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ON QUANTUM FIELDS AND SYSTEMS WITH ORDERED GROUND STATE

Abstract. In the first part of the work, we consider problems of the field theory approach to the quantum Hall effect (QHE) and the anyon superconductivity. In the remaining part, we study the convexity property of the effective potential and some features of the high temperature behaviour of field theory models with two scalar fields. Describing the QHE, we adapt the Chern–Simons gauge theory in the holomorphic gauge which seems to incorporate Laughlin's picture of the incompressible quantum fluid and Jain's composite fermion approach. Further, the 2+1 dimensional relativistic field theory is used to describe the Meissner effect in the planar superconducting matter. Using Bogolubov's concept of quasiaverages and analyzing the boundary conditions in the path integral representation of the Green functions generating funcitonal, a self-consistent calculation of the effective potential is proposed. Studying the effective potentials for the two Higgs models, it is demonstrated that the symmetry can be spontaneously broken in some high temperature interval, being exact at zero and very high temperatures.

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Key words and Phrases. Chern-Simons theory, quantum Hall effect, similarity transformation, complex gauge potentials and holomorphic quantization, anyon superconductivity, structure functions, effective potential, convexity, ground state, symmetry breaking.

რეზიუმე. ნაშრომის პირველ ნაწილში განხილულია ჰოლის კვანტური ეფექტისა (ჰკე) და ანიონური ზეგამტარობის საკითხებისადმი ველის თეორიისეული მიდგომა. დანარჩენ ნაწილში შესწავლილია ეფექტური პოტენციალის ამოზნექილობის თვისება და ორი სკალარული ველის შემცველი მოდელების მაღალტემპერატურული ყოფაქცევის საკითხები. ჰკე-ის აღწერისას გამოყენებულია ჩერნ-საიმონსის ყალიბური თეორია ჰოლომორფულ ყალიბში, რაც, როგორც ჩანს, მოიცავს უკუმშვადი სითხის ლაფლინისეულ სურათსა და ჯეინის შედგენილი ფერმიონების მიდგომას. შემდგომ $2+1$ განზომილებიანი რელატივისტური ველის თეორია გამოყენებულია ბრტყელ ზეგამტარ ნივთიერებაში მეისნერის ეფექტის აღსაწერად. ბოგოლიუბოვის კვაზისაშუალოების კონცეფციის გამოყენებითა და გრინის ფუნქციების მაწარმოებელი ფუნქციონალის კონტინუალური ინტეგრალით სასაზღვრო პირობების ანალიზის დახმარებით დადგენილია ეფექტური პოტენციალის გამოთვლის თვითშეთანხმებული ვარიანტი. შესწავლილია ეფექტური პოტენციალი ჰიგსის ორი ველის შემცველ მოდელებში და ნაჩვენებია, რომ შესაძლებელია სიმეტრია დარღვეულ იქნას მაღალი ტემპერატურების გარკვეულ ინტერვალში ისე, რომ იგი დარჩეს ზუსტი ნულოვანი და ზემაღალი ტემპერატურებისათვის.

Preface

The aim of the present monograph is to present a systematic review of some problems concerning the quantum field description of systems with non-trivial (ordered) ground states.

It is widely appreciated that in studying different dynamical systems, it is important to know not only individual properties of the constituents and their interactions, but also collective phenomena which provide an overall ordering. The mutual balance between order and disorder, transitions from one kind of ordering to another define finally the most profound properties of matter. This is a deep and quite general principle, applicable not only in physics or chemistry, but also in biology and even in social science.

The notion of order is intimately related to the symmetry properties: a higher order corresponds to a lower symmetry and *vice versa*. This is clearly seen for interacting high energy fundamental particles and condensed matter systems. In both cases, the knowledge of the ground state (vacuum) and its elementary excitations solves most important physical problems. This unity of basic concepts formally can be expressed using the language of the quantum field theory. As impressive examples of this assertion, one can consider the modern theory of spontaneously broken gauge symmetries and the theory of superconducting (or superfluid) matter.

The discovery of the quantum Hall effect (QHE) provoked a great theoretical interest to the studies of strongly correlated ground states with a new type of ordering. A field theory model pretending to describe this phenomenon must be able to reproduce the picture of an incompressible quantum fluid, the ground state wave function and the spectrum of elementary excitations, and give room to hierarchical constructions.

As a most reliable candidate for such a theory, one can consider Chern–Simons (topological) gauge theory in three space-time dimensions, which is in a close connection with the conformal field theory, and at the same time can play a significant role in the studies of planar condensed matter systems.

The Chern–Simons theory was applied to another condensed matter phenomenon: the anyon superconductivity. Here the role of statistical gauge field is to organize a matter in such a way that there is a gap in the charged particle spectrum and a massless pole in the electromagnetic response function, leading to the Meissner effect.

Chapters 1–4 of the present review account for applications of the Chern–Simons field theory to studies of planar systems, the quantum Hall effect and the anyon superconductivity.

In Chapter 1, after a general introduction, we calculate the correlation functions and the effective action for a system of planar fermions interacting

with the Maxwell and the Chern–Simons gauge fields. In Chapter 2, we consider the incompressible Hall fluid from the point of view of algebraic classification. The accent is made on a similarity between integral and fractional QHEs. This similarity can be expressed via non-unitary singular transformation equivalent to the introduction of a complex statistical gauge potential. This approach is helpful in order to reconstruct ground state wave functions and incorporates the picture of composite electrons carrying magnetic fluxes.

In Chapter 3, we develop a scheme of canonical quantization of the Chern–Simons theory with non-compact (Abelian and non-Abelian) gauge groups. The proposed framework of holomorphic gauge quantization clarifies the results of previous Chapter 2, and can be used both to determine ground state wave functions and to develop a dynamical description.

In Chapter 4, we study a planar condensed matter system exhibiting the Meissner effect: an anyon superconductor. Considering the problem, first we apply the formalism of thermo field dynamics and then analyze high temperature properties of a relativistic version of the model.

Field-theoretic description of systems with a non-invariant vacuum can be devided into two parts. The first part concerns axiomatic problems, such as a Hilbert space realization of commutation relations. The second part deals with the practical quantization, calculations of Green functions, the determination of the order parameter *etc.*

In Chapter 5, we try to give satisfactory answer to the apparent puzzle which was noticed in computations of the functional generating Green functions: general consideration has predicted a convex form of the effective potential, while existing schemes of calculation were giving a non-convex one. Here we discuss a self-consistent procedure based on Bogolubov's concept of quasiaverages and the analysis of the boundary conditions in the path integral representation of the functional generating Green functions.

The mechanism of the spontaneous symmetry breaking depends in the crucial way on the scalar sector of the theory. In simplest models, the symmetry broken at zero temperature is restored after heating up the system in the same way as ferromagnetics lost spontaneous magnetization above the Curie temperature. In the final Chapter 6, we consider a different type of symmetry temperature bahaviour, when the symmetry, being exact at low and high temperatures, is spontaneously broken for some finite temperature interval.

