1, \ldots, y_n$ does not belong to $R(\Lambda)$, therefore $M_n = R(Y)$. Thus $\mathbf{D} = R(\Lambda) \oplus R(Y)$ and consequently the operator $\{\Lambda, Y\}$ has its inverse by virtue of Banach theorem.

Necessity. From invertibility of $\{\Lambda, Y\}$, we have $\mathbf{D} = R(\Lambda) \oplus R(Y)$. Consequently the operator Λ is normally solvable by virtue of Theorem 1.5 and $\dim \ker \Lambda^* = n$. Besides $\dim \ker \Lambda = 0$. Therefore $\operatorname{ind} \Lambda = -n$. Assumption $\det \lambda Y = 0$ leads to the conclusion that a nontrivial combination of the elements $y_1, \ldots, y_n$ belongs to $R(\Lambda)$. ■

Theorem 1.9. *A linear bounded operator* $[\delta, r] : \mathbf{D} \to \mathbf{B} \times \mathbf{R}^n$ *has a bounded inverse if and only if the following conditions are satisfied.*

a) *The operator* $\delta : \mathbf{D} \to \mathbf{B}$ *is a Noether one and* $\operatorname{ind} \delta = n$.

b) $\dim \ker \delta = n$.

c) *If* $x_1, \ldots, x_n$ *is a basis of* $\ker \delta$ *and* $X = (x_1, \ldots, x_n)$, *then* $\det rX \neq 0$.

Proof. Sufficiency. From a) and b) it follows that $\dim \ker \delta^* = 0$. Thus $R(\delta) = \mathbf{B}$. Each solution of the equation $\delta x = z$ has the form

$$x = \sum_{i=1}^{n} c_i x_i + v,$$

where $c_i = \text{const}$, $i = 1, \ldots, n$, and v is any solution of this equation. By virtue of c), the system

$$\delta x = z, \quad rx = \beta$$

has a unique solution for each pair $z \in \mathbf{B}$, $\beta \in \mathbf{R}^n$. Consequently the operator $[\delta, r]$ has the bounded inverse.

Necessity. Let $[\delta, r]^{-1} = \{\Lambda, Y\}$. From the equality $\delta\Lambda = I$, by virtue of Theorem 1.7 it follows that δ is a Noether operator and by virtue of

Theorem 1.3, $\operatorname{ind}\delta = n$. As far as $R(\delta) = \mathbf{B}$, we have $\dim\ker\delta^* = 0$ and consequently $\dim\ker\delta = n$. If $\det rX = 0$, then the homogeneous system

$$\delta x = 0, \quad rx = 0$$

has a nontrivial solution. This gives a contradiction to the invertibility of the operator $[\delta, r]$. ■

§ 2. Linear Equation and Linear Boundary Value Problem

The Cauchy problem

$$(\mathcal{L}x)(t) \stackrel{def}{=} \dot{x}(t) - P(t)x(t) = f(t), \quad x(a) = \alpha, \quad t \in [a, b],$$

is uniquely solvable for $\alpha \in \mathbf{R}^n$ and any summable f if the elements of the $n \times n$-matrix P are summable. Thus, the representation of the solution

$$x(t) = X(t) \int_a^t X^{-1}(s) f(s) ds + X(t)\alpha$$

of the problem (the Cauchy formula), where X is a fundamental matrix such that $X(a)$ is the identity matrix, is also a representation of the general solution of the equation $\mathcal{L}x = f$. The Cauchy formula is the base for investigations on various problems in the theory of ordinary differential equations. The Cauchy problem for functional differential equation is not solvable generally speaking, but some boundary value problems may be solvable. Therefore the boundary value problem plays the same role in the theory of functional differential equations as the Cauchy problem does in the theory of ordinary differential equations.

We will call the equation

$$\mathcal{L}x = f \tag{2.1}$$

a *linear abstract functional-differential equation* if $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ is a linear operator, $\mathbf{D}$ and $\mathbf{B}$ are Banach spaces and the space $\mathbf{D}$ is isomorphic to the direct product $\mathbf{B} \times \mathbf{R}^n$ ($\mathbf{D} \simeq \mathbf{B} \times \mathbf{R}^n$).

Let $\mathcal{J} = \{\Lambda, Y\} : \mathbf{B} \times \mathbf{R}^n \to \mathbf{D}$ be a linear isomorphism and $\mathcal{J}^{-1} = [\delta, r]$. Everywhere below the norms in the spaces $\mathbf{B} \times \mathbf{R}^n$ and $\mathbf{D}$ are defined by

$$\big\|\{z, \beta\}\big\|_{\mathbf{B}\times\mathbf{R}^n} = \|z\|_{\mathbf{B}} + |\beta|, \quad \|x\|_{\mathbf{D}} = \|\delta x\|_{\mathbf{B}} + |rx|.$$

By such a definition of the norms, the isomorphism $\mathcal{J}$ is an isometrical one. Therefore

$$\big\|\{\Lambda, Y\}\big\|_{\mathbf{B}\times\mathbf{R}^n\to\mathbf{D}} = 1, \quad \big\|[\delta, r]\big\|_{\mathbf{D}\to\mathbf{B}\times\mathbf{R}^n} = 1.$$

Since

$$\|\Lambda z\|_{\mathbf{D}} = \big\|\{\Lambda, Y\}\{z, 0\}\big\|_{\mathbf{D}} \le \big\|\{\Lambda, Y\}\big\| \, \big\|\{z, 0\}\big\|_{\mathbf{B}\times\mathbf{R}^n} = \|z\|_{\mathbf{B}},$$

$\|\Lambda\|_{\mathbf{B}\to\mathbf{D}} = 1$. Analogously it is established that $\|Y\|_{\mathbf{R}^n\to\mathbf{D}} = 1$. Next we have

$$\|\delta x\|_{\mathbf{B}} \le \|x\|_{\mathbf{D}}$$

and, if $rx = 0$,

$$\|\delta x\|_{\mathbf{B}} = \|x\|_{\mathbf{D}}.$$

Therefore $\|\delta\|_{\mathbf{D}\to\mathbf{B}} = 1$. Analogously $\|r\|_{\mathbf{D}\to\mathbf{R}^n} = 1$.

We will assume that the operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ is bounded. Applying $\mathcal{L}$ to the both parts of (1.2), we get the decomposition

$$\mathcal{L}x = Q\delta x + Arx. \tag{2.2}$$

Here $Q = \mathcal{L}\Lambda : \mathbf{B} \to \mathbf{B}$ is a *principal part*, and $A = \mathcal{L}Y : \mathbf{R}^n \to \mathbf{B}$ is a *finite-dimensional part* of $\mathcal{L}$.

As examples of (2.1) in the case where $\mathbf{D}$ is a space $\mathbf{D}^n$ of absolutely continuous functions $x : [a,b] \to \mathbf{R}^n$ and $\mathbf{B}$ is a space $\mathbf{L}^n$ of summable functions $z : [a,b] \to \mathbf{R}^n$, we can take an ordinary differential equation

$$\dot{x}(t) - P(t)x(t) = f(t), \quad t \in [a,b], \tag{2.3}$$

where the columns of the matrix P belong to $\mathbf{L}^n$, or an integro-differential equation

$$\dot{x}(t) - \int_a^b H_1(t,s)\dot{x}(s)ds - \int_a^b H(t,s)x(s)ds = f(t), \quad t \in [a,b]. \tag{2.4}$$

We will assume the elements $h_{ij}(t,s)$ of the matrix $H(t,s)$ to be measurable on $[a,b] \times [a,b]$, the functions $\int_a^b h_{ij}(t,s)ds$ to be summable on $[a,b]$, and the integral operator

$$(H_1 z)(t) = \int_a^b H_1(t,s)z(s)ds$$

on $\mathbf{L}^n$ into $\mathbf{L}^n$ to be completely continuous. The corresponding operators $\mathcal{L}$ for these equations have the representation in the form (2.2)

$$(\mathcal{L}x)(t) = \dot{x}(t) - P(t)\int_a^t \dot{x}(s)ds - P(t)x(a)$$

for (2.3), and

$$(\mathcal{L}x)(t) = \dot{x}(t) - \int_a^b \left\{H_1(t,s) + \int_s^b H(t,\tau)d\tau\right\}\dot{x}(s)ds - \int_a^b H(t,s)ds\, x(a)$$

for (2.4).

Theorem 2.1. *An operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ is a Noether one if and only if the principal part $Q : \mathbf{B} \to \mathbf{B}$ of $\mathcal{L}$ is a Noether operator. In this case,* $\operatorname{ind} \mathcal{L} = \operatorname{ind} Q + n$.

Proof. If $\mathcal{L}$ is a Noether operator, $Q = \mathcal{L}\Lambda$ is also Noether as the product of Noether operators and $\operatorname{ind} \mathcal{L} = \operatorname{ind} Q + n$ (Theorems 1.8 and 1.3).

If Q is a Noether operator, $Q\delta$ is also Noether. Consequently $\mathcal{L} = Q\delta + Ar$ is also Noether (Theorem 1.9 and 1.2). ■

Due to Theorem 2.1, the equality $\operatorname{ind} \mathcal{L} = n$ is equivalent to the fact that Q is a Fredholm operator. The operator $Q : \mathbf{B} \to \mathbf{B}$ is a Fredholm one if and only if it is representable in the form $Q = P^{-1} + V$ $(Q = P_1^{-1} + V_1)$, where P^{-1} is an inverse to a bounded operator P and V is a completely continuous operator (P_1^{-1} is an inverse to the bounded P_1 and V_1 is a finite-dimensional operator) [6]. An operator $Q = (I + V) : \mathbf{B} \to \mathbf{B}$ is a Fredholm one, if a certain degree V^m of V is completely continuous [6]. If the operator V itself is completely continuous, the operator $Q = I + V$ is said to be a *canonical Fredholm operator.*

In the above given examples $Q = I - K$, where K is an integral operator. For (2.3),

$$(Kz)(t) = \int_a^t P(t)z(s)ds$$

and it is a completely continuous operator. For (2.4),

$$(Kz)(t) = \int_a^b \left\{ H_1(t,s) + \int_s^b H(t,\tau)d\tau \right\} z(s)ds.$$

Here K^2 is a completely continuous operator. The property of being completely continuous of these operators may be established by V. Maksimov's Theorem 6.1 [7, 1] which is given below.

Theorem 2.2. *Let $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ be a Noether operator,* $\operatorname{ind} L = n$. *Then* $\dim\ker \mathcal{L} \geq n$ *and* $\dim\ker \mathcal{L} = n$ *if and only if the equation* (2.1) *is solvable for each $f \in \mathbf{B}$.*

Proof. Recall that $\dim\ker \mathcal{L} - \dim\ker \mathcal{L}^* = n$. Besides the equation $\mathcal{L}x = f$ is solvable for each $f \in \mathbf{B}$ if and only if $\dim\ker \mathcal{L}^* = 0$ (Theorem 1.1). ■

We call the vector $X = (x_1, \dots, x_\nu)$ whose components constitute a basis for the kernel of $\mathcal{L}$ *the fundamental vector* of the equation $\mathcal{L}x = 0$ and the components $x_1, \dots, x_\nu$ we call *the fundamental system* of solutions of this equation.

Let $l = [l^1, \dots, l^m] : \mathbf{D} \to \mathbf{R}^m$ be a linear bounded vector-functional, $\alpha = \operatorname{col}\{\alpha^1, \dots, \alpha^m\} \in \mathbf{R}^m$. The system

$$\mathcal{L}x = f, \quad lx = \alpha \tag{2.5}$$

is called a *linear boundary value problem.*

If $R(\mathcal{L}) = \mathbf{B}$ and $\dim\ker\mathcal{L} = n$, the question of solvability of (2.5) is the one of solvability of a linear system of algebraic equations with the matrix $lX = (l^i x_j)$, $i = 1, \ldots, m$; $j = 1, \ldots, n$. Indeed, since the general solution of the equation $\mathcal{L}x = f$ has the form

$$x = \sum_{j=1}^{n} c_j x_j + v,$$

where v is any solution of this equation, $c_1, \ldots, c_n$ are arbitrary constants, the problem (2.5) is solvable if and only if the algebraic system

$$\sum_{j=1}^{n} l^i x_j c_j = \alpha^i - l^i v, \quad i = 1, \ldots, m,$$

in respect of $c_1, \ldots, c_n$ is solvable. In such a way the problem (2.5) has a unique solution for each $f \in \mathbf{B}$, $\alpha \in \mathbf{R}^m$ if and only if $m = n$ and $\det lX \neq 0$. The determinant $\det lX$ is said to be the *determinant of the problem* (2.5).

By applying the operator l to the two parts of the equality (1.2), we get the decomposition

$$lx = \Phi\delta x + \Psi r x, \tag{2.6}$$

where $\Phi : \mathbf{B} \to \mathbf{R}^m$ is a linear bounded vector-functional. We will denote the matrix defined by the linear operator $\Psi : \mathbf{R}^n \to \mathbf{R}^m$ also by Ψ.

Using the representations (2.2) and (2.6), we can rewrite the problem (2.5) in the form of the equation

$$\begin{pmatrix} Q & A \\ \Phi & \Psi \end{pmatrix} \begin{pmatrix} \delta x \\ r x \end{pmatrix} = \begin{pmatrix} f \\ \alpha \end{pmatrix}. \tag{2.5}$$

The operator

$$\begin{pmatrix} Q^* & \Phi^* \\ A^* & \Psi^* \end{pmatrix} : \mathbf{B}^* \times (\mathbf{R}^m)^* \to \mathbf{B}^* \times (\mathbf{R}^n)^*$$

is adjoint to the operator

$$\begin{pmatrix} Q & A \\ \Phi & \Psi \end{pmatrix} : \mathbf{B} \times \mathbf{R}^n \to \mathbf{B} \times \mathbf{R}^m.$$

Taking into account the isomorphism between the spaces $\mathbf{B}^* \times (\mathbf{R}^n)^*$ and $\mathbf{D}^*$, we therefore call the equation

$$\begin{pmatrix} Q^* & \Phi^* \\ A^* & \Psi^* \end{pmatrix} \begin{pmatrix} \omega \\ \gamma \end{pmatrix} = \begin{pmatrix} g \\ \eta \end{pmatrix}$$

to be adjoint to the problem (2.5).

Lemma 2.1. *The operator* $[\delta, l] : \mathbf{D} \to \mathbf{B} \times \mathbf{R}^m$ *is a Noether one with* $\operatorname{ind}[\delta, l] = n - m$.

Proof. We have $[\delta, l] = [\delta, 0] + [0, l]$, where the symbol "0" denotes a null-operator on the corresponding space. The operator $[0, l] : \mathbf{D} \to \mathbf{B} \times \mathbf{R}^m$ is compact because the finite-dimensional operator $l : \mathbf{D} \to \mathbf{R}^m$ is compact. Compact perturbations does not change the index of the operator (Theorem 1.2). Therefore it is sufficient to prove Lemma only for the operator $[\delta, 0]$.

The direct product $\mathbf{B} \times \{0\}$ is the range of values of $[\delta, 0]$. The homogeneous adjoint equation to the problem $[\delta, 0]x = \{f, 0\}$ is reducible to one equation $\omega = 0$ in the space $\mathbf{B}^* \times (\mathbf{R}^m)^*$. The solutions of this equation are the pairs $\{0, \gamma\}$. Therefore $\dim \ker[\delta, 0]^* = m$.

Thus $[\delta, 0] : \mathbf{D} \to \mathbf{B} \times \mathbf{R}^m$ is a Noether operator with $\operatorname{ind}[\delta, 0] = n - m$. ■

Rewrite the problem (2.5) in the form of one equation

$$[\mathcal{L}, l]x = \{f, \alpha\}. \tag{2.5}$$

Theorem 2.3. *The problem* (2.5) *is a Noether one if and only if the principal part* $Q : \mathbf{B} \to \mathbf{B}$ *of* $\mathcal{L}$ *is a Noether operator and also* $\operatorname{ind}[\mathcal{L}, l] = \operatorname{ind} Q + n - m$.

Proof. The operator $[\mathcal{L}, l]$ has the representation

$$[\mathcal{L}, l] = \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} [\delta, l] + [Ar, 0],$$

where $I : \mathbf{R}^m \to \mathbf{R}^m$ is an identical operator, and the symbol "0" denotes the null operator on the corresponding space. Indeed,

$$\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} [\delta, l]x + [Ar, 0]x = \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} \operatorname{col}\{\delta x, lx\} + \operatorname{col}\{Arx, 0\} = \\ = \operatorname{col}\{Q\delta x + Arx, lx\}.$$

The operator $Q : \mathbf{B} \to \mathbf{B}$ is Noether if and only if the operator

$$\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} : \mathbf{B} \times \mathbf{R}^m \to \mathbf{B} \times \mathbf{R}^m$$

is Noether with

$$\operatorname{ind} \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} = \operatorname{ind} Q.$$

Therefore the operator

$$\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} [\delta, l] : \mathbf{D} \to \mathbf{B} \times \mathbf{R}^m$$

is a Noether one if and only if Q is a Noether operator and also

$$\operatorname{ind} \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} [\delta, l] = \operatorname{ind} \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} + \operatorname{ind}[\delta, l] = \operatorname{ind} Q + n - m$$

(Theorems 1.3 and 1.7). The product $Ar : \mathbf{D} \to \mathbf{B}$ is compact. Hence the operator $[Ar, 0] : \mathbf{D} \to \mathbf{B} \times \mathbf{R}^m$ is also compact. Now we get the conclusion of the Theorem from the fact that compact perturbation does not violate the property of being Noether operator and does not change the index. ■

It should be noted the following Corollaries from Theorem 2.3 under the assumption of $\mathcal{L}$ being a Noether operator.

Corollary 2.1. *The problem* (2.5) *is a Fredholm one if and only if* $\operatorname{ind} Q = m - n$.

Corollary 2.2. *The problem* (2.5) *is solvable if and only if the right hand side* $\{f, \alpha\}$ *is orthogonal to all the solutions* $\{\omega, \gamma\}$ *of the homogeneous adjoint equation*

$$Q^*\omega + \Phi^*\gamma = 0,$$
$$A^*\omega + \Psi^*\gamma = 0.$$

The condition of being orthogonal has the form

$$\langle \omega, f \rangle + \langle \gamma, \alpha \rangle = 0.$$

Everywhere below we assume that the operator $\mathcal{L}$ is Noether with $\operatorname{ind} \mathcal{L} = n$ which means that Q is a Fredholm operator. Under such an assumption, by virtue of Corollary 2.1 the problem (2.5) is a Fredholm one if and only if $m = n$.

The functionals $l^1, \ldots, l^m$ are assumed to be linear independent.

We will call the special case of (2.5) when $l = r$ a *principal boundary value problem.* The equation $[\delta, r]x = \{f, \alpha\}$ is just the problem which is the base of the isomorphism $\mathcal{J}^{-1} = [\delta, r]$ between $\mathbf{D}$ and $\mathbf{B} \times \mathbf{R}^n$.

Theorem 2.4. *The principal boundary value problem*

$$\mathcal{L}x = f, \quad rx = \alpha \tag{2.7}$$

is uniquely solvable if and only if the principal part $Q : \mathbf{B} \to \mathbf{B}$ *of* $\mathcal{L}$ *has the bounded inverse* $Q^{-1} : \mathbf{B} \to \mathbf{B}$. *The solution* x *of (2.7) admits the representation*

$$x = \Lambda Q^{-1} f + (Y - \Lambda Q^{-1} A)\alpha. \tag{2.8}$$

Proof. Using the decomposition (2.2), we can rewrite (2.7) in the form

$$Q\delta x + Arx = f, \quad rx = \alpha.$$

If Q is invertible, then

$$\delta x = Q^{-1} f - Q^{-1} A\alpha.$$

An application to this equality of the operator Λ yields (2.8) because $\Lambda\delta = I - Yr$.

If Q is not invertible and y is a nontrivial solution of the equation $Qy = 0$, the homogeneous problem

$$\mathcal{L}x = 0, \quad rx = 0$$

has a nontrivial solution x, for instance $x = \Lambda y$. ■

From (2.8) one can see that the vector $X = Y - \Lambda Q^{-1} A$ is fundamental and also $rX = E$ (Here A denotes the vector defying the finite-dimensional operator $A : \mathbf{R}^n \to \mathbf{B}$).

Theorem 2.5. *The following assertions are equivalent.*
a) $R(\mathcal{L}) = \mathbf{B}$.
b) $\dim \ker \mathcal{L} = n$.
c) *There exists a vector-functional* $l : \mathbf{D} \to \mathbf{R}^n$ *such that the problem* (2.5) *is uniquely solvable for each* $f \in \mathbf{B}$, $\alpha \in \mathbf{R}^n$.

Proof. The equivalence of the assertions a) and b) was established while proving Theorem 2.2.

Let $\dim \ker \mathcal{L} = n$ and $l = [l^1, \dots, l^n]$, the system $l^1, \dots, l^n$ be biorthogonal to the basis $x_1, \dots, x_n$ of the kernel of $\mathcal{L} : l^i x_j = \delta_{ij}$, $i, j = 1, \dots, n$ and δ_{ij} be the Kronecker symbol. Then the problem (2.5) with such l has the unique solution

$$x = X(\alpha - lv) + v,$$

where $X = (x_1, \dots, x_n)$ and v is any solution of $\mathcal{L}x = f$. This is seen by taking into account that $lX = E$. Conversely, if (2.5) is uniquely solvable for each f and α, then the solutions of the problems

$$\mathcal{L}x = 0, \quad lx = \alpha_i, \quad \alpha_i \in \mathbf{R}^n, \quad i = 1, \dots, n,$$

can be taken as the basis $x_1, \dots, x_n$ provided the matrix $(\alpha_1, \dots, \alpha_n)$ is invertible. Thus the equivalence of the assertions b) and c) is proved. ■

§ 3. Green Operator

We will consider here the boundary value problem

$$\mathcal{L}x = f, \quad lx = \alpha \tag{3.1}$$

under the assumption that the dimension m of l (the number of the boundary conditions) is equal to n. By virtue of Corollary 2.2, such a condition is necessary for unique solvability of the problem (3.1). Recall that we assume $\mathcal{L}$ to be a Noether operator with $\operatorname{ind} \mathcal{L} = n$ ($\operatorname{ind} Q = 0$). If $m = n$, then the problem (3.1) is Fredholm ($[\mathcal{L}, l] : \mathbf{D} \to \mathbf{B} \times \mathbf{R}^n$ is a Fredholm operator). Consequently, for this problem the assertions "the problem has a unique solution for some right hand side $\{f, \alpha\}$ (the problem is uniquely solvable)", "the problem is solvable for each $\{f, \alpha\}$ (the problem is solvable everywhere)", "the problem is everywhere and uniquely solvable" are equivalent.

Let (3.1) be uniquely solvable and denote $[\mathcal{L}, l]^{-1} = \{G, X\}$. Then the solution x of the problem (3.1) has the representation

$$x = Gf + X\alpha.$$

The operator $G : \mathbf{B} \to \mathbf{D}$ is called *the Green operator of the problem* (3.1), the vector $X = (x_1, \ldots, x_n)$ is a fundamental vector for the equation $\mathcal{L}x = 0$ and also $lX = E$.

It should be noted that Λ is the Green operator of the problem $\delta x = f$, $rx = \alpha$.

Theorem 3.1. *A linear bounded operator $G : \mathbf{B} \to \mathbf{D}$ is Green operator of the boundary value problem* (3.1) *if and only if the following conditions are fulfilled.*

a) *G is a Noether operator with* $\operatorname{ind} G = -n$.

b) $\ker G = \{0\}$.

Proof. $\{G, X\} : \mathbf{B} \times \mathbf{R}^n \to \mathbf{D}$ is a one-to-one mapping if G is the Green operator of (3.1). So a) and b) are fulfilled by virtue of Theorem 1.8. Conversely, let G be such that a) and b) are fulfilled. Then $\dim \ker G^* = n$. If $l^1, \ldots, l^n$ constitute a basis of $\ker G^*$ and $l = [l^1, \ldots, l^n]$, then $R(G) = \ker l$. G is the Green operator of (3.1), where

$$\mathcal{L}x = G^{-1}(x - Ulx) + Vlx,$$

G^{-1} is the inverse to $G : \mathbf{B} \to \ker l$, $U = (u_1, \ldots, u_n)$ with $u_i \in \mathbf{D}$ is a vector such that $lU = E$, and $V = (v_1, \ldots, v_n)$ with $v_i \in \mathbf{B}$ is an arbitrary vector. ■

Theorem 3.2. *Let the problem* (3.1) *be uniquely solvable and G be the Green operator of this problem. Let further $U = (u_1, \ldots, u_n)$, $u_i \in \mathbf{D}$, $lU = E$. Then the vector*

$$X = U - G\mathcal{L}U$$

is a fundamental one to the equation $\mathcal{L}x = 0$.

Proof. $\dim \ker \mathcal{L} = n$ by virtue of Theorem 2.5 and the unique solvability of (3.1). The components of X are linear independent because $lX = E$. The equality $\mathcal{L}X = 0$ can be checked immediately. ■

Theorem 3.3. *Let G and G_1 be the Green operators of the problems*

$$\mathcal{L}x = f, \quad lx = \alpha$$

and

$$\mathcal{L}x = f, \quad l_1x = \alpha.$$

Let further X be the fundamental vector of $\mathcal{L}x = 0$. Then

$$G = G_1 - X(lX)^{-1}lG_1.$$

Proof. The general solution of $\mathcal{L}x = f$ has the representation

$$x = Xc + G_1 f,$$

where $c \in \mathbf{R}^n$ is an arbitrary vector. Define c in such a way that $lx = 0$. We have

$$0 = lx = lXc + lG_1 f.$$

Hence

$$c = -(lX)^{-1} lG_1 f$$

and the solution x of the half homogeneous problem $\mathcal{L}x = f$, $lx = 0$ has the form

$$x = (G_1 - X(lX)^{-1} lG_1) f = Gf. \quad \blacksquare$$

In the investigation of particular boundary value problems and some properties of Green operator, it is useful to employ the "elementary Green operator" W_l which is possible to construct for any boundary condition $lx = \alpha$. Beforehand we will prove the following proposition.

Lemma 3.1. *For any linear bounded vector-functional* $l = [l^1, \ldots, l^n] : \mathbf{D} \to \mathbf{R}^n$ *with linear independent components, there exists a vector* $U = (u_1, \ldots, u_n)$ *with* $u_i \in \mathbf{D}$ *such that* $\det rU \neq 0$ *and* $\det lU \neq 0$.

Proof. Let U_1 and U_2 be n-dimensional vectors such that $\det rU_1 \neq 0$ and $lU_2 = E$. Let further

$$U = U_1 + \mu U_2$$

where μ is a numerical parameter. The function $\psi(\mu) = \det rU$ is continuous and $\psi(0) \neq 0$. Hence $\psi(\mu) \neq 0$ on an interval $(-\mu_0, \mu_0)$. The polynomial $P(\mu) = \det lU = \det(lU_1 + \mu E)$ has no more than n roots. Consequently, there exists such a $\mu_1 \in (-\mu_0, \mu_0)$ that $P(\mu_1) \neq 0$. For $U = U_1 + \mu_1 U_2$ we have: $\det rU \neq 0$ and $\det lU \neq 0$. $\blacksquare$

Let $U = (u_1, \ldots, u_n)$, $u_i \in \mathbf{D}$, $\det rU \neq 0$, $lU = E$. Define the operator $W_l : \mathbf{B} \to \mathbf{D}$ by:

$$W_l = \Lambda - U\Phi, \tag{3.2}$$

where $U : \mathbf{R}^n \to \mathbf{D}$ is a finite-dimensional operator corresponding to U and $\Phi : \mathbf{B} \to \mathbf{R}^n$ is the principal part of the vector-functional l (see the equality (2.6)). Let further $\mathcal{L}_0 : \mathbf{D} \to \mathbf{B}$ be defined by

$$\mathcal{L}_0 x = \delta x - \delta U (rU)^{-1} rx. \tag{3.3}$$

Theorem 3.4. *W_l is the Green operator of the boundary value problem*

$$\mathcal{L}_0 x = f, \quad lx = \alpha. \tag{3.4}$$

Proof. The principal boundary value problem for the equation $\mathcal{L}_0 x = f$ is uniquely solvable. Consequently, the dimension of the fundamental vector for $\mathcal{L}_0 x = 0$ is equal to n. By immediate substitution, we get $\mathcal{L}_0 U = 0$. The problem (3.4) is solvable because $lU = E$. We have

$$\begin{aligned}\mathcal{L}_0 W_l f &= \delta(\Lambda f - U\Phi f) - \delta U(rU)^{-1} r(\Lambda f - U\Phi f) = \\ &= f - \delta U\Phi f + \delta U(rU)^{-1} rU\Phi f = f, \\ lW_l f &= \Phi\delta(\Lambda f - U\Phi f) + \Psi r(\Lambda f - U\Phi f) = \\ &= \Phi f - lU\Phi f = 0. \quad \blacksquare\end{aligned}$$

The collection of all Green operators corresponding to a given vector-functional $l : \mathbf{D} \to \mathbf{R}^n$ is of the form

$$G = W_l \Gamma, \tag{3.5}$$

where Γ is a linear homeomorphism of $\mathbf{B}$ into $\mathbf{B}$. Indeed, if $\Gamma : \mathbf{B} \to \mathbf{B}$ is a homeomorphism, then by virtue of Theorem 3.1 $W_l\Gamma$ is a Green operator of (3.1). Conversely, any Green operator $G : \mathbf{B} \to \ker l$ may be represented by (3.5), where $\Gamma = W_l^{-1}G$, $W_l^{-1} : \ker l \to \mathbf{B}$ is the inverse to $W_l : \mathbf{B} \to \ker l$.

Theorem 3.5. *The collection of all Green operators $G : \mathbf{B} \to \mathbf{D}$ is defined by*

$$G = (\Lambda - Uv)\Gamma,$$

where $U = (u_1, \dots, u_n)$, $u_i \in \mathbf{D}$, $\det rU \neq 0$, $v : \mathbf{B} \to \mathbf{R}^n$ is a linear bounded vector-functional, and Γ is a linear homeomorphism of the space $\mathbf{B}$ into $\mathbf{B}$.

Proof. $W = \Lambda - Uv$ is the Green operator of the problem (3.4), where $lx = v\delta x + [E - v\delta U](rU)^{-1} rx$. Indeed,

$$\begin{aligned}\mathcal{L}_0 W f &= \delta(\Lambda - Uv)f - \delta U(rU)^{-1} r(\Lambda - Uv)f = \\ &= f - \delta Uvf + \delta U(rU)^{-1} rUvf = f, \\ lWf &= v\delta(\Lambda - Uv)f + [E - v\delta U](rU)^{-1} r(\Lambda - Uv)f = \\ &= vf - v\delta Uvf - [E - v\delta U]vf = 0.\end{aligned}$$

Now the assertion of the Theorem follows from the representation (3.5). $\blacksquare$

In the investigation of boundary value problems, an important part belongs to the so called “W-method” [8] which is based on an expedient choice of an auxiliary “model” equation $\mathcal{L}_1 x = f$. The foundation to this method is laid by the following assertion.

Theorem 3.6. *Let a model boundary value problem*

$$\mathcal{L}_1 x = f, \quad lx = 0$$

be uniquely solvable and $W : \mathbf{B} \to \mathbf{D}$ be the Green operator of this problem. The problem (3.1) *is uniquely solvable, if and only if the operator $\mathcal{L}W : \mathbf{B} \to$*

B *has the continuous inverse* $[\mathcal{L}W]^{-1}$. *In this case, the Green operator* G *of the problem* (3.1) *has the representation*

$$G = W[\mathcal{L}W]^{-1}.$$

Proof. There is a one-to-one correspondence between the set of solutions $z \in$ **B** of the equation $\mathcal{L}Wz = f$ and the set of solutions $x \in$ **D** of the problem (3.1) with homogeneous boundary conditions $lx = 0$ the correspondence is defined by $x = Wz$ and $z = \mathcal{L}_1 x$. Consequently the problem (3.1) is uniquely solvable and also the solution x of the problem (3.1) for $\alpha = 0$ has the representation $x = W[\mathcal{L}W]^{-1}f$. Thus $G = W[\mathcal{L}W]^{-1}$. ■

In the applications of Theorem 3.6 one may put $W = W_l$, where W_l is defined by (3.2). Let the operator $U : \mathbf{R}^n \to \mathbf{D}$ be defined as above by the vector $U = (u_1, \dots, u_n)$, $u_i \in \mathbf{D}$, $\det rU \neq 0$, $lU = E$. Let further $\Phi : \mathbf{B} \to \mathbf{R}^n$ be the principal part of $l : \mathbf{D} \to \mathbf{R}^n$. Define the operator $F : \mathbf{B} \to \mathbf{B}$ by $F = \mathcal{L}U\Phi$.