Our scientific interests were formed under the fruitful influence of A. Tavkhelidze. I should like to take this opportunity to thank him for valuable discussion and a permanent support.

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The results reported in Chapter 4 and partially in Chapter 1 have been obtained in collaboration with G.Tsitsishvili. I express him my gratitude.

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CHAPTER 1
QUANTUM PARTICLES IN 2+1 DIMENSIONS

1. Introduction

The Pauli exclusion principle and the underlying spin-statistics theorem [117] reflect one of the most profound properties of matter:

- Particles (or condensed matter quasiparticles) with integer spin satisfy the Bose–Einstein statistics, and half-integer spin excitations satisfy Fermi–Dirac statistics.
- The wave functions of N-identical particles are described by one-dimensional representations of the permutation group S_N: symmetric wave functions correspond to bosons and antisymmetric ones to fermions.

Higher dimensional representations are known as corresponding to the parastatistics and have been exploited in a quark model by Greenberg [66].

As it is known from the seminal papers by Bogolubov, Struminsky and Tavkhelidze [17] and Han and Nambu [75], the use of these representations is equivalent to the introduction of a new quantum number – color, which is the starting point of a modern theory of hadron interactions – quantum chromodynamics.

The situation with regard to spin and statistics is different when particles move in the plane (2+1 dimensional space-time). In that case, the rotation group associated with R^2 is Abelian SO(2), and the spin is not quantized. An other observation is that now the statistics is determined by irreducible representations of the braid group B_N (and not of the permutation group S_N), which is the fundamental group of configuration space of a system of identical particles moving in the two-dimensional physical space [104] (for details see, e.g., [9], [55], [105]).

The braid group B_N is an infinite nonabelian group having one-dimensional and higher dimensional representations. One-dimensional unitary representations are given by

$$\chi(\vartheta) = e^{i\vartheta} \qquad (0 \leq \vartheta \leq 2\pi).$$

The quantity ϑ is a real parameter which is identified with a statistics provided the Hamiltonian does not contain long-range interaction between particles: under the interchange of two particles, the wave function changes according to

$$\psi(\mathbf{r}_1, \mathbf{r}_2) \to \psi'(\mathbf{r}_1, \mathbf{r}_2) = e^{i\vartheta}\psi(\mathbf{r}_1, \mathbf{r}_2).$$

Hence quantum excitations living in two spatial dimensions can obey the fractional (ϑ) statistics. Cases $\vartheta = 0, \pi$ correspond to the Bose and Fermi particles. Elementary excitations with interpolating statistics are called anyons [137], [6].

The spin of the ϑ statistics anyon is $S = \frac{\vartheta}{2\pi}$. One of the essential features of anyons is the violation of the discrete symmetries of parity and time reversal (unless $\vartheta = 0$ or $\vartheta = \pi$). Remind that in d=2 dimensions. the parity reverses the sign of only one of the coordinates.

A simple illustration of anyonic excitations are point particles having both electric charge e and magnetic flux ϕ under the condition that there exists only charge - flux interaction, but not charge - charge or flux - flux interactions. If one slowly moves one particle around another by a full positively oriented loop, the wave function acquires a Dirac [28] - Bohm-Aharonov phase factor [2], from which one gets the statistical parameter $\vartheta = \left(\frac{\phi}{\phi_0}\right)\pi$, where $\phi_0 = hc/e$ is the elementary magnetic flux quantum.

This construction can be realized if one considers a system of particles (fermions or bosons) moving in the effective non-local vector potential [6]

$$a_I^i(\mathbf{r}_1, \ldots, \mathbf{r}_N) = -\frac{\hbar c}{e}\frac{\vartheta}{\pi}\frac{\partial}{\partial x_I^i}\sum_{I\neq J}\varphi_{IJ}, \quad \varphi_{IJ}(\mathbf{r}_I - \mathbf{r}_J) = \arctan\frac{y_I - y_J}{x_I - x_J},$$

where φ_{IJ} is the winding angle of the particle J with respect to the particle I (note that $\mathbf{a} = 0$ when $N = 1$). The magnetic field associated to $\mathbf{a}_I(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ is equal to

$$B_I = \frac{\phi_0\vartheta}{\pi}\sum_{J\neq I}\delta^{(2)}(\mathbf{r} - \mathbf{r}').$$

It means that each particle sees the $(N-1)$ others as vortices carrying a magnetic flux $\phi = \frac{\vartheta}{\pi}\phi_0$.

Since statistics plays the key role in the organization of matter, it is natural that in two-dimensional systems one must expect new phenomena caused by anyonic excitations.

Quasiparticles with fractional statistics can exist in condensed matter systems, such as the materials with a layered structure or interfaces where the electrons are largely confined to move in the planes.

The first possibility is realized in the copper-oxid ($Cu-O$) crystals which are known to be the high temperature superconductors. The second one is related to planar electron systems moving in a strong perpendicular magnetic field and exhibiting the quantum Hall effect.

Below we will make some short comments on the properties of multi-anyon systems and their possible relevance to two-dimensional condensed matter systems (see [139]).

- Anyons occupy the so called Landau levels with finite energy spacing between levels (gap); each level is highly degenerate. The exact filling of this bands is especially favorable energetically and produces a particularly stable or rigid state.
- Anyons can be considered as fermions interacting with statistical gauge field. Taking into account the field fluctuations, it can be

shown that there is a massless pole in some two-point functions corresponding to the charged Golstone mode. This circumstance together with the gap in the fermion spectrum can generate a new mechanism of superconductivity [53].

- The third peculiarity is that the N-particle wave function is a multivalued quantity:

$$\tilde{\Psi}(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \prod_{I<J} (z_I - z_J)^{\vartheta/\pi} \cdot \Phi(\mathbf{r}_1, \ldots, \mathbf{r}_N),$$

where $\Phi(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ is a single-valued function. ($z_I = x_I + iy_I$, $\bar{z}_I = x_I - iy_I$ are the complex coordinates of the I^{th} particle.) This type of wave functions is needed to describe the quantum Hall states – an incompressible two dimensional quantum fluid with the fractionally charged excitations obeying the fractional statistics (see [99]).

The notions of anyons and of fractional statistics can be formulated in the framework of the local quantum field theory in terms of a single-valued matter field interacting with the Chern–Simons (CS) gauge field. Remind that the Chern–Simons action is a topological (the metric independent) invariant in 2+1 dimensional space-time:

$$\mathcal{S}_{CS} = \int d^3x \frac{1}{2} \kappa \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda.$$

The constant κ determines the ϑ-statistics of particles interacting with the Chern–Simons field.

Our objective in this chapter is to develop the field-theoretic formalism accommodate for studies of electrodynamical and thermal properties of planar fermions coupled to a statistical gauge field. In the subsequent parts, we will use this formalism in order to describe some features of the Hall fluid and two-dimensional superconductors.

In Section 2, we shortly review the transition from a quantum-mechanical to a field-theoretic description of anyons and introduce the statistical gauge field. Further, in Section 3, we consider a system of interacting Chern–Simons, Maxwell and matter fields. In Section 4, we define the generating functional. In Sections 5 and 6, we calculate the current-current correlation functions for planar fermions in a magnetic field. Then, in Section 7, we write down the relevant physical quantities, like the polarization operator and the response function.

2. Chern–Simons Description of the Free Anyons

The Hamiltonian describing the collection of N anyons in 2+1 dimensions is given by

$$H = \sum_{I=1}^{N} \frac{1}{2m} \left(\mathbf{p}_I - \frac{e}{c\hbar} \mathbf{a}_I \right)^2.$$

Here

$$a_I^i(\mathbf{r}_1,\ldots,\mathbf{r}_N) = -\frac{c\hbar}{e}\frac{\vartheta}{\pi}\epsilon^{im}\sum_{I\neq J}\frac{r_I^m - r_J^m}{|\mathbf{r}_I - \mathbf{r}_J|^2} \qquad i = 1,2$$

is a nonlocal gauge potential generating the Bohm–Aharonov phase interaction with a statistical parameter ϑ. Anyons carry the charge e and the flux $(\vartheta/\pi)\phi_0$.

It is easy to write down the second-quantized form of this many-particle Hamiltonian. For this purpose, introduce a Shrödinger field $\psi(\mathbf{r})$ and a fictitious gauge field potential $\mathbf{a}(\mathbf{r})$ satisfying the conditions (in the units $c = \hbar = 1$)

$$\begin{aligned}\partial_i a^i(\mathbf{r}) &= 0,\\ \frac{e}{2\vartheta}\epsilon_{ij}\partial_i a^j(\mathbf{r}) &= \psi^\dagger(\mathbf{r})\psi(\mathbf{r}).\end{aligned} \tag{1.1}$$

The solution to (1.1) is the field

$$a^i(\mathbf{r}) = -\frac{\vartheta}{e\pi}\int \mathrm{d}\mathbf{r}'\frac{\epsilon^{im}(r^m - r'^m)}{|\mathbf{r}-\mathbf{r}'|^2}\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}').$$

Now the Hamiltonian reads as

$$\begin{aligned}\mathcal{H} &= -\frac{1}{2m}\int \mathrm{d}\mathbf{r}\psi^\dagger(\mathbf{r})\mathbf{D}^2\psi(\mathbf{r})\\ D_k &= \partial_k + iea_k(\mathbf{r}).\end{aligned}$$

This Hamiltonian naturally arises if one considers the Lagrangian density

$$\mathcal{L} = i\psi^\dagger(\mathbf{r},t)\left(\partial_0 + iea_0\right)\psi(\mathbf{r},t) - \frac{1}{2m}|D_k\psi|^2 + \frac{1}{2}\kappa\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda. \tag{1.2}$$

Our metric for three-dimensional space-time is $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1)$ and Levy-Civita tensor $\varepsilon^{012} = +1$. The Chern–Simons constant and the statistical parameter are related by the equation $\kappa = -\frac{e^2}{2\vartheta}$.

Although $\mathcal{L}_{CS}$ is not invariant under the gauge transformation

$$a_\mu \to a_\mu + \partial_\mu\lambda, \qquad \psi \to e^{-ie\lambda}\psi, \tag{1.3}$$

it changes by the divergence $\mathcal{L}_{CS} \to \mathcal{L}_{CS} - \frac{1}{2}\kappa\varepsilon^{\mu\nu\rho}\partial_\mu\left(a_\rho\partial_\nu\lambda\right)$ and the equations of motion are gauge invariant.

The equations of motion for the Chern–Simons field

$$\begin{aligned}\kappa\varepsilon^{0\nu\lambda}\partial_\nu a_\lambda &= eJ^0 = e\psi^\dagger\psi,\\ \kappa\varepsilon^{k\nu\lambda}\partial_\nu a_\lambda &= eJ^k = e\frac{i}{2m}\left[\psi^\dagger D^k\psi - (D^k\psi)^\dagger\psi\right]\end{aligned}$$

can be easily solved in the Coulomb gauge

$$a^\mu = -\frac{e}{\kappa}\varepsilon^{\mu n\lambda}\partial_n^{-1}J_\lambda.$$

Substituting this solutions into (1.2), we arrive to the Lagrangian

$$\mathcal{L} = -\frac{1}{2m}|D_k\psi|^2 + i\psi^\dagger\dot{\psi} - \frac{e^2}{2\kappa}J_m\epsilon^{mn}\partial_n^{-1}J_0 - \frac{e^2}{2\kappa}J_0\epsilon^{mn}\partial_n^{-1}J_m, \quad (1.4)$$

where the gauge fields are totally excluded in favor of the matter field.

For further purposes, it is convenient to introduce the average density ρ and define the currents

$$\hat{j}_0 = \psi^\dagger\psi_0 - \rho,$$
$$j_k = \frac{i}{2m}[\psi^\dagger\nabla_k\psi - (\nabla\psi)^\dagger\psi],$$

where the "shortened" covariant derivative operators refer to the background Chern–Simons field

$$\nabla_k = \partial_k - i\frac{e^2}{\kappa}\epsilon^{kn}\partial_n^{-1}\rho.$$

Substituting all this in (1.4), after some rearrangement we get

$$\mathcal{L} = i\psi^\dagger\dot{\psi} - \frac{1}{2m}|\nabla_k\psi|^2 - \frac{e^2}{2\kappa}\left[j_k\epsilon^{kn}\partial_n^{-1}\hat{j}_0 + \hat{j}_0\epsilon^{nk}\partial_n^{-1}j_k\right] -$$
$$- \frac{e^4}{2m\kappa^2}\left[\rho\partial_n^{-1}\hat{j}_0\partial_n^{-1}\hat{j}_0 + \hat{j}_0\partial_n^{-1}\hat{j}_0\partial_n^{-1}\hat{j}_0\right]. \quad (1.5)$$

This, generally speaking non-local, Lagrangian contains in addition to usual two-particle interactions also a three-particle one, given by the last term in (1.5).

Introduction of the average density ρ is equivalent to the separation of the Chern–Simons gauge field into background and fluctuating parts:

$$a_k = \bar{a}_k + a'_k = -\frac{e}{\kappa}\epsilon_{km}\partial_m^{-1}\rho - \frac{e}{\kappa}\epsilon_{km}\partial_m^{-1}\hat{j}_0,$$

and it is a starting point for the mean field approximation.