Corollary 3.1. *The boundary value problem* (3.1) *is uniquely solvable if and only if the operator* $(Q - F) : \mathbf{B} \to \mathbf{B}$ *has the bounded inverse. The Green operator of this problem has the representation*

$$G = W_l(Q - F)^{-1}. \tag{3.6}$$

The proof follows from the fact that W_l is the Green operator of the model problem $\mathcal{L}_0 x = z$, $lx = 0$, where $\mathcal{L}_0$ is defined by (3.3) and

$$\mathcal{L}W_l = \mathcal{L}\Lambda - \mathcal{L}U\Phi = Q\delta\Lambda - Ar\Lambda - \mathcal{L}U\Phi = Q - \mathcal{L}U\Phi = Q - F.$$

The following assertions characterize some properties of the Green operator of the problem (3.1) connected with the properties of the principal part Q of $\mathcal{L}$.

Theorem 3.7. *Assume that a boundary value problem* (3.1) *is uniquely solvable. Let* $P : \mathbf{B} \to \mathbf{B}$ *be a linear bounded operator with bounded inverse* P^{-1}. *The Green operator of this problem has the representation*

$$G = W_l(P + H), \tag{3.7}$$

where $H : \mathbf{B} \to \mathbf{B}$ *is compact, if and only if the principal part* Q *of* $\mathcal{L}$ *may be represented in the form* $Q = P^{-1} + V$, *where* $V : \mathbf{B} \to \mathbf{B}$ *is compact.*

Proof. Let $G = W_l(Q - F)^{-1}$ (see (3.6)), $Q = P^{-1} + V$. Define $V_1 = V - F$. Then

$$(Q - F)^{-1} = (P^{-1} + V_1)^{-1} = (I + PV_1)^{-1}P = (I + H_1)P = P + H,$$

where $H : \mathbf{B} \to \mathbf{B}$ and $H_1 : \mathbf{B} \to \mathbf{B}$ are compact operators.
Conversely, if $(Q - F)^{-1} = P + H$ then

$$Q = F + (P + H)^{-1} = F + (I + P^{-1}H)^{-1}P^{-1} =$$
$$= F + (I + V_1)P^{-1} = P^{-1} + V,$$

where $V : \mathbf{B} \to \mathbf{B}$ and $V_1 : \mathbf{B} \to \mathbf{B}$ are compact operators. ■

Theorem 3.8. *A linear bounded operator $G : \mathbf{B} \to \mathbf{D}$ is the Green operator of the problem* (3.1)*, where $Q = P^{-1} + V$, if and only if $\ker G = \{0\}$ and*

$$G = \Lambda P + T \tag{3.8}$$

with a compact operator $T : \mathbf{B} \to \mathbf{D}$.

Proof. If G is the Green operator and $Q = P^{-1} + V$ then (3.8) immediately follows from (3.7) and (3.2).

Conversely, if G has the form (3.8), then G is a Noether operator with $\operatorname{ind} G = -n$. By virtue of Theorem 3.1, G is the Green operator of the problem (3.1). From $\mathcal{L}G = I$ it follows that $QP + \mathcal{L}T = I$. Hence $Q = P^{-1} + V$, where $V = -\mathcal{L}TP^{-1}$. ■

We now state two Corollaries of Theorem 3.8.

Corollary 3.2. *The representation $\delta G = P + H$, where $H : \mathbf{B} \to \mathbf{B}$ is a compact operator and $P : \mathbf{B} \to \mathbf{B}$ is a linear bounded operator with a bounded inverse P^{-1} is possible if and only if G is the Green operator of the problem* (3.1)*, where $Q = P^{-1} + V$, V is a compact operator.*

Proof. If $\delta G = P + H$, then

$$G = \Lambda P + \Lambda H + YrG$$

and due to Theorem 3.8, $Q = P^{-1} + V$.

Conversely, if $Q = P^{-1} + V$, then $G = \Lambda P + T$ and, consequently, $\delta G = P + \delta T$. ■

Corollary 3.3. *The operator δG is canonical Fredholm if and only if the principal part Q of $\mathcal{L}$ is canonical Fredholm.*

§ 4. Boundary Value Problems which are Not Everywhere and Uniquely Solvable

We assume as above that $\operatorname{ind} \mathcal{L} = n$ ($\operatorname{ind} Q = 0$) and, in addition, that the equation $\mathcal{L}x = 0$ has an n-dimensional fundamental vector X. By Theorem 2.5, the equation $\mathcal{L}x = f$ is solvable for each $f \in \mathbf{B}$.

We will consider the boundary value problem

$$\mathcal{L}x = f, \quad lx = \alpha \tag{4.1}$$

without the assumption that the number m of boundary conditions is equal to n.

Denote $\rho = \operatorname{rank} lX$. In the case $\rho > 0$, we may assume without loss of generality that the determinant of the rank ρ composed of the elements in the left top of the matrix lX does not vanish. Let us choose a fundamental vector as follows. In the case $\rho > 0$, the elements $x_1, \dots, x_\rho$ are selected such that $l^i x_j = \delta_{ij}$, $i, j = 1, \dots, \rho$ (δ_{ij} is the symbol of Kronecker). If $0 \le \rho < n$, the homogeneous problem $\mathcal{L}x = 0$, $lx = 0$ has $n - \rho$ linearly independent solutions $u_1, \ldots, u_{n-\rho}$. In the capacity of fundamental vector, everywhere below we will take $X = (u_1, \ldots, u_n)$ if $\rho = 0$, $X = (x_1, \ldots, x_\rho, u_1, \ldots, u_{n-\rho})$ if $0 < \rho < n$, and $X = (x_1, \ldots, x_n)$ if $\rho = n$.

Recall that the problem (4.1) can not be Fredholm if $m \neq n$ (Corollary 2.1) and the question on solvability of the problem (4.1) is the question on solvability of a linear algebraic system with the matrix lX.

Consider the cases corresponding to all possible relations between the numbers n, m and ρ.

The case $n = m = \rho$ was investigated in the previous sections.

If $\rho = m < n$, the problem is solvable (but not uniquely) for any $f \in \mathbf{B}$, $\alpha = \{\alpha^1, \ldots, \alpha^m\} \in \mathbf{R}^m$. To obtain the representation of the solution in this case, we can supplement the functionals $l^1, \ldots, l^m$ by such additional $l^{m+1}, \ldots, l^n$ that

$$\det \left(l^{m+i} u_j\right)_{i,j=1}^{n-m} \neq 0.$$

The determinant of the problem

$$\mathcal{L}x = f, \quad l^1 x = \alpha^1, \ldots, l^n, x = \alpha^n$$

does not vanish and therefore this problem is uniquely solvable. Using the Green operator G of this problem, we can represent the solutions of (4.1) in the form

$$x = Gf + \sum_{i=1}^{m} \alpha^i x_i + \sum_{i=1}^{n-m} c_i u_i,$$

where $c_1, \ldots, c_{n-m}$ are arbitrary constants.

In all the other cases (4.1) is not everywhere solvable. The conditions of solvability can be obtained by using the Green operator of any uniquely solvable boundary value problem for the equation $\mathcal{L}x = f$. Such a problem exists by virtue of Theorem 2.5.

Let $\rho = n < m$. In this case, the homogeneous problem $\mathcal{L}x = 0$, $lx = 0$ has only the trivial solution. Thus if the problem (4.1) is solvable, the solution is unique and is a solution of the problem

$$\mathcal{L}x = f, \quad l^i x = \alpha^i, \quad i = 1, \ldots, n$$

(recall our convention $(l^i x_j)_{i,j=1}^n = E$). If G is the Green operator of the latter problem, the solution of the problem (4.1) in the cases of its solvability has the representation

$$x = Gf + \sum_{i=1}^{n} \alpha^i x_i,$$

and the necessary and sufficient condition of solvability of the problem (4.1) obtains the form

$$\alpha^j = l^j G f + \sum_{i=1}^{n} \alpha^i l^j x_i, \quad j = n+1, \ldots, m.$$

If $\rho < n \le m$ or $\rho < m < n$, the solution of the problem (4.1) can not be unique. Let us choose functionals $\bar{l}^{\rho+1}, \dots, \bar{l}^n$ such that $\det(\bar{l}^{\rho+i} u_j)_{i,j=1}^{n-\rho} \ne 0$. Then the problem

$$\mathcal{L}x = f, \quad \bar{l}^i x = \alpha^i, \quad i = 1, \dots, n$$

for $\rho = 0$ or the problem

$$\mathcal{L}x = f, \quad l^i x = \alpha^i, \quad i = 1, \dots, \rho, \quad \bar{l}^{\rho+j} x = \alpha^{\rho+j}, \quad j = 1, \dots, n-\rho$$

for $\rho > 0$ is uniquely solvable. Using the Green operator G of this problem, we may write the solutions of (4.1) in the case of its solvability in the form

$$x = Gf + \sum_{i=1}^{n} c_i u_i$$

for $\rho = 0$, and in the form

$$x = Gf + \sum_{i=1}^{\rho} \alpha^i x_i + \sum_{i=1}^{n-\rho} c_i u_i$$

for $\rho > 0$. Here $c_1, \dots, c_{n-\rho}$ are arbitrary constants. The necessary and sufficient condition of solvability of (4.1) obtains the form

$$\alpha^j = l^j G f, \quad j = 1, \dots, m$$

for $\rho = 0$, and

$$\alpha^j = l^j G f + \sum_{i=1}^{\rho} \alpha^i l^j x_i, \quad j = \rho+1, \dots, m$$

for $\rho > 0$.

In the theory of ordinary differential equation, it is widely used the so called "generalized Green function" for representation of solutions of the linear boundary value problem in the case, where the solution is not unique. The construction of such a function (the kernel of the integral operator, the generalized Green operator) is based on a well known structure of E. Schmidt [9]. This structure permits to construct for an irreversible operator H a finite-dimensional operator F^0 such that there exists the bounded inverse $(H + F^0)^{-1}$. The classical scheme of the construction of generalized Green operators for differential equations is entirely extended for abstract functional-differential equations. We will dwell here on this scheme.

Due to Corollary 3.1, the Fredholm operator $Q - F = \mathcal{L}W_l : \mathbf{B} \to \mathbf{B}$ is irreversible if $\rho < m = n$. In this case, the half homogeneous problem

$$\mathcal{L}x = f, \quad lx = 0 \tag{4.2}$$

is solvable if and only if the function f is orthogonal to all the elements of the basis of the kernel of $(Q - F)^*$. Using the procedure which will be given below, we will construct an operator F^0 such that the operator $Q - F + F^0$

will have the inverse $\Gamma = (Q - F + F^0)^{-1}$. The product $G^0 = W_l\Gamma$ has the property: if the problem (4.2) is solvable, then the solutions of this problem may be represented in the form

$$x = G^0 f + \sum_{i=1}^{n-\rho} c_i u_i, \tag{4.3}$$

where $u_i = W_l y_i$, $y_1, \ldots, y_{n-\rho}$ is the basis of the kernel of the operator $Q-F$ and $c_1, \ldots, c_{n-\rho}$ are arbitrary constants. This operator $G^0 : \mathbf{B} \to \ker l$ is said to be the *generalized Green operator* of the problem (4.2). By virtue of (3.5), G^0 is an ordinary Green operator of a certain boundary value problem

$$\mathcal{L}^0 x = f, \quad lx = \alpha. \tag{4.4}$$

To construct the operator F^0, let us choose any system $\varphi_1, \ldots, \varphi_{n-\rho}$ of functionals from the space $\mathbf{B}^*$ biorthogonal to $y_1, \ldots, y_{n-\rho}$ ($\langle \varphi_i, y_j \rangle = \delta_{ij}$, $i, j = 1, \ldots, n-\rho$) and a system $z_1, \ldots, z_{n-\rho}, z_i \in \mathbf{B}$, biorthogonal to the basis $\omega_1, \ldots, \omega_{n-\rho}$ of the kernel of $(Q-F)^*$. Schmidt's structure defines the operator $F^0 : \mathbf{B} \to \mathbf{B}$ by

$$F^0 y = \sum_{i=1}^{n-\rho} \langle \varphi_i, y \rangle z_i.$$

By virtue of Schmidt's Lemma [9], there exists a bounded inverse $\Gamma = (Q - F + F^0)^{-1}$. Also, if y satisfies the equation $(Q - F + F^0)y = f$ and conditions of orthogonality $\langle \omega_i, f \rangle = 0$, $i = 1, \ldots, n-\rho$, then $(Q-F)y = f$. Indeed, in this case we get from $(Q-F)y = f - F^0 y$ that

$$\langle \omega_i, f - F^0 y \rangle = 0, \quad i = 1, \ldots, n-\rho.$$

Hence

$$\langle \omega_i, f \rangle - \langle \omega_i, F^0 y \rangle = -\left\langle \omega_i, \sum_{j=1}^{n-\rho} c_j z_j \right\rangle = 0, \quad i = 1, \ldots, n-\rho,$$

where c_i are some arbitrary constants. But the latter equality is possible only if $c_1 = \cdots = c_{n-\rho} = 0$. Therefore $F^0 y = 0$ and consequently $(Q-F)y = f$ and $x = W_l \Gamma f$ is a solution of (4.2). Hence we get the representation (4.3).

Remark 4.1. In the construction of a generalized Green operator one, can use instead of W_l defined by (3.6) the Green operator of any model problem $\mathcal{L}_1 x = f$, $lx = 0$ (see Theorem 3.6).

The not everywhere solvable problem (4.1) may become everywhere solvable by some generalization of the notion of the solution. For instance, the solution of (4.4) for the equation $\mathcal{L}^0 x = f$ constructed on the base of Schmidt's structure may be considered as a kind of such a generalization. A generalization of the notion of solution of (4.1) was used in [10, 11] by extending of the space $\mathbf{D}$. In this connection, the construction of a generalized (extended) everywhere solvable boundary value problem requires sometimes additional boundary conditions. So, the problem

$$\dot{x}(t) = f(t), \quad x(a) - x(b) = 0$$

has absolutely continuous solutions not for any summable f. If we declare the solution to be a function admitting a finite discontinuity at a fixed point $\tau \in (a, b)$, then the extended problem

$$\dot{y}(t) = f(t), \quad y(a) - y(b) = \alpha, \quad y(\xi) = \beta, \quad \xi \in (a, b),$$

has a unique solution for each f, α and β. Indeed, in this case the fundamental system of solutions of the equation $\dot{y}(t) = 0$ consists of two functions $y_1 = 1$ and $y_2 = \chi_{[\tau,b]}(t)$ ($\chi_{[\tau,b]}(t)$ is a characteristic function of $[\tau, b]$). The determinant of the problem

$$\Delta = \begin{vmatrix} 0 & 1 \\ -1 & \chi_{[\tau,b]}(\xi) \end{vmatrix} \neq 0.$$

Next we will prove under the assumption that the space $\mathbf{D}$ admits a finite-dimensional extension that for any not everywhere solvable problem (4.1), it is possible to construct an extended problem which is uniquely solvable.

The problem (4.1) is not everywhere solvable, if $\rho = n < m$, $\rho < n \le m$ or $\rho < m < n$. These cases are characterized by the inequality $m - \rho > 0$.

Let the space $\mathbf{D}$ be embedded into a Banach space $\widetilde{\mathbf{D}}$ in such a way that $\widetilde{\mathbf{D}} = \mathbf{D} \oplus M^\mu$, where M^μ is a finite-dimensional subspace of the dimension μ. Any linear extension $\widetilde{\mathcal{L}} : \widetilde{\mathbf{D}} \to \mathbf{B}$ of $\mathcal{L}$ is a Noether operator with $\operatorname{ind} \widetilde{\mathcal{L}} = \operatorname{ind} \mathcal{L} + \mu = n + \mu$. (Theorem 1.6). As far as $R(\mathcal{L}) = \mathbf{B}$, also $R(\widetilde{\mathcal{L}}) = \mathbf{B}$, therefore $\dim \ker \widetilde{\mathcal{L}} = n + \mu$.

Let $\widetilde{\mathcal{L}} : \widetilde{\mathbf{D}} \to \mathbf{B}$ and $\tilde{l} : \widetilde{\mathbf{D}} \to \mathbf{R}^m$ be a linear extension of $\mathcal{L}$ and l. Consider the boundary value problem

$$\widetilde{\mathcal{L}} y = f, \quad \tilde{l} y = \alpha \tag{4.5}$$

in the space $\widetilde{\mathbf{D}}$. Since $\dim \ker \widetilde{\mathcal{L}} = n + \mu$, this problem can be uniquely and everywhere solvable only if $\mu = m - n$. If $\mu > m - n$, it is necessary to add to m boundary conditions some more $\mu + n - m$ conditions.

The problem (4.5) if $\mu + n - m = 0$, and the problem

$$\widetilde{\mathcal{L}} y = f, \quad \tilde{l} y = \alpha, \quad \tilde{l}_1 y = \alpha_1, \tag{4.6}$$

if $m + n - \mu > 0$ is called *an extended boundary value problem.* Here $\tilde{l}_1 : \widetilde{\mathbf{D}} \to \mathbf{R}^{\mu+n-m}$ is any bounded vector-functional.

As it was noted above, the inequality $\mu \ge m - n$ is necessary for unique solvability of the extended problem.

Everywhere below $y_1, \ldots, y_\mu$ are such elements of a fundamental system of the equation $\widetilde{\mathcal{L}} y = 0$ which do not belong to $\mathbf{D}$.

For the beginning, consider an extended problem for a uniquely solvable problem (4.1).

Theorem 4.1. *Let* $m = n$, *the problem* (4.1) *be uniquely solvable and* $\widetilde{\mathbf{D}} = \mathbf{D} \oplus M^\mu$. *For any linear extensions* $\widetilde{\mathcal{L}} : \widetilde{\mathbf{D}} \to \mathbf{B}$, $\tilde{l} : \widetilde{\mathbf{D}} \to \mathbf{R}^n$ *of* $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ *and* $l : \mathbf{D} \to \mathbf{R}^n$, *there exists a vector-functional* $\tilde{l}_1 : \widetilde{\mathbf{D}} \to \mathbf{R}^\mu$ *such that the problem* (4.6) *is uniquely solvable.*

Proof. For any linear extension $\tilde{l}$ of vector-functional l, we have $\tilde{l}X = lX$. Therefore $\det \tilde{l}X \neq 0$. Let us choose $y_1, \dots, y_\mu$ such that $\tilde{l}y_i = 0$, $i = 1, \dots, \mu$. This is possible since letting

$$y_i = \bar{y}_i - \sum_{j=1}^{n} c_j x_j$$

for a fundamental system $x_1, \dots, x_n, \bar{y}_1, \dots, \bar{y}_\mu$ of the solutions of the equation $\widetilde{\mathcal{L}}y = 0$, we get for constants $c_1, \dots, c_n$ the system

$$\sum_{j=1}^{n} c_j \tilde{l}^k x_j = \tilde{l}^k \bar{y}_i, \quad k = 1, \dots, n$$

with the determinant not equal to zero. Let us take now a system of functionals $\tilde{l}^{n+i} : \widetilde{\mathbf{D}} \to \mathbf{R}^1$, $i = 1, \dots, \mu$, such that

$$\Delta = \det(\tilde{l}^{n+i} y_j)_{i,j=1}^{\mu} \neq 0.$$

Then the determinant of the problem (4.6) with $\tilde{l}_1 = [\tilde{l}^{n+1}, \dots, \tilde{l}^{n+\mu}]$ is equal to $\Delta \cdot \det lX \neq 0$. ■

Any element $y \in \widetilde{\mathbf{D}}$ has the representation

$$y = \pi y + \sum_{i=1}^{\mu} z_i \lambda^i y, \tag{4.7}$$

where $\pi : \widetilde{\mathbf{D}} \to \mathbf{D}$ is a projector, $z_1, \dots, z_\mu$ is a basis of M^μ, $\lambda = [\lambda^1, \dots, \lambda^\mu] : \widetilde{\mathbf{D}} \to \mathbf{R}^\mu$ is a vector-functional such that $\lambda x = 0$ for each $x \in \mathbf{D}$ and $\lambda^i z_j = \delta_{ij}$, $i, j = 1, \dots, \mu$. From (4.7) it follows that any linear extension $\widetilde{\mathcal{L}} : \widetilde{\mathbf{D}} \to \mathbf{B}$ of the operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ has the representation

$$\widetilde{\mathcal{L}}y = \mathcal{L}\pi y + \sum_{i=1}^{\mu} a_i \lambda^i y, \tag{4.8}$$

where $a_i = \widetilde{\mathcal{L}} z_i$, and also for any $a_i \in \mathbf{B}$, $i = 1, \dots, \mu$, the last equality defines a linear extension of $\mathcal{L}$ on the space $\widetilde{\mathbf{D}}$. Analogously, the representation

$$\tilde{l}y = l\pi y + \Gamma \lambda y, \tag{4.9}$$

where $\Gamma = (\gamma_{ij})$ is a numerical $m \times n$-matrix, defines the general form of the linear extension $\tilde{l} : \widetilde{\mathbf{D}} \to \mathbf{R}^m$ of the vector-functional $l : \mathbf{D} \to \mathbf{R}^m$.

In what follows, $m - \rho > 0$. The next assertion recommends for uniquely solvable problem a more precise estimate of the number μ then the above given inequality $\mu \geq m - n$.

Theorem 4.2. *Let $\widetilde{\mathbf{D}} = \mathbf{D} \oplus M^{\mu}$. If the problem* (4.1) *has a uniquely solvable extended problem, then $\mu \geq m - \rho$.*

Proof. Let $\mu < m - \rho$. If $\rho = n$, then $\mu < m - n$. Therefore only the case $\rho < n$ needs the proof.

Let $\widetilde{\mathcal{L}}$ and $\tilde{l}$ be any linear extensions on the spaces $\widetilde{\mathbf{D}}$ of $\mathcal{L}$ and l, respectively. If $\mu = m - n$, then the determinant of the problem (4.5), the determinant of the order m, is equal to zero because it has non-zero elements only at the columns corresponding to $x_1, \dots, x_\rho, y_1, \dots, y_\mu$, if $\rho > 0$ or $y_1, \dots, y_\mu$, if $\rho = 0$. The number of such columns is equal to $\rho + \mu < m$.

Let $\mu > m-n$. Then the determinant of the problem (4.6) is equal to zero. Indeed, the cofactors of the minors of the $(\mu + n - m)$-th order composed of the elements of the rows corresponding to the vector-functional $\tilde{l}_1$ are determinants of the m-th order. These determinants are equal to zero. ■

Theorem 4.3. *Let $\widetilde{\mathbf{D}} = \mathbf{D} \oplus M^{m-\rho}$. For any linear extension $\widetilde{\mathcal{L}} : \widetilde{\mathbf{D}} \to \mathbf{B}$ of the operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ there exists a linear extension $\tilde{l} : \widetilde{\mathbf{D}} \to \mathbf{R}^m$ of the vector-functional $l : \mathbf{D} \to \mathbf{R}^m$, and in the case $\rho < n$ a vector-functional $\tilde{l}_1 : \widetilde{\mathbf{D}} \to \mathbf{R}^{n-\rho}$ such that the extended problem* (4.5) *if $\rho = n$ or the extended problem* (4.6) *if $\rho < n$ is uniquely solvable.*

Proof. The operator $\widetilde{\mathcal{L}}$ admits the representation (4.8), where $\mu = m - \rho$. Denote by v_i any solution of the equation

$$\mathcal{L}x = -a_i,$$

and let $y_i = v_i + z_i$, $i = 1, \dots, m - \rho$. Thus $u_1, \dots, u_n, y_1, \dots, y_m$ if $\rho = 0$, $x_1, \dots, x_\rho, u_1, \dots, u_{n-\rho}, y_1, \dots, y_{m-\rho}$ if $0 < \rho < n$ and $x_1, \dots, x_n, y_1, \dots, y_{m-n}$ if $\rho = n$ is a fundamental system of solutions of the equation

$$\widetilde{\mathcal{L}}y = 0$$

Let $0 < \rho \leq n$. Denote $Y = (x_1, \dots, x_\rho, y_1, \dots, y_{m-\rho})$. We will show that it is possible to choose a $m \times (m - \rho)$-matrix Γ for the corresponding extension (4.9) of the vector-functional l such that $\det \tilde{l}Y \neq 0$. Due to special choice of $x_1, \dots, x_\rho$, we have $\tilde{l}^i x_j = \delta_{ij}$, $i, j = 1, \dots, \rho$ for any extension $\tilde{l}$. Further, $\pi Y = (x_1, \dots, x_\rho, v_1, \dots, v_{m-\rho})$; $\lambda x_i = 0$, $i = 1, \dots, \rho$; $\lambda^i y_j = \delta_{ij}$, $i, j = 1, \dots, m - \rho$. Therefore

$$\tilde{l}Y = l\pi Y + \Gamma\lambda Y =$$

$$= \begin{pmatrix} 1 & 0 & \dots & 0 & l^1 v_1 & \dots & l^1 v_{m-\rho} \\ 0 & 1 & \dots & 0 & l^2 v_1 & \dots & l^2 v_{m-\rho} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & l^\rho v_i & \dots & l^\rho v_{m-\rho} \\ l^{\rho+1} x_1 & l^{\rho+1} x_2 & \dots & l^{\rho+1} x_\rho & l^{\rho+1} v_1 & \dots & l^{\rho+1} v_{m-\rho} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ l^m x_1 & l^m x_2 & \dots & l^m x_\rho & l^m v_1 & \dots & l^m v_{m-\rho} \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & \dots & 0 & \gamma_{11} & \dots & \gamma_{1,m-\rho} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \gamma_{m1} & \dots & \gamma_{m,m-\rho} \end{pmatrix}.$$

The matrix Γ may be chosen, for instance, as follows. Let $\gamma_{ij} = -l^i v_j$ for $i = 1, \dots, \rho$, $j = 1, \dots, m-\rho$, and the numbers $\gamma_{\rho+i,j}$, $i, j = 1, \dots, m-\rho$, be chosen such that

$$\Delta = \det \left(l^{\rho+i} v_j + \gamma_{\rho+i,j} \right)_{i,j=1}^{m-\rho} \neq 0.$$

Then $\det \tilde{l} Y = \Delta \neq 0$.

If $\rho = n$, the theorem is proved because the problem (4.5) with the constructed extension $\tilde{l}$ is uniquely solvable.

If $0 < \rho < n$, we choose in addition a vector-functional

$$\tilde{l}_1 = [\tilde{l}^{m+1}, \dots, \tilde{l}^{m+n-\rho}] : \widetilde{\mathbf{D}} \to \mathbf{R}^{n-\rho}$$

such that

$$\Delta_1 = \det \left(\tilde{l}^{m+i} u_j \right)_{i,j=1}^{n-\rho} \neq 0.$$

The determinant of the problem (4.6) with the above constructed extension $\tilde{l}$ and the vector-functional $\tilde{l}_1$ is equal to $\Delta_1 \cdot \det \tilde{l} Y \neq 0$.

If $\rho = 0$, let $Y = (y_1, \dots, y_m)$. In this case,

$$\tilde{l} Y = \left(l^i v_j + \gamma_{ij} \right)_{i,j=1}^{m}.$$

Let us choose γ_{ij} such that $\det \tilde{l} Y \neq 0$ and further, as above, take a vector-functional

$$\tilde{l}_1 = \left[\tilde{l}^{m+1}, \dots, \tilde{l}^{m+n} \right] : \widetilde{\mathbf{D}} \to \mathbf{R}^n$$

such that

$$\Delta_1 = \det \left(\tilde{l}^{m+i} u_j \right)_{i,j=1}^{n} \neq 0.$$

Then the determinant of the problem (4.6) will be equal to $\Delta_1 \cdot \det \tilde{l} Y \neq 0$. ■

Denote by $\widetilde{G}$ the Green operator of the extended problem (the problem (4.5) if $\rho = n$ or (4.6) if $\rho < n$). Then the solution of the problem admits the representation

$$y = \widetilde{G} f + Z (\tilde{l}_\rho Z)^{-1} \alpha_\rho,$$

where Z is a fundamental vector of the equation $\widetilde{\mathcal{L}} y = 0$; $\tilde{l}_\rho = \tilde{l}$, $\alpha_\rho = \alpha$ if $\rho = n$ and $\tilde{l}_\rho = [\tilde{l}, \tilde{l}_1]$, $\alpha_\rho = \{\alpha, \alpha_1\}$, if $\rho < n$.

Theorems 4.2. and 4.3 provide the minimal number $\mu = m - \rho$ for which there exists a uniquely solvable extended problem to the problem (4.1). If $\mu > m - \rho$, the uniquely solvable extended problem also exists by virtue of Theorem 4.1. If the rank of the matrix lX is unknown, then we can take $\mu = m$ for the construction of uniquely solvable extended problem. It will demand n additional boundary conditions. The inequality $\mu \geq m - \rho$ could be used for the estimation of the rank of the matrix lX : if for a certain μ there exists a uniquely solvable extended problem, then $\operatorname{rank} lX \geq m - \mu$.

§ 5. Continuous Dependence on Parameters of Solution of the Boundary Value Problem

One of the central places in the theory of differential equations belongs to the question on conditions guaranteeing continuous dependence on the parameters λ, α of the solution of the Cauchy problem

$$\dot{x}(t) = f(t, x(t), \lambda), \quad x(a) = \alpha.$$

J. Kurzweil [12] has approached this question in the following generalized formulation: under which conditions does the sequence $\{x_k\}$ of the solutions of the problems

$$\dot{x}(t) = f_k(t, x(t)), \quad x(a) = \alpha_k, \quad k = 1, 2, \ldots$$

converge to the solution x_0 of the "limiting case"

$$\dot{x}(t) = f_0(t, x(t)), \quad x(a) = \alpha_0$$

of the problems? The general linear boundary value problem

$$\mathcal{L}x = f, \quad lx = \alpha$$

was studied in [1]. Here $\mathcal{L} : \mathbf{D}^n \to \mathbf{L}^n$ and $l : \mathbf{D}^n \to \mathbf{R}^n$ are linear operators, $\mathbf{D}^n$ and $\mathbf{L}^n$ are Banach spaces of n-dimensional vector functions defined on $[a, b]$, absolutely continuous and summable, respectively. The conditions of convergence to the solution x_0 of the limiting case

$$\mathcal{L}_0 x = f_0, \quad l_0 x = \alpha_0$$

of the solutions x_k of the problems

$$\mathcal{L}_k x = f_k, \quad l_k x = \alpha_k, \quad k = 1, 2, \ldots$$

are formulated.

An analogous question for linear abstract functional differential equation was considered by A. V. Anokhin on the base of general theory of G. M. Vainikko [13]. Each operator $\mathcal{L}_k$ and vector-functional l_k are defined on their own space in Anokhin's setting of the question. Anokhin's theorem was published in [14] without proof. We offer below the thorough proof of this theorem.

We will formulate here the definitions and propositions of Vainikko's paper [13] which are required for the proof of the main theorem. We provide these results of G. M. Vainikko in the form we are in need of. In the brackets there are indicated those general propositions of the paper [13] on the base of which the theorems stated below are formulated.

Let $\mathbf{E}_0$ and $\mathbf{E}_k$, $k = 1, 2, \dots$, be Banach spaces.

Definition 5.1. A system $\mathcal{P} = (\mathcal{P}_k)$, $k = 1, 2, \dots$, of linear bounded operators $\mathcal{P}_k : \mathbf{E}_0 \to \mathbf{E}_k$ is said to be connecting for $\mathbf{E}_0$ and $\mathbf{E}_k$, $k = 1, 2, \dots$, if

$$\lim_{k\to\infty} \|\mathcal{P}_k u\|_{\mathbf{E}_k} = \|u\|_{\mathbf{E}_0}$$

for any $u \in \mathbf{E}_0$.

Observe that the norms of the operators $\mathcal{P}_k$ are bounded in common ($\sup\limits_k \| \mathcal{P}_k \| < \infty$) due to the principle of uniform boundedness.

Definition 5.2. The sequence $\{u_k\}$, $u_k \in \mathbf{E}_k$, is said to be $\mathcal{P}$-convergent to $u_0 \in \mathbf{E}_0$ (this fact is denoted by $u_k \xrightarrow{\mathcal{P}} u_0$) if

$$\lim_{k\to\infty} \|u_k - \mathcal{P}_k u_0\|_{\mathbf{E}_k} = 0.$$

Observe that from the $\mathcal{P}$-convergence $u_k \xrightarrow{\mathcal{P}} u_0$ follows, in particular, that $\lim_{k\to\infty} \|u_k\|_{\mathbf{E}_k} = \|u_0\|_{\mathbf{E}_0}$.

Definition 5.3. The sequence $\{u_k\}$, $u_k \in \mathbf{E}_k$, is said to be $\mathcal{P}$-compact if any of its subsequences includes a $\mathcal{P}$-convergent subsequence.