Consider now the corresponding generating functional

$$Z \propto \int D\psi D\psi^\dagger D\mathbf{a}'\delta[\partial_i a_i]\delta[e(\psi^\dagger\psi - \rho) - \kappa\varepsilon_{mn}\partial_m a'_n]e^{i\int dx[i\psi^\dagger\dot{\psi} - \frac{1}{2m}|D_k\psi|^2]}$$

which can be written in a more conventional form

$$Z \propto \int D\psi D\psi^\dagger D\mathbf{a}' Da_0\delta(\partial_i a'_i)e^{i\int dx\mathcal{L}},$$

where the Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = i\psi^\dagger\dot{\psi} - \frac{1}{2m}|\nabla_k\psi|^2 + ej_k a'_k - \frac{e^2}{2m}(\psi^\dagger\psi - \rho)a'_k a'_k - e\psi^\dagger\psi a_0$$
$$- \frac{e^2\rho}{2m}a'_k a'_k + \frac{\kappa}{2}\varepsilon^{\mu\nu\lambda}a'_\mu\partial_\nu a'_\lambda. \quad (1.6)$$

3. Interacting System

Consider a matter interacting with an external electromagnetic field A_μ^{ext} and dynamical Maxwell and Chern-Simons gauge fields. The corresponding Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_{\text{Gauge}} + \mathcal{L}_{\text{Matter}}.$$

The gauge Lagrangian contains kinetic terms for the Maxwell and Chern–Simons fields and takes into account the interaction with a background charge density n_e:

$$\mathcal{L}_{\text{Gauge}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + en_e(A_0 + A_0^{ext}) + \frac{1}{2}\kappa\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda.$$

The matter Lagrangian

$$\mathcal{L}_{\text{Matter}} = i\psi^\dagger\mathcal{D}_0\psi - \frac{1}{2m}|\mathcal{D}_k\psi|^2$$

in the case of non-relativistic spinless particles, and

$$\mathcal{L}_{\text{Matter}} = \bar{\psi}(i\gamma^\mu\mathcal{D}_\mu - \sigma m)\psi$$

for the Dirac fermions ($\sigma = \pm 1$).

The covariant derivatives are defined by

$$\mathcal{D}_\mu = \partial_\mu + ie(A_\mu^{ext} + A_\mu + a_\mu).$$

The equations of motion derived from this Lagrangian are given by

$$\begin{aligned}
\partial_\nu F^{\mu\nu} &= eJ^\mu - en_e\delta_{\mu\nu},\\
\kappa\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda &= eJ^\mu,\\
i\frac{\partial}{\partial t}\psi &= -\frac{1}{2m}\mathcal{D}_k\mathcal{D}_k\psi + e(A_0^{ext} + A_0 + a_0)\psi.
\end{aligned}$$

Here J^μ are gauge invariant conserved currents,

$$\begin{aligned}
eJ^\mu(x) &= -\frac{\delta S_{\text{Matter}}}{\delta A_\mu(x)},\\
J^0 &= \psi^\dagger\psi,\\
J_k &= \frac{i}{2m}[\psi^\dagger\partial_k\psi - \partial_k\psi^\dagger\psi] - \frac{e}{m}(A_k^{ext} + A_k + a_k)\psi^\dagger\psi
\end{aligned}$$

Decompose the Chern–Simons field into time-independent background and fluctuating part $a_\mu = \bar{a}_\mu + a'_\mu$ such that, $\bar{a}_0 = 0$, $\epsilon_{ik}\partial_i\bar{a}_k = \frac{e}{\kappa}n_e$. The external electromagnetic field is also the sum of a fixed background and a small perturbation $A_\mu^{ext}(x) = \bar{A}_\mu(x) + \mathcal{A}_\mu(x)$. Introduce the shortened

covariant derivatives, referred to the background gauge fields $D_\mu = \partial_\mu + ie(\bar{A}_\mu + \bar{a}_\mu)$. Using the equations

$$\varepsilon^{\mu\nu\lambda}\bar{a}_\mu(x)\partial_\nu\bar{a}_\lambda(x) = 0, \qquad \bar{a}_0 = 0,$$
$$\varepsilon^{\mu\nu\lambda}a'_\mu(x)\partial_\nu\bar{a}_\lambda(x) = \frac{en_e}{\kappa}a'_0(x)$$

and

$$\varepsilon^{\mu\nu\lambda}\bar{a}_\mu(x)\partial_\nu a'_\lambda(x) = \frac{en_e}{\kappa}a'_0(x) + \partial_\nu[\varepsilon^{\mu\nu\lambda}\bar{a}_\mu(x)a'_\lambda(x)],$$

the Lagrangian can be written as

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + en_e(A_0^{ext} + A_0 + a'_0) + \frac{1}{2}\kappa\varepsilon^{\mu\nu\lambda}a'_\mu\partial_\nu a'_\lambda + \\ &+ i\psi^\dagger D_0\psi - \frac{1}{2m}|D_k\psi|^2 - e\psi^\dagger\psi(A_0^{ext} + A_0 + a'_0) + \\ &+ \frac{ie}{2m}\left[\psi^\dagger D_k\psi - h.c\right](\mathcal{A}_k + A_k + a'_k) - \\ &- \frac{e^2}{2m}\left(\mathcal{A}_k + A_k + a'_k\right)^2\psi^\dagger\psi. \end{aligned} \tag{1.7}$$

4. Generating Functionals

Next introduce the Minkowsky space Green functions generating functional

$$\begin{aligned} Z[j] \equiv e^{iW[j]} &= \langle T\exp\left[i\int dxj(x)\phi(x)\right]\rangle = \\ &= \mathcal{N}\int D\phi\exp\left[i\int dx\left(L + j(x)\phi(x)\right)\right] = \\ &= \langle 0|T\exp\left[i\int dx\left(L_{int} + j(x)\phi(x)\right)\right]|0\rangle. \end{aligned} \tag{1.8}$$

In (1.8), under $\phi(x)$ we mean a dynamical field (gauge or matter), $< \cdots >$ means the average with respect to the Heisenberg representation vacuum and $< 0|\cdots|0 >$ is the corresponding average in the interaction picture.

The effective action is defined by the Legendre transformation

$$\Gamma[\phi] = W[j] - \int dxj(x)\phi(x),$$

where

$$\frac{\delta W[j]}{\delta j(x)} = \phi(x) \quad \frac{\delta\Gamma[\phi]}{\delta\phi(x)} = -j(x).$$

Up to the second order terms,

$$W[j] = W[0] + \int dx\left(\frac{\delta W}{\delta j(x)}\right)_0 j(x) + \frac{1}{2}\int dx\int dyj(x)\left(\frac{\delta^2 W[j]}{\delta j(x)\delta j(y)}\right)_0 j(y),$$

where

$$j(x) = \int dy D^{-1}(x,y)[\phi(y) - \phi_0], \quad \phi_0 = \left(\frac{\delta W[j]}{\delta j(x)}\right)_{j=0}$$

and

$$D(x,y) = \frac{i}{Z[0]}\langle 0|T\phi(x)\phi(y)e^{i\int du\mathcal{L}_{int}(u)}|0\rangle_c \tag{1.9}$$

is a corresponding propagator. In the same order, for the effective action one has

$$\Gamma[\phi] = W[0] - \frac{1}{2}\int dx \int dy(\phi(x)-\phi_0)D(x,y)^{-1}(\phi(y)-\phi_0) + \cdots .$$

Introduce a free two-point function

$$D_0(x,y) = i < 0|T\phi(x)\phi(y)|0 >$$

and a polarization operator $\mathcal{P}(x,y)$ which is the sum of one-particle irreducible two-point Green functions. These quantities satisfy the equation

$$D(x,y) = D_0(x,y) + \int du \int dv D_0(x,u)\mathcal{P}(u,v)D(v,y), \tag{1.10}$$

and consequently

$$D^{-1}(x,y) = D_0^{-1}(x,y) - \mathcal{P}(x,y).$$

Using the last equality, the effective action can be written in the form of a free action plus an interaction term

$$\Gamma[\phi] = -\frac{1}{2}\int dx \int dy(\phi(x)-\phi_0)\{D_0^{-1}(x,y) - \mathcal{P}(x,y)\}(\phi(y)-\phi_0) + \cdots \tag{1.11}$$

(we assume, that $W[0] = 0$).

Consider a gauge theory with the interaction Lagrangian

$$\mathcal{L}_{int} = -eA^{\mu}(x)I_{\mu}(x|A).$$

In the Dirac theory,

$$I_{\mu}(x) = \bar{\psi}(x)\gamma^{\mu}\psi(x),$$

and in the Schrödinger case,

$$I_0(x) = \psi^{\dagger}(x)\psi(x),$$

$$I_k(x) = \frac{i}{2m}(\psi^{\dagger}(x)\nabla_k\psi(x) - h.c) - \frac{e}{2m}\psi^{\dagger}(x)\psi(x)A_k(x).$$

The gauge field propagator can be determined using (1.9) and is given by the expression

$$D_{\mu\nu}(x,y) = D_{0\mu\nu}(x,y) +$$
$$+i^2 e \int dudv \langle |TA_\mu(x)A_\nu(y)A_\lambda(u)A_\rho(v)|0\rangle_c < 0|\frac{\delta I_\lambda(u)}{\delta A_\rho(v)}|0 >_{A=0} -$$
$$-ie^2/2 \int dudv \langle 0|TA_\mu(x)A_\nu(y)A_\lambda(u)A_\rho(v)|0\rangle_c \langle 0|Tj_\lambda(u)j_\rho(v)|0\rangle_c +$$
$$+\mathcal{O}(e^3). \tag{1.12}$$

Using the expansion

$$J_\mu(x) = j_\mu(x) + \int dy \left(\frac{\delta I_\mu(x)}{\delta A_\nu(y)}\right)_{A=0} A_\nu(y),$$

one gets for the polarization operator

$$P_{\mu\nu}(x,y) = ie^2\langle 0|j_\mu(x)j_\nu(y)|0\rangle_c +$$
$$+ 2e^2\langle 0|\left(\frac{\delta I_\mu(x)}{\delta A^\nu(y)}\right)|0\rangle_{A=0} + \mathcal{O}(e^4) \tag{1.13}$$

(remind that $\langle 0|T\cdots|0\rangle$ is average with respect to the free Lagrangian).

In the non-relativistic theory, the polarization operator is defined by the current correlators:

$$\begin{aligned} P_{00}(x,y) &= ie^2\langle 0|Tj_0(x)j_0(y)|0\rangle_c + \mathcal{O}(e^4), \\ P_{0k}(x,y) &= ie^2\langle 0|Tj_0(x)j_k(y)|0\rangle_c + \mathcal{O}(e^4), \\ P_{kl}(x,y) &= ie^2\langle 0|Tj_k(x)j_l(y)|0\rangle_c - \\ &\quad - \frac{e^2}{m}\delta_{kl}\delta(x-y)\langle 0|\psi^\dagger\psi|0\rangle + \mathcal{O}(e^4). \end{aligned} \tag{1.14}$$

Plugging in this values for the polarization tensor, we find that up to the $\mathcal{O}(e^4)$ terms, the effective action (1.11) is expressed by

$$\Gamma[\mathcal{A}_\mu] = \Gamma_0[\mathcal{A}^\mu - A_0^\mu] +$$
$$+\frac{1}{2}\int dx \int dy (\mathcal{A}^\mu(x) - A_0^\mu)P_{\mu\nu}(x,y)(\mathcal{A}^\nu(y) - A_0^\nu) + \cdots ,$$

where A_0^μ is the background value of the gauge field.

In the 2+1 dimensions, the current correlators

$$\Gamma_{\mu\nu}(x,y) = i\langle T(j_\mu(x)j_\nu(x'))\rangle_c$$

have the following Fourier representation

$$\Gamma_{\mu\nu}(x,y) = \int \frac{d\mathbf{q}d\omega}{(2\pi)^3} e^{-i\omega(x_0-y_0)} e^{i\mathbf{q}(\mathbf{x}-\mathbf{y})}\Gamma_{\mu\nu}(\mathbf{q},\omega).$$

In the momentum space, the correlators can be expressed in terms of invariant structure functions [120]:

$$\Gamma_{00}(\omega,\mathbf{q}) = \mathbf{q}^2\Pi_0(\omega,\mathbf{q}),$$
$$\Gamma_{i0}(\omega,\mathbf{q}) = \omega q_i\Pi_0(\omega,\mathbf{q}) + i\epsilon_{ij}q_j\Pi_1(\omega,\mathbf{q}),$$
$$\Gamma_{ik}(\omega,\mathbf{q}) = \delta_{ik}\omega^2\Pi_0(\omega,\mathbf{q}) + i\varepsilon_{ik}\omega\Pi_1(\omega,\mathbf{q}) + (\delta_{ik}\mathbf{q}^2 - q_iq_k)\Pi_2(\omega,\mathbf{q}) + \delta_{ik}\Pi_3.$$

The effective action reads as

$$S_{eff}[\mathcal{F}] = S_0 + \frac{1}{2}\int dxdy\mathcal{F}^\mu(x)P_{\mu\nu}(x,y)\mathcal{F}^\nu(y),$$

where S_0 is a free action, $\mathcal{F}_\mu = A_\mu + a_\mu + A^{ext}_\mu$ is a gauge field, and

$$P_{\mu\nu}(x,y) = e^2\Gamma_{\mu\nu}(x,y) - \frac{e^2}{m}\delta_{\mu k}\delta_{\nu l}\delta_{kl} < j_0 > . \tag{1.15}$$

Consequently,

$$S_{eff}[\mathcal{F}] = S_0 + $$
$$+ \frac{e^2}{2}\int dx[(\partial_0\mathcal{F}_k - \partial_k\mathcal{F}_0)\hat{\Pi}_0(\partial_0\mathcal{F}_k - \partial_k\mathcal{F}_0) + \varepsilon^{\mu\nu\lambda}\mathcal{F}_\mu\hat{\Pi}_1\partial_\nu\mathcal{F}_\lambda +$$
$$+ \epsilon_{im}\partial_i\mathcal{F}^m\hat{\Pi}_2\epsilon_{kn}\partial_k\mathcal{F}^n + \mathcal{F}_k\big(\hat{\Pi}_3 - \frac{< j_0 >}{m}\big)\mathcal{F}_k].$$

Here $\hat{\Pi}_a = \Pi_a(i\partial_0, -\frac{1}{2}\ell^2\Delta)\,(a = 0,1,2,3)$ are differential operators defined by the corresponding momentum space structure functions (see also [120], [116]).