Let further $\mathbf{F}_0$ and $\mathbf{F}_k$, $k = 1, 2, \dots$, be Banach spaces; $\mathcal{P} = (\mathcal{P}_k)$, $k = 1, 2, \dots$, be a connecting system for $\mathbf{E}_0$ and $\mathbf{E}_k$; $\mathcal{Q} = (\mathcal{Q}_k)$, $k = 1, 2, \dots$, be a connecting system for $\mathbf{F}_0$ and $\mathbf{F}_k$; $A_k : \mathbf{E}_k \to \mathbf{F}_k$; $k = 0, 1, \dots$, be linear bounded operators.

Definition 5.4. A sequence $\{A_k\}$ is said to be $\mathcal{PQ}$-convergent to A_0 (this fact is denoted by $A_k \xrightarrow{\mathcal{PQ}} A_0$) if the sequence $\{A_k u_k\}$ is $\mathcal{Q}$-convergent to $A_0 u_0$ for any $\mathcal{P}$-convergent to $u_0 \in \mathbf{E}_0$ sequence $\{u_k\}$, $u_k \in \mathbf{E}_k$.

Theorem 5.1 (Proposition 2.1). *If* $A_k \xrightarrow{\mathcal{PQ}} A_0$, *then* $\sup\limits_k \|A_k\| < \infty$.

If a sequence $\{\gamma_k\}$ of the elements of a Banach space converges to γ_0 by the norm, we will denote this fact henceforth by $\gamma_k \to \gamma_0$.

Theorem 5.2 (Proposition 3.5 and Theorem 4.1). *Let the sequences* $\{B_k\}$ *and* $\{C_k\}$ *of linear bounded operators* $B_k : \mathbf{E}_k \to \mathbf{F}_k$, $C_k : \mathbf{E}_k \to \mathbf{F}_k$, $k = 1, 2, \dots$, *be* $\mathcal{PQ}$*-convergent to* B_0 *and* C_0, *respectively. Let further the following conditions be fulfilled.*

1. $R(B_0) = \mathbf{F}_0$, *there exist continuous inverses* B_k^{-1}, $k = 1, 2, \dots$, *and* $\sup\limits_k \|B_k^{-1}\| < \infty$.

2. *The sequence* $\{C_k u_k\}$ *is* $\mathcal{Q}$*-compact for any bounded sequence* $\{u_k\}$, $u_k \in \mathbf{E}_k$ $(\sup\limits_k \|u_k\|_{\mathbf{E}_k} < \infty)$.

3. The operators $A_k = B_k + C_k$, $k = 0, 1, \dots,$ *are Fredholm,* $\ker A_0 = \{0\}$.

Then for $k = 0$ *and all sufficiently large* k*, there exist bounded inverses* A_k^{-1} *and*

$$A_k^{-1} y_k \xrightarrow{\mathcal{P}} A_0^{-1} y_0 \quad \text{if} \quad y_k \xrightarrow{\mathcal{Q}} y_0 \quad \left(A_k^{-1} \xrightarrow{\mathcal{QP}} A_0^{-1}\right).$$

Remark 5.1. Condition 1 of Theorem 5.2 is equivalent to Condition 1*. There exist bounded inverses $B_k^{-1} : \mathbf{F}_k \to \mathbf{E}_k$, $k = 0, 1, \dots,$ and $B_k^{-1} \xrightarrow{\mathcal{QP}} B_0^{-1}$.

The implication $1^* \Rightarrow 1$ is obvious. Let us prove the implication $1 \Rightarrow 1^*$.

As it was shown in [13] (Proposition 3.3), conditions imposed on the operators B_k guarantee the existence of a $\gamma > 0$ such that

$$\|B_0 u\|_{\mathbf{F}_0} \geq \gamma \|u\|_{\mathbf{E}_0}$$

for any $u \in \mathbf{E}_0$, and from $R(B_0) = \mathbf{F}_0$ it follows the existence of bounded inverse B_0^{-1}.

Let $y_k \xrightarrow{\mathcal{Q}} y_0$, $y_k \in \mathbf{F}_k$. We have

$$\begin{aligned}\left\|B_k^{-1} y_k - \mathcal{P}_k B_0^{-1} y_0\right\|_{\mathbf{E}_k} \leq \left\|B_k^{-1} y_k - B_k^{-1} \mathcal{Q}_k y_0\right\|_{\mathbf{E}_k} + \\ + \left\|B_k^{-1} \mathcal{Q}_k y_0 - \mathcal{P}_k B_0^{-1} y_0\right\|_{\mathbf{E}_k}.\end{aligned}$$

$$\left\|B_k^{-1} y_k - B_k^{-1} \mathcal{Q}_k y_0\right\|_{\mathbf{E}_k} \leq \|B_k^{-1}\| \left\|y_k - \mathcal{Q}_k y_0\right\|_{\mathbf{F}_k} \to 0.$$

Denote $B_0^{-1} y_0 = u_0$. Then

$$\left\|B_k^{-1} \mathcal{Q}_k y_0 - \mathcal{P}_k B_0^{-1} y_0\right\|_{\mathbf{E}_k} \leq \|B_k^{-1}\| \left\|\mathcal{Q}_k B_0 u_0 - B_k \mathcal{P}_k u_0\right\|_{\mathbf{F}_k} \to 0$$

since $\mathcal{P}_k u_0 \xrightarrow{\mathcal{P}} u_0$ and $B_k \xrightarrow{\mathcal{PQ}} B_0$. ■

Let $\mathbf{D}_k$ and $\mathbf{B}_k$ be Banach spaces, $\mathbf{D}_k$ be isomorphic to the direct product $\mathbf{B}_k \times \mathbf{R}^n$,

$$\{\Lambda_k, Y_k\} : \mathbf{B}_k \times \mathbf{R}^n \to \mathbf{D}_k \quad \left([\delta_k, r_k] = \{\Lambda_k, Y_k\}^{-1}\right)$$

be isomorphisms, and

$$\|u\|_{\mathbf{D}_k} = \|\delta_k u\|_{\mathbf{B}_k} + |r_k u|, \quad k = 0, 1, \dots.$$

Let further $\mathcal{H} = (\mathcal{H}_k)$ be a connecting system for $\mathbf{B}_0$, $\mathbf{B}_k$ and $\mathcal{P} = (\mathcal{P}_k)$ be a connectig system for $\mathbf{D}_0$, $\mathbf{D}_k$, $k = 1, 2, \dots$. By $\mathcal{H}_0$ and $\mathcal{P}_0$ we denote the identical operators in the spaces $\mathbf{B}_0$ and $\mathbf{D}_0$, respectively.

Consider sequences $\{\mathcal{L}_k\}$, $\{l_k\}$ of bounded linearly Noether operators $\mathcal{L}_k : \mathbf{D}_k \to \mathbf{B}_k$, $\operatorname{ind} \mathcal{L}_k = n$, and bounded linear vector-functionals $l_k : \mathbf{D}_k \to \mathbf{R}^n$ with linear independent components, $k = 0, 1, \dots$. We will assume that $\mathcal{L}_k \xrightarrow{\mathcal{PH}} \mathcal{L}_0$ and $l_k u_k \to l_0 u_0$ provided $u_k \xrightarrow{\mathcal{P}} u_0$.

Let the boundary value problem

$$\mathcal{L}_0 x = f, \quad l_0 x = \alpha \tag{5.1}$$

be uniquely solvable. Consider the question on conditions which guarantee unique solvability for all sufficiently large k of the problems

$$\mathcal{L}_k x = f, \quad l_k x = \alpha \tag{5.2}$$

as well as the convergence $x_k \xrightarrow{\mathcal{P}} x_0$ for any sequences $\{f_k\}$ and $\{\alpha_k\}$ with $f_k \xrightarrow{\mathcal{H}} f_0$ and $\alpha_k \to \alpha_0$. Here x_k is the solution of the problem

$$\mathcal{L}_k x = f_k, \quad l_k x = \alpha_k, \tag{5.3}$$

x_0 is the solution of the problem

$$\mathcal{L}_0 x = f_0, \quad l_0 x = \alpha_0. \tag{5.4}$$

We will assume the spaces $\mathbf{B}_k$, $k = 1, 2, \ldots$, to be isomorphic to $\mathbf{B}_0$ and the operators $\mathcal{H}_k : \mathbf{B}_0 \to \mathbf{B}_k$ of the connecting system for $\mathbf{B}_0$ and $\mathbf{B}_k$ to be isomorphisms and $\sup\limits_k \|\mathcal{H}_k^{-1}\| < \infty$.

Define the connecting system $\mathcal{Q} = (\mathcal{Q}_k)$ of the isomorphisms of the spaces $\mathbf{B}_0 \times \mathbf{R}^n$ and $\mathbf{B}_k \times \mathbf{R}^n$ by

$$\mathcal{Q}_k\{f, \alpha\} = \big\{\mathcal{H}_k f, \alpha\big\}, \quad \{f, \alpha\} \in \mathbf{B}_0 \times \mathbf{R}^n,$$
$$\mathcal{Q}_k^{-1}\{f, \alpha\} = \big\{\mathcal{H}_k^{-1} f, \alpha\big\}, \quad \{f, \alpha\} \in \mathbf{B}_k \times \mathbf{R}^n.$$

Thus if $f_k \xrightarrow{\mathcal{H}} f_0$ and $\alpha_k \to \alpha_0$, then $\{f_k, \alpha_k\} \xrightarrow{\mathcal{Q}} \{f_0, \alpha_0\}$. It is easy to see that

$$\|\mathcal{Q}_k\| = \max\big\{\|\mathcal{H}_k\|, 1\big\}, \quad \|\mathcal{Q}_k^{-1}\| = \max\big\{\|\mathcal{H}_k^{-1}\|, 1\big\}.$$

We will choose the connecting system $\mathcal{P} = (\mathcal{P}_k)$ for the spaces $\mathbf{D}_0$ and $\mathbf{D}_k$ in such a way that the operators $\mathcal{P}_k$ have bounded inverses and $\sup\limits_k \|\mathcal{P}_k^{-1}\| < \infty$. For instance,

$$\mathcal{P}_k = \Lambda_k \mathcal{H}_k \delta_0 + Y_k r_0 = \big\{\Lambda_k, Y_k\big\}\mathcal{Q}_k[\delta_0, r_0].$$

Then

$$\mathcal{P}_k^{-1} = \Lambda_0 \mathcal{H}_k^{-1}\delta_k + Y_0 r_k = \{\Lambda_0, Y_0\}\mathcal{Q}_k^{-1}[\delta_k, r_k],$$
$$\|\mathcal{P}_k\| = \|\mathcal{Q}_k\|, \quad \|\mathcal{P}_k^{-1}\| = \|\mathcal{Q}_k^{-1}\|.$$

This system is a connecting one for $\mathbf{D}_0$ and $\mathbf{D}_k$. Really,

$$\delta_k \mathcal{P}_k u = \mathcal{H}_k \delta_0 u, \quad r_k \mathcal{P}_k u = r_0 u$$

for any $u \in \mathbf{D}_0$. Therefore

$$\|\mathcal{P}_k u\|_{\mathbf{D}_k} = \|\mathcal{H}_k \delta_0 u\|_{\mathbf{B}_k} + |r_0 u| \to \|\delta_0 u\|_{\mathbf{B}_0} + |r_o u| = \|u\|_{\mathbf{D}_0}.$$

(The possibility of choosing of $\mathcal{P}_k$ will be considered more extensive in the end of this section).

We will prove the following Theorem 5.3 under the assumptions:

a) There exists a connecting system $\mathcal{H} = (\mathcal{H}_k)$ of isomorphisms for the spaces $\mathbf{B}_0$ and $\mathbf{B}_k$ such that

$$\sup_k \|\mathcal{H}_k^{-1}\| < \infty.$$

b) The connecting system $\mathcal{P} = (\mathcal{P}_k)$ for $\mathbf{D}_0$ and $\mathbf{D}_k$ is chosen such that the operators $\mathcal{P}_k : \mathbf{B}_0 \to \mathbf{B}_k$ are isomorphisms and

$$\sup_k \|\mathcal{P}_k^{-1}\| < \infty.$$

c) $\mathcal{L}_k \overset{\mathcal{P}\mathcal{H}}{\to} \mathcal{L}_0$ and $l_k u_k \to l_0 u_0$, if $u_k \overset{\mathcal{P}}{\to} u_0$.

Theorem 5.3. *Let the problem* (5.1) *be uniquely solvable. Then the problems* (5.2) *are uniquely solvable for all sufficiently large* k. *For any sequences* $\{f_k\}$, $\{\alpha_k\}$ *with* $f_k \overset{\mathcal{H}}{\to} f_0$, $\alpha_k \to \alpha_0$, *the solutions* x_k *of the problems* (5.3) *are* $\mathcal{P}$*-convergent to the solution* x_0 *of the problem* (5.4) *if and only if there exists a vector-functional* $l : \mathbf{D}_0 \to \mathbf{R}^n$ *such that the problems*

$$\mathcal{H}_k^{-1}\mathcal{L}_k\mathcal{P}_k x = f, \quad lx = \alpha \tag{5.5}$$

are uniquely solvable for $k = 0$ *and all sufficiently large* k *by any right hand side* $\{f, \alpha\} \in \mathbf{B}_0 \times \mathbf{R}^n$ *and* $v_k \to v_0$, *where* $v_k \in \mathbf{D}_0$ *are the solutions of the problems* (5.5) *holds.*

Let us rewrite the problems (5.1) - (5.4) in the form

$$[\mathcal{L}_0, l_0]x = \{f, \alpha\}, \tag{5.1}$$
$$[\mathcal{L}_k, l_k]x = \{f, \alpha\}, \tag{5.2}$$
$$[\mathcal{L}_k, l_k]x = \{f_k, \alpha_k\}, \tag{5.3}$$
$$[\mathcal{L}_0, l_0]x = \{f_0, \alpha_0\}. \tag{5.4}$$

Then the Theorem 5.3 may be stated as follows.

Let the operator $[\mathcal{L}_0, l_0] : \mathbf{D}_0 \to \mathbf{B}_0 \times \mathbf{R}^n$ *be continuously invertiple. Then the operators* $[\mathcal{L}_k, l_k] : \mathbf{D}_k \to \mathbf{B}_k \times \mathbf{R}^n$ *are continuously invertiply for all sufficiently large* k *and also*

$$[\mathcal{L}_k, l_k]^{-1} \overset{\mathcal{Q}\mathcal{P}}{\to} [\mathcal{L}_0, l_0]^{-1},$$

if and only if there exists a vector-functional $l : \mathbf{D}_0 \to \mathbf{R}^n$ *such that the operators*

$$[\mathcal{H}_k^{-1}\mathcal{L}_k\mathcal{P}_k, l] : \mathbf{D}_0 \to \mathbf{B}_0 \times \mathbf{R}^n$$

are continuously invertiply for $k = 0$ *and all sufficiently large* k *and*

$$[\mathcal{H}_k^{-1}\mathcal{L}_k\mathcal{P}_k, l]^{-1}\{f, \alpha\} \to [\mathcal{L}_0, l]^{-1}\{f, \alpha\}$$

for any $\{f, \alpha\} \in \mathbf{B}_0 \times \mathbf{R}^n$.

Beforehand we will prove two lemmas.

Denote $\mathcal{M}_k = \mathcal{H}_k^{-1}\mathcal{L}_k\mathcal{P}_k$.

Lemma 5.1. *$\mathcal{M}_k u \to \mathcal{L}_0 u$ for any $u \in \mathbf{D}_0$ if and only if $\mathcal{L}_k \overset{\mathcal{PH}}{\to} \mathcal{L}_0$.*

Proof. Let $\mathcal{L}_k \overset{\mathcal{PH}}{\to} \mathcal{L}_0$. Since $\mathcal{P}_k u \overset{\mathcal{P}}{\to} u$ and $\sup\limits_k \|\mathcal{H}_k^{-1}\| < \infty$, we have

$$\mathcal{M}_k u - \mathcal{L}_0 u = \mathcal{H}_k^{-1}\big(\mathcal{L}_k \mathcal{P}_k u - \mathcal{H}_k \mathcal{L}_0 u\big) \to 0.$$

Conversely, let $\mathcal{M}_k u \to \mathcal{L}_0 u$ for any $u \in \mathbf{D}_0$ and $u_k \overset{\mathcal{P}}{\to} u_0$. We have

$$\begin{aligned}\mathcal{L}_k u_k - \mathcal{H}_k \mathcal{L}_0 u_0 &= \mathcal{H}_k \mathcal{M}_k \mathcal{P}_k^{-1} u_k - \mathcal{H}_k \mathcal{L}_0 u_0 = \\ &= \mathcal{H}_k \Big\{ \mathcal{M}_k \big(\mathcal{P}_k^{-1} u_k - u_0\big) + \big(\mathcal{M}_k u_0 - \mathcal{L}_0 u_0\big) \Big\} \to 0,\end{aligned}$$

since $\mathcal{P}_k^{-1} u_k \to u_0$, $\mathcal{M}_k u_0 \to \mathcal{L}_0 u_0$, $\sup\limits_k \|\mathcal{H}_k\| < \infty$, $\sup\limits_k \|\mathcal{M}_k\| < \infty$. ■

Denote

$$\begin{aligned}\Phi_k &= \big[\mathcal{L}_k, l\mathcal{P}_k^{-1}\big] : \mathbf{D}_k \to \mathbf{B}_k \times \mathbf{R}^n, \\ F_k &= \big[\mathcal{H}_k^{-1} \mathcal{L}_k \mathcal{P}_k, l\big] : \mathbf{D}_0 \to \mathbf{B}_0 \times \mathbf{R}^n \quad (\Phi_0 = F_0).\end{aligned}$$

Lemma 5.2. *The operators Φ_k and F_k are (or are not) continuously invertible simultaneously; $\Phi_k^{-1} \overset{\mathcal{QP}}{\to} \Phi_0^{-1}$ if and only if $F_k^{-1} y \to F_0^{-1} y$ for any $y \in \mathbf{B}_0 \times \mathbf{R}^n$.*

Proof. Simultaneous invertipility follows from the representation $\Phi_k = \mathcal{Q}_k F_k \mathcal{P}_k^{-1}$. Let $F_k^{-1} y \to F_0^{-1} y$ for any $y \in \mathbf{B}_0 \times \mathbf{R}^n$ and $y_k \overset{\mathcal{Q}}{\to} y_0$, $y_k \in \mathbf{B}_k \times \mathbf{R}^n$. We have

$$\Phi_k^{-1} y_k - \mathcal{P}_k \Phi_0^{-1} y_0 = \mathcal{P}_k F_k^{-1} \mathcal{Q}_k^{-1}\big(y_k - \mathcal{Q}_k y_0\big) + \mathcal{P}_k\big(F_k^{-1} y_0 - F_0^{-1} y_0\big).$$

Hence it follows that $\Phi_k^{-1} \overset{\mathcal{QP}}{\to} \Phi_0^{-1}$.

Conversely, let $\Phi_k^{-1} \overset{\mathcal{QP}}{\to} \Phi_0^{-1}$. We have

$$F_k^{-1} y - F_0^{-1} y = \mathcal{P}_k^{-1}\big(\Phi_k^{-1} \mathcal{Q}_k y - \mathcal{P}_k \Phi_0^{-1} y\big).$$

Hence $F_k^{-1} y \to F_0^{-1} y$. ■

The proof of Theorem 5.3. Sufficiency. Let us represent the operator $[\mathcal{L}_k, l_k]$ in the form

$$[\mathcal{L}_k, l_k] = \big[\mathcal{L}_k, l\mathcal{P}_k^{-1}\big] + \big[0, l_k - l\mathcal{P}_k^{-1}\big].$$

Since $\mathcal{L}_k \overset{\mathcal{PH}}{\to} \mathcal{L}_0$ and $l\mathcal{P}_k^{-1} u_k \to l u_0$ if $u_k \overset{\mathcal{P}}{\to} u_0$, we have:

$$\Phi_k = \big[\mathcal{L}_k, l\mathcal{P}_k^{-1}\big] \overset{\mathcal{PQ}}{\to} [\mathcal{L}_0, l] = \Phi_0.$$

By virtue of Lemma 5.2, for all sufficiently large k there exist continuous inverses

$$\Phi_k^{-1} = \big[\mathcal{L}_k, l\mathcal{P}_k^{-1}\big]^{-1} : \mathbf{B}_k \times \mathbf{R}^n \to \mathbf{D}_k$$

and $\Phi_k^{-1} \overset{\mathcal{QP}}{\to} \Phi_0^{-1}$. Thus, taking into account Theorem 5.1, Condition 1 is fulfilled for the sequence $\{\Phi_k\}$.

Next consider the sequence of the operators

$$C_k = \left[0, l_k - l\mathcal{P}_k^{-1}\right] : \mathbf{D}_k \to \mathbf{B}_k \times \mathbf{R}^n, \quad k = 1, 2, \dots .$$

Let $u_k \overset{\mathcal{P}}{\to} u_0$. Then $l_k u_k \to l_0 u_0$ due to the assumption c) of Theorem, and $l\mathcal{P}_k^{-1} u_k - l u_0 \to 0$ since $\mathcal{P}_k^{-1} u_k \to u_0$. Therefore

$$C_k \overset{\mathcal{PQ}}{\to} C_0 = [0, l_0 - l].$$

If the sequence $\{u_k\}$, $u_k \in \mathbf{D}_k$, is bounded, from the estimate

$$\left|(l_k - l\mathcal{P}_k^{-1}) u_k\right| \le \|l_k - l\mathcal{P}_k^{-1}\| \, \|u_k\|_{\mathbf{D}_k}$$

and the boundedness in common of the norms $\|l_k - l\mathcal{P}_k^{-1}\|$ it follows boundedness in $\mathbf{R}^n$ and consequently compactness of the sequence $\{(l_k - l\mathcal{P}_k^{-1}) u_k\}$. So the sequence $\{C_k u_k\}$ is $\mathcal{Q}$-compact. Thus Condition 2 of Theorem 5.2 is fulfilled for the operators C_k.

Further, since $A_k = [\mathcal{L}_k, l_k] = \Phi_k + C_k$ are Fredholm operators, the equality $\ker A_0 = \{0\}$ follows from the unique solvability of the problem (5.1). Thus by virtue of Theorem 5.2, continuous inverses $A_k^{-1} = [\mathcal{L}_k, l_k]^{-1}$ exist, and

$$[\mathcal{L}_k, l_k]^{-1} \overset{\mathcal{QP}}{\to} [\mathcal{L}_0, l_0]^{-1}.$$

Necessity. Let us show that we can take l_0 as of the vector-functional l. In other words, the operators

$$F_k = \left[\mathcal{H}_k^{-1} \mathcal{L}_k \mathcal{P}_k, l_0\right] : \mathbf{D}_0 \to \mathbf{B}_0 \times \mathbf{R}^n$$

have continuous inverses F_k^{-1} for all sufficiently large k and $F_k^{-1} y \to F_0^{-1} y$ for any $y \in \mathbf{B}_0 \times \mathbf{R}^n$. By virtue of Lemma 5.2, it is sufficient to verify that for all sufficiently large k, the operators

$$\Phi_k = \left[\mathcal{L}_k, l_0 \mathcal{P}_k^{-1}\right] : \mathbf{D}_k \to \mathbf{B}_k \times \mathbf{R}^n$$

have continuous inverses with $\Phi_k^{-1} \overset{\mathcal{QP}}{\to} \Phi_0^{-1}$. We have

$$\Phi_k = [\mathcal{L}_k, l_k] + \left[0, l_0 \mathcal{P}_k^{-1} - l_k\right].$$

Under the condition

$$B_k = [\mathcal{L}_k, l_k] \overset{\mathcal{PQ}}{\to} [\mathcal{L}_0, l_0] = B_0,$$

for $k = 0$ and all sufficiently large k there exist continuous inverses B_k^{-1} and $B_k^{-1} \overset{\mathcal{QP}}{\to} B_0^{-1}$.

Further we have

$$C_k = \left[0, l_0 \mathcal{P}_k^{-1} - l_k\right] \overset{\mathcal{PQ}}{\to} [0, 0] = C_0.$$

Indeed, if $u_k \overset{\mathcal{P}}{\to} u_0$, then

$$\left(l_0 \mathcal{P}_k^{-1} - l_k\right) u_k = l_0 \mathcal{P}_k^{-1} \left(u_k - \mathcal{P}_k u_0\right) - \left(l_k u_k - l_0 u_0\right) \to 0.$$

$\mathcal{Q}$-compactness of the sequence $\{C_k u_k\}$ can be proved like it was done while proving the sufficiency.

$\Phi_k = [\mathcal{L}_k, l_0\mathcal{P}_k^{-1}] = B_k + C_k$ are Fredholm operators and $\ker \Phi_0 = \{0\}$. Thus there for all sufficiently large k continuous inverses Φ_k^{-1} exist with $\Phi_k^{-1} \overset{\mathcal{QP}}{\to} \Phi_0^{-1}$. ■

The condition $v_k \to v_0$ in the statement of Theorem 5.3 may be changed by another equivalent one due to the following Theorem 5.4.

Let $M_k : \mathbf{D}_0 \to \mathbf{B}_0$, $k = 0, 1, \ldots$, be linear bounded operators, $M_k u \to M_0 u$ for any $u \in \mathbf{D}_0$, and let there exist a linear bounded vector-functional $l : \mathbf{D}_0 \to \mathbf{R}^n$ such that for each $k = 0, 1, \ldots$, the boundary value problem

$$M_k x = f, \quad lx = \alpha \tag{5.6}$$

is uniquely and everywhere solvable. Denote by v_k the solution of this problem and by z_k the solution of the half homogeneous problem

$$M_k x = f, \quad lx = 0$$

Let G_k be the Green operator of the latter problem.

Theorem 5.4. *The following assertions are equivalent.*

a) $v_k \to v_0$ *for any* $\{f, \alpha\} \in \mathbf{B}_0 \times \mathbf{R}^n$.

b) $\sup\limits_k \|z_k\|_{\mathbf{D}_0} < \infty$ *for any* $f \in \mathbf{B}_0$.

c) $G_k f \to G_0 f$ *for any* $f \in \mathbf{B}_0$.

Proof. The implication $a) \Rightarrow b)$ is obvious.

The implication $b) \Rightarrow c)$. The Green operator $G_k : \mathbf{B}_0 \to \ker l$ is an inverse to $M_k : \ker l \to \mathbf{B}_0$. From b) it follows that $\sup\limits_k \|G_k\| < \infty$. Thus by virtue of Remark 5.1, we have c).

Implication $c) \Rightarrow a)$. The solution v_k has the representation

$$v_k = G_k f + X_k \alpha,$$

where X_k is the fundamental vector of the equation $M_k x = 0$ and $lX_k = E$. By virtue of Theorem 3.2,

$$X_k = U - G_k M_k U,$$

where $U = (u_1, \ldots, u_n)$, $u_i \in \mathbf{D}_0$, $lU = E$. Thus $X_k\alpha \to X_0\alpha$ for any $\alpha \in \mathbf{R}^n$, and consequently $v_k \to v_0$. ■

Next we dwell on the question of choosing the connecting systems of isomorphisms $\mathcal{H}_k : \mathbf{B}_0 \to \mathbf{B}_k$ and $\mathcal{P}_k : \mathbf{D}_0 \to \mathbf{D}_k$. It is natural to subordinate the operators $\mathcal{P}_k$ and $\mathcal{H}_k$ to the following request of agreement

$$u_k \overset{\mathcal{P}}{\to} u_0 \Leftrightarrow \delta_k u_k \overset{\mathcal{H}}{\to} \delta_0 u_0 \quad \text{and} \quad r_k u_k \to r_0 u_0. \tag{5.7}$$

Theorem 5.5. *Let*

$$\left\|(\mathcal{H}_k\delta_0-\delta_k\mathcal{P}_k)u\right\|_{\mathbf{B}_k}\to 0 \quad \textit{and} \quad (r_0-r_k\mathcal{P}_k)u\to 0 \quad \forall u\in\mathbf{D}_0. \quad (5.8)$$

Then (5.7) *holds.*

Proof. The assertion follows from the inequalities

$$\begin{aligned}
\left\|\delta_k u_k-\mathcal{H}_k\delta_0 u_0\right\|_{\mathbf{B}_k} &\le \left\|\delta_k(u_k-\mathcal{P}_k u_0)\right\|_{\mathbf{B}_k}+\left\|(\delta_k\mathcal{P}_k-\mathcal{H}_k\delta_0)u_0\right\|_{\mathbf{B}_k},\\
|r_k u_k-r_0u_0| &\le \left|r_k(u_k-\mathcal{P}_k u_0)\right|+\left|(r_k\mathcal{P}_k-r_0)u_0\right|,\\
\left\|u_k-\mathcal{P}_k u_0\right\|_{\mathbf{D}_k} &= \left\|\delta_k(u_k-\mathcal{P}_k u_0)\right\|_{\mathbf{B}_k}+\left|r_k(u_k-\mathcal{P}_k u_0)\right| \le\\
&\le \left\|\delta_k u_k-\mathcal{H}_k\delta_0 u_0\right\|_{\mathbf{B}_k}+\left\|(\mathcal{H}_k\delta_0-\delta_k\mathcal{P}_k)u_0\right\|_{\mathbf{B}_k}+\\
&+|r_k u_k-r_0u_0|+\left|(r_0-r_k\mathcal{P}_k)u_0\right|. \ \blacksquare
\end{aligned}$$

Conversley, if $\delta_k u_k\xrightarrow{\mathcal{H}}\delta_0 u_0$ and $r_k u_k\to r_0 u_0$, where $u_k\xrightarrow{\mathcal{P}}u_0$, the limiting relations (5.8) are fulfilled. This follows from $\mathcal{P}_k u\xrightarrow{\mathcal{P}}u$.

Thus (5.7) are fulfilled if and only if the limiting relations (5.8) hold, in particular, if $\delta_k\mathcal{P}_k=\mathcal{H}_k\delta_0$ and $r_k\mathcal{P}_k=r_0$. Applying Λ_k to the first of these relations, we get

$$(I-Y_k r_k)\mathcal{P}_k=\Lambda_k\mathcal{H}_k\delta_0.$$

Hence, taking into account the second equality,

$$\mathcal{P}_k=\Lambda_k\mathcal{H}_k\delta_0+Y_k r_0=\{\Lambda_k,Y_k\}\mathcal{Q}_k[\delta_0,r_0].$$

CHAPTER II
APPLICATIONS OF THE GENERAL THEORY

The theory of abstract functional differential equation $\mathcal{L}x = f$ considers wide classes of equations from a unified point of view. The unity of the consideration is defined by the property of the principal part Q of $\mathcal{L}$ being Fredholm.

The ideas and methods of the general theory permit new approaches to many problems and adoption of standard schemes and theorems to their study. This is why the application of the abstract theory begins to play a serious role in modern investigations connected with various equations, in particular, with ordinary differential equations and systems with aftereffect.

The Chapter is devoted to some typical applications. In §§6,7, using a uniform scheme, the fundamentals of the theory of equations on the space of absolutely continuous and piecewise absolutely continuous vector-functions are presented. In §8, by the same scheme a concise theory of the n-th order scalar equations of the is presented. In §9, some singular equations are considered. A special choice of the space $\mathbf{D} \simeq \mathbf{B} \times \mathbf{R}^n$ ensures the principal part $Q : \mathbf{B} \to \mathbf{B}$ of $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ of being a Fredholm operator. In such a way, the theorems of Chapter 1 became applicable to the equation. In §10, a new approach to the minimization of square functionals is proposed. The role of the choosing of the space for the existence of the minimum is emphasized. Some efficient tests of existence of the unique point of minimum are given in terms of parameters of the functional.

§ 6. Systems of Ordinary Functional Differential Equations

The equation

$$\mathcal{L}x = f \tag{6.1}$$

with a linear operator $\mathcal{L}$ acting from the space $\mathbf{D}^n$ of absolutely continuous functions $x : [a, b] \to \mathbf{R}^n$ into the space $\mathbf{L}^n$ of summable functions $z : [a, b] \to \mathbf{R}^n$ is called a linear ordinary functional differential equation

$$\|x\|_{\mathbf{D}^n} = \|\dot{x}\|_{\mathbf{L}^n} + |x(a)|, \quad \|z\|_{\mathbf{L}^n} = \int_a^b |z(s)| ds.$$

As examples of (6.1), we can present the equations (2.3), (2.4) and also

$$(\mathcal{L}x)(t) \stackrel{def}{=} \dot{x}(t) + \int_a^b d_s R(t, s) x(s) = f(t), \quad t \in [a, b], \tag{6.2}$$

under the assumption that the elements $r_{ij}(t, s)$ of the $n \times n$-matrix $R(t, s)$ are measurable on $[a, b] \times [a, b]$, the functions $\operatorname{var}_{s \in [a,b]} r_{ij}(t, s)$ are summable on $[a, b]$ and $R(t, b) \equiv 0$.

If the isomorphism $\mathcal{J} : \mathbf{L}^n \times \mathbf{R}^n \to \mathbf{D}^n$ is defined on the base of the representation

$$x(t) = \int_a^t \dot{x}(s)ds + x(a) \quad \left((\Lambda z)(t) = \int_a^t z(s)ds, \quad (Y\beta)(t) = \beta\right),$$

then the principal parts Q of the operators $\mathcal{L}$ for the equations (2.3), (2.4) and (6.2) have the forms

$$(Q_1 z)(t) = z(t) - \int_a^t P(t)z(s)ds,$$

$$(Q_2 z)(t) = z(t) - \int_a^b \left\{ H_1(t,s) + \int_s^b H(t,\tau)d\tau \right\} z(s)ds,$$

$$(Q_3 z)(t) = z(t) - \int_a^b R(t,s)z(s)ds,$$

respectively, and the equations may be represented as follows

$$\dot{x}(t) - \int_a^t P(t)\dot{x}(s)ds - P(t)x(a) = f(t),$$

$$\dot{x}(t) - \int_a^b \left\{ H_1(t,s) + \int_s^b H(t,\tau)d\tau \right\} \dot{x}(s)ds - \left\{ \int_a^b H(t,\tau)d\tau \right\} x(a) = f(t),$$

$$\dot{x}(t) - \int_a^b R(t,s)\dot{x}(s)ds - R(t,a)x(a) = f(t).$$

The principal parts of these equations have the form $Q = I - K$, where $K : \mathbf{L}^n \to \mathbf{L}^n$ is an integral operator. The compactness of K (of K^2 if Q_2) can be established by means of the following test.

Theorem 6.1 ([1]). *Let the elements $k_{ij}(t,s)$, $i,j = 1,\dots,n$, of the matrix $K(t,s)$ be measurable on $[a,b]\times[a,b]$, for almost every $t \in [a,b]$ the functions $k_{ij}(t,\cdot)$ have finite one sided limits at each point $s \in [a,b]$ and there exist summable functions $v_{ij} : [a,b] \to \mathbf{R}^1$ such that $|k_{ij}(\cdot,s)| \le v_{ij}(\cdot)$ for each $s \in [a,b]$. Then the integral operator*

$$(Kz)(t) = \int_a^b K(t,s)z(s)ds$$

acts in the space $\mathbf{L}^n$ and is compact.

The equation with "deviated argument"

$$\begin{gathered}\dot{x}(t) - B(t)\dot{x}[g(t)] - P(t)x[h(t)] = v(t), \quad t \in [a,b], \\ x(\xi) = \varphi(\xi), \quad \dot{x}(\xi) = \psi(\xi), \quad \text{if} \quad \xi \notin [a,b],\end{gathered} \tag{6.3}$$

is also an equation of the kind (6.1). The necessity of introduction of the so called "initial functions" φ and ψ is due to the fact that the solution is defined only on $[a,b]$. In order to rewrite (6.3) in the form (6.1), we must use the linear *operation of interior superposition* defined by

$$(S_r y)(t) = \begin{cases} y[r(t)], & \text{if} \quad r(t) \in [a,b], \\ 0, & \text{if} \quad r(t) \notin [a,b]. \end{cases} \tag{6.4}$$

Define also the function θ^r by

$$\theta^r(t) = \begin{cases} 0, & \text{if} \quad r(t) \in [a,b], \\ \theta[r(t)], & \text{if} \quad r(t) \notin [a,b]. \end{cases} \tag{6.5}$$

Using these notation, we can rewrite (6.3) in the form $\mathcal{L}x = f$ defining $\mathcal{L}$ and f by

$$(\mathcal{L}x)(t) = \dot{x}(t) - B(t)(S_g\dot{x})(t) - P(t)(S_h x)(t)$$

and

$$f(t) = v(t) + B(t)\psi^g(t) + P(t)\varphi^h(t).$$

The linear operator $S_h : \mathbf{D}^n \to \mathbf{L}^n$ can be represented in the form

$$(S_h x)(t) = \int_a^b \chi_h(t,s)\dot{x}(s)ds + \chi_h(t,a)x(a),$$

where $\chi_h(t,s)$ is the characteristic function of the set

$$\Big\{(t,s) \in [a,b] \times [a,b] : \ a \le s \le h(t) \le b\Big\}.$$

Denoting

$$(Sz)(t) = B(t)(S_g z)(t), \quad (Kz)(t) = \int_a^b P(t)\chi_h(t,s)z(s)ds,$$

we obtain the principal part Q of $\mathcal{L}$ corresponding to (6.3) in the form $Q = I - S - K$.

By virtue of Theorem 6.1, the integral operator $K : \mathbf{L}^n \to \mathbf{L}^n$ with

$$K(t,s) = \chi_h(t,s)P(t)$$

is compact. Thus if (6.3) can be solved in respect to the derivative $\dot{x}$ ($Sz = 0$ for all $z \in \mathbf{L}^n$), then Q is a canonical Fredholm operator. If that is not the case, the question arises when $S_g : \mathbf{L}^n \to \mathbf{L}^n$ is continuous and $Q = I - S - K$ is Fredholm? During the last 20 years a large series of investigations necessary for completion of the general theory of the equation (6.3) was devoted to these questions. The main results of these investigations are thoroughly treated in [1]. Some of them will be stated below.

If the equation (6.3) is solved in respect to the derivative, (6.3) can be written in the form (6.2) by denoting

$$R(t,s) = -\sigma(t,s)P(t),$$

where $\sigma(t,s)$ is the characteristic function of the set

$$\Big\{(t,s) \in [a,b]\times[a,b] :\ a \le s \le h(t) < b\Big\} \cup \Big\{(t,s) \in [a,b]\times[a,b) :\ h(t) = b\Big\}.$$

A natural generalization of (6.3) is the equation

$$\dot{x}(t) - \sum_{i=1}^{k} B_i(t)(S_{g_i}\dot{x})(t) + \int_a^b d_s R(t,s)x(s) = f(t). \tag{6.6}$$

If $g_i(t) \le t$, $i = 1, \dots, k$, and $R(t,s) = 0$ for $t < s$, then (6.6) is called the equation with retarded argument.

The solution of the equation (6.3) was defined by many authors as a continuous prolongation on $[a,b]$ of the initial function φ. In other words, the conditions $x(a) = \varphi(a)$, $x(b) = \varphi(b)$ of "continuous junction" was demanded. There is no need of such conditions from the point of view of correctness of all the operations in the left side of (6.3). But these conditions cause many complications in the conception of the equation and turn out to be a serious obstacle in applying of the general theory of the equation

$$\mathcal{L}x = f$$

with the linear operator $\mathcal{L} : \mathbf{D}^n \to \mathbf{L}^n$ to the equation (6.3). Beginning from the works [15, 16], the most part of authors refused from the necessity of the continuous junction condition. It is relevant to note that the refusal from old conceptions, connected with continuous junction, does not mean forbidding the boundary conditions $x(a) = \varphi(a)$, $x(b) = \varphi(b)$ and the modern theory generalizes the results of previous investigations but does not contradict them.

In the book [17], the equation with "distributed retardation"

$$\dot{x}(t) + \int_0^{\sigma(t)} d_s g(t,s)x(t-s) = v(t), \quad t \in [a,b], \quad \sigma(t) \ge 0,$$

$$x(\xi) = \varphi(\xi), \quad \text{if} \quad \xi < a$$

was studied along with the necessary condition $x(a) = \varphi(a)$. We can write this equation in the form

$$\begin{gathered}\dot{x}(t) + \int\limits_{-\infty}^{t} d_s R(t,s)x(s) = v(t), \quad t \in [a,b], \\ x(\xi) = \varphi(\xi), \quad \text{if} \quad \xi \le a.\end{gathered} \tag{6.7}$$

Thus, (6.7) is a boundary value problem for the equation

$$(\mathcal{L}x)(t) \stackrel{def}{=} \dot{x}(t) + \int\limits_{a}^{t} d_s R(t,s)x(s) = f(t) \tag{6.8}$$

with boundary condition $x(a) = \varphi(a)$ and

$$f(t) = v(t) - \int\limits_{-\infty}^{a} d_s R(t,s)\varphi(s).$$

The equation (6.8) (without boundary conditions) under the assumption of the mentioned book [17] turns out to be an equation $\mathcal{L}x = f$ with the linear operator $\mathcal{L} : \mathbf{D}^n \to \mathbf{L}^n$. The principal part

$$(Qz)(t) = z(t) - \int\limits_{a}^{t} R(t,s)z(s)ds$$

of $\mathcal{L}$ has a bounded inverse by virtue of the compactness of the integral Volterra operator with the kernel $R(t,s)$. Thus by virtue of Theorem 2.4, the equation (6.8) with boundary conditions $x(a) = \alpha$ is uniquely solvable. Consequently (Theorem 2.5) the dimension of the fundamental system of the homogeneous equation $\mathcal{L}x = 0$ is equal to n. It is relevant to point out that in the book [17], the "infinite-dimensional fundamental system" of solutions of the homogeneous equation is determined. This does not contradict to what has been said above because, due to A. D. Myshkis, the homogeneous equation corresponding to (6.7) is said to be this equation with $v(t) \equiv 0$.

The equation (6.6) arises some problems about properties of S_g in the spaces of summable functions. We will state here the main facts of the theory of the operator $S_g : \mathbf{L}^n \to \mathbf{L}^n$. The thorough treatment of the matter can be found in [1].

The values of the function $g : [a,b] \to \mathbf{R}^1$ which do not belong to $[a,b]$ have no effect on the construction of the operator S_g. Thus this function can be defined arbitrarily on the set

$$\Big\{ t \in [a,b] : \ g(t) \notin [a,b] \Big\}.$$

Let mes be Lebesgue measure, $e \subset [a,b]$,

$$g^{-1}(e) = \Big\{ t \in [a,b] : \ g(t) \in e \Big\}.$$

Theorem 6.2 ([18, 19]). *The operator S_g is continuous in the space $\mathbf{L}^n$ if and only if*

$$\sup_{\substack{e \subset [a,b] \\ \mathrm{mes}\, e > 0}} \frac{\mathrm{mes}\, g^{-1}(e)}{\mathrm{mes}\, e} = M < \infty \tag{6.9}$$

and

$$\| S_g \| = M.$$

It is useful to point out that the condition

$$\mathrm{mes}\, e = 0 \Rightarrow \mathrm{mes}\, g^{-1}(e) = 0 \tag{6.10}$$

is necessary for (6.9). The condition (6.10) (the so called condition of "non-hovering") does not hold, in particular, if $g(t) = \mathrm{const}$ on a set $e \subset [a,b]$ of positive measure. (6.10) is fulfilled, for instance, if g is piecewise strictly monotonic and on each segment of monotonicity has an absolutely continuous inverse function g^{-1}. For a strictly monotonic g, we have

$$\|S_g\| = \sup_{\substack{e \subset [a,b] \\ \mathrm{mes}\, e > 0}} \frac{\mathrm{mes}\, g^{-1}(e)}{\mathrm{mes}\, e} = \operatorname*{vraisup}_{s \in [a,b] \cap g([a,b])} \left| \frac{dg^{-1}}{ds}(s) \right|.$$

From this and Theorem 6.2 it follows, in particular, that the operator S_g with $g(t) = \frac{1}{2}t^2$, $[a,b] = [0,1]$ is not continuous on $\mathbf{L}^n$.

For the equation (6.6), the principal part Q of $\mathcal{L}$ has the form

$$Q = I - S - K,$$

where $K : \mathbf{L}^n \to \mathbf{L}^n$ is an integral operator and

$$(Sz)(t) = \sum_{i=1}^{k} B_i(t)(S_{g_i} z)(t). \tag{6.11}$$

The operator $S : \mathbf{L}^n \to \mathbf{L}^n$ defined by (6.11) is continuous if $S_{g_i} : \mathbf{L}^n \to \mathbf{L}^n$ are continuous and the elements of $n \times n$-matrixes B_i are bounded in essential. For more sophistical conditions of continuity of the operator $S : \mathbf{L}^n \to \mathbf{L}^n$, we refer to [20] .

It ought to be pointed out that the operator $S : \mathbf{L}^n \to \mathbf{L}^n$ is never compact (but in the case $Sz = 0$ for all $z \in \mathbf{L}^n$).

Theorem 6.3 ([21, 22]). *Let $S : \mathbf{L}^n \to \mathbf{L}^n$ be a bounded operator, and $K : \mathbf{L}^n \to \mathbf{L}^n$ be a compact one. Then the operator $I - S - K$ is Fredholm if and only if there exists the bounded inverse $(I - S)^{-1} : \mathbf{L}^n \to \mathbf{L}^n$.*

In connection with this Theorem, it is interesting to point out the following.

Let

$$\mathcal{L}x = (I - S - K)\dot{x} + Ax(a)$$

and

$$P = (I - S)^{-1}$$

be bounded. Then the equation $\mathcal{L}x = f$ may be transformed to the form

$$\mathcal{L}_1 x \stackrel{def}{=} (I - K_1)\dot{x} + A_1 x(a) = f_1,$$

where $K_1 = PK$, $A_1 = PA$, $f_1 = Pf$. If $K : \mathbf{L}^n \to \mathbf{L}^n$ is compact, then the principal part of $\mathcal{L}_1$ is a Fredholm operator.

Equations with Volterra operator $\mathcal{L}$ or, as it is called, "the equations with aftereffect" represent a special allurement for some investigators.

Definition 6.1. Let $\mathbf{X}$ and $\mathbf{Y}$ be linear spaces of measurable functions $x : [a, b] \to \mathbf{R}^n$ and $y : [a, b] \to \mathbf{R}^n$, respectively. A linear operator $F : \mathbf{X} \to \mathbf{Y}$ is said to be the Volterra one if for each $c \in (a, b)$, $(Fx)(t) = 0$ almost everywhere on $[a, c]$ for all such $x \in \mathbf{X}$ that $x(t) = 0$ almost everywhere on $[a, c]$.

The Volterra property of $\mathcal{L}$ permits to consider the solution x of the equation $\mathcal{L}x = f$ on every segment $[a, c] \subset [a, b]$ disregarding the values of $x(t)$ and $(\mathcal{L}x)(t)$ for $t > c$. Thus we can study the evolution of the process described by the equation $\mathcal{L}x = f$ with Volterra operator $\mathcal{L}$.

A highly general representative of the equation with aftereffect is the equation of the form

$$\mathcal{L}x \stackrel{def}{=} (I - S - K)\dot{x} + Ax(a) = f,$$

where S and K are Volterra operators. The equation (6.3) is of such a kind if $g(t) \leq t$, $h(t) \leq t$.

Let us dwell on some results about equations with aftereffect.

By virtue of Theorem 2.4, the Cauchy problem

$$\mathcal{L}x = f, \quad x(a) = \alpha \tag{6.12}$$

is uniquely solvable if and only if there exists the bounded inverse $Q^{-1} : \mathbf{L}^n \to \mathbf{L}^n$. In this case, the solution of (6.12) admits the representation

$$x = \Lambda Q^{-1} f + \big(E - \Lambda Q^{-1} A\big)\alpha.$$

Here $(\Lambda Q^{-1} f)(t) = \int_a^t (Q^{-1} f)(s)ds$ is the Green operator of the problem (6.12), and $X = E - \Lambda Q^{-1} A$ is the fundamental matrix of solutions of the homogeneous equation $\mathcal{L}x = 0$.

If Q^{-1} is a Volterra operator, the product ΛQ^{-1} of Volterra operators is also Volterra. Any bounded operator $C : \mathbf{L}^n \to \mathbf{D}^n$ is an integral one. Thus in the case of Volterra Q^{-1}, we have

$$(Cf)(t) \stackrel{def}{=} \int_a^t (Q^{-1}f)(s)ds = \int_a^t C(t,s)f(s)ds.$$

By analogy with the theory of differential equations, the operator C is called the Cauchy operator and $C(t,s)$ is called the Cauchy matrix.

The following statement gives us a condition of Volterra invertibility of $Q = I - S - K$.

Theorem 6.4 ([23, 1]). *Let $Q = I - S - K$. Assume that $S : \mathbf{L}^n \to \mathbf{L}^n$ is the Volterra operator defined by (6.11), $K : \mathbf{L}^n \to \mathbf{L}^n$ is a compact Volterra operator, the spectral radius $\rho(S)$ of S is less then 1 ($\rho(S) < 1$). Then there exists a bounded inverse $Q^{-1} : \mathbf{L}^n \to \mathbf{L}^n$ and Q^{-1} is also Volterra.*

Under the condition of Theorem 6.4, the general solution of the equation $\mathcal{L}x = f$ admits the representation

$$x(t) = \int_a^t C(t,s)f(s)ds + X(t)x(a). \tag{6.13}$$

By analogy with the theory of differential equations, this representation is called the Cauchy formula.

Let us remark that only for differential equation the Cauchy matrix is defined by the fundamental matrix, namely $C(t,s) = X(t)X^{-1}(s)$.

In order to formulate some estimates of the spectral radius $\rho(S)$ of the Volterra operator $S : \mathbf{L}^n \to \mathbf{L}^n$ defined by (6.11), let us fix the numbers $\tau_i > 0$ and denote

$$\omega_i = \Big\{ t \in [a,b] : \ t - g_i(t) \le \tau_i, \ g_i(t) \in [a,b] \Big\}.$$

Let us stipulate also that

$$\operatorname*{vraisup}_{t \in \omega} y(t) = 0$$

if ω is an empty set. By $|B|$ we denote the norm of the matrix B compatible with the norm $|\cdot|$ in $\mathbf{R}^n$.

Theorem 6.5 ([23, 1]). *Let $g_i(t) \le t$, $M_i = \|S_{g_i}\|$, $i = 1, \ldots, k$. Then*

$$\rho(S) \le \sum_{i=1}^{k} M_i \operatorname*{vraisup}_{t \in \omega_i} |B(t)|.$$

Thus, in particular, the existence of such a number $\tau > 0$ that $t - g_i(t) \geq \tau$, $i = 1, \ldots, k$, means that the sets ω_i are empty for $\tau_i < \tau$ and, consequently, $\rho(S) = 0$.

Let us return to the general linear boundary value problem

$$(\mathcal{L}x)(t) \stackrel{def}{=} (Q\dot{x})(t) + A(t)x(a) = f(t), \quad lx = \alpha \tag{6.14}$$

under the assumption that $Q : \mathbf{L}^n \to \mathbf{L}^n$ is a Fredholm operator.

The decomposition (2.6) of a bounded linear vector-functional $l = [l^1, \ldots, l^m] : \mathbf{D}^n \to \mathbf{R}^m$ has the form

$$lx = \int_a^b \Phi(s)\dot{x}(s)ds + \Psi x(a),$$

where the elements of the $m \times n$-matrix Φ are measurable and bounded in essential, and the $m \times n$-matrix Ψ is constant. Rewrite the problem (6.14) in the form

$$\begin{pmatrix} Q & A \\ \Phi & \Psi \end{pmatrix} \begin{pmatrix} \dot{x} \\ x(a) \end{pmatrix} = \begin{pmatrix} f \\ \alpha \end{pmatrix} \tag{6.14}$$

defining the operators $A : \mathbf{R}^n \to \mathbf{L}^n$ and $\Phi : \mathbf{L}^n \to \mathbf{R}^m$ by

$$(A\beta)(t) = A(t)\beta, \quad \Phi y = \int_a^b \Phi(s)y(s)ds.$$

The operator $\Psi : \mathbf{R}^n \to \mathbf{R}^m$ is defined by the matrix Ψ.

Let ω be a linear bounded functional on $\mathbf{L}^n$ and $\omega(s) = \{\omega^1(s), \ldots, \omega^n(s)\}$ be a row vector with measurable and bounded in essential components which realizes this functional. Then

$$A^*\omega = \int_a^b \omega(s)A(s)ds.$$

Assume further

$$(\Phi^*\gamma)(t) = \gamma\Phi(t), \quad \Psi^*\gamma = \gamma\Psi,$$

whose γ denotes a linear functional on $\mathbf{R}^m$ and simultaneously the row vector which realizes this functional. Thus the equation

$$\begin{pmatrix} Q^* & \Phi^* \\ A^* & \Psi^* \end{pmatrix} \begin{pmatrix} \omega \\ \gamma \end{pmatrix} = \begin{pmatrix} g \\ \eta \end{pmatrix}$$

which is adjoint to the problem (6.14) is realized in the form of the system

$$\begin{gathered} (Q^*\omega)(t) + \gamma\Phi(t) = g(t), \\ \int_a^b \omega(s)A(s)ds + \gamma\Psi = \eta. \end{gathered} \tag{6.15}$$

The condition of orthogonality of the right-hand side $\{f, \alpha\}$ of the equation (6.14) to the solution $\{\omega, \gamma\}$ of the homogeneous adjoint equation

$$(Q^*\omega)(t) + \gamma\Phi(t) = 0,$$
$$\int_a^b \omega(s)A(s)ds + \gamma\Psi = 0$$

obtains the form

$$\int_a^b \omega(s)f(s)ds + \gamma\alpha = 0.$$

The substitution

$$y(t) = \int_a^t \omega(s)ds + \gamma$$

into (6.14) in the case $m = n$ leads to the boundary value problem

$$\begin{gathered}(Q^*\dot{y})(t) + y(a)\Phi(t) = g(t),\\ \int_a^b \dot{y}(s)A(s)ds + y(a)\Psi = \eta.\end{gathered} \tag{6.16}$$

This problem is naturally said to be the boundary value problem adjoined to (6.14). The solution of this problem is a row vector $y = \{y^1, \dots, y^n\}$ with absolutely continuous components y^i and bounded in essential derivatives $\frac{d}{dt}y^i$. The condition of orthogonality of $\{f, \alpha\}$ to the solution y of the homogeneous adjoint problem has the form

$$\int_a^b \dot{y}(s)f(s)ds + y(a)\alpha = 0.$$

Consider, as an example, the "periodic" boundary value problem

$$(Q\dot{x})(t) + A(t)x(a) = f(t), \quad \int_a^b \dot{x}(s)ds = \alpha. \tag{6.17}$$

The problem

$$\begin{gathered}(Q^*\dot{y})(t) + y(a) = 0,\\ \int_a^b \dot{y}(s)A(s)ds = 0\end{gathered} \tag{6.18}$$

is homogeneous adjoint to (6.17). The problem (6.17) is uniquely solvable if and only if (6.18) has only the trivial solution. Therefore the linear independence of the columns of the matrix A is a necessary condition for unique solvability of (6.17). Thus the linear independence of the columns of the matrices $P(t)$, $\int_a^b H(t,\tau)d\tau$, $R(t,a)$ and $P(t)\chi_h(t,a)$ is necessary for unique solvability of the periodic problem for the equations (2.6), (2.4), (6.2) and (6.3), respectively.

Let $Q = I - S - K$, where S is defined by (6.11) and K is an integral compact operator. Then

$$(K^*y)(t) = \int_a^b y(s)K(s,t)ds,$$

$$(S^*y)(t) = \sum_{i=1}^k \frac{d}{dt}\int_a^b y(s)B_i(s)\sigma_i(t,s)ds,$$

where $\sigma_i(t,s)$ is the characteristic function of the set [1]

$$\Big\{(t,s) \in [a,b] \times [a,b] : \ a \le g_i(s) \le t\Big\}.$$

If the problem (6.14) is uniquely solvable, then $m = n$ (Corollary 2.1) and $x = Gf$ is the solution of this problem for $\alpha = 0$. Here $G : \mathbf{L}^n \to \mathbf{D}^n$ is the Green operator of the considered problem. This operator as every bounded operator acting from $\mathbf{L}^n$ into $\mathbf{D}^n$ is an integral one:

$$(Gf)(t) = \int_a^b G(t,s)f(s)ds.$$

The kernel $G(t,s)$ of this operator is called Green matrix.

By virtue of Theorem 3.2, the matrix X defined by

$$X(t) = U(t) - \int_a^b G(t,s)(\mathcal{L}U)(s)ds$$

is a fundamental matrix of the solutions of the homogeneous equation $\mathcal{L}x = 0$ provided $n \times n$-matrix U with columns from $\mathbf{D}^n$ satisfies the condition $lU = E$.

Thus, the Green matrix of any problem for the equation $\mathcal{L}x = f$ being available, we may write the general solution of this equation in explicit form

$$x = Gf + Xc, \quad c \in \mathbf{R}^n.$$

It follows from Theorem 3.3 that the Green functions $G(t,s)$ and $G_1(t,s)$ of various boundary value problems for the same equation $\mathcal{L}x = f$ with different vector-functionals l and l_1 are connected by the relation

$$G(t,s) = G_1(t,s) - X(t)(lX)^{-1}V(s),$$

where X is a fundamental matrix of $\mathcal{L}x = 0$, $n \times n$-matrix V is the kernel of the integral representation

$$lG_1 f = \int_a^b V(s)f(s)ds$$

of the vector-functional $lG_1 : \mathbf{L}^n \to \mathbf{R}^n$.

Denote by U an $n \times n$-matrix with columns from $\mathbf{D}^n$ such that $\det U(a) \neq 0$, $lU = E$. By virtue of Lemma 3.1, such a matrix exists for any bounded vector-functional $l : \mathbf{D}^n \to \mathbf{R}^n$ with linearly independent components.

The Green matrix $W_l(t,s)$ of the problem

$$\dot{x}(t) - \dot{U}(t)[U(a)]^{-1}x(a) = f(t), \quad lx = \alpha \tag{6.19}$$

is defined by

$$W_l(t,s) = E\chi_{[a,t]}(s) - U(t)\Phi(s)$$

due to Theorem 3.4. Using the representation (3.6), one may investigate some properties of Green functions by means of the "primary" Green function $W_l(t,s)$ of the problem (6.19).

Theorem 6.6 (1]). *Let the problem* (6.14) *be uniquely solvable,* $Q = I - K$,

$$(Kz)(t) = \int_a^b K(t,s)z(s)ds$$

be a compact operator in the space $\mathbf{L}^n$. *Then the Green matrix* $G(t,s)$ *of the problem* (6.14) *possesses the properties:*

a) *for almost every* $s \in (a,b)$, $G(\cdot,s)$ *is absolutely continuous on* $[a,s)$ *and* $(s,b]$, *and besides*

$$G(s+0,s) - G(s-0,s) = E.$$

b)

$$\frac{d}{dt}\int_a^b G(t,s)f(s)ds = \int_a^b \frac{\partial}{\partial t}G(t,s)f(s)ds + f(t)$$

for each $f \in \mathbf{L}^n$.

c) *for almost every* $s \in (a,b)$, $G(\cdot,s)$ *satisfies*

$$\frac{\partial}{\partial t}G(t,s) - \int_a^b K(t,\tau)\frac{\partial}{\partial \tau}G(\tau,s)d\tau + A(t)G(a,s) = K(t,s),$$

$$\int_a^b \Phi(\tau)\frac{\partial}{\partial \tau}G(\tau,s)d\tau + \Psi G(a,s) = -\Phi(s).$$

The Green matrix has a much more complicated structure if $Q = I - S - K$, with S defined by (6.11). The results of studying of such matrices have only fragmentary character as yet.

§ 7. Systems With Impulse Effect

Denote by $\mathbf{DS}^n(m) = \mathbf{DS}^n[a, t_1, \ldots, t_m, b]$ the space of piecewise absolutely continuous functions $y : [a, b] \to \mathbf{R}^n$ representable in the form

$$y(t) = \int_a^t \dot{y}(s)ds + y(a) + \sum_{i=1}^{m} \chi_{[t_i, b]}(t)\Delta y(t_i).$$

Here t_i are fixed points, $a < t_1 < \cdots < t_m < b$, $\Delta y(t_i) = y(t_i) - y(t_i - 0)$, $\chi_{[t_i, b]}(t)$ is the characteristic function of the segment $[t_i, b]$. Thus, the elements $x \in \mathbf{DS}^n(m)$ are the functions absolutely continuous on each $[a, t_1)$, $[t_i, t_{i+1})$, $i = 1, \ldots, m-1$, $[t_m, b]$ and right-continuous at the points $t_1, \ldots, t_m$. The space $\mathbf{DS}^n(m)$ is isomorphic to the product $\mathbf{L}^n \times \mathbf{R}^{n+nm}$, the isomorphism

$$\mathcal{J} = \{\Lambda, Y\} : \mathbf{L}^n \times \mathbf{R}^{n+nm} \to \mathbf{DS}^n(m)$$

is defined by

$$(\Lambda z)(t) = \int_a^t z(s)ds, \quad Y(t) = \Big(E_n, \chi_{[t_1, b]}(t)E_n, \ldots, \chi_{[t_m, b]}(t)E_n\Big),$$

E_n is identical $n \times n$-matrix. The inverse

$$\mathcal{J}^{-1} = [\delta, r] : \mathbf{DS}^n(m) \to \mathbf{L}^n \times \mathbf{R}^{n+nm}$$

is defined by

$$\delta y = \dot{y}, \quad ry = \big(y(a), \Delta y(t_1), \ldots, \Delta y(t_m)\big).$$

If

$$\|y\|_{\mathbf{DS}^n(m)} = \|\dot{y}\|_{\mathbf{L}^n} + \|ry\|_{\mathbf{R}^{n+nm}},$$

then $\mathbf{DS}^n(m)$ is a Banach space.

The space $\mathbf{D}^n$ is continuously imbedded into $\mathbf{DS}^n(m)$, and also

$$\mathbf{DS}^n(m) = \mathbf{D}^n \oplus M^{nm},$$

where M^{nm} is a finite-dimensional subspace with dimension nm. Thus any linear operator on $\mathbf{DS}^n(m)$ is a linear extension on this space of a linear operator $\mathcal{L}$ defined on $\mathbf{D}^n$. To emphasize this fact, we will denote by $\widetilde{\mathcal{L}}$ the linear operator defined on $\mathbf{DS}^n(m)$. If $\mathcal{L} : \mathbf{D}^n \to \mathbf{L}^n$ is a Noether operator, $\operatorname{ind} \mathcal{L} = n$ (which is always supposed), then the linear extension $\widetilde{\mathcal{L}} : \mathbf{DS}^n(m) \to \mathbf{L}^n$ is as well a Noether operator, and also

$$\operatorname{ind} \widetilde{\mathcal{L}} = \operatorname{ind} \mathcal{L} + nm = n + nm.$$

A linear bounded operator $\widetilde{\mathcal{L}} : \mathbf{DS}^n(m) \to \mathbf{L}^n$ has the representation

$$(\widetilde{\mathcal{L}}y)(t) = (Q\dot{y})(t) + A_0(t)y(a) + \sum_{i=1}^{m} A_i(t)\Delta y(t_i), \tag{7.1}$$

where $Q = \widetilde{\mathcal{L}}\Lambda$, $A_0 = \widetilde{\mathcal{L}}(E_n)$, $A_i = \widetilde{\mathcal{L}}(\chi_{[t_i,b]}E_n)$, $i = 1,\ldots,m$. Hence any operator defined by (7.1) is a linear extension on $\mathbf{DS}^n(m)$ of an operator $\mathcal{L} : \mathbf{D}^n \to \mathbf{L}^n$ such that

$$(\mathcal{L}x)(t) = (Q\dot{x})(t) + A_0(t)x(a)$$

for any matrices $A_1,\ldots,A_m$ with columns from $\mathbf{L}^n$.

All the assertions of the theory of abstract functional differential equations are valid for the equation

$$\widetilde{\mathcal{L}}y = f \tag{7.2}$$

with Noether operator $\widetilde{\mathcal{L}} : \mathbf{DS}^n(m) \to \mathbf{L}^n$, $\operatorname{ind}\widetilde{\mathcal{L}} = n + nm$. In particular, it is necessary for unique solvability of a boundary value problem that the number of boundary conditions be equal to $n + nm$.

The studying of the equation (7.2) and boundary value problems for such an equation was started by A. V. Anokhin [24].

The solution of the principal boundary value problem

$$\widetilde{\mathcal{L}}y = f, \quad ry = \alpha$$

(in the case of its unique solvability) has the form

$$y(t) = \int_a^t (Q^{-1}f)(s)ds + \left[Y(t) - \int_a^t (Q^{-1}A)(s)ds\right]\alpha,$$

where $A = (A_0, A_1, \ldots, A_m)$. The matrix

$$\widetilde{X}(t) = Y(t) - \int_a^t (Q^{-1}A)(s)ds$$

is a fundamental one of the solutions of the homogeneous equation $\widetilde{\mathcal{L}}y = 0$. For this matrix, $r\widetilde{X} = E_{n+nm}$ (E_{n+nm} is an unity $(n+nm)\times(n+nm)$-matrix).

If the boundary value problem

$$\widetilde{\mathcal{L}}y = f, \quad \widetilde{l}y = \alpha \tag{7.3}$$

is uniquely solvable, its solution has the form

$$y(t) = (\widetilde{G}f)(t) + \widetilde{X}(t)\alpha.$$

Here $\widetilde{X}$ is such a fundamental matrix of $\widetilde{\mathcal{L}}y = 0$ that $\widetilde{l}\widetilde{X} = E_{n+mn}$, $\widetilde{G} : \mathbf{L}^n \to \mathbf{DS}^n(m)$ is the Green operator of this problem. This operator is an integral one. This follows from the fact that the equality

$$y(t) = (Fz)(t)$$

for the linear bounded operator $F : \mathbf{L}^n \to \mathbf{DS}^n(m)$ defines at every point $t \in [a, b]$ a bounded linear vector-functional on the space $\mathbf{L}^n$.