The important physical quantity which describes the electromagnetic and thermal properties of a bulk matter is the linear response function. The standard derivation of Kubo's formula for the linear response uses the adiabatic switching on of an external perturbation and the approximate solution of Schrödinger equation with a time dependent Hamilton operator (see, e.g., [52]).

The formal expression for the linear response can be obtained using the formula relating current operators in the Heisenberg and *in*-representations [15]

$$J^\mu_H(x|A) = S^{-1}\big(TJ^\mu_{in}(x|A)S\big), \tag{1.16}$$

where the S-operator is given by the chronological product

$$S = Te^{-ie\int dxA_\mu(x)I^\mu(x|A)}.$$

For the weak external classical electromagnetic field $A^\nu(x)$, one gets a known result:

$$J^\mu_H(x|A) = J^\mu_{in}(x|A) - ie\int dy\theta(x_0 - y_0)\left[j^\mu(x), j^\nu(y)\right]A_\nu(y) \tag{1.17}$$

(here $j_\mu(x) = J_\mu(x|A=0) = I_\mu(x|A=0)$).

Consider the case of matter interacting with the Chern–Simons and external electromagnetic fields, (i.e., there is no quantum Maxwell field). As interaction Lagrangian, take

$$\mathcal{L}_{\rm int} = -eA^\mu(x) I_\mu(x|a+A).$$

The gauge invariant current is given by

$$eJ^\mu(x) = -\frac{\delta S_{\rm Matter}}{\delta A_\mu(x)} = -\frac{\delta S_{\rm Matter}}{\delta a_\mu(x)} = \kappa\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda(x). \tag{1.18}$$

The last equality is a result of the Euler-Lagrange equation for the Chern–Simons field. Consider a small variation of the external field: $A_\mu \to A_\mu + \delta A_\mu$. The corresponding change in the current is given by:

$$\begin{aligned}\delta J^\mu(x) &= \int d^3y \left(\frac{\delta J^\mu(x)}{\delta A_\nu(y)}\right)\delta A_\nu(y) = \\ &= \int d^3y \left(\frac{\delta J^\mu(x)}{\delta a_\nu(y)}\right)\delta A_\nu(y).\end{aligned}$$

Using (1.18), we get:

$$\delta J^\mu(x) = \frac{\kappa}{e}\varepsilon^{\mu\nu\lambda}\partial_\nu \delta A_\lambda(x). \tag{1.19}$$

The time component of this equation permits to express the density variation with the help of the magnetic field

$$\delta J^0(x) = -\frac{\kappa}{e}\epsilon^{ik}\partial_i \delta A^k(x) = -\frac{\kappa}{e}\delta B(x). \tag{1.20}$$

From the linear response formula, we have

$$\delta J_0(x) = ie\int d^3y D^R_{0k}(x-y)\delta A_k(y),$$

where $D^R_{0k}(x-y)$ is the retarded commutator of the current operators (see (1.17)). In the momentum space, this quantity looks like

$$D^R_{0k}(q) = q_0 q_k \Pi^R_0(q) - i\epsilon_{lm}q_m\Pi^R_1(q),$$

where $\Pi^R_0(q)$ and $\Pi^R_1(q)$ are some structure functions. Considering the limit of the time-independent external field, one obtains

$$\delta J_0(x) = e\int d^3y \frac{d^3q}{(2\pi)^3} e^{-iq(x-y)}\Pi^R_1(q)\delta B(y). \tag{1.21}$$

Comparing (1.20) and (1.21), we get

$$-\frac{\kappa}{e^2}\delta B(\mathbf{r}) = \int d^3y \int \frac{d^3q}{(2\pi)^3} e^{-iq(x-y)}\Pi^R_1(q)\delta B(\mathbf{y}).$$

In the case of the uniform magnetic field $\delta B = const$, one can extract the threshold value of the structure function

$$\Pi_1^R(0) = -\frac{\kappa}{e^2} = \frac{1}{2\vartheta}, \tag{1.22}$$

where ϑ is the statistics parameter. Note that this relation is obtained without reference to perturbative approximation, and it is in fact a low energy theorem. It must be noticed that the previous derivation of this relation was based on the summation of certain diagrams [110], [111].

5. Current Correlators

In this section, we will calculate the current correlation function for a system of planar electrons in a perpendicular magnetic field B. This magnetic field can be, as the external one (B), generated by a given external current distribution, or the Chern–Simons magnetic field produced by the average fermion density ($\bar{b} = -(e/\kappa)\rho$). Introduce first the fermion Green function (see, e.g., [52]). At zero temperature, it is defined as a ground state average:

$$\begin{aligned} G(x,x') &= i\langle\Phi_0|T(\psi(x)\psi^\dagger(x'))|\Phi_0\rangle = \langle\mathbf{r}|G(t-t')|\mathbf{r}'\rangle \\ &= -\int\frac{dE}{2\pi}e^{-iE(t-t')}\langle\mathbf{r}|\mathcal{G}(E)|\mathbf{r}'\rangle. \end{aligned} \tag{1.23}$$

In the magnetic field, fermions fill from the bottom to up the Landau levels with finite energy gap between them (see Appendix A). One-particle states with a given energy can be represented as $|n\rangle \otimes |\xi\rangle$. The integer index n labels the Landau level with the energy $E_n \sim (n + 1/2)$, a and ξ labels the eigenvalue of some observable commuting with Hamiltonian (e.g., one of the momentum components or the orbital angular momentum). The choice of the basis is dictated by physical conditions. In what follows, we consider the rectangular geometry, and as a good quantum number, select the p-momentum along the x-axis.

Each Landau level is highly degenerate with the state density

$$n_B = \frac{|eB|}{2\pi}.$$

Supposing that there are N levels exactly filled, the kernel $\mathcal{G}(E)$ is given by

$$\mathcal{G}(E) = \sum_{n=0}^{N-1}\frac{|n\rangle\langle n|}{E - E_n - i\delta} + \sum_{n=N}^{\infty}\frac{|n\rangle\langle n|}{E - E_n + i\delta}. \tag{1.24}$$

The states $|n\rangle$ are generated by the oscillator rising and lowering operators a and a^+ ($[a, a^+] = 1$):

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^+)^n|0\rangle.$$

The corresponding one-particle Hamiltonian is given by

$$H_{1\mathrm{p}} = \frac{1}{m\ell^2}(a^+a + 1/2).$$

The energy of each Landau level is proportional to the background magnetic field

$$E_n = \frac{1}{m\ell^2}(n + 1/2) \equiv \frac{\omega_c}{m}\Big(n + \frac{1}{2}\Big),$$

where the magnetic length $\ell^2 = 1/|eB|$, and ω_c is the cyclotron frequency. (Below we consider the case $eB > 0$).