Let

$$\widetilde{\mathcal{L}} : \mathbf{DS}^n(m) \to \mathbf{L}^n \quad \text{and} \quad \widetilde{l} : \mathbf{DS}^n(m) \to \mathbf{R}^n$$

be linear extensions of

$$\mathcal{L} : \mathbf{D}^n \to \mathbf{L}^n \quad \text{and} \quad l : \mathbf{D}^n \to \mathbf{R}^n.$$

Theorem 7.1. *If one of the boundary value problems*

$$\mathcal{L}x = f, \quad lx = \alpha \tag{7.4}$$

and

$$\widetilde{\mathcal{L}}y = f, \quad \widetilde{l}y = \alpha, \quad \Delta y(t_i) = \gamma_i, \ \ i = 1, \ldots, m \tag{7.5}$$

is uniquely solvable, then the other one is also solvable. If these problems are uniquely solvable then the Green operator of (7.4) *is also the Green operator of* (7.5).

Proof. The problem

$$\widetilde{\mathcal{L}}y = f, \quad \widetilde{l}y = \alpha, \quad \Delta y(t_i) = 0, \ i = 1, \ldots, m,$$

and (7.4) are equivalent. The problems (7.4) and (7.5) are both Fredholm. Consequently, the unique solvability of (7.4) or (7.5) for any right-hand side implies the unique solvability of the other one for each right-hand side. If $x = Gf$ is the unique solution of (7.4) for $\alpha = 0$, this x is also the unique solution of (7.5) for $\alpha = 0$, $\gamma_i = 0$, $i = 1, \ldots, m$. Thus G is the Green operator of (7.5). ∎

Assume that the principal part Q of $\mathcal{L} : \mathbf{D}^n \to \mathbf{L}^n$ has the form $Q = I - K$, where $K : \mathbf{L}^n \to \mathbf{L}^n$ is a compact operator. In this event (Theorem 6.6) for almost every $s \in (a, b)$ the Green matrix $G(\cdot, s)$ of (7.4) satisfies the matrix equations

$$(\widetilde{\mathcal{L}}Z)(t) \stackrel{def}{=} \dot{Z}(t) - \int_a^b K(t,\tau)\dot{Z}(\tau)d\tau + A_0(t)Z(a) - K(t,s)\Delta Z(s) = 0,$$

$$\widetilde{l}Z \stackrel{def}{=} \int_a^b \Phi(\tau)\dot{Z}(\tau)d\tau + \Psi Z(a) + \Phi(s)\Delta Z(s) = 0,$$

where

$$\Delta Z(s) = Z(s) - Z(s-0),$$

and besides the condition

$$G(s,s) - G(s-0,s) = E_n$$

is fulfilled (we may assume that $G(\cdot,s)$ is right-continuous at the point s). Thus, the Green matrix $G(\cdot,s)$ of (7.4) is the solution of the matrix boundary value problem

$$\widetilde{\mathcal{L}}Z = 0, \quad \widetilde{l}Z = 0, \quad \Delta Z(s) = E_n$$

for almost every $s \in (a,b)$ if the linear extensions of $\mathcal{L}$ and l on the space $\mathbf{DS}^n[a,s,b]$ are constructed as follows

$$(\widetilde{\mathcal{L}}y)(t) = \dot{y}(t) - \int_a^b K(t,\tau)\dot{y}(\tau)d\tau + \\ + A_0(t)y(a) - K(t,s)\Delta y(s), \tag{7.6}$$

$$\widetilde{l}y = \int_a^b \Phi(\tau)\dot{y}(\tau)d\tau + \Psi y(a) + \Phi(s)\Delta y(s). \tag{7.7}$$

In respect of extensions (7.6) and (7.7), observe the following. Let

$$(\mathcal{L}x)(t) \overset{def}{=} \dot{x}(t) + P(t)(S_h x)(t) = \\ = \dot{x}(t) + \int_a^b P(t)\chi_h(t,\tau)\dot{x}(\tau)d\tau + P(t)\chi_h(t,a)x(a),$$

where $\chi_h(t,\tau)$ is the characteristic function of the set

$$\Big\{(t,\tau) \in [a,b]\times[a,b] : \; \tau \le h(t) \le b\Big\}.$$

Then the extension (7.6) preserves its original form

$$(\widetilde{\mathcal{L}}y)(t) = \dot{y}(t) + P(t)(S_h y)(t).$$

This follows from the representation

$$(S_h y)(t) = \int_a^b \chi_h(t,\tau)\dot{y}(\tau)d\tau + y(a) + \chi_h(t,s)\Delta y(s).$$

Analogously, for the vector-functional

$$lx \overset{def}{=} \chi(\xi) = \int_a^b \chi_{[a,\xi]}(\tau)\dot{x}(\tau)d\tau + x(a) \quad \big(\xi \in [a,b]\big)$$

the original form preserves after the extension (7.7):

$$\tilde{l}y = \int\limits_a^b \chi_{[a,\xi]}(\tau)\dot{y}(\tau)d\tau + y(a) + \chi_{[a,\xi]}(s)\Delta y(s) = y(\xi).$$

In the general case, the form of the operators $\mathcal{L}$ and l may be changed after extension. Let

$$(\mathcal{L}x)(t) \overset{def}{=} \dot{x}(t) + \int\limits_a^b d_\tau R(t,\tau)x(\tau) =$$

$$= \dot{x}(t) - \int\limits_a^b R(t,\tau)\dot{x}(\tau)d\tau - R(t,a)x(a).$$

Without loss of generality one may assume that $R(t,\cdot)$ is left-continuous at any point $s \in (a,b)$. Then the extension (7.6) can be written in the form

$$(\widetilde{\mathcal{L}}y)(t) = \dot{y}(t) + \int\limits_a^s d_\tau R(t,\tau)y(\tau) + \int\limits_s^b d_\tau R(t,\tau)y(\tau).$$

Indeed,

$$\int\limits_a^s d_\tau R(t,\tau)y(\tau) = R(t,s)y(s-0) - R(t,a)y(a) - \int\limits_a^s R(t,\tau)\dot{y}(\tau)d\tau,$$

$$\int\limits_s^b d_\tau R(t,\tau)y(\tau) = -R(t,s)y(s) - \int\limits_s^b R(t,\tau)\dot{y}(\tau)d\tau.$$

From this

$$\int\limits_a^s d_\tau R(t,\tau)y(\tau) + \int\limits_s^b d_\tau R(t,\tau)y(\tau) =$$

$$= -\int\limits_a^b R(t,\tau)\dot{y}(\tau)d\tau - R(t,a)y(a) - R(t,s)\Delta y(s).$$

Let us consider by using Theorem 5.3 the conditions which guarantee the continuous dependence of the solution of the problem (7.3) with respect to parameters of the problem, in particular, with respect to the arrangement of the points $a, t_1, \dots, t_m, b$.

For each $k = 0, 1, \dots$, let us choose such a system of points $a^k = t_0^k < t_1^k < \dots < t_{m+1}^k = b^k$ that

$$\lim_{k\to\infty} t_i^k = t_i^0, \quad i = 0, 1, \dots, m+1.$$

Denote

$$\mathbf{D}_k = \mathbf{DS}^n[a^k, t_1^k, \ldots, t_m^k, b^k], \quad \mathbf{B}_k = \mathbf{L}^n[a^k, b^k].$$

The element $y \in \mathbf{D}_k$ has the representation

$$y(t) = \int\limits_{a^k}^{t} \dot{y}(s)ds + y(a^k) + \sum_{i=1}^{m} \chi_{[t_i^k, b^k]}(t)\Delta y(t_i^k).$$

The space $\mathbf{D}_k$ is isomorphic to the direct product $\mathbf{B}_k \times \mathbf{R}^{n+nm}$, the isomorphism $\mathcal{J}_k = \{\Lambda_k, Y_k\} : \mathbf{B}_k \times \mathbf{R}^{n+nm} \to \mathbf{D}_k$ being defined by

$$(\Lambda_k z)(t) = \int\limits_{a^k}^{t} z(s)ds, \quad Y_k(t) = \big(E_n, \chi_{[t_1^k, b^k]}(t)E_n, \ldots, \chi_{[t_m^k, b^k]}(t)E_n\big).$$

$$\mathcal{J}_k^{-1} = [\delta_k, r_k], \quad \delta_k y = \dot{y}, \quad r_k y = \big(y(a^k), \Delta y(t_1^k), \ldots, \Delta y(t_m^k)\big),$$

$$\|y\|_{\mathbf{D}_k} = \|\dot{y}\|_{\mathbf{B}_k} + \|r_k y\|_{\mathbf{R}^{n+nm}}.$$

Define the functions $\omega_k : [a^k, b^k] \to [a^0, b^0]$ by

$$\omega_0(t) = t, \quad \omega_k(t) = \sum_{i=0}^{m} \Big[\frac{t_{i+1}^0 - t_i^0}{t_{i+1}^k - t_i^k}(t - t_i^k) + t_i^0\Big]\chi_{[t_i^k, t_{i+1}^k]}(t),$$
$$k = 1, 2, \ldots .$$

This function has the inverse

$$\omega_k^{-1}(t) = \sum_{i=0}^{m} \Big[\frac{t_{i+1}^k - t_i^k}{t_{i+1}^0 - t_i^0}(t - t_i^0) + t_i^k\Big]\chi_{[t_i^0, t_{i+1}^0]}(t), \quad t \in [a^0, b^0].$$

Define $\mathcal{H}_k : \mathbf{B}_0 \to \mathbf{B}_k$ by

$$(\mathcal{H}_k z)(t) = z[\omega_k(t)].$$

Then

$$(\mathcal{H}_k^{-1} z)(t) = z[\omega_k^{-1}(t)].$$

We have

$$\|\mathcal{H}_k z\|_{\mathbf{B}_k} = \sum_{i=0}^{m} \int\limits_{t_i^k}^{t_{i+1}^k} \Big| z\Big[\frac{t_{i+1}^0 - t_i^0}{t_{i+1}^k - t_i^k}(t - t_i^k) + t_i^0\Big]\Big| dt =$$

$$= \sum_{i=0}^{m} \frac{t_{i+1}^k - t_i^k}{t_{i+1}^0 - t_i^0} \int\limits_{t_i^0}^{t_{i+1}^0} |z(\tau)| d\tau.$$

We have from this that

$$\|\mathcal{H}_k z\|_{\mathbf{B}_k} \le \max_i \frac{t_{i+1}^k - t_i^k}{t_{i+1}^0 - t_i^0} \|z\|_{\mathbf{B}_0}, \quad \|\mathcal{H}_k\| = \max_i \frac{t_{i+1}^k - t_i^k}{t_{i+1}^0 - t_i^0},$$

$$\lim_{k\to\infty} \|\mathcal{H}_k z\|_{\mathbf{B}_k} = \|z\|_{\mathbf{B}_0}$$

for all $z \in \mathbf{B}_0$.

Quite similarly

$$\|\mathcal{H}_k^{-1}\| = \max_i \frac{t_{i+1}^0 - t_i^0}{t_{i+1}^k - t_i^k}.$$

Define $(\mathcal{P}_k y)(t) = y[\omega_k(t)]$, $y \in \mathbf{D}_0$. We have

$$(\mathcal{P}_k y)(t) = \int\limits_{a^0}^{\omega_k(t)} \dot{y}(s)ds + y(a^0) + \sum_{i=1}^m \chi_{[t_i^0, b^0]}[\omega_k(t)]\Delta y(t_i^0) =$$

$$= \int\limits_{a^k}^{t} \frac{d}{ds}\big(y[\omega_k(s)]\big)ds + y(a^0) + \sum_{i=1}^m \chi_{[t_i^k, b^k]}(t)\Delta y(t_i^0).$$

Thus $\mathcal{P}_k y \in \mathbf{D}_k$, $r_k \mathcal{P}_k y = r_0 y$, $(\mathcal{P}_k^{-1} y)(t) = y[\omega_k^{-1}(t)]$. Further we have:

$$\|\mathcal{P}_k y\|_{\mathbf{D}_k} = \left\| \frac{d}{dt}\,\mathcal{P}_k y \right\|_{\mathbf{B}_k} + \|r_k \mathcal{P}_k y\|_{|boldR^{n+nm}} =$$

$$= \sum_{i=0}^m \int\limits_{t_i^k}^{t_{i+1}^k} \left| \dot{y}\left[\frac{t_{i+1}^0 - t_i^0}{t_{i+1}^k - t_i^k}\,(t - t_i^k) + t_i^0 \right] \right| \frac{t_{i+1}^0 - t_i^0}{t_{i+1}^k - t_i^k}\,dt + \|r_0 y\|_{\mathbf{R}^{n+nm}} =$$

$$= \sum_{i=0}^m \int\limits_{t_i^0}^{t_{i+1}^0} |\dot{y}(\tau)|d\tau + \|r_0 y\|_{\mathbf{R}^{n+nm}} = \|y\|_{\mathbf{D}_0}.$$

Hence the systems $\{\mathcal{H}_k\}$ and $\{\mathcal{P}_k\}$ are connecting systems of isomorphisms for the spaces $\mathbf{B}_0$, $\mathbf{B}_k$ and $\mathbf{D}_0$, $\mathbf{D}_k$, satisfying the conditions of Theorem 5.3.

Let $\widetilde{\mathcal{L}}_k : \mathbf{DS}^n[a^k, t_1^k, \dots, t_m^k, b^k] \to \mathbf{L}^n[a^k, b^k]$ be a linear bounded Noether operator with $\operatorname{ind}\widetilde{\mathcal{L}}_k = n + nm$ and $\widetilde{l}_k : \mathbf{DS}^n[a^k, t_1^k, \dots, t_m^k, b^k] \to \mathbf{R}^{n+nm}$ be a linear bounded vector-functional, $k = 0, 1, \dots$. Under the assumption that $\widetilde{\mathcal{L}}_k \xrightarrow{\mathcal{P}H} \widetilde{\mathcal{L}}_0$, $\widetilde{l}_k u_k \to \widetilde{l}_0 u_0$ whenever $u_k \xrightarrow{\mathcal{P}} u_0$, the following assertion is valid.

Theorem 7.2. *Let the boundary value problem*

$$\widetilde{\mathcal{L}}_0 y = f, \quad \widetilde{l}_0 y = \alpha$$

be uniquely solvable.

The problems

$$\widetilde{\mathcal{L}}_k y = f, \quad \widetilde{l}_k y = \alpha$$

are uniquely solvable for all sufficiently large k, and for all $f_k \xrightarrow{\mathcal{H}} f_0$ and $\alpha_k \to \alpha_0$, the sequence $\{y_k\}$ of solutions y_k of the problems

$$\widetilde{\mathcal{L}}_k y = f_k, \quad \widetilde{l}_k y = \alpha_k$$

has the property $y_k \xrightarrow{\mathcal{P}} y_0$, *where* y_0 *is the solution of*

$$\widetilde{\mathcal{L}_0} y = f_0, \quad \widetilde{l_0} y = \alpha_0$$

if and only if

a) *there exists a vector-functional*

$$l : \mathbf{DS}^n\left[a^0, t_1^0, \ldots, t_m^0, b^0\right] \to \mathbf{R}^{n+nm}$$

such that the problems

$$\mathcal{H}_k^{-1} \widetilde{\mathcal{L}_k} \mathcal{P}_k y = f, \quad ly = \alpha \tag{7.8}$$

are uniquely solvable for $k = 0$ *and all sufficiently large* k,

b) *for any right-hand side* $\{f, \alpha\} \in \mathbf{L}^n[a^0, b^0] \times \mathbf{R}^{n+nm}$, *the convergence* $v_k \to v_0$ *holds, where* $v_k \in \mathbf{DS}^n[a^0, t_1^0, \ldots, t_m^0, b^0]$ *are the solutions of the problem* (7.8).

§ 8. Equations of the n-th Order

Denote by $\mathbf{W}^n$ the space of the functions $x : [a, b] \to \mathbf{R}^1$ with absolutely continuous derivative of the $(n-1)$-th order. Such a space is isomorphic to $\mathbf{L} \times \mathbf{R}^n$, where $\mathbf{L}$ is the space of summable functions $z : [a, b] \to \mathbf{R}^1$. The isomorphism $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L} \times \mathbf{R}^n \to \mathbf{W}^n$ may be constructed on the ground of the equality

$$x(t) = \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} x^{(n)}(s) ds + \sum_{i=0}^{n-1} \frac{(t-a)^i}{i!} x^{(i)}(a)$$

for any element $x \in \mathbf{W}^n$. Then

$$(\Lambda z)(t) = \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} z(s) ds, \quad (Y\beta)(t) = \sum_{i=0}^{n-1} \frac{(t-a)^i}{i!} \beta^{i+1},$$

$$\beta = \left\{\beta^1, \ldots, \beta^n\right\}; \quad \mathcal{J}^{-1} = [\delta, r] : \mathbf{W}^n \to \mathbf{L} \times \mathbf{R}^n,$$

$$(\delta x)(t) = x^{(n)}(t), \quad rx = \operatorname{col}\left\{x(a), \ldots, x^{(n-1)}(a)\right\},$$

$$\|x\|_{\mathbf{W}^n} = \|x^{(n)}\|_{\mathbf{L}} + \left|\operatorname{col}\left\{x(a), \ldots, x^{(n-1)}(a)\right\}\right|.$$

The decomposition (2.2) of a linear operator $\mathcal{L} : \mathbf{W}^n \to \mathbf{L}$ under such a choice of isomorphism has the form

$$(\mathcal{L}x)(t) = (Qx^{(n)})(t) + \sum_{i=0}^{n-1} p_i(t) x^{(i)}(a). \tag{8.1}$$

The decomposition of the component $l^i : \mathbf{W}^n \to \mathbf{R}^1$ of the vector-functional $l = [l^1, \ldots, l^m] : \mathbf{W}^n \to \mathbf{R}^m$ has the form

$$l^i x = \int\limits_a^b \varphi^i(s) x^{(n)}(s) ds + \sum_{j=0}^{n-1} \psi_j^i x^{(j)}(a),$$

where φ^i are measurable and bounded in essential functions, ψ_j^i are constants.

A sufficiently general representative of the equation $\mathcal{L}x = f$ with the linear operator $\mathcal{L} : \mathbf{W}^n \to \mathbf{L}$ has the form

$$(\mathcal{L}x)(t) \stackrel{def}{=} x^{(n)}(t) - \sum_{i=1}^{k} b_i(t)\big(S_{g_i} x^{(n)}\big)(t) + \\ + \sum_{i=0}^{n-1} \int\limits_a^b x^{(i)}(s) d_s r_i(t,s) = f(t). \tag{8.2}$$

We will assume below that the coefficients b_i are measurable and essentially bounded on $[a,b]$, the functions g_i satisfy the conditions of Theorem 6.2 about continuity of $S_{g_i} : \mathbf{L} \to \mathbf{L}$, $r_i(t,s)$ are measurable on the square $[a,b] \times [a,b]$, $\mathrm{var}_{s \in [a,b]}\, r_i(t,s)$ are summable on $[a,b]$, and $r_i(t,b) \equiv 0$.

Using (8.1), rewrite (8.2) in the form

$$(Qx^{(n)})(t) + \sum_{i=0}^{n-1} p_i(t) x^{(i)}(a) = f(t).$$

Here the principal part $Q = \mathcal{L}\Lambda$ of the operator $\mathcal{L}$ is defined by $Q = I - S - K$, where

$$(Sz)(t) = \sum_{i=1}^{k} b_i(t)\big(S_{g_i} z\big)(t), \tag{8.3}$$

$$(Kz)(t) = \sum_{i=0}^{n-1} \int\limits_a^b \frac{d^i}{ds^i} (\Lambda z)(s) d_s r_i(t,s) = \int\limits_a^b K(t,s) z(s) ds,$$

$$K(t,s) = \sum_{i=0}^{n-2} \int\limits_s^b \frac{(\tau - s)^{n-i-2}}{(n-i-2)!} r_i(t,\tau) d\tau + r_{n-1}(t,s), \quad \text{if} \quad n \geq 2,$$

$$K(t,s) = r_0(t,s), \quad \text{if} \quad n = 1.$$

The coefficients $p_i(t) = (\mathcal{L}y_i)(t)$ of the finite-dimensional part $\mathcal{L}Y$, where $y_i(t) = \dfrac{(t-a)^i}{i!}$ are the components of the vector $Y = (y_0, \ldots, y_{n-1})$, are defined by

$$p_0(t) = -r_0(t,a),$$

$$p_i(t) = -r_i(t,a) - \sum_{j=0}^{i-1} \int\limits_a^b \frac{(\tau - a)^{i-j-1}}{(i-j-1)!} r_j(t,\tau) d\tau, \quad i = 1, \dots, n-1.$$

Under the above assumption, $S : \mathbf{L} \to \mathbf{L}$ is continuous and $K : \mathbf{L} \to \mathbf{L}$ is compact by virtue of Theorem 6.1. Hence $Q = I - S - K$ is Fredholm if and only if there exists the continuous inverse $(I - S)^{-1}$ (Theorem 6.3). The condition $\|S\| < 1$ or $\rho(S) < 1$ guarantee such invertibility.

Under the chosen isomorphism between $\mathbf{W}^n$ and $\mathbf{L} \times \mathbf{R}^n$, the principal boundary value problem turns out to be the Cauchy one

$$\mathcal{L}x = f, \quad x^{(i)}(a) = \alpha^i, \quad i = 0, 1, \dots, n-1. \tag{8.4}$$

This problem is uniquely solvable if and only if there exists the bounded inverse Q^{-1}.

If $\mathcal{L}$ is a Volterra operator, then the equation (8.2) possesses some specific properties which put it into a special position among other equations from the point of view of theory and application. Theorem 6.4 permits to formulate the following conditions of Volterra invertibility of the operator Q of the equation (8.2) with Volterra operator $\mathcal{L}$.

Theorem 8.1. *Let $g_i(t) \le t$, $i = 1, \dots, k$; $r_i(t,s) = 0$ for $a \le t < s \le b$, $i = 0, \dots, n-1$. Suppose that the spectral radius of the operator $S : \mathbf{L} \to \mathbf{L}$ defined by* (8.3) *is less than* 1. *Then the Cauchy problem*

$$\mathcal{L}x = f, \quad x^{(i)}(a) = 0, \quad i = 0, \dots, n-1,$$

for the equation (8.2) *is uniquely solvable and the solution admits the representation*

$$x(t) \stackrel{def}{=} (Cf)(t) = \int\limits_a^t C(t,s) f(s) ds. \tag{8.5}$$

Under the assumptions of Theorem 8.1, the representation (2.8) of the solution of the Cauchy problem (8.4) for (8.2) obtains the form

$$x(t) = \int\limits_a^t C(t,s) f(s) ds + \sum_{i=0}^{n-1} \left[\frac{(t-a)^i}{i!} - \int\limits_a^t C(t,s) p_i(s) ds \right] \alpha^i,$$

where $p_0(s) = -r_0(s,a)$,

$$p_i(s) = -r_i(s,a) - \sum_{j=0}^{i-1} \int\limits_a^s \frac{(\tau - a)^{i-j-1}}{(i-j-1)!} r_j(s,\tau) d\tau, \quad i = 1, \dots, n-1.$$

The integral Volterra operator defined by (8.5) is said to be the Cauchy operator and the function $C(t,s)$ is said to be the Cauchy function of the equation (8.2).

Let $\mathcal{L} : \mathbf{W}^n \to \mathbf{L}$ be a linear bounded Noether operator of the index n. Consider the boundary value problem

$$\mathcal{L}x = f, \quad l^i x = \alpha^i, \quad i = 1, \dots, n, \tag{8.6}$$

with linearly independent functionals $l^1, \dots, l^n$. If the problem is uniquely solvable, then the solution in the case $\alpha^1 = \dots = \alpha^n = 0$ admits the integral representation

$$x(t) \stackrel{def}{=} (Gf)(t) = \int\limits_a^b G(t,s) f(s) ds.$$

The kernel $G(t,s)$ of the operator G (the Green operator) is called to be the Green function.

By virtue of Theorem 3.6, the problem (8.6) is uniquely solvable if and only if there exists the bounded inverse $[\mathcal{L}W]^{-1}$, where $W : \mathbf{L} \to \mathbf{W}^n$ is the Green operator of any model problem

$$\mathcal{L}_1 x = z, \quad l^i x = 0, \quad i = 1, \dots, n.$$

As a model equation $\mathcal{L}_1 x = f$, it is possible to take the one with

$$\begin{aligned}(\mathcal{L}_1 x)(t) \stackrel{def}{=} \frac{1}{w(a)} &\begin{vmatrix} u_1(a) & \dots & u_n(a) & x(a) \\ \dots & \dots & \dots & \dots \\ u_1^{(n-1)}(a) & \dots & u_n^{(n-1)}(a) & x^{(n-1)}(a) \\ u_1^{(n)}(t) & \dots & u_n^{(n)}(t) & x^{(n)}(t) \end{vmatrix} = \\ &= x^{(n)}(t) + \sum_{i=0}^{n-1} p_i(t) x^{(i)}(a),\end{aligned}$$

provided the system $u_1, \dots, u_n$ is chosen such that

$$w(a) = \begin{vmatrix} u_1(a) & \dots & u_n(a) \\ \dots & \dots & \dots \\ u_1^{(n-1)}(a) & \dots & u_n^{(n-1)}(a) \end{vmatrix} \neq 0, \quad \begin{vmatrix} l^1 u_1 & \dots & l^1 u_n \\ \dots & \dots & \dots \\ l^n u_1 & \dots & l^n u_n \end{vmatrix} \neq 0.$$

Such a system exists due to Lemma 3.1.

As an example, consider the two-point boundary value problem

$$\begin{gathered}(\mathcal{L}x)(t) \stackrel{def}{=} \ddot{x}(t) + p(t)\big(S_h x\big)(t) = f(t), \\ x(a) = \alpha^1, \quad x(b) = \alpha^2.\end{gathered} \tag{8.7}$$

As a model problem, we may use

$$\ddot{x}(t) = z(t), \quad x(a) = 0, \quad x(b) = 0. \tag{8.8}$$

The Green function $W(t,s)$ in this case may be constructed in the explicit form

$$W(t,s)=\begin{cases}-\dfrac{(s-a)(b-t)}{b-a}\,, & \text{if}\quad a\le s\le t\le b,\\ -\dfrac{(t-a)(b-s)}{b-a}\,, & \text{if}\quad a\le t<s\le b.\end{cases}\tag{8.9}$$

Assuming $W(t,s)=0$ outside the square $[a,b]\times[a,b]$, we have $\mathcal{L}W=I-\Omega$, where

$$(\Omega z)(t)=-\int\limits_a^b p(t)W[h(t),s]z(s)ds,$$

$$\|\Omega\|\le\int\limits_a^b |p(t)|\max_{s\in[a,b]}\big|W[h(t),s]\big|dt.$$

For every $t\in[a,b]$, the function $|W(t,s)|$ achieves its maximum at the point $s=t$. Thus

$$|W(t,s)|\le\frac{(t-a)(b-t)}{b-a}\,.$$

Consequently,

$$\|\Omega\|\le\int\limits_a^b |p(t)|\sigma_h(t)\frac{[h(t)-a][b-h(t)]}{b-a}\,dt,$$

where

$$\sigma_h(t)=\begin{cases}1, & \text{if}\quad h(t)\in[a,b],\\ 0, & \text{if}\quad h(t)\notin[a,b].\end{cases}$$

Hence the problem (8.7) is uniquely solvable if

$$\int\limits_a^b |p(t)|\sigma_h(t)[h(t)-a]\,[b-h(t)]dt<b-a.\tag{8.10}$$

This inequality holds, in particular, if

$$\int\limits_a^b |p(t)|\sigma_h(t)dt\le\frac{4}{b-a}\,.$$

In the theory of differential equations, the last condition is known as the inequality of Lyapunov and Zhukovskı.

Various estimates of Green function $G(t,s)$ is one of the central questions in the theory of boundary value problem. The conditions which guarantee the property $G(t,s)\ge 0$ ($G(t,s)\le 0$) call a special interest of many authors. Under such conditions the famous Theorem of Chaplygin is valid. This theorem guarantees for the solution x of the problem (8.6) the estimate of the form $x(t) \leq y(t)$ ($x(t) \geq y(t)$) if y satisfies the inequality

$$(\mathcal{L}y)(t) - f(t) = \varphi(t) \geq 0, \quad l^i y = \alpha^i, \quad i = 1, \ldots, n.$$

Really, the difference $\eta = y - x$ satisfies

$$\mathcal{L}\eta = \varphi, \quad l^i \eta = 0, \quad i = 1, \ldots, n,$$

and consequently

$$\eta(t) = \int_a^b G(t,s)\varphi(s)ds.$$

This difference $\eta = y - x$ is positive (negative) if the Green function does not assume negative (positive) values on the square $[a,b] \times [a,b]$.

An operator $A : \mathbf{L} \to \mathbf{L}$ is called isotone (antitone) if $(Az)(t) \geq 0$ ($(Az)(t) \leq 0$) almost everywhere on $[a,b]$ for each $z \in \mathbf{L}$ such that $z(t) \geq 0$ almost everywhere on $[a,b]$.

The criterion for the Green operator to be isotone or antitone may be formulated as follows.

Theorem 8.2. *The problem* (8.6) *is uniquely solvable and its Green operator is isotone* (*antitone*) *if and only if*

a) *There exists a model problem*

$$\mathcal{L}_1 x = z, \quad l^i x = 0, \quad i = 1, \ldots, n,$$

with isotone (antitone) Green operator W.

b) *The operator $(I - \mathcal{L}W) = \Omega : \mathbf{L} \to \mathbf{L}$ is isotone.*

c) *The successive approximations for the equation $\mathcal{L}Wz = f$ are convergent in $\mathbf{L}$.*

Proof. Sufficiency. Unique solvability of (8.6) follows from c) by virtue of Theorem 3.6. The solution x of (8.6) and the solution z of the equation $z = \Omega z + f$ are connected by $x = Wz$, $z = \mathcal{L}_1 x$. Under the assumption c), $z = f + \Omega f + \Omega^2 f + \cdots$. Thus for every $f \in \mathbf{L}$, $f(t) \geq 0$, the inequality $z(t) \geq f(t)$ holds. Hence

$$x(t) \stackrel{def}{=} (Gf)(t) = (Wz)(t) \geq 0 \quad ((Gf)(t) \leq 0).$$

The necessity follows if we put $\mathcal{L}_1 = \mathcal{L}$. ■

Remark 8.1. It follows from the proof of Theorem 8.2 that the difference $G - W$ is an isotone (antitone) operator because $(Wz)(t) \geq (Wf)(t)$ $(Wz)(t) \leq (Wf)(t))$.

To exemplify Theorem 8.2, consider the problem (8.7) under the assumption $p(t) \geq 0$. The Green operator W of the model problem (8.8) is antitone, Ω is isotone and if (8.10) holds, all the conditions of Theorem 8.2 are fulfilled. Consequently, the Green operator G of the problem (8.7) is antitone. Since $W(t,s) < 0$ on the open square $(a,b) \times (a,b)$, the Green function $G(t,s) < 0$ in $(a,b) \times (a,b)$ due to Remark 8.1.

On the basis of the representation (3.6) of the Green operator, the following assertion is proved in [1].

Theorem 8.3. *Let the principal part Q of the operator $\mathcal{L} : \mathbf{W}^n \to \mathbf{L}$ be of the form $Q = I - K$ with a compact operator $K : \mathbf{L} \to \mathbf{L}$*

$$(Kz)(t) = \int_a^b K(t,s)z(s)ds. \tag{8.11}$$

Let further the boundary value problem

$$\begin{gathered}(\mathcal{L}x)(t) \stackrel{def}{=} x^{(n)}(t) - \int_a^b K(t,s)x^{(n)}(s)ds + \sum_{k=0}^{n-1} p_k(t)x^{(k)}(a) = f(t), \\ l^i x = \alpha^i, \quad i = 1, \ldots, n,\end{gathered} \tag{8.12}$$

be uniquely solvable. Then the Green function $G(t,s)$ of this problem possesses the following properties.

a) *For almost every $s \in (a,b)$, the function $G(\cdot,s)$ has absolutely continuous derivative of the $(n-1)$-th order on $[a,s)$ and $(s,b]$, and also*

$$\frac{\partial^{n-1}}{\partial t^{n-1}} G(t,s)\Big|_{t=s+0} - \frac{\partial^{n-1}}{\partial t^{n-1}} G(t,s)\Big|_{t=s-0} = 1.$$

b)

$$\frac{d^n}{dt^n} \int_a^b G(t,s)f(s)ds = f(t) + \int_a^b \frac{\partial^n}{\partial t^n} G(t,s)f(s)ds$$

for any $f \in \mathbf{L}$.

c) *For almost every $s \in (a,b)$, the function $G(\cdot,s)$ satisfies the equalities*

$$\begin{gathered}\frac{\partial^n}{\partial t^n} G(t,s) - \int_a^b K(t,\tau)\frac{\partial^n}{\partial \tau^n} G(\tau,s)d\tau + \\ + \sum_{k=0}^{n-1} p_k(t) \frac{\partial^k}{\partial t^k} G(t,s)\Big|_{t=a} = K(t,s),\end{gathered}$$

$$\int_a^b \varphi^i(\tau) \frac{\partial^n}{\partial \tau^n} G(\tau,s)d\tau + \sum_{k=0}^{n-1} \psi_k^i \frac{\partial^k}{\partial t^k} G(t,s)\Big|_{t=a} = -\varphi^i(s), \quad i = 1, \ldots, n.$$

In [10], it is suggested a detailed scheme of investigation of the linear functional-differential equation of the n-th order in the space of functions $y : [a,b] \to \mathbf{R}^1$ with possible discontinuity of derivatives of various order at finite number of fixed points of the interval (a,b). Theorem 8.3 permits to consider the section $G(\cdot, s)$ as a solution of the proper boundary value problem in the space of functions with possible discontinuity of the derivative of the $(n-1)$-th order at the point $s \in (a,b)$. We confine ourselves below to this special case.