The current correlators are defined by the average value of the T-product

$$\begin{aligned}\Gamma_{\mu\nu}(x,y) &= i\langle\Phi_0|T(j_\mu(x)j_\nu(y))|\Phi_0\rangle = \\ &= \frac{1}{(2\pi)^3}\int d\omega d\mathbf{q} e^{-i\omega(x_0-y_0)+i\mathbf{qx}-\mathbf{y}}\Gamma_{\mu\nu}(\omega,\mathbf{q}). \qquad (1.25)\end{aligned}$$

The current operators are given by

$$\begin{aligned}j_0(x) &= \psi^\dagger(x)\psi(x) \\ j_k(x) &= \frac{i}{2m}\left(\psi^\dagger(x)\nabla_k\psi(x) - \nabla_k\psi^\dagger(x)\psi(x)\right).\end{aligned}$$

The covariant derivative operator is $\nabla_k = \partial_k + ieA_k$. The vector-potential A_k generates the static and homogeneous magnetic field $B = \epsilon_{mn}\partial_m A^n$.

The current correlators had been calculated several times, using various methods and different approximations [53], [120], [24], [76], [110], [111], [134], [78]. Below we will follow the presentation described in the papers [40], [41].

Introduce the bilocal objects

$$j_\mu(x,x') = -[\Lambda_\mu(x) + \Lambda_\mu^\dagger(x')]T(\psi(x)\psi^\dagger(x')),$$

where the operators Λ_μ are defined by

$$\Lambda_0 = 1/2, \qquad \Lambda_k(x) = \frac{i}{2m}D_k(\mathbf{r}).$$

Redefine the currents by the limiting procedure

$$j_\mu = \lim_{(t',\mathbf{r}')\to\ (t^+,\mathbf{r})} j_\mu(x,x').$$

Using the fermion Green function, we can write down

$$\langle j_\mu\rangle = i\ \lim[\Lambda_\mu(x) + \Lambda_\mu^\dagger(x')]G(x,x').$$

For the Fourier representation, one obtains

$$\langle j_\mu(x)\rangle = \frac{i}{2\pi\ell^2}\int d\mathbf{q}\delta(\mathbf{q})\,\mathrm{T}r\left[\{\Lambda_\mu, Z(\mathbf{q})\}G(0^+)\right].$$

The $\mathrm{T}r$ operation is taken over the basis of one particle states $|n\,p>=|n>\otimes|p>$. It is not difficult to show that the anticommutator can be expressed as

$$\{\Lambda_\mu, Z(\mathbf{q}\} = Y_\mu(\mathbf{q})Z(\mathbf{q}), \tag{1.26}$$

where

$$Y_0 = 1, \quad Y_n = -\frac{i}{m\ell^2}\epsilon_{nk}\frac{\partial}{\partial q^k}.$$

In (1.26), $Z(\mathbf{q})$ is the coherent state operator

$$Z(\mathbf{q}) = e^{-\ell(q_+ a^+ - q_- a)} \tag{1.27}$$
$$\left(q_\pm = \frac{1}{\sqrt{2}}(q_x \pm i q_y), \quad \kappa = \frac{1}{2}\ell^2\mathbf{q}^2\right).$$

The matrix elements of this operator can be easily calculated and are given by

$$\langle n|Z(\mathbf{q})|n+\alpha\rangle = \sqrt{\frac{n!}{(n+\alpha)!}}(\ell q_-)^\alpha L_n^\alpha(\kappa)e^{-\kappa/2},$$
$$\langle n+\alpha|Z(\mathbf{q}|n\rangle = \sqrt{\frac{n!}{(n+\alpha)!}}(-\ell q_+)^\alpha L_n^\alpha(\kappa)e^{-\kappa/2},$$
$$\langle n|Z(\mathbf{q}|n\rangle = L_n^0(\kappa)e^{-\kappa/2}.$$

In the above equations, $L_n^\alpha(\kappa)$ are the adjoint Laguerre polynomials defined by

$$L_n^\alpha(z) = \frac{1}{n!}\left[e^z z^{-\alpha}\frac{d^n}{dz^n}\left(e^{-z}z^n\right)\right] =$$
$$= (n+\alpha)!\sum_{k=0}^{n}(-1)^k\frac{1}{k!(\alpha+k)!(n-k)!}z^k.$$

The operator $Z(\mathbf{q})$ is useful in order to calculate the Fourier transformations

$$\int d\mathbf{r}e^{-i\mathbf{k}\mathbf{r}}\langle\mathbf{r}|A(a^\dagger,a)|\mathbf{r}\rangle = \frac{2\pi}{\ell^2}tr[Z(\mathbf{k})A],$$
$$\int d\mathbf{r}\int d\mathbf{r}'e^{-i\mathbf{k}\mathbf{r}}e^{i\mathbf{k}'\mathbf{r}'}\langle\mathbf{r}|A(a^\dagger,a)|\mathbf{r}'\rangle\langle\mathbf{r}'|B(a^\dagger,a)|\mathbf{r}\rangle =$$
$$= \frac{2\pi}{\ell^2}\delta(\mathbf{k}-\mathbf{k}')tr[Z(\mathbf{k})AZ^\dagger(\mathbf{k}')B],$$

where A and B are some ladder operator depending quantities.

Consequently,

$$\langle j_\mu(x)\rangle = \frac{i}{2\pi\ell^2}\{Y_\mu(\mathbf{q})\,\mathrm{T}r[Z(\mathbf{q})G(0^+)]\}_{\mathbf{q}=0} = \frac{i}{2\pi\ell^2}\delta_{\mu 0}\mathrm{T}r[G(0^+)].$$

In (1.25), $j_\mu(x)$ is a divergenceless current. At the same time, due to the anomaly, the correlator is not conserved: the 3-divergence of this object is given by

$$\partial_x^\mu \Gamma_{\mu\nu}(x,y) = i\,\delta(x_0-y_0)\langle[j_0(x),j_\nu(y)]\rangle = \frac{i}{m}\delta_{\nu k}\frac{\partial}{\partial x^k}\delta(x-y)\langle j_0\rangle.$$

In the momentum space, the correlators can be expressed in terms of invariant structure functions:

$$\begin{aligned}
\Gamma_{00}(\omega,\mathbf{q}) &= \mathbf{q}^2\Pi_0(\omega,\mathbf{q}),\\
\Gamma_{i0}(\omega,\mathbf{q}) &= \omega q_i\Pi_0(\omega,\mathbf{q}) + i\epsilon_{ij}q_j\Pi_1(\omega,\mathbf{q}),\\
\Gamma_{ik}(\omega,\mathbf{q}) &= \delta_{ik}\omega^2\Pi_0(\omega,\mathbf{q}) + i\epsilon_{ik}\omega\Pi_1(\omega,\mathbf{q}) +\\
&\quad + (\delta_{ik}\,\mathbf{q}^2 - q_iq_k)\Pi_2(\omega,\mathbf{q}) + \delta_{ik}\Pi_3.
\end{aligned} \tag{1.28}$$