Let s be a fixed point of the interval (a,b). Denote by $\mathbf{WS}^n[a,s,b]$ the space of functions $y : [a,b] \to \mathbf{R}^1$ which are representable in the form

$$y(t) = \int_a^t \frac{(t-\tau)^{n-1}}{(n-1)!}\, y^{(n)}(\tau)d\tau + \\ + \sum_{i=0}^{n-1} \frac{(t-a)^i}{i!}\, y^{(i)}(a) + \frac{(t-s)^{n-1}}{(n-1)!}\, \chi_{[s,b]}(t)\Delta y^{(n-1)}(s),$$

where $\Delta y^{(n-1)}(s) = y^{(n-1)}(s) - y^{(n-1)}(s-0)$, $\chi_{[s,b]}$ is the characteristic function of the segment $[s,b]$. On the ground of this representation, the isomorphism $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L} \times \mathbf{R}^{n+1} \to \mathbf{WS}^n[a,s,b]$ might be constructed:

$$(\Lambda z)(t) = \int_a^t \frac{(t-\tau)^{n-1}}{(n-1)!}\, z(\tau)d\tau,$$

$$(Y\beta)(t) = \sum_{i=0}^{n-1} \frac{(t-a)^i}{i!}\, \beta^i + \frac{(t-s)^{n-1}}{(n-1)!}\, \chi_{[s,b]}(t)\beta^n,$$

$$\beta = \operatorname{col}\{\beta^0, \beta^1, \ldots, \beta^n\}.$$

Under the norm

$$\|y\|_{\mathbf{WS}^n[a,s,b]} = \|y^{(n)}\|_{\mathbf{L}} + \left\| \operatorname{col}\{y(a), \ldots, y^{n-1}(a), \Delta y^{(n-1)}(s)\} \right\|_{\mathbf{R}^{n+1}},$$

the space $\mathbf{WS}^n[a,s,b]$ will be a Banach one.

Let $\widetilde{\mathcal{L}} : \mathbf{WS}^n[a,s,b] \to \mathbf{L}$ and $\widetilde{l}^i : \mathbf{WS}^n[a,s,b] \to \mathbf{R}^1$ be linear extensions of $\mathcal{L} : \mathbf{W}^n \to \mathbf{L}$ and $l^i : \mathbf{W}^n \to \mathbf{R}^1$, respectively. Analogously to the case of systems with impulse perturbation (Theorem 7.1), it may be established that the problem (8.6) as well as the problem

$$\widetilde{\mathcal{L}}y = f, \quad \widetilde{l}^i y = \alpha^i, \quad i = 1, \ldots, n, \quad \Delta y^{(n-1)}(s) = \alpha^{n+1}$$

are (or are not) uniquely solvable simultaneously. In the case of their unique solvability, the Green functions of both problems coincide.

The section $G(\cdot,s)$ of the Green function of the problem (8.12) is an element of $\mathbf{WS}^n[a,s,b]$. Defining $\widetilde{\mathcal{L}}$ and $\widetilde{l}^i$ by

$$(\widetilde{\mathcal{L}}y)(t) = y^{(n)}(t) - \int_a^b K(t,\tau)y^{(n)}(\tau)d\tau + \\ + \sum_{k=0}^{n-1} p_k(t)y^{(k)}(a) - K(t,s)\Delta y^{(n-1)}(s), \tag{8.13}$$

$$\widetilde{l}^i y = \int_a^b \varphi^i(\tau)y^{(n)}(\tau)d\tau + \sum_{k=0}^{n-1} \psi_k^i y^{(k)}(a) + \\ + \varphi^i(s)\Delta y^{(n-1)}(s), \tag{8.14}$$

we may rephrase Theorem 8.3 as follows.

Let the principal part Q of the operator $\mathcal{L} : \mathbf{W}^n \to \mathbf{L}$ be of the form $Q = I - K$ with a compact $K : \mathbf{L} \to \mathbf{L}$ defined by (8.11). Let further the problem (8.12) be uniquely solvable. Then for almost every $s \in (a,b)$, the section $G(\cdot,s)$ of the Green function is a solution of the boundary value problem

$$(\widetilde{\mathcal{L}}y)(t) = 0, \quad \widetilde{l}^i y = 0, \quad i = 1,\dots,n, \quad \Delta y^{(n-1)}(s) = 1. \tag{8.15}$$

Thus the question on unique solvability of the problem (8.12) and on the Green function having a fixed sign can be reduced to the question on unique solvability and the solution of the problem (8.15) having a fixed sign for each $s \in (a,b)$. In this connection, we will understand the Green function $G(t,s)$ as a function which is a solution of (8.15) for each $s \in (a,b)$. Let us remind that the Green function $G(t,s)$ as a kernel of the integral operator $G : \mathbf{L} \to \mathbf{W}^n$ permits for each fixed $t \in [a,b]$ a deliberate change on the set of measure zero.

The fact of the Green function having a fixed sign sometimes might be established on the ground of the following Theorem 8.4.

Let us fix a point $\theta \in [a,b]$ such that the functionals $l^1,\dots,l^n,l^{n+1}$, where $l^{n+1}x = x(\theta)$, are linearly independent. Define the linear extensions $\mathcal{L}$ and l^i by (8.13) and (8.14).

Theorem 8.4. *Let the problem* (8.12) *be uniquely solvable. The Green function of this problem possesses the property $G(\theta,s) \neq 0$ if and only if the boundary value problem*

$$\widetilde{\mathcal{L}}y = 0, \quad \widetilde{l}^i y = 0, \quad i = 1,\dots,n, \quad y(\theta) = 0 \tag{8.16}$$

has only the trivial solution.

Proof. Let $x_1, \dots, x_n$ be a fundamental system of solutions of the equation $\mathcal{L}x = 0$ and

$$\Delta = \begin{vmatrix} l^1 x_1 & \dots & l^1 x_n \\ \dots & \dots & \dots \\ l^n x_1 & \dots & l^n x_n \end{vmatrix}$$

be the determinant of the problem (8.12). Due to the assumption, $\Delta \neq 0$. Denote $g_s(t) = G(t,s)$. The functions $x_1, \dots, x_n, g_s$ form a fundamental system of solutions of the equation $\widetilde{\mathcal{L}}y = 0$. The determinant of the problem (8.16) obtains the form

$$\widetilde{\Delta} = \begin{vmatrix} l^1 x_1 & \dots & l^1 x_n & 0 \\ \dots & \dots & \dots & \dots \\ l^n x_1 & \dots & l^n x_n & 0 \\ x_1(\theta) & \dots & x_n(\theta) & g_s(\theta) \end{vmatrix} = G(\theta, s)\Delta.$$

Hence it follows the conclusion of the Theorem. ■

Let us return to the problem (8.7) under the assumption $p(t)\sigma_h(t) \leq 0$, $t \in [a,b]$. This problem is uniquely solvable if and only if the problem

$$(\widetilde{\mathcal{L}}y)(t) \stackrel{def}{=} \ddot{y}(t) + p(t)(S_h y)(t) = f(t), \quad y(a) = y(b) = \Delta\dot{y}(s) = 0$$

is uniquely solvable in the space $\mathbf{WS}^2[a,s,b]$. If both these problems are uniquely solvable, then they have the same Green function $G^-(t,s)$, and besides for each $s \in (a,b)$, the section $G^-(\cdot,s)$ of the Green function is the solution of the problem

$$\widetilde{\mathcal{L}}y = 0, \quad y(a) = y(b) = 0, \quad \Delta\dot{y}(s) = 1.$$

The last problem is equivalent to the equation

$$y(t) = -\int\limits_a^b W(t,\tau)p(\tau)(S_h y)(\tau)d\tau + \omega_s(t), \tag{8.17}$$

where $W(t,s)$ is the Green function of the problem

$$\ddot{y} = f, \quad y(a) = y(b) = \Delta\dot{y}(s) = 0$$

in the space $\mathbf{WS}^2[a,s,b]$, $\omega_s(\cdot) = W(\cdot,s)$ ($W(t,s)$ is the Green function of the problem (8.8) in the space W^2). The equation (8.17) may be considered in the space $\mathbf{C}$ of continuous functions because all the continuous solutions of this equation belong to $\mathbf{WS}^2[a,s,b]$. Thus the question on unique solvability and the Green function of the problem (8.7) having a fixed sign is reducible to the question on unique solvability and the solution of the equation (8.17) in the space $\mathbf{C}$ having a fixed sign.

Denoting

$$(Hy)(t) = -\int\limits_a^b W(t,\tau)p(\tau)(S_h y)(\tau)d\tau,$$

rewrite (8.17) in the form

$$y - Hy = \omega_s. \tag{8.17}$$

H is an antitone operator. Applying to both parts of the last equation the operator $I + H$, we obtain the equation

$$y - H^2 y = \varphi \quad (\varphi = \omega_s + H\omega_s) \tag{8.18}$$

with the isotone operator H^2. If $\|H\|_{\mathbf{C}\to\mathbf{C}} < 1$, then both equations (8.17) and (8.18) are equivalent and besides the successive approximations for these equations converge. Consequently, under the condition $\|H\|_{\mathbf{C}\to\mathbf{C}} < 1$ the inequality $\varphi(t) \leq 0$, $t \in [a, b]$, guarantees the estimate

$$g_s = \varphi + H^2\varphi + \cdots \leq \varphi.$$

for the solution $g_s(\cdot) = G^-(\cdot, s)$ of the equation (8.17). From this estimate and (8.17), we have

$$W(t,s) \leq G^-(t,s) \leq 0, \quad (t,s) \in [a,b] \times (a,b).$$

Thus, assuming $W(t,s) = 0$ outside the square $[a,b] \times [a,b]$, we may formulate.

Lemma 8.1. *Let the following conditions be fulfilled.*
a) $p(t)\sigma_h(t) \leq 0$, $t \in [a,b]$.
b) $\|H\|_{\mathbf{C}\to\mathbf{C}} < 1$.
c) *For each fixed* $s \in (a,b)$,

$$\varphi(t) = W(t,s) - \int_a^b W(t,\tau)p(\tau)W[h(\tau),s]d\tau \leq 0, \quad t \in [a,b].$$

Then the problem (8.7) *is uniquely solvable and the Green function of this problem does not assume positive values in the square* $[a,b] \times (a,b)$.

Following [25], we will show that the inequality

$$\int_a^b |p(\tau)|\sigma_h(\tau)d\tau \leq \frac{1}{b-a}$$

guarantees the fulfilment of the conditions b) and c) of Lemma 8.1. Since

$$\|H\|_{\mathbf{C}\to\mathbf{C}} = \max_{t\in[a,b]} \int_a^b |W(t,\tau)|\,|p(\tau)|\sigma_h(\tau)d\tau < (b-a)\int_a^b |p(\tau)|\sigma_h(\tau)d\tau,$$

b) is fulfilled.

For verification of the fulfilment of the condition c), we will use the estimates

$$|W(t,s)| \le \frac{(t-a)(b-t)}{b-a}, \quad |W(t,s)| \le \frac{(s-a)(b-s)}{b-a}$$

following from (8.9).

Let $a < t < s$. Then

$$\varphi(t) = -\frac{(t-a)(b-s)}{b-a} + \int_a^b |W(t,\tau)| \cdot |p(\tau)| \cdot |W[h(\tau), s]| d\tau <$$

$$< \int_a^b \frac{(b-t)(t-a)}{b-a} |p(\tau)| \sigma_h(\tau) \frac{(s-a)(b-s)}{b-a} d\tau - \frac{(t-a)(b-s)}{b-a} \le$$

$$\le \frac{(t-a)(b-s)}{b-a} \left\{ (b-a) \int_a^b |p(\tau)| \sigma_h(\tau) d\tau - 1 \right\} \le 0.$$

Analogously it may be shown that $\varphi(t) < 0$, $t \in [s,b]$.

Next consider the problem (8.7) in the general case, where the coefficient p may change the sign. Let $p = p^+ - p^-$, $p^+(t) \ge 0$, $p^-(t) \ge 0$. Then the equation $\mathcal{L}x = f$ can be rewritten in the form

$$\ddot{x}(t) - p^-(t)(S_h x)(t) = -p^+(t)(S_h x)(t) + f(t).$$

Let us assume that for the auxiliary problem

$$\ddot{x}(t) - p^-(t)(S_h x)(t) = z(t), \quad x(a) = x(b) = 0,$$

the conditions of Lemma 8.1 are fulfilled. Under these conditions the Green operator $(G^- z)(t) = \int_a^b G^-(t,s) z(s) ds$ of the auxiliary problem is antitone. For the problem (8.7), $\mathcal{L}G^- = I - \Omega$, where

$$(\Omega z)(t) = -p^+(t) \int_a^b G^-[h(t), s] z(s) ds$$

is an isotone operator. Thus, by virtue of Theorem 8.2, the inequality $\|\Omega\|_{\mathbf{L}\to\mathbf{L}} < 1$ guarantees unique solvability of the problem (8.7) and antitonicity of the Green operator of this problem. Since

$$|G^-(t,s)| \le |W(t,s)| \quad \text{and} \quad \max_{(t,s)\in[a,b]\times[a,b]} |W(t,s)| = \frac{b-a}{4},$$

we have

$$\|\Omega\|_{\mathbf{L}\to\mathbf{L}} < \frac{b-a}{4} \int_a^b p^+(t) \sigma_h(t) dt.$$

So we can formulate the following test.

Theorem 8.5. *The problem* (8.7) *is uniquely solvable and the Green operator of this problem is antitone if*

$$\int_a^b p^-(t)\sigma_h(t)dt \le \frac{1}{b-a}\,, \quad \int_a^b p^+(t)\sigma_h(t)dt \le \frac{4}{b-a}\,.$$

We will illustrate the application of Theorem 8.4 on the example of the following periodic problem

$$\begin{gathered}(\mathcal{L}x)(t) \stackrel{def}{=} \ddot{x}(t) + \int_a^b x(\tau)d_\tau r(t,\tau) = f(t),\\ x(b)-x(a)=0, \quad \dot{x}(b)-\dot{x}(a)=0\end{gathered} \tag{8.19}$$

under the assumption that $r(t,\tau)$ is measurable on the square $[a,b]\times[a,b]$, $\mathrm{var}_{\tau\in[a,b]}\, r(\cdot,\tau)$ is summable on $[a,b]$ and $r(t,b)\equiv 0$.

Theorem 8.6. *Let the problem* (8.19) *be uniquely solvable and*

$$\int_a^b \mathrm{var}_{\tau\in[a,b]}\, r(t,\tau)dt < \frac{1}{b-a}\,.$$

Then the Green function $G(t,s)$ of this problem has the same sign at each point of the square $[a,b]\times(a,b)$.

Here we give only a scheme of the proof of this Theorem. The thorough proof for a more general equation is produced in [26].

By virtue of Theorem 8.4, unique solvability of the problem

$$\begin{gathered}(\widetilde{\mathcal{L}}y)(t) \stackrel{def}{=} \ddot{y}(t) + \int_a^b y(\tau)d_\tau r(t,\tau) = f(t),\\ y(b)-y(a)=0, \quad \dot{y}(b)-\dot{y}(a)=0, \quad y(\theta)=0\end{gathered} \tag{8.20}$$

in the space $\mathbf{WS}^2[a,s,b]$ for each $s\in(a,b)$ and any $\theta\in[a,b]$ guarantees that $G(t,s)$ does not have zeros at any point of the square $[a,b]\times(a,b)$. In [1] (Theorem 3.4.5), conditions are formulated which guarantee continuity of the function $G(t,\cdot)$ on the interval (a,b) for each $t\in[a,b]$. These conditions are fulfilled for the problem (8.19). Thus, from unique solvability of (8.20) it follows that the sign of $G(t,s)$ is the same at each point of the square $[a,b]\times(a,b)$. The unique solvability of (8.20) may be established on the base of Theorem 3.6. As a model problem, we will use the following one

$$\ddot{y}=z, \quad y(b)-y(a)=0, \quad \dot{y}(b)-\dot{y}(a)=0, \quad y(\theta)=0. \tag{8.21}$$

The functions 1, t and $(t-s)\chi_{[s,b]}(t)$ compose a fundamental system of solutions of the equation $\ddot{y}=0$ in the space $\mathbf{WS}^2[a,s,b]$. The determinant of the problem (8.21)

$$\begin{vmatrix} 0 & b-a & b-s \\ 0 & 0 & 1 \\ 1 & \theta & (\theta-s)\chi_{[s,b]}(\theta) \end{vmatrix} = -(b-a) \neq 0.$$

Consequently, the problem (8.21) is uniquely solvable. The Green function of this problem was constructed in [26]:

$$W_{\theta,s}(t,\tau) = \chi_{[a,t]}(\tau)(t-\tau) - \chi_{[a,\theta]}(\tau)(\theta-\tau) - \\ - \chi_{[s,b]}(t)(t-s) + \chi_{[s,b]}(\theta)(\theta-s) + \frac{\tau-s}{b-a}(t-\theta).$$

In [1] it was derived the estimate

$$|W_{\theta,s}(t,\tau)| \leq b-a, \quad (t,\tau) \in [a,b]\times[a,b], \quad s \in (a,b), \ \ \theta \in [a,b].$$

We have $\widetilde{\mathcal{L}}W_{\theta,s} = I - \Omega$, where $W_{\theta,s}$ is the Green operator of the problem (8.21),

$$(\Omega z)(t) = \int_a^b \left\{ \int_a^b W_{\theta,s}(\xi,\tau)z(\tau)d\tau \right\} d_\xi r(t,\xi) = \\ = \int_a^b \left\{ \int_a^b W_{\theta,s}(\xi,\tau)\, d_\xi r(t,\xi) \right\} z(\tau)d\tau,$$

$$\|\Omega\|_{\mathbf{L}\to\mathbf{L}} \leq (b-a)\int_a^b \operatorname{var}_{\xi\in[a,b]} r(t,\xi)dt < 1.$$

From this it follows the unique solvability of the problem (8.20). ■

§ 9. Singular Equations

The set of functions to which solutions of an equation belong sometimes is chosen without proper reason: the space of continuous or summable functions, or some other well known space is often used. But an unsuccessful choice of the set may cause much trouble in utilizing traditional schemes and standard theorems. Below we will discuss some reasons and examples connected with the question of choosing the proper Banach space in which it would be advisable to seek the solution of a given equation.

Let $\mathcal{L}x = f$ be an equation with the linear operator $\mathcal{L}: \mathbf{D}_0 \to \mathbf{B}_0$, $\mathbf{D}_0$ be isomorphic to $\mathbf{B}_0 \times \mathbf{R}^n$, and $\mathcal{J}_0 = \{\Lambda_0, Y_0\} : \mathbf{B}_0 \times \mathbf{R}^n \to \mathbf{D}_0$ be the isomorphism. If the principal part $\mathcal{L}\Lambda_0 : \mathbf{B}_0 \to \mathbf{B}_0$ of the operator $\mathcal{L}$ is not Fredholm, we have not available standard schemes for investigation of the equation. In this case, it is reasonable to call the equation "singular".

Yet one may try to find or construct another space $\mathbf{D} \simeq \mathbf{B} \times \mathbf{R}^n$ with the isomorphism $\mathcal{J} = \{\Lambda, Y\} : \mathbf{B} \times \mathbf{R}^n \to \mathbf{D}$ so that the principal part $\mathcal{L}\Lambda$ of the operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ would be a Fredholm or even an invertible operator. Then the equation ceases to be singular and one may apply to this equation the theorems of the above developed general theory.

Let us note that the property of the principal part being Fredholm characterizes many intrinsic specifics of the equation. For instance, this property is necessary for unique solvability of any boundary value problem

$$\mathcal{L}x = f, \quad lx = \alpha$$

for each $\{f, \alpha\} \in \mathbf{B} \times \mathbf{R}^n$.

Considering the same equation in various spaces, we change correspondingly the notion of this equation. The classical theory of differential equations does not use the notions of spaces and operators in these spaces and in that theory the investigation of singular equations begins with the definition of the notion of solution as a function satisfying in one sense or another the equation and possessing certain properties. Thus the set is chosen to which solutions belong. In our reasoning, we act analogously by choosing a Banach space on which the operator $\mathcal{L}$ is defined. In addition, we offer some recommendation about construction of the spaces $\mathbf{D}$ on which the operator $\mathcal{L}$ possesses necessary properties.

A. I. Shindyapin [27] has considered the equation

$$(\mathcal{L}x)(t) \overset{def}{=} \dot{x}(t) - (S\dot{x})(t) - (K\dot{x})(t) - A(t)x(a) = f(t) \tag{9.1}$$

with unbounded composition operator $S : \mathbf{L} \to \mathbf{L}$ (defined by (6.11)) and unbounded integral operator $K : \mathbf{L} \to \mathbf{L}$. Thus the equation (9.1) in the space of absolutely continuous functions $x : [a, b] \to \mathbf{R}^1$ is singular. A. I. Shindyapin has constructed a special space $\mathbf{B}$, more narrow than $\mathbf{L}$, so that both operators S and K are bounded in this space, and has considered the equation (9.1) in the space $\mathbf{D} \simeq \mathbf{B} \times \mathbf{R}^1$, where the isomorphism $\mathcal{J} = \{\Lambda, Y\} : \mathbf{B} \times \mathbf{R}^1 \to \mathbf{D}$ is defined by

$$(\Lambda z)(t) = \int_a^t z(s)ds, \quad (Y\beta)(t) = \beta, \quad \{z, \beta\} \in \mathbf{B} \times \mathbf{R}^1.$$

Under some natural conditions, the principal part $Q = \mathcal{L}\Lambda$ of the operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ has the bounded inverse. Therefore under these conditions the equation (9.1) has a one-dimensional fundamental vector $X(t)$ and the general solution of this equation has the form

$$x(t) = \int_a^t (Q^{-1}f)(s)ds + cX(t), \quad c = \mathrm{const}\,.$$

It is relevant to observe that severe constraints are needed for the composition operator $S : \mathbf{L} \to \mathbf{L}$ to be bounded. Therefore some works arose [28] concerning the operator in special spaces, where this operator could be bounded without some constraints necessary for boundedness in Lebesque spaces.

E. I. Bravi [29, 30] has been studying the equation

$$(\mathcal{L}x)(t) \stackrel{def}{=} \pi(t)x^{(n)}(t) + \sum_{k=0}^{n-1} p_k(t)(S_{h_k}x^{(k)})(t) = f(t), \quad t \in [a,b],$$

with summable coefficients p_k. Here the singularity arises due to zeros of the coefficient π (π has finite numbers of zeros, the multiplicity of each is not greater than $n-1$). The results of E. I. Bravi are a far-reaching generalization of those of S. M. Labovskiĭ [31] concerning the equation

$$(\mathcal{L}x)(t) \stackrel{def}{=} t(1-t)\ddot{x}(t) + p(t)(S_h x)(t) = f(t), \quad t \in [0,1], \tag{9.2}$$

with measurable h and summable p,f. We will dwell on this equation.

The principal part $Q = \mathcal{L}\Lambda_0 : \mathbf{L} \to \mathbf{L}$ of the operator $\mathcal{L} : \mathbf{W}^2 \to \mathbf{L}$, where $\mathbf{W}^2$ is a traditional space for the second order equation is not a Fredholm operator. Really, let $\mathcal{J}_0 = \{\Lambda_0, Y_0\} : \mathbf{L} \times \mathbf{R}^2 \to \mathbf{W}^2$ be the isomorphism,

$$(\Lambda_0 z)(t) = \int_0^t (t-s)z(s)ds.$$

Then

$$Q_0 = \mathcal{L}\Lambda_0 = P + V, \quad (Pz)(t) = t(1-t)z(t), \quad (Vz)(t) = p(t)(S_h\Lambda_0 z)(t).$$

The operator $V : \mathbf{L} \to \mathbf{L}$ is compact, but the range of values $R(P)$ of the operator $P : \mathbf{L} \to \mathbf{L}$ is not closed. Thus the principal part $Q_0 : \mathbf{L} \to \mathbf{L}$ of $\mathcal{L} : \mathbf{W}^2 \to \mathbf{L}$ is not even Noether. Therefore we will consider the equation (9.2) in another space $\mathbf{D} = \Lambda\mathbf{L} \oplus Y\mathbf{R}^2$ defined by

$$(\Lambda z)(t) = \int_0^1 \Lambda(t,s)z(s)ds, \quad (Y\beta)(t) = (1-t)\beta^1 + t\beta^2, \quad \beta = \operatorname{col}\{\beta^1, \beta^2\},$$

where

$$\Lambda(t,s) = \frac{G_0(t,s)}{s(1-s)}, \quad G_0(t,s) = \begin{cases} s(t-1), & \text{if } \ 0 \le s \le t \le 1, \\ t(s-1), & \text{if } \ 0 \le t < s \le 1, \end{cases}$$

($G_0(t,s)$ is the Green function of the boundary value problem $\ddot{x}(t) = z(t)$, $x(0) = x(1) = 0$). The space $\mathbf{D}$ is isomorphic to $\mathbf{L} \times \mathbf{R}^2$, $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L} \times \mathbf{R}^2 \to \mathbf{D}$ is an isomorphism, the inverse $\mathcal{J}^{-1} = [\delta, r] : \mathbf{D} \to \mathbf{L} \times \mathbf{R}^2$ is defined by

$$(\delta x)(t) = t(1-t)\ddot{x}(t), \quad rx = \{x(0), x(1)\};$$

$$\|x\|_{\mathbf{D}} = \|\delta x\|_{\mathbf{L}} + |x(0)| + |x(1)|.$$

The element $x \in \mathbf{D}$ is defined by $x = \Lambda z + Y\beta$ and consequently is characterized by the following properties.

a) The function x is continuous on $[0,1]$.

b) The derivative $\dot{x}$ is absolutely continuous on the interval $(0,1)$.

c) The product $t(1-t)\ddot{x}(t)$ is summable on $[0,1]$.

Under such a choice of the space $\mathbf{D}$, the operator $\mathcal{L} : \mathbf{D} \to \mathbf{L}$ is Noether, and

$$(Qz)(t) \stackrel{def}{=} (\mathcal{L}\Lambda z)(t) = z(t) - (Kz)(t) = z(t) + p(t)\int_0^1 \Lambda[h(t), s]z(s)ds.$$

Here and below we suppose $\Lambda(t,s) = 0$ outside the square $[0,1]\times[0,1]$. The operator $K : \mathbf{L} \to \mathbf{L}$ is compact (Theorem 6.1) and therefore $Q : \mathbf{L} \to \mathbf{L}$ is canonical Fredholm.

If $\|K\|_{\mathbf{L}\to\mathbf{L}} < 1$, then there exists the bounded Q^{-1}. Due to theorem 2.4, in this case the principal boundary value problem

$$\mathcal{L}x = f, \quad x(0) = 0, \quad x(1) = 0 \tag{9.3}$$

is uniquely solvable and the general solution of the equation $\mathcal{L}x = f$ has the form

$$x(t) = \int_0^1 G(t,s)f(s)ds + c_1x_1(t) + c_2x_2(t),$$

where $G(t,s)$ is the Green function of (9.3), x_1, x_2 constitute a fundamental system of solutions of the homogeneous equation $\mathcal{L}x = 0$ in the space $\mathbf{D}$, and c_1, c_2 are constants.

Since $|\Lambda(t,s)| \le 1$, the estimate $\|K\|_{\mathbf{L}\to\mathbf{L}} < 1$ follows from the inequality

$$\int_0^1 |p(s)|\sigma_h(s)ds \le 1,$$

where

$$\sigma_h(s) = \begin{cases} 1, & \text{if} \quad h(s) \in [0,1], \\ 0, & \text{if} \quad h(s) \notin [0,1]. \end{cases}$$

If $\|K\|_{\mathbf{L}\to\mathbf{L}} < 1$ and besides $p(t) \ge 0$, $G(t,s) \le 0$ in the square $[0,1] \times [0,1]$. Really, in this case $z = f + Kf + K^2 f + \cdots$ is the solution of the equation $Qz = f$. From this and the isotonical property of K, we have $z(t) \ge f(t)$ if $f(t) \ge 0$. Consequently, for the solution

$$x(t) = \int_0^1 G(t,s)f(s)ds$$

of (9.3), we have $x(t) \le 0$ if $f(t) \ge 0$, $t \in [0,1]$.

To the conclusion, we observe that the properties a), b), c) were laid to the ground of the definition of the solution of (9.2) in the works of I. T. Kiguradze [32, 33].

The case where the coefficient π has zeros inside $[a,b]$ demands a more complicated construction. We will illustrate the situation on the example of the equation

$$(\mathcal{L}x)(t) \stackrel{def}{=} t\ddot{x}(t) + p(t)(S_h x)(t) = f(t), \quad t \in [a,b],$$

$a < 0 < b$, $p, f \in \mathbf{L}$, h is a measurable function.

As in the previous example, the principal part of $\mathcal{L} : \mathbf{W}^2 \to \mathbf{L}$ is not a Fredholm operator. As the space $\mathbf{D}$ on which it is reasonable to consider the operator $\mathcal{L}$ we will take the space of solutions of the three-point impulse model boundary value problem

$$t\ddot{x}(t) = z(t), \quad x(a) = \beta^1, \quad x(b) = \beta^2, \quad x(0) = \beta^3. \tag{9.4}$$

We will suppose that the solution of this problem is a function $x : [a,b] \to \mathbf{R}^1$ such that $\dot{x}$ is absolutely continuous on $[a,0)$ and $[0,b]$ and $t\ddot{x}(t)$ is summable on $[a,b]$. Thus the homogeneous equation $t\ddot{x}(t) = 0$ has three linearly independent solutions

$$u_1(t) = \frac{t}{a}\chi_{[a,0)}(t), \quad u_2(t) = \frac{a-t}{a}\chi_{[a,0)}(t) + \frac{b-t}{b}\chi_{[0,b]}(t),$$
$$u_3(t) = \frac{t}{b}\chi_{[0,b]}(t),$$

and the nonhomogeneous equation $t\ddot{x}(t) = z(t)$ has solutions for any $z \in \mathbf{L}$, for instance

$$x(t) = (\Lambda z)(t) = \int_a^b \Lambda(t,s) z(s) ds,$$

where

$$\Lambda(t,s) = \begin{cases} -\dfrac{t(s-a)}{as}, & \text{if } \ a \le s \le t < 0, \\ -\dfrac{t-a}{a}, & \text{if } \ a \le t < s \le 0, \\ \dfrac{t-b}{b}, & \text{f } \ 0 \le s \le t \le b, \\ \dfrac{t(s-b)}{bs}, & \text{if } \ 0 \le t < s \le b, \\ 0 & \quad \text{at all other points.} \end{cases}$$

Since the determinant of the model problem

$$\begin{vmatrix} u_1(a) & u_2(a) & u_3(a) \\ u_1(b) & u_2(b) & u_3(b) \\ u_1(0) & u_2(0) & u_3(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0,$$

this problem has for any $\{z, \beta\} \in \mathbf{L} \times \mathbf{R}^3$ the unique solution $x = \Lambda z + Y\beta$, where $\beta = \mathrm{col}\{\beta^1, \beta^2, \beta^3\}$,

$$(Y\beta)(t) = \beta^1 u_1(t) + \beta^2 u_2(t) + \beta^3 u_3(t).$$

Let us take $\mathbf{D} = \Lambda\mathbf{L} \oplus Y\mathbf{R}^3$, where $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L} \times \mathbf{R}^3 \to \mathbf{D}$ is the isomorphism, the inverse $\mathcal{J}^{-1} = [\delta, r]$ being defined by

$$(\delta x)(t) = t\ddot{x}(t), \quad rx = \{x(a), x(b), x(0)\}.$$

The principal part of $\mathcal{L} : \mathbf{D} \to \mathbf{L}$ has the form $Q = I + K$, where

$$(Kz)(t) = \int_a^b p(t)\Lambda[h(t), s]z(s)ds.$$

If the operator $Q : \mathbf{L} \to \mathbf{L}$ has the bounded inverse, then the principal boundary value problem

$$\mathcal{L}x = f, \quad x(a) = \alpha^1, \quad x(b) = \alpha^2, \quad x(0) = \alpha^3$$

is uniquely solvable (Theorem 2.4), and the general solution of the equation $\mathcal{L}x = f$ admits the representation

$$x(t) = \int_a^b G(t, s)f(s)ds + c_1x_1(t) + c_2x_2(t) + c_3x_3(t),$$

where $G(t, s)$ is the Green function of this problem, x_1, x_2, x_3 constitute a fundamental system of solutions of $\mathcal{L}x = 0$, c_i are constants.

Consider an example of singularity of the other kind. Define the operation θ by

$$(\theta x)(t) = \begin{cases} \ddot{x}(t), & \text{if} \quad t \in [1, 2], \\ 0, & \text{if} \quad t \in [0, 1), \end{cases}$$

and let us study the equation

$$(\mathcal{L}x)(t) \stackrel{def}{=} (\theta x)(t) + \dot{x}(t) + (Tx)(t) = f(t), \quad t \in [0, 2], \tag{9.5}$$

with a linear operator $T : \mathbf{W}^2 \to \mathbf{L}$.

The principal part of the operator $\mathcal{L} : \mathbf{W}^2 \to \mathbf{L}$ is not Fredholm even under the assumption that $T : \mathbf{W}^2 \to \mathbf{L}$ is a compact operator. We will define the operator $\mathcal{L}$ on a more wide space $\mathbf{D}$, assuming that T allows an extension onto this space. We will construct the space $\mathbf{D}$ as follows.

Let us take as a model the problem

$$\begin{gathered}(\mathcal{L}_0x)(t) \stackrel{def}{=} (\theta x)(t) + \chi_{[0,1)}(t)\dot{x}(t) = z(t), \quad t \in [0, 2], \\ x(0) = \beta^1, \quad x(1) = \beta^2, \quad \dot{x}(1) = \beta^3.\end{gathered} \tag{9.6}$$

This problem splits up into two ones which are integrable in the explicit form

$$
\begin{gathered}
\dot{x}(t) = z(t), \quad t \in [0,1), \quad x(0) = \beta^1, \\
\ddot{x}(t) = z(t), \quad t \in [1,2], \quad x(1) = \beta^2, \quad \dot{x}(1) = \beta^3.
\end{gathered}
$$

As a solution of the model problem, we may take the function

$$
\begin{aligned}
x(t) = \chi_{[0,1)}(t)\Big\{ \int_0^t z(s)ds + \beta^1 \Big\} &+ \\
+ \chi_{[1,2]}(t)\Big\{ \int_0^t \chi_{[1,2]}(s)(t-s)z(s)ds &+ \beta^2 + \beta^3(t-1) \Big\}.
\end{aligned}
$$

Denote $(\Lambda z)(t) = \int_0^2 \Lambda(t,s)z(s)ds$, where

$$
\Lambda(t,s) = \begin{cases} 1, & \text{if } \; 0 \le s \le t < 1, \\ t-s, & \text{if } \; 1 \le s \le t \le 2, \\ 0 & \quad \text{at all other points.} \end{cases}
$$

Let further

$$
\begin{gathered}
(Y\beta)(t) = \beta^1 u_1(t) + \beta^2 u_2(t) + \beta^3 u_3(t), \quad \beta = \operatorname{col}\{\beta^1, \beta^2, \beta^3\}, \\
u_1(t) = \chi_{[0,1)}(t), \quad u_2(t) = \chi_{[1,2]}(t), \quad u_3(t) = \chi_{[1,2]}(t)(t-1).
\end{gathered}
$$

The solution of the model problem has the form $x = \Lambda z + Y\beta$.

Next, define the space $\mathbf{D}$ by $\mathbf{D} = \Lambda\mathbf{L} \oplus Y\mathbf{R}^3$. This space consists of the functions $x : [0,2] \to \mathbf{R}^1$ with possible discontinuity at $t = 1$. These functions are absolutely continuous on $[0,1)$ and have absolutely continuous derivatives on $[1,2]$. $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L} \times \mathbf{R}^3 \to \mathbf{D}$ is the isomorphism, $\mathcal{J}^{-1} = [\delta, r]$, where

$$
\delta x = \mathcal{L}_0 x, \quad rx = \{x(0), x(1), \dot{x}(1)\}.
$$

The norm may be defined by

$$
\|x\|_{\mathbf{D}} = \|\mathcal{L}_0 x\|_{\mathbf{L}} + |x(0)| + |x(1)| + |\dot{x}(1)|.
$$

Since $\mathcal{L}x = \mathcal{L}_0 x + \chi_{[1,2]}\dot{x} + Tx$,

$$
(Qz)(t) = z(t) + \chi_{[1,2]}(t) \int_0^t \chi_{[1,2]}(s)z(s)ds + (T\Lambda z)(t).
$$

If the product $T\Lambda : \mathbf{L} \to \mathbf{L}$ is compact, the principal part $Q : \mathbf{L} \to \mathbf{L}$ will be canonical Fredholm. If $\|K\|_{\mathbf{L}\to\mathbf{L}} < 1$, where

$$(Kz)(t) = (T\Lambda z)(t) + \chi_{[1,2]}(t) \int\limits_0^t \chi_{[1,2]}(s) z(s) ds,$$

then the principal boundary value problem

$$\mathcal{L}x = f, \quad x(0) = \alpha^1, \quad x(1) = \alpha^2, \quad \dot{x}(1) = \alpha^3$$

is uniquely solvable. In this case (Theorem 2.5), the homogeneous equation $\mathcal{L}x = 0$ has a three-dimensional fundamental system of solutions x_1, x_2, x_3, and the general solution of the equation $\mathcal{L}x = f$ in the space $\mathbf{D}$ has the representation

$$x(t) = \int\limits_0^2 G(t,s) f(s) ds + c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t),$$

where $G(t,s)$ is the Green function of the principal boundary value problem, $c_i = \text{const}$.

Denote $\Delta x(t) = x(t) - x(t-0)$. The subspace $\mathbf{D}_0 = \{x \in \mathbf{D} : \Delta x(1) = 0\}$ of the space $\mathbf{D}$ is constituted only of continuous functions. The homogeneous equation $\mathcal{L}_0 x = 0$ has two linearly independent solutions

$$y_1(t) = 1 - \chi_{[1,2]}(t)(t-1), \quad y_2(t) = \chi_{[1,2]}(t)(t-1)$$

in the space $\mathbf{D}_0$. The equation $\mathcal{L}_0 x = z$ has for any $z \in \mathbf{L}$ solutions belonging to $\mathbf{D}_0$, for instance,

$$v(t) = \chi_{[0,1)}(t) \int\limits_0^t z(s) ds + \\ + \chi_{[1,2]}(t) \left\{ \int\limits_0^t \chi_{[1,2]}(s)(t-s) z(s) ds + \int\limits_0^1 z(s) ds \right\}.$$

Thus the general solution of the model equation $\mathcal{L}_0 x = z$ in the space $\mathbf{D}_0$ may be represented in the form

$$x(t) = v(t) + c_1 y_1(t) + c_2 y_2(t), \tag{9.7}$$

where c_1, c_2 are constants.

Since

$$\begin{vmatrix} y_1(0) & y_2(0) \\ y_1(2) & y_2(2) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0,$$

the two-point boundary value problem

$$\mathcal{L}_0 x = z, \quad x(0) = 0, \quad x(2) = 0$$

is uniquely solvable in the space $\mathbf{D}_0$. The Green function $W(t,s)$ of this problem can be constructed by finding the constants c_1, c_2 in (9.7) such that $x(0) = x(2) = 0$. We have

$$x(t) \overset{def}{=} (Wz)(t) = \int_0^2 W(t,s)z(s)ds,$$

where

$$W(t,s) = \begin{cases} 1, & \text{if } \ 0 \le s \le t < 1, \\ 2-t, & \text{if } \ 1 \le t \le 2, \ \ 0 \le s < 1, \\ -(2-t)(s-1), & \text{if } \ 1 \le s \le t \le 2, \\ -(2-s)(t-1), & \text{if } \ 1 \le t < s \le 2, \\ 0 & \quad \text{at all other points.} \end{cases}$$

Notice that it is possible to construct $W(t,s)$ on the ground of the representation

$$x(t) = (\Lambda z)(t) + \beta^1 u_1(t) + \beta^2 u_2(t) + \beta^3 u_3(t)$$

of the solution (9.6) by demanding the fulfillment of the conditions $x(0) = \Delta x(1) = x(2) = 0$.

Thus the space $\mathbf{D}_0$ is defined by $\mathbf{D}_0 = W\mathbf{L} \oplus Y_0\mathbf{R}^2$, where

$$\begin{aligned} (Y_0\beta)(t) &= \big[1 - \chi_{[1,2]}(t)(t-1)\big]\beta^1 + \chi_{[1,2]}(t)(t-1)\beta^2, \\ \beta &= \operatorname{col}\left\{\beta^1, \beta^2\right\}, \end{aligned}$$

$\mathcal{J}_0 = \{W, Y_0\} : \mathbf{L} \times \mathbf{R}^2 \to \mathbf{D}_0$ is the isomorphism, $\mathcal{J}_0^{-1} = [\mathcal{L}_0, r_0]$, $r_0 x = \{x(0), x(2)\}$. The two-point boundary value problem

$$\mathcal{L}x = f, \quad x(0) = \alpha^1, \quad x(2) = \alpha^2$$

is the principal boundary value problem for the equation $\mathcal{L}x = f$ in the space $\mathbf{D}_0$. This problem is uniquely solvable if and only if the operator $Q = \mathcal{L}W : \mathbf{L} \to \mathbf{L}$ has the bounded inverse.

§ 10. Minimization of Square Functionals

The problem of minimization of functionals is unsolvable in the frame of classical calculus of variations if the given functional has not a minimum on the traditional sets of functions. The question on the expedient choice of the set on which the functional must be defined was put up by Hilbert and, as it was emphasized by the authors of the book [34], each class of functionals must be studied in the proper “own” space.

The scheme proposed below permits to approach in a new fashion the problem of minimization, expands the possibility of the calculus of variations and leads to tests for the existence of the minimum for some classes of problems in terms of the problem.

The scheme has been developing on the base of the theory of abstract functional differential equations in the works of the Perm Seminar [35-38]. Let $\mathbf{D}$ be a Banach space of functions $x : [a,b] \to \mathbf{R}^1$ isomorphic to the direct product $\mathbf{L}_2 \times \mathbf{R}^n$, $\mathbf{L}_2$ be the Banach space of square summable functions $z : [a,b] \to \mathbf{R}^1$, $\|z\|_{\mathbf{L}_2} = \{\int_a^b z^2(s)ds\}^{\frac{1}{2}}$. Denote by $T_{ji} : \mathbf{D} \to \mathbf{L}_2$, $j = 1,2$; $i = 1,\dots,\mu$, $T_0 : \mathbf{D} \to \mathbf{L}_2$ linear bounded operators. Let further $l = [l^1,\dots,l^N] : \mathbf{D} \to \mathbf{R}^N$ be a bounded linear vector-functional with linearly independent components, $N \geq n$, $\omega \in \mathbf{L}$.

Consider the problem on existence of an element $x \in \mathbf{D}$ on which the square functional

$$\mathcal{I}(x) = \int_a^b \left\{ \sum_{i=1}^{\mu} (T_{1i}x)(s)(T_{2i}x)(s) + (T_0x)(s) + \omega(s) \right\} ds$$

with additional conditions $l^i x = \alpha^i$, $i = 1,\dots,N$, has the minimum.

The problem on the minimum of the functional $\mathcal{I}$ on the set $\mathbf{D}_\alpha = \{x \in \mathbf{D} : l^1x = \alpha^1,\dots,l^Nx = \alpha^N\}$ contains the problems of classical calculus of variations and many other new ones.

The approach to this problem is based on the substitution

$$x = \Gamma z + u, \tag{10.1}$$

where $u \in \mathbf{D}_\alpha$ is a fixed element and $\Gamma : \mathbf{L}_2 \to \mathbf{D}$ is a linear operator such that $\mathbf{D}_\alpha = \Gamma\mathbf{L}_2 + \{u\}$. By means of this substitution, the considered problem may be reduced to the well-known problem of the unconditional minimum of the functional $\mathcal{I}_1(z) = \mathcal{I}(\Gamma z + u)$ on the space $\mathbf{L}_2$.

Beforehand we will note the following. The isomorphism $\mathcal{J} : \mathbf{L}_2 \times \mathbf{R}^n \to \mathbf{D}$ for the given space $\mathbf{D}$ may be defined on the base of any uniquely solvable boundary value problem in this space. In particular, it might be constructed on the base of the problem

$$\mathcal{L}_1 x = z, \quad l^i x = \beta^i, \quad i = 1,\dots,n \tag{10.2}$$

with boundary conditions defined by any n components of the given vector-functional $l = [l^1,\dots,l^N]$. Theorem 3.4 guarantees the existence of a linear operator $\mathcal{L}_1 : \mathbf{D} \to \mathbf{L}_2$ such that the problem (10.2) is uniquely solvable.

Let Λ be the Green operator for the problem (10.2) and $Y = (y_1,\dots,y_n)$ be a fundamental vector of the equation $\mathcal{L}_1 x = 0$, i.e., $l^i y_j = \delta_{ij}$, $i,j = 1,\dots,n$. Then we can set $\mathcal{J} = \{\Lambda, Y\}$, $\mathcal{J}^{-1} = [\delta, r]$, where $\delta = \mathcal{L}_1$, $r = [r^1,\dots,r^n] = [l^1,\dots,l^n]$. Therefore we may suppose without loss of generality that $r = [l^1,\dots,l^n]$.

If $N = n$, we will suppose $\Gamma = \Lambda$ and $u = Y\alpha$ in (10.1). In the case $N > n$, we will use the following construction.

Let a system of elements $v_1, \dots, v_N \in \mathbf{D}$ be biorthogonal to the system of functionals $l^1, \dots, l^N$ ($l^i v_j = \delta_{ij}$, $i, j = 1, \dots, N$). Let further

$$\Gamma z = \Lambda z - \sum_{k=n+1}^{N} v_k l^k \Lambda z, \quad u = \sum_{i=1}^{N} v_i \alpha^i.$$

We will show that $\Gamma \mathbf{L}_2 + \{u\} = \mathbf{D}_\alpha$.

It is sufficient to see that $\Gamma \mathbf{L}_2 = \mathbf{D}_0 \overset{def}{=} \{x \in \mathbf{D} : lx = 0\}$. The inclusion $\Gamma \mathbf{L}_2 \subset \mathbf{D}_0$ is verifying immediately. Let us show that $\mathbf{D}_0 \subset \Gamma \mathbf{L}_2$, that is, for each $x \in \mathbf{D}_0$ there exists a $z \in \mathbf{L}_2$ such that $\Gamma z = x$.

Define the degenerate operator $F : \mathbf{D}_0 \to \mathbf{D}_0$ by

$$F\xi = \sum_{k=n+1}^{N} v_k l^k \xi.$$

Let us fix $x \in \mathbf{D}_0$ and consider the equation

$$\xi - F\xi = x. \tag{10.3}$$

The unit is an eigen-value of F because the functions $v_{n+1}, \dots, v_N$ are solutions to the equation $\xi = F\xi$. Due to definition of the set $\mathbf{D}_0$, any $x \in \mathbf{D}_0$ is orthogonal to the functionals $l^{n+1}, \dots, l^N$. This system is a basis of the kernel of the operator $(I - F)^*$ adjoint to $I - F$. Thus the equation (10.3) has solutions. Let ξ_0 be one of them and $z = \delta \xi_0$. Then

$$\Gamma z = \Lambda \delta (x + F\xi_0) - F \Lambda \delta (x + F\xi_0) = x + F\xi_0 - Fx - F^2 \xi_0 = x,$$

because from (10.3) it follows that $Fx = F\xi_0 - F^2 \xi_0$.

Denote

$$Q_{ji} = T_{ji}\Gamma, \quad Q_0 = T_0 \Gamma.$$

Due to the substitution (10.1), we have:

$$\begin{aligned}
\mathcal{I}(x) = \mathcal{I}(\Gamma z + u) \overset{def}{=} \mathcal{I}_1(z) = \int_a^b \sum_{i=1}^{\mu} (Q_{1i} z)(s)(Q_{2i} z)(s) ds + \\
+ \int_a^b \sum_{i=1}^{\mu} \big\{ (Q_{1i} z)(s)(T_{2i} u)(s) + (Q_{2i} z)(s)(T_{1i} u)(s) \big\} ds + \\
+ \int_a^b \sum_{i=1}^{\mu} (T_{1i} u)(s)(T_{2i} u)(s) ds + \\
+ \int_a^b \big\{ (Q_0 z)(s) + (T_0 u)(s) \big\} ds + \int_a^b \omega(s) ds.
\end{aligned}$$

Using the equality

$$\int_a^b (Az)(s)(Bz)(s)ds = \int_a^b (A^*Bz)(s)z(s)ds$$

and denoting

$$<\varphi, \psi> = \int_a^b \varphi(s)\psi(s)ds,$$

we may write

$$\mathcal{I}_1(z) = \frac{1}{2} < Hz, z > - < f, z > + g, \tag{10.4}$$

where

$$H = \sum_{i=1}^{\mu} (Q_{1i}^* Q_{2i} + Q_{2i}^* Q_{1i}), \tag{10.5}$$

$$f = -\sum_{i=1}^{\mu} (Q_{1i}^* T_{2i} + Q_{2i}^* T_{1i})u - Q_0^*(1),$$

$$g = \int_a^b \Big\{ \sum_{i=1}^{\mu} (T_{1i}u)(s)(T_{2i}u)(s) + (T_0u)(s) + \omega(s) \Big\} ds.$$

Thus $H : \mathbf{L}_2 \to \mathbf{L}_2$ is a self-adjoint operator, $f \in \mathbf{L}_2$, $g = \text{const}$.

Following the adopted terminology, we will call the operator $H : \mathbf{L}_2 \to \mathbf{L}_2$ positive definite if $< Hz, z > \geq 0$ for all $z \in \mathbf{L}_2$. The positive definite operator H is called to be strictly positive definite if $< Hz, z > = 0$ only for $z = 0$.

In order to formulate and prove the main result about the problem in consideration, we will use the following definitions.

A point $x_0 \in \mathbf{D}$ ($z_0 \in \mathbf{L}_2$) is called the point of local minimum of functional $\mathcal{I}$ ($\mathcal{I}_1$), if there exists an $\varepsilon > 0$ such that $\mathcal{I}(x) \geq \mathcal{I}(x_0)$ ($\mathcal{I}_1(z) \geq \mathcal{I}_1(z_0)$) for all $x \in \mathbf{D}_\alpha$ ($z \in \mathbf{L}_2$) satisfying $\|x - x_0\|_{\mathbf{D}} < \varepsilon$ ($\|z - z_0\|_{\mathbf{L}_2} < \varepsilon$). If $\mathcal{I}(x) \geq \mathcal{I}(x_0)$ ($\mathcal{I}_1(z) \geq \mathcal{I}_1(z_0)$) holds for all $x \in \mathbf{D}_\alpha$ ($z \in \mathbf{L}_2$), x_0 (z_0) is called the point of global minimum. The value $\mathcal{I}(x_0)$ ($\mathcal{I}_1(z_0)$) is called local or correspondingly global minimum of the functional.

From the equality $\mathcal{I}(x) - \mathcal{I}(x_0) = \mathcal{I}_1(z) - \mathcal{I}_1(z_0)$ for $x_0 = \Gamma z_0 + u$ and for $x = \Gamma z + u$, it immediately follows that x_0 is the point of the global minimum of the functional $\mathcal{I}$ if and only if z_0 is the point of global minimum of the functional $\mathcal{I}_1$.

Theorem 10.1. *Any local minimum of the functional $\mathcal{I}$ is the global one.*

The functional $\mathcal{I}$ has a point of minimum x_0 on the set $\mathbf{D}_\alpha = \{x \in \mathbf{D} : l^i x = \alpha^i,\ i = 1, \ldots, N\}$ if and only if the operator $H : \mathbf{L}_2 \to \mathbf{L}_2$ defined by

(10.5) *is positive definite and the equation* $Hz = f$ *has a solution* $z_0 \in \mathbf{L}_2$. *In this case,* $x_0 = \Gamma z_0 + u$.

The proof follows from the next two Lemmas.

Lemma 10.1. *Any local minimum of the functional* $\mathcal{I}_1$ *on the space* $\mathbf{L}_2$ *is the global one.*

The element $z_0 \in \mathbf{L}_2$ *is the point of minimum of* $\mathcal{I}_1$ *if and only if the operator* $H : \mathbf{L}_2 \to \mathbf{L}_2$ *defined by* (10.5) *is positive definite and* z_0 *is a solution to the equation* $Hz = f$.

Proof. Let z_0 be a point of local minimum. It means that there exists an $\varepsilon > 0$ such that $\mathcal{I}_1(z) - \mathcal{I}_1(z_0) \geq 0$ if $\|z - z_0\|_{\mathbf{L}_2} < \varepsilon$. Let us fix $\xi \in \mathbf{L}_2$ and let $\gamma_0 > 0$ be a number such that $\|\gamma_0 \xi\|_{\mathbf{L}_2} < \varepsilon$. From (10.4), we have

$$\mathcal{I}_1(z_0 + \gamma\xi) - \mathcal{I}_1(z_0) = \frac{\gamma^2}{2} < H\xi, \xi > +\gamma < Hz_0 - f, \xi > . \qquad (10.6)$$

The quadratic binomial $\frac{\gamma^2}{2} < H\xi, \xi > +\gamma < Hz_0 - f, \xi >$ takes no negative values if $\gamma \in (-\gamma_0, \gamma_0)$. It means that this binomial takes no negative values for any γ. Consequently, z_0 is a point of global minimum. Besides, due to the arbitrary choice of ξ, we deduce that $Hz_0 - f = 0$ and $< H\xi, \xi > \geq 0$ for any $\xi \in \mathbf{L}_2$.

The converse assertion follows from (10.6). ■

Lemma 10.2. *If* x_0 *is the point of a local minimum of the functional* $\mathcal{I}$ *on the set* $\mathbf{D}_\alpha$ *and* $x_0 = \Gamma z_0 + u$, *then* z_0 *is the point of minimum of the functional* $\mathcal{I}_1$.

Proof. Let $\varepsilon > 0$ be such that $\mathcal{I}(x) - \mathcal{I}(x_0) \geq 0$ if $\|x - x_0\|_{\mathbf{D}} < \varepsilon$. Any $x \in \mathbf{D}_\alpha$ has the representation $x = \Gamma z + u$. Since $\|x - x_0\|_{\mathbf{D}} \leq \|\Gamma\|_{\mathbf{L}_2 \to \mathbf{D}} \|z - z_0\|_{\mathbf{L}_2}$,

$$\mathcal{I}_1(z) - \mathcal{I}_1(z_0) = \mathcal{I}(x) - \mathcal{I}(x_0) \geq 0, \quad \text{if} \quad \|z - z_0\|_{\mathbf{L}_2} \leq \frac{\varepsilon}{\|\Gamma\|_{\mathbf{L}_2 \to \mathbf{D}}}.$$

Consequently, z_0 is a point of minimum of the functional $\mathcal{I}_1$. ■

It is known that a self-adjoint $H : \mathbf{L}_2 \to \mathbf{L}_2$ is positive definite if and only if its spectrum $\sigma(H)$ does not contain negative numbers: $\sigma(H) \subset [0, +\infty)$, and it is strictly positive definite if $\sigma(H) \subset (0, +\infty)$. Therefore if $H = 2(I - K)$, then H is positive (strictly positive) definite if and only if $\sigma(K) \subset (-\infty, 1]$, $(\sigma(K) \subset (-\infty, 1))$. If K is isotonic and $\rho(K)$ is the spectral radius of K, then $\sigma(K) \subset [-\rho(K), \rho(K)]$. Consequently, in the case of isotonic K, the operator $H = 2(I - K)$ is strictly positive definite if and only if $\rho(K) < 1$. Thus, from the said and taking into account that $\rho(K) = \|K\|_{\mathbf{L}_2}$, we are in a position to state the following

Corollary 10.1. *Let the operator* $H : \mathbf{L}_2 \to \mathbf{L}_2$ *be defined by* (10.5) *and* $H = 2(I - K)$. *For the existence of unique point* $x_0 \in \mathbf{D}_\alpha$ *of minimum of the functional* $\mathcal{I}$ *it is sufficient, and in the case of isotonic* K *it is also necessary that* $\|K\|_{\mathbf{L}_2 \to \mathbf{L}_2} < 1$.

Denote $f_0 = -Q_0^*(1)$ and define $\mathcal{L} : \mathbf{D} \to \mathbf{L}_2$ by

$$\mathcal{L} = \sum_{i-1}^{\mu} \big(Q_{1i}^* T_{2i} + Q_{2i}^* T_{1i} \big).$$

Theorem 10.2. $x_0 \in \mathbf{D}_\alpha$ *is the point of minimum of the functional* $\mathcal{I}$ *if and only if*

a) x_0 *is a solution to the boundary value problem*

$$\mathcal{L}x = f_0, \quad lx = \alpha, \tag{10.7}$$

where $\alpha = \operatorname{col}\{\alpha^1, \ldots, \alpha^N\}$.

Remark 10.1. b) The operator $H : \mathbf{L}_2 \to \mathbf{L}_2$ defined by (10.5) is positive definite.

Proof. Let $x_0 \in \mathbf{D}_\alpha$ be a solution to (10.6). There exists $z_0 \in \mathbf{L}_2$ such that $x_0 = \Gamma z_0 + u$, besides

$$Hz_0 = \mathcal{L}\Gamma z_0 = \mathcal{L}(x_0 - u) = f_0 + f - f_0 = f.$$

Consequently, z_0 is a solution to $Hz = f$. By virtue of Theorem 10.1, x_0 is the point of minimum of the functional $\mathcal{I}$.

Conversely, if z_0 is a solution to $Hz = f$, then $x_0 = \Gamma z_0 + u$ is the point of minimum of $\mathcal{I}$ and satisfies to (10.6). Really, $lx_0 = \alpha$, $\mathcal{L}x_0 = \mathcal{L}(\Gamma z_0 + u) = Hz_0 - f + f_0 = f_0$. ■

Remark 10.2. It is natural to call the equation $\mathcal{L}x = f_0$ Euler's equation and the boundary condition $lx = \alpha$ corresponds to "natural boundary condition" in the classical calculus of variations.

Corollary 10.1 permits sometimes to reduce the problem on the minimum of a functional to proper estimation of the spectral radius (the norm) of the operator $K : \mathbf{L}_2 \to \mathbf{L}_2$. Such an estimation meets a good deal of difficulty in many cases. But sometimes it is possible to construct an operator in the space $\mathbf{C}$ of continuous functions with the same spectral radius as the one of K.

We give below a well-known Lemma 10.3 and an addition to the results of [39, 40] in the form of Lemma 10.4.

Lemma 10.3. *The spectral radius of a linear bounded isotonic operator* $A : \mathbf{C} \to \mathbf{C}$ *is less than* 1 *if and only if there exists a continuous function* v *such that*

$$v(t) > 0, \quad r(t) \overset{def}{=} v(t) - (Av)(t) > 0, \quad t \in [a, b].$$

Proof. Necessity is obvious: as the function v one can take the solution of the equation $x - Ax = 1$.

Sufficiency. Define the isotonic operator $F : \mathbf{C} \to \mathbf{C}$ by $Fx = \frac{1}{v}A(vx)$. We have

$$\rho(F) \leq \|F\|_{\mathbf{C}\to\mathbf{C}} = \max_{t\in[a,b]}[F(1)](t) = \max_{t\in[a,b]} \frac{(Av)(t)}{v(t)} < 1.$$

For each λ there exists a one-to-one mapping between the set of solutions x to the equation $\lambda x = Ax + f$ and the set of solutions y to the equation $\lambda y = Fy + \frac{1}{v}f$: $x = vy$, $y = \frac{1}{v}x$. Therefore the spectrums of F and A coincide. Thus $\rho(A) < 1$. ∎

The condition of Lemma 10.3 about strong positiveness of r and v meets some difficulties in application of this Lemma. But some additional requests on the properties of the operator A permit to weaken this condition.

Lemma 10.4. *Let a linear bounded isotonic operator $A : \mathbf{C} \to \mathbf{C}$ have the property $(A\xi)(a) = (A\xi)(b) = 0$ for each $\xi \in \mathbf{C}$. Let further a continuous function v satisfy the inequalities*

$$v(t) > 0, \quad r(t) \stackrel{def}{=} v(t) - (Av)(t) > 0, \quad t \in (a,b).$$

Then $\rho(A) < 1$.

Proof. Denote

$$\xi = 1 - A(1), \quad v_\varepsilon = v + \varepsilon\xi, \quad r_\varepsilon \stackrel{def}{=} v_\varepsilon - Av_\varepsilon = r + \varepsilon\psi,$$

where $\varepsilon > 0$, $\psi = \xi - A\xi$. It follows from $\xi(t) \geq 0$ that $v_\varepsilon(t) > 0$ on $[a,b]$, and from $\psi(t) \geq 0$ that $r_\varepsilon(t) > 0$ on $[a,b]$. If both inequalities $\xi(t) \geq 0$ and $\psi(t) \geq 0$ are fulfilled, $\rho(A) < 1$ by virtue of Lemma 10.3.

Let ψ change its sign on $[a,b]$. Denote $\omega = \{t \in [a,b] : \psi(t) < 0\}$. The inequality $r_\varepsilon(t) > 0$ is fulfilled on $[a,b] \setminus \omega$. Denote by τ the first zero of ψ to the right of a and denote by θ the first zero of ψ to the left of b ($\tau > a$ and $\theta < b$ since $\psi(a) = \psi(b) = 1$). Denote

$$m_1 = \min_{t\in[\tau,\theta]} r(t), \quad m_2 = \min_{t\in[\tau,\theta]} \psi(t).$$

For $\varepsilon_1 \in (0, \frac{m_1}{-m_2})$, we have the inequality

$$r(t) + \varepsilon_1\psi(t) > r(t) + \frac{m_1}{-m_2}\psi(t) \geq r(t) - m_1 \geq 0$$

on ω. Consequently, $r(t) + \varepsilon_1\psi(t) > 0$ on the whole segment $[a,b]$.

If ξ changes its sign, we can analogously choose $\varepsilon_2 > 0$ such that $v(t) + \varepsilon_2\xi(t) > 0$ on $[a,b]$. Thus we obtain the inequalities $v_\varepsilon(t) > 0$ and $r_\varepsilon(t) > 0$ on $[a,b]$ if $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Therefore $\rho(A) < 1$ by virtue of Lemma 10.3. ∎

Below we will apply to thee examples the above scheme of investigation of functionals in the space $\mathbf{D} \simeq \mathbf{L}_2 \times \mathbf{R}^n$. For the first example, a simplest functional is taken making possible to construct the point of minimum in the explicit form. This example illustrates also the possibility of choosing various spaces on which the functional has a minimum. In this connection, it is emphasized that minimums on different spaces may differ. For the second example, the functional is taken which was investigated in particular cases in [41] on the base of the classical calculus of variations. This example illustrates the advantage of our scheme before classical methods. The Euler's equation for the third example turned out to be singular by using traditional spaces. A particular case of this functional was investigated in [42], where a special minimizing sequence was constructed and its convergence was proved. The space $\mathbf{D} \simeq \mathbf{L}_2 \times \mathbf{R}^2$ for this functional was constructed using the above scheme of investigation of the singular equation (9.2).

Example 10.1. Consider the functional

$$\mathcal{I}(x) = \int_0^1 \big\{\dot{x}^2(s) - q(s)\dot{x}(s) - p(s)x(s)\big\}ds$$

with conditions $x(0) = \alpha^1$, $x(1) = \alpha^2$.

If q is absolutely continuous, then the classical methods from elementary textbooks are applicable. The classical Euler's equation in this case has the form

$$\ddot{x}(t) = \frac{1}{2}[\dot{q}(t) - p(t)],$$

and consequently the point of minimum is defined by

$$x_0(t) = \frac{1}{2}\int_0^1 W(t,s)[\dot{q}(s) - p(s)]ds + \alpha^1(1-t) + \alpha^2 t,$$

where

$$W(t,s) = \begin{cases} -s(1-t), & \text{if } \ 0 \le s \le t \le 1, \\ -t(1-s), & \text{if } \ 0 \le t < s \le 1, \end{cases}$$

is the Green function of the problem $\ddot{x} = z$, $x(0) = 0$, $x(1) = 0$. Thus

$$x_0(t) = \frac{1}{2}\Big[\int_0^t q(s)ds - t\int_0^1 q(s)ds - (t-1)\int_0^t sp(s)ds -$$

$$- t\int_t^1 (s-1)p(s)ds\Big] + \alpha^1(1-t) + \alpha^2 t.$$

Thus $\mathcal{I}(x_0) = -\frac{p^2}{48}$, if $\alpha^1 = \alpha^2 = 0$, $p = \text{const}$, $q = \text{const}$.

Next consider the same problem using the above scheme and various spaces $\mathbf{D} \simeq \mathbf{L}_2 \times \mathbf{R}^n$.

Since $T_{11}x = T_{21}x \stackrel{def}{=} Tx = \dot{x}$, $T_0 x = -q\dot{x} - px$, we have $Q_{11} = Q_{12} = T\Gamma \stackrel{def}{=} Q$, $H = 2Q^*Q$. In any case of $\mathbf{D}$, the operator $H : \mathbf{L}_2 \to \mathbf{L}_2$ is positive definite since

$$< Hz, z >= 2 < Q^*Qz, z >= 2 < Qz, Qz > .$$

1). Let $\mathbf{D} = \mathbf{W}_2^2$ be the space of the functions $x : [a, b] \to \mathbf{R}^1$ with absolutely continuous derivative $\dot{x}$ and $\ddot{x} \in \mathbf{L}_2$. Define the isomorphism $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L}_2 \times \mathbf{R}^2 \to \mathbf{W}_2^2$ by

$$(\Lambda z)(t) = \int_0^1 W(t, s)z(s)ds, \quad (Y\beta)(t) = \beta^1(1-t) + \beta^2 t,$$
$$\beta = \mathrm{col}\{\beta^1, \beta^2\}.$$

In this case, $\Gamma = \Lambda$, $u = \alpha^1(1-t) + \alpha^2 t$, and

$$(\Lambda z)(t) = (t-1)\int_0^t sz(s)ds - t\int_t^1 (1-s)z(s)ds.$$

After direct calculations we have:

$$(Qz)(t) = -\int_t^1 z(s)ds + \int_0^1 sz(s)ds, \quad (Q^*z)(t) = -\int_0^t z(s)ds + t\int_0^1 z(s)ds,$$

$$(Q_0 z)(t) = q(t)\int_t^1 z(s)ds - q(t)\int_0^1 sz(s)ds +$$
$$+ (1-t)p(t)\int_0^t sz(s)ds + tp(t)\int_t^1 (1-s)z(s)ds,$$

$$(Q_0^* z)(t) = \int_0^t q(s)z(s)ds - t\int_0^1 q(s)z(s)ds +$$
$$+ t\int_t^1 (1-s)p(s)z(s)ds + (1-t)\int_0^t sp(s)z(s)ds,$$

$$f_0(t) = -\int_0^t q(s)ds + t\int_0^1 q(s)ds +$$

$$+t\int_t^1 (s-1)p(s)ds + (t-1)\int_0^t sp(s)ds.$$

Next, $\mathcal{L} = 2Q^*T$ and the equation $\mathcal{L}x = f_0$ obtains the form

$$2[-x(t) + x(0) + tx(1) - tx(0)] =$$

$$= -\int_0^t q(s)ds + t\int_0^1 q(s)ds + t\int_t^1 (s-1)p(s)ds + (t-1)\int_0^t sp(s)ds.$$

By virtue of Theorem 10.2, the unique point of minimum is again the function

$$x_0(t) = \frac{1}{2}\Bigg[\int_0^t q(s)ds - t\int_0^1 q(s)ds - t\int_t^1 (s-1)p(s)ds -$$

$$-(t-1)\int_0^1 sp(s)ds\Bigg] + \alpha^1(1-t) + \alpha^2 t.$$

Direct differentiation shows that $x_0 \in \mathbf{W}_2^2$ if and only if $p, \dot{q} \in \mathbf{L}_2$. Therefore, without this condition, the functional $\mathcal{I}$ has no minimum on the space $\mathbf{W}_2^2$.

Let us note that the double differentiation of the equation leads to the classical Euler's equation $\ddot{x}(t) = \frac{1}{2}[\dot{q}(t) - p(t)]$.

2). Supposing $p, q \in \mathbf{L}_2$, we may look for the minimum of the functional $\mathcal{I}$ in the space $\mathbf{D} = \mathbf{W}_2^1 \simeq \mathbf{L}_2 \times \mathbf{R}^1$ of the absolutely continuous functions with $\dot{x} \in \mathbf{L}_2$ which is larger than $\mathbf{W}_2^2$. Each element of this space has the representation

$$x(t) = (\Lambda z)(t) + (Y\beta)(t) \overset{def}{=} \int_0^t z(s)ds + \beta, \quad \{z, \beta\} \in \mathbf{L}_2 \times \mathbf{R}^1.$$

In the case under consideration, $N = 2 > n = 1$, and therefore we suppose in the substitution (10.1)

$$(\Gamma z)(t) = \int_0^t z(s)ds - t\int_0^1 z(s)ds, \quad u(t) = \alpha^1(1-t) + \alpha^2 t.$$

We have

$$Qz = Q^*z = z - \int_0^1 z(s)ds, \quad Hz = 2\Bigg[z - \int_0^1 z(s)ds\Bigg],$$

$$(Q_0^* z)(t) = -q(t)z(t) + \int_0^1 [q(s) + sp(s)]z(s)ds - \int_t^1 p(s)z(s)ds,$$

$$f_0(t) = f(t) = q(t) - \int_0^1 [q(s) + sp(s)]ds + \int_t^1 p(s)ds.$$

Any solution of $Hz = f$ has the form

$$z_0 = \frac{1}{2} f + c, \quad c = \text{const}\,.$$

Nevertheless, the point $x_0 = \Gamma z_0 + u$ of minimum is unique since $\Gamma c = 0$.

The uniqueness of the point x_0 follows also from the consideration of the problem (10.7) which has the form

$$(\mathcal{L}x)(t) \stackrel{def}{=} \dot{x}(t) - \int_0^1 \dot{x}(s)ds = f_0(t),$$

$$x(0) = \alpha^1, \quad x(1) = \alpha^2.$$

The functions $x_1 = 1$, $x_2 = t$ constitute a fundamental system of solutions of $\mathcal{L}x = 0$ and the determinant

$$\begin{vmatrix} x_1(0) & x_2(0) \\ x_1(1) & x_2(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1.$$

Since for the given f_0 the equation $\mathcal{L}x = f_0$ has a solution, the problem has a unique solution for any α^1, α^2.

Direct computation shows that the functional $\mathcal{I}$ has the minimum at the same point x_0 as in previous case.

3). Finally consider the problem on the minimum of the functional $\mathcal{I}$ in the space $\mathbf{D} \simeq \mathbf{L}_2 \times \mathbf{R}^2$ of the functions $x : [0,1] \to \mathbf{R}^1$ which are absolutely continuous on $[0, c)$ and $[c, 1]$ and $\dot{x} \in \mathbf{L}^2$. The isomorphism between $\mathbf{D}$ and $\mathbf{L}_2 \times \mathbf{R}^2$ may be constructed on the basis of the impulse boundary value problem

$$\dot{x}(t) = z(t), \quad x(0) = \beta^1, \quad x(1) = \beta^2$$

in the space $\mathbf{D}$. The solution of this problem has the form

$$x(t) = (\Lambda z)(t) + (Y\beta)(t) \stackrel{def}{=} \int_0^t z(s)ds -$$

$$- \chi_{[c,1]}(t) \int_0^1 z(s)ds + \beta^1 \chi_{[0,c)}(t) + \beta^2 \chi_{[c,1]}(t).$$

Let us put $\Gamma = \Lambda$. Then $Qz = Q^*z = z$, $Hz = 2z$,

$$(Q_0^* z)(t) = -q(t)z(t) + \int_c^t p(s)z(s)ds,$$

$$f(t) = q(t) - \int_c^t p(s)ds.$$

The solution of $Hz = f$ is $z_0 = \frac{1}{2}f$. In the case $\alpha^1 = \alpha^2 = 0$, the point of the minimum has the form

$$x_0(t) = \frac{1}{2}(\Gamma f)(t) = \frac{1}{2}\Big\{ \int_0^t q(s)ds + \int_0^t sp(s)ds -$$

$$- t\int_c^t p(s)ds - \chi_{[c,1]}(t)\Big[\int_0^1 q(s)ds + \int_0^1 sp(s)ds - \int_c^1 p(s)ds \Big]\Big\}.$$

If p and q are constants,

$$\mathcal{I}(x_0) = -\frac{1}{4}\Big[q^2 + pq(2c-1) + p^2\Big(c^2 - c + \frac{1}{3}\Big)\Big].$$

Thus the minimum depends on the position of the point c of discontinuity. If $q = 0$, then $\mathcal{I}(x_0) = -\frac{1}{4}p^2(c^2 - c + \frac{1}{3})$. $\mathcal{I}(x_0) = -\frac{p^2}{48}$ for $c = \frac{1}{2}$. If $c \to 0$ or $c \to 1$, then $\mathcal{I}(x_0) \to -\frac{p^2}{12}$.

Example 10.2. Next consider the functional

$$\mathcal{I}(x) = \int_0^\omega \Big\{ \frac{x^2(\omega)}{\omega} + \dot{x}^2(s) - p(s)x[h(s)]x[g(s)] + \mu(s)\dot{x}(s) + \nu(s)x(s) \Big\} ds,$$

$$x(\xi) = \varphi(\xi), \quad \text{if} \quad \xi \notin [0, \omega]$$

with "periodic" condition $lx \overset{def}{=} x(0) - x(\omega) = \alpha$. Suppose that $\rho, \mu, \nu \in \mathbf{L}_2$, the functions h, g are measurable, and the initial function $\varphi : (-\infty, +\infty) \setminus [0, \omega] \to \mathbf{R}^1$ is piecewise continuous.

In the case, where $h(t) \equiv g(t) \equiv t$ and the coefficients are sufficiently smooth, the problem on the existence – uniqueness of the point of minimum of the functional $\mathcal{I}$ was investigated in [41]. In this connection, the methods of classical calculus of variations were used. We will consider the problem following the general scheme given above.

Using the notation (6.4) and (6.5), rewrite the functional in the form

$$\mathcal{I}(x) = \int_0^\omega \Big\{ \frac{x^2(\omega)}{\omega} + \dot{x}^2(s) - p(s)(S_h x)(s)(S_g x)(s) \Big\} ds -$$

$$-\int_0^\omega p(s)\{\varphi^g(s)(S_hx)(s)+\varphi^h(s)(S_gx)(s)+$$
$$+\varphi^h(s)\varphi^g(s)+\mu(s)\dot{x}(s)+\nu(s)x(s)\}ds.$$

It is natural to look for the point of minimum of this functional in the space $\mathbf{D}=\mathbf{W}_2^1$ of absolutely continuous functions $x:[0,\omega]\to\mathbf{R}^1$ with $\dot{x}\in\mathbf{L}_2$. The isomorphism $\mathcal{J}=\{\Lambda,Y\}:\mathbf{L}_2\times\mathbf{R}^1\to\mathbf{W}_2^1$ will be constructed on the basis of the general solution $x=\Lambda z+Y\beta$ of the model boundary value problem

$$(\mathcal{L}_0x)(t)\overset{def}{=}\dot{x}(t)+\frac{x(\omega)}{\omega}=z(t),\quad rx\overset{def}{=}x(0)-x(\omega)=\beta.$$

One can see directly that the solution of this problem with $z\in\mathbf{L}_2$ is defined by

$$(Y\beta)(t)=\Big(2-\frac{t}{\omega}\Big)\beta,\quad(\Lambda z)(t)=\int_0^\omega\Lambda(t,s)z(s)ds,$$

where

$$\Lambda(t,s)=\begin{cases}2-\dfrac{t}{\omega}, & \text{if}\quad 0\le s\le t\le\omega,\\ 1-\dfrac{t}{\omega}, & \text{if}\quad 0\le t<s\le\omega,\\ 0 & \quad\text{outside the square }\ [0,\omega]\times[0,\omega].\end{cases}$$

Let us dwell beforehand on the problem about the minimum of the "curtailed" functional

$$\mathcal{I}_0x=\int_0^\omega\Big\{\dot{x}^2(s)-p(s)(S_hx)(s)(S_gx)(s)+\frac{x^2(\omega)}{\omega}\Big\}ds$$

with condition $x(0)-x(\omega)=0$. We have

$$(T_{11}x)(t)=(T_{21}x)(t)=\dot{x}(t),\quad(T_{12}x)(t)=-p(t)(S_hx)(t),$$
$$(T_{22}x)(t)=(S_gx)(t),\quad T_{13}x=T_{23}x=\frac{1}{\sqrt{\omega}}x(\omega).$$

Let us set $\Gamma=\Lambda$. Thus

$$Q_{11}z=Q_{11}^*z=Q_{21}z=Q_{21}^*z=z-\frac{1}{\omega}\int_0^\omega z(s)ds,$$

$$(Q_{12}z)(t)=-p(t)(S_h\Lambda z)(t)=-p(t)\int_0^\omega\Lambda[h(t),s]z(s)ds,$$

$$(Q_{12}^*z)(t) = -\int_0^\omega p(s)\Lambda[h(s),t]z(s)ds,$$

$$(Q_{22}z)(t) = (S_g\Lambda z)(t) = \int_0^\omega \Lambda[g(t),s]z(s)ds,$$

$$(Q_{22}^*z)(t) = \int_0^\omega \Lambda[g(s),t]z(s)ds,$$

$$Q_{13}z = Q_{23}z = Q_{13}^*z = Q_{23}^*z = \frac{1}{\sqrt{\omega}}\int_0^\omega z(s)ds,$$

$$f(t) = f_0(t) \equiv 0.$$

$$(\mathcal{L}x)(t) = 2\dot{x}(t) + 2x(\omega) - \\ - \int_0^\omega p(s)\big\{\Lambda[g(s),t](S_hx)(s) + \Lambda[h(s),t](S_gx)(s)\big\}ds.$$

Let us represent $\mathcal{L}$ in the form

$$\mathcal{L}x = 2(\mathcal{L}_0x - Px),$$

where

$$(Px)(t) = \frac{1}{2}\int_0^\omega p(s)\big\{\Lambda[g(s),t](S_hx)(s) + \Lambda[h(s),t](S_gx)(s)\big\}ds + \\ + x(\omega)\Big(\frac{1}{\omega} - 1\Big).$$

The operator $P : \mathbf{W}_2^1 \to \mathbf{L}_2$ is completely continuous. This follows from the complete continuity in the space $\mathbf{L}_2$ of the integral operator with the kernel $p(s)\Lambda[g(s),t]$ and the boundedness of S_h as the operator acting from $\mathbf{W}_2^1$ into $\mathbf{L}_2$. Let us represent the operator $H : \mathbf{L}_2 \to \mathbf{L}_2$ in the form

$$Hz = \mathcal{L}\Lambda z = 2(z - Kz),$$

where $K = P\Lambda$.

The operator $H : \mathbf{L}_2 \to \mathbf{L}_2$ is Fredholm because of the complete continuity of $K : \mathbf{L}_2 \to \mathbf{L}_2$. Therefore the existence-uniqueness of the point of minimum of the functional $\mathcal{I}$ does not depend on its linear summands and the number α. The summands and α define the right-hand side of the equation $Hz = f$ and does not influence the construction of H. Thus it is sufficient to consider the problem of existence-uniqueness of the point of minimum of the functional $\mathcal{I}$ only for the curtailed functional $\mathcal{I}_0$. Besides, the condition $\rho(K) < 1$ is sufficient, and in the case of isotonicity of K, is also necessary (Corollary 10.1) for the existence of unique point of minimum of the functional $\mathcal{I}$.

The problem (10.7) for $\mathcal{I}_0$ has the form

$$\mathcal{L}x = 0, \quad lx = 0. \tag{10.8}$$

Denote

$$A = \Lambda P.$$

The problem is equivalent to the equation $x = Ax$ in the space $\mathbf{W}_2^1$. Any continuous solution of the equation $x = Ax$ belongs to $\mathbf{W}_2^1$ by virtue of the property of Λ. For each λ, there is the one-to-one mapping $z = \mathcal{L}_0 x$, $x = \Lambda z$ between the set of solutions $x \in \mathbf{C}$ of the equation $\lambda x = Ax$ and the set of solutions $z \in \mathbf{L}_2$ of the equation $\lambda z = Kz$. Thus the spectrums of the compact operators $A : \mathbf{C} \to \mathbf{C}$ and $K : \mathbf{L}_2 \to \mathbf{L}_2$ coincide.

The inequalities $p(t) \geq 0$ and $\omega \leq 1$ guarantee the isotonicy of P, K and A since $\Lambda(t,s) > 0$ on the square $[0,\omega] \times [0,\omega]$. Under the assumptions of these inequalities, the following Vallée-Poussin-like [43, 44] theorem is valid.

Theorem 10.3. *Let $p(t) \geq 0$, $t \in [0,\omega]$, $\omega \leq 1$. Then the following assertions are equivalent.*

a) *There exists the unique point $x_0 \in \mathbf{W}_2^1$ of minimum of functional $\mathcal{I}$.*

b) *The spectral radius of $K : \mathbf{L}_2 \to \mathbf{L}_2$ is less than* 1.

c) *The spectral radius of $A : \mathbf{C} \to \mathbf{C}$ is less than* 1.

d) *There exists $v \in \mathbf{W}_2^1$ such that*

$$v(t) \geq 0, \quad \eta(t) \overset{def}{=} (\mathcal{L}v)(t) \geq 0, \quad t \in [0,\omega],$$

besides

$$\theta \overset{def}{=} v(0) - v(\omega) \geq 0, \quad \theta + \int_0^\omega \eta(s)ds > 0.$$

e) *The problem* (10.8) *is uniquely solvable and the Green operator G of the problem is isotonic.*

f) *There exists a solution ξ of the homogeneous equation $\mathcal{L}x = 0$ such that $\xi(0) - \xi(\omega) > 0$, $\xi(t) > 0$, $t \in [0,\omega]$.*

Proof. The implication a)⇒b) follows from Corollary 10.1.

The implication b)⇔ c) was established above.

The implication d)⇒c). The function v satisfies to $\mathcal{L}x = \eta$, $lx = \theta$. Consequently, $v - Av = r$, where

$$r(t) = \frac{1}{2}(\Lambda\eta)(t) + \left(2 - \frac{t}{\omega}\right)\theta > 0, \quad t \in [0,\omega].$$

From this $v(t) > (Av)(t) > 0$, $t \in [0,\omega]$. Therefore $\rho(A) < 1$ by virtue of Lemma 10.3.

The implication c)⇒d) (c)⇒f)) can be proved by taking the solution x of the half homogeneous problem $\mathcal{L}x = 0$, $lx = 1$ in the capacity of v (the solution ξ). Really, the last problem is equivalent to the equation $x = Ax+y$, where $y = 2 - \frac{t}{\omega}$. The solution of the equation is strictly positive:

$$x(t) = y(t) + (Ay)(t) + (A^2y)(t) + \cdots > y(t) > 0, \quad t \in [0,\omega].$$

The implication c)⇒e) follows from the fact that the solution of the problem $\mathcal{L}x = f$, $lx = 0$

$$x(t) = (Gf)(t) = r(t) + (Ar)(t) + (A^2r)(t) + \cdots \quad \Big(r = \frac{1}{2}\Lambda f\Big)$$

is strictly positive on $[0,\omega]$, if $f(t) \geq 0$, $f(t) \not\equiv 0$.

The implication e)⇒d) (f)⇒d)) can be obtained by taking

$$v(t) = \int\limits_0^\omega G(t,s)ds \quad (v(t) = u(t)). \ \blacksquare$$

Remark 10.3. The significance of Theorem 10.3 can be seen, in particular, in the possibility of reducing the problem of minimum of the functional $\mathcal{I}$ to establishing of some properties of Euler's equation: the existence of a positive solution of the homogeneous equation (the assertion f)) or the validity of the assertion d) for a functional differential inequality like the theorem of Vallée-Poussin [43] for the ordinary differential equation of the second order. The rational choice of the function v in the assertion d) leads to tests of the existence of minimum in the terms of parameters of the functional $\mathcal{I}$.

Denoting

$$\sigma_h(t) = \begin{cases} 1, & \text{if} \quad h(t) \in [0,\omega], \\ 0, & \text{if} \quad h(t) \notin [0,\omega], \end{cases}$$

we can formulate the following test derived from Theorem 10.3.

Corollary 10.2. *Let $p(t) \geq 0$, $t \in [0,\omega]$, $\omega \leq 1$. Then the inequality*

$$\int\limits_0^\omega p(s)\sigma_h(s)\sigma_g(s)\Big[4 - \frac{g(s)+h(s)}{\omega}\Big] \leq 2 \tag{10.9}$$

guarantees the existence of a unique point of minimum in the space $\mathbf{W}_2^1$ of the functional $\mathcal{I}$.

Proof. Let us set $v(t) \equiv 1$ in the assertion d) of Theorem 10.3. Then

$$\mathcal{L}(1) = 2\Big[1 - \frac{1}{2}\int\limits_0^\omega p(s)\big\{\Lambda[g(s),t]\sigma_h(s) + \Lambda[h(s),t]\sigma_g(s)\big\}ds\Big] >$$

$$> 2\Big[1 - \frac{1}{2}\int\limits_0^\omega p(s)\sigma_h(s)\sigma_g(s)\Big\{4 - \frac{g(s)+h(s)}{\omega}\Big\}ds\Big] \geq 0$$

if (10.9) holds. ■

Remark 10.4. A special case of the functional $\mathcal{I}_0$, where $p(t) \equiv 1$ and $h(t) \equiv g(t) \equiv t$ was thoroughly investigated in [41]. It was shown there, in particular, that the inequality

$$\omega < \arcsin \frac{4}{5} \tag{10.10}$$

guarantees the existence of a unique point of minimum. For this case, we derive from (10.9) only $\omega \le \frac{2}{3}$. The inequality (10.10) follows from Theorem 10.3 if we choose in the assertion d)

$$v(t) = \cos t + \sin t \frac{1 - \cos\omega}{\sin\omega}.$$

Then for $t \in [0, \omega]$, we have

$$v(t) > 0, \quad (\mathcal{L}v)(t) = 2\Big(1 - \sin\omega - \frac{(1-\cos\omega)^2}{\sin\omega}\Big) > 0$$

if (10.10) holds.

In the case $\omega = \arcsin\frac{4}{5}$, the homogeneous problem (10.8) has the nontrivial solution

$$v(t) = \cos t + \sin t \frac{1-\cos\omega}{\sin\omega} = \cos t + \frac{1}{2}\sin t.$$

Thus the estimate (10.10) which guarantees the existence of a unique point of minimum is best possible.

Example 10.3. Consider the functional

$$\mathcal{I}(x) = \int_0^1 \big\{[s(1-s)\ddot{x}(s)]^2 - p(s)(S_h x)(s)(S_g x)(s)\big\}ds$$

with boundary conditions $x(0) = \alpha^1$, $x(1) = \alpha^2$. Assume that $p \in \mathbf{L}_2$ and the functions $h, g : [0,1] \to \mathbf{R}^1$ are measurable. Using the space $\mathbf{W}_2^2$, we meet the fact that the Euler's equation $\mathcal{L}x = f$ turns out to be singular. Therefore, as in the case of the equation (9.2), we introduce the space $\mathbf{D}$ whose elements x have the properties:

a) The function x is continuous on $[0,1]$.
b) The derivative $\dot{x}$ is absolutely continuous in the interval $(0,1)$.
c) The product $t(1-t)\ddot{x}(t)$ is square integrable on $[0,1]$.

Such a space is defined by $\mathbf{D} = \Lambda\mathbf{L}_2 \oplus Y\mathbf{R}^2$, where

$$(\Lambda z)(t) = \int_0^1 \Lambda(t,s)z(s)ds,$$

$$(Y\beta)(t) = (1-t)\beta^1 + t\beta^2, \quad \beta = \mathrm{col}\{\beta^1, \beta^2\},$$

$$\Lambda(t,s)=\begin{cases}\dfrac{t-1}{1-s}, & \text{if}\quad 0\le s\le t\le 1,\\ -\dfrac{t}{s}, & \text{if}\quad 0\le t<s\le 1,\\ 0 & \text{outside the square } [0,1]\times[0,1].\end{cases}$$

We observe that $\Lambda(t,s)$ is the Green function of the singular problem

$$t(1-t)\ddot{x}(t)=z(t),\quad x(0)=\beta^1,\quad x(1)=\beta^2.$$

The space $\mathbf{D}$ is isomorphic to the product $\mathbf{L}_2\times\mathbf{R}^2$, $\mathcal{J}=\{\Lambda,Y\}:\mathbf{L}_2\times\mathbf{R}^2\to\mathbf{D}$ is the isomorphism and the inverse $\mathcal{J}^{-1}=[\delta,r]:\mathbf{D}\to\mathbf{L}_2\times\mathbf{R}^2$ is defined by

$$(\delta x)(t)=t(1-t)\ddot{x}(t),\quad rx=\big\{x(0),x(1)\big\}.$$

Following the general scheme, we have

$$\Gamma=\Lambda,\quad u(t)=(1-t)\alpha^1+t\alpha^2,\quad T_{11}=T_{21}=\delta,$$
$$Q_{11}=Q_{21}=Q_{11}^*=Q_{21}^*=I,\quad (T_{12}x)(t)=-p(t)(S_hx)(t),$$
$$(T_{22}x)(t)=(S_gx)(t),\quad (Q_{12}z)(t)=-p(t)\int_0^1\Lambda[h(t),s]z(s)ds,$$
$$(Q_{22}z)(t)=\int_0^1\Lambda[g(t),s]z(s)ds,$$
$$(Q_{12}^*z)(t)=-\int_0^1 p(s)\Lambda[h(s),t]z(s)ds,$$
$$(Q_{22}^*z)(t)=\int_0^1\Lambda[g(s),t]z(s)ds,\quad f_0(t)\equiv 0,$$
$$Hz=2\big(z+Q_{12}^*Q_{22}z+Q_{22}^*Q_{12}z\big)=2(z-Kz),$$

where

$$(Kz)(t)=\int_0^1 K(t,s)z(s)ds,$$
$$K(t,s)=\frac{1}{2}\int_0^1 p(\tau)\big\{\Lambda[h(\tau),t]\Lambda[g(\tau),s]+\Lambda[g(\tau),t]\Lambda[h(\tau),s]\big\}d\tau,$$
$$\mathcal{L}x\overset{def}{=}\sum_{i=1}^{2}(Q_{1i}^*T_{2i}+Q_{2i}^*T_{1i})x=2(\delta x-Px)$$

and

$$(Px)(t) = \frac{1}{2}\int_0^1 p(s)\big\{\Lambda[h(s),t](S_g x)(s) + \Lambda[g(s),t](S_h x)(s)\big\}ds.$$

The problem (10.7) has the form

$$\mathcal{L}x = 0, \quad x(0) = \alpha^1, \quad x(1) = \alpha^2. \tag{10.11}$$

It is equivalent to the equation

$$\Lambda\mathcal{L}x \stackrel{def}{=} 2(x - Ax) = u$$

in the space $\mathbf{C}$. Here $A = \Lambda P$. Thus the problem (10.11) is uniquely solvable if and only if $I - A$ has the inverse.

The equalities $z = \delta x$, $x = \Lambda z$ establish a one-to-one mapping between the sets of solutions $x \in \mathbf{C}$ of equation $\lambda x = Ax$ and of the solutions $z \in \mathbf{L}_2$ of the equation $\lambda z = Kz$. Therefore the spectrums of the compact operators $A : \mathbf{C} \to \mathbf{C}$ and $K : \mathbf{L}_2 \to \mathbf{L}_2$ coincide.

The inequality $\rho(K) < 1$ guarantees by virtue of Corollary 10.1 the existence of a unique point of minimum. We have $\rho(K) = \rho(A) \le \|A\|_{\mathbf{C}\to\mathbf{C}}$. Since $|\Lambda(t,s)| \le 1$, $\rho(A) < 1$ if

$$\int_0^1 |p(s)|\big\{\sigma_h(s) + \sigma_g(s)\big\}ds \le 2. \tag{10.12}$$

Theorem 10.4. *Let $p(t) \ge 0$, $t \in [0,1]$. Then the following assertions are equivalent.*

a) *The functional $\mathcal{I}$ has a unique point $x_0 \in \mathbf{D}$ of minimum.*

b) *The spectral radius of $K : \mathbf{L}_2 \to \mathbf{L}_2$ is less than* 1.

c) *The spectral radius of $A : \mathbf{C} \to \mathbf{C}$ is less than* 1.

d) *There exists $v \in \mathbf{C}$ such that for any $t \in [0,1]$, the inequalities*

$$v(t) \ge 0, \quad \eta(t) \stackrel{def}{=} (\mathcal{L}v)(t) \le 0,$$

hold; besides

$$v(0) + v(1) - \int_0^1 \eta(s)ds > 0.$$

e) *The problem* (10.11) *is uniquely solvable and the Green operator G of the problem is antitonic.*

f) *There exists a positive on $[0,1]$ solution $y \in \mathbf{D}$ of the homogeneous equation $\mathcal{L}x = 0$.*

Proof. If $p(t) \geq 0$, the operators K and A are isotonic.

The implication a)⇔b) follows directly from Corollary 10.1.

The implication b)⇔c) was established above.

The implication d)⇒c). The function v is a solution of the problem

$$\mathcal{L}x = \eta, \quad x(0) = v(0), \quad x(1) = v(1),$$

and consequently satisfies the equation $v - Av = r$, where

$$r(t) = \frac{1}{2}(\Lambda\eta)(t) + (1-t)v(0) + tv(1) \geq 0.$$

The operator A satisfies the conditions of Lemma 10.4. By virtue of the Lemma, we obtain $\rho(A) < 1$.

The implication c)⇒d) (c)⇒f)) may be obtained by taking in the capacity of the function v (the solution y) the solution x of the half homogeneous problem

$$\mathcal{L}x = 0, \quad x(0) = 1, \quad x(1) = 1,$$

which is equivalent to the equation

$$x(t) - (Ax)(t) = (1-t) + t \equiv 1.$$

The solution of the last equation

$$x = 1 + A(t) + A^2(t) + \cdots$$

is strictly positive on [0,1].

The implication c)⇒e) follows from the fact that the solution of the problem

$$\mathcal{L}x = f, \quad x(0) = 0, \quad x(1) = 0$$

has the representation

$$x(t) = (Gf)(t) = r(t) + (Ar)(t) + (A^2r)(t) + \cdots,$$

where $r = \frac{1}{2}\Lambda f$. This solution is strictly positive in $(0,1)$ if $f(t) \geq 0$, $f(t) \not\equiv 0$.

The implication e)⇒d) (f)⇒d)) can be obtained by taking

$$v(t) = -\int_0^1 G(t,s)ds, \quad (v(t) \equiv y(t)). \ \blacksquare$$

Corollary 10.3. *Let $h(t) \equiv g(t) \equiv t$, $p(t) \geq 0$, $t \in [0,1]$. Then the functional $\mathcal{I}$ has a unique point of minimum in the space $\mathbf{D}$ if*

$$\operatorname{vraisup}_{t\in[0,1]} p(t) \leq 4. \tag{10.13}$$

Proof. Let us take $v(t) = t(1-t)$ in the assertion d) of Theorem 10.3. We have

$$(\mathcal{L}v)(t) = 2\left\{2t(t-1) - \int_0^1 p(s)\Lambda(s,t)s(1-s)ds\right\},$$

$$\Lambda(s,t)s(1-s) = \begin{cases} s(t-1), & \text{if } 0 \le s \le t \le 1, \\ t(s-1), & \text{if } 0 \le t < s \le 1. \end{cases}$$

Since

$$-\int_0^1 \Lambda(s,t)s(1-s)ds = \frac{1}{2}t(1-t),$$

we have

$$-\int_0^1 p(s)\Lambda(s,t)s(1-s)ds \le 2t(1-t).$$

Consequently, $(\mathcal{L}v)(t) \le 0$. ■

We observe that using the inequality (10.12) to the case on the hand, we get the estimate $\operatorname{vraisup}_{t\in[0,1]} p(t) \le 1$ to guarantee the existence of a unique point of minimum. A more exact estimate was obtained at the expense of the choice of the function v with regard to the specific character of the problem.

Going back now to the last two examples, we will suppose $h = g$ and refuse from the condition $p(t) \ge 0$. Let further $p = p^+ - p^-$, where $p^+(t) \ge 0$, $p^-(t) \ge 0$ and denote by $\mathcal{I}^+$ the functional obtained from $\mathcal{I}$ by replacing p by p^+. It is obvious that $\mathcal{I}x \ge \mathcal{I}^+x$ for each $x \in \mathbf{D}$. Therefore the boundedness from below of $\mathcal{I}^+$ implies the boundedness of $\mathcal{I}$. Consequently, the inequalities (10.9),(10.12) and (10.13) under the assumption that $h = g$ and p is replaced by p^+ guarantee the existence of the minimum of $\mathcal{I}$. We can not guarantee the uniqueness of the point of minimum in this case.

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