From the current conservation, it follows that Π_3 is in fact a Schwinger term:

$$q^\mu\Gamma_{\mu\nu}(\omega,\mathbf{q}) = q^l\Pi_3\delta_{l\nu} = \frac{1}{m}q^l\langle j_0\rangle\delta_{l\nu},$$

i.e.,

$$\Pi_3 = \frac{\langle j_0\rangle}{m}.$$

Performing the Fourier transformation, one obtains [120]

$$\begin{aligned}
\Gamma_{\mu\nu}(x,y) = \frac{i}{(2\pi)^3\ell^2}\int d\mathbf{k}_1\int d\mathbf{k}_2 e^{i\mathbf{k}\mathbf{x}}e^{i\mathbf{k}'\mathbf{y}}\delta(\mathbf{k}+\mathbf{k}')\times\\
\times\,\mathrm{T}r\,[\{Z(\mathbf{k}),\Lambda_\mu\}G(x_0-y_0)\{Z(\mathbf{k}'),\Lambda_\nu\}G(y_0-x_0)].
\end{aligned}$$

Here $G(x_0-y_0)$ is the kernel of the fermion Green function. Applying (1.26), we get the representation

$$\Gamma_{\mu\nu}(\mathbf{q},\omega) = \frac{i}{2\pi\ell^2}[Y_\mu(\mathbf{q})Y_\nu^\dagger(\mathbf{q}')\Gamma(\mathbf{q},\mathbf{q}',\omega)]_{\mathbf{q}=\mathbf{q}'}.$$

Introduce the variables

$$\kappa = \frac{\ell^2}{2}\mathbf{q}^2,\quad \zeta = \frac{\ell^2}{2}(\mathbf{q}\cdot\mathbf{q}' + i\mathbf{q}\wedge\mathbf{q}')$$

and the notation

$$d_n^\alpha(\omega) = \int\frac{dE}{2\pi}\langle n+\alpha|\mathcal{G}(E)|n+\alpha\rangle\langle n|\mathcal{G}(E-\omega)|n\rangle = \bar{d}_n^\alpha(-\omega). \tag{1.29}$$

We see that the kernel

$$\Gamma(\mathbf{q},\mathbf{q}',\omega) = \int\frac{dE}{2\pi}\mathrm{Tr}[Z(\mathbf{q})\mathcal{G}(E)Z^\dagger(\mathbf{q}')\mathcal{G}(E-\omega)]$$

can be written as a sum

$$\Gamma(\mathbf{q},\mathbf{q}',\omega) = i\sum_{n=0}^{\infty}\sum_{\alpha+0}^{\infty}\left(1-\frac{\delta_{\alpha 0}}{2}\right)\frac{n!}{(n+\alpha)!}e^{-(\kappa+\kappa')/2}L_n^\alpha(\kappa)L_n^\alpha(\kappa')\times$$
$$\times\left(\zeta^\alpha d_n^\alpha(\omega)+\bar{\zeta}^\alpha\bar{d}_n^\alpha(\omega)\right)\equiv\sum_{n\alpha}\left(1-\frac{\delta_{\alpha 0}}{2}\right)\frac{n!}{(n+\alpha)!}\sigma_{n\alpha}(\mathbf{q},\mathbf{q}',\omega).$$

The action of the operators $Y_\mu(\mathbf{q})$ on $\sigma_{n\alpha}(\mathbf{q},\mathbf{q}',\omega)$ is given by the expressions

$$\begin{aligned}[Y_i(\mathbf{q})\sigma_{n\alpha}]_{\mathbf{q}=\mathbf{q}'} &= \frac{1}{2m}q_i\left[\alpha\kappa^{\alpha-1}\Phi_n^\alpha(\kappa)\Phi_n^\alpha(\kappa)\right]\left(d_n^\alpha(\omega)-\bar{d}_n^\alpha(\omega)\right)\\ &+\frac{i}{2m}\epsilon_{ij}q_j\left[\kappa^\alpha\frac{d\Phi_n^\alpha(\kappa)}{d\kappa}\Phi_n^\alpha(\kappa)+\right.\\ &\left.+\frac{1}{2}\alpha\kappa^{\alpha-1}\Phi_n^\alpha(\kappa)\Phi_n^\alpha(\kappa)\right]\left(d_n^\alpha(\omega)+\bar{d}_n^\alpha(\omega)\right), \qquad (1.30)\\ \left[Y_i(\mathbf{q})Y_k^\dagger(\mathbf{q}')\sigma_{n\alpha}\right]_{\mathbf{q}=\mathbf{q}'} &= \frac{1}{2m^2\ell^2}\delta_{ik}\left[\alpha^2\kappa^{\alpha-1}\Phi_n^\alpha(\kappa)\Phi_n^\alpha(\kappa)\right]\left(d_n^\alpha(\omega)+\bar{d}_n^\alpha(\omega)\right)+\\ &+\frac{i}{m^2\ell^2}\epsilon_{ik}\left[\frac{1}{2}\alpha^2\kappa^{\alpha-1}\Phi_n^\alpha(\kappa)\Phi_n^\alpha(\kappa)+\right.\\ &\left.+\alpha\kappa^\alpha\frac{d\Phi_n^\alpha(\kappa)}{d\kappa}(\kappa)\Phi_n^\alpha(\kappa)\right]\left(d_n^\alpha(\omega)-\bar{d}_n^\alpha(\omega)\right)+\\ &+\frac{1}{m^2}(\delta_{ik}\mathbf{q}^2-q_iq_k)\left[\frac{d\Phi_n^\alpha(\kappa)}{d\kappa}\frac{d\Phi_n^\alpha(\kappa)}{d\kappa}+\right.\\ &\left.+\alpha\kappa^{\alpha-1}\frac{d\Phi_n^\alpha(\kappa)}{d\kappa}\Phi_n^\alpha(\kappa)\right]\left(d_n^\alpha(\omega)+\bar{d}_n^\alpha(\omega)\right). \qquad (1.31)\end{aligned}$$

Here $\Phi_n^\alpha(\kappa)=e^{-\kappa/2}L_n^\alpha(\kappa)$. In deriving this relations, we have used

$$\partial_i\equiv\frac{\partial}{\partial q^i}=q^{'i}+\partial_\kappa+\frac{1}{2}q^{'i}(\partial_\zeta+\partial_{\bar{\zeta}})+\frac{i}{2}\epsilon^{im}q^{'m}(\partial_\zeta-\partial_{\bar{\zeta}}),$$
$$\partial_i'\equiv\frac{\partial}{\partial q^{'i}}=q^{'i}+\partial_\kappa+\frac{1}{2}q^i(\partial_\zeta+\partial_{\bar{\zeta}})-\frac{i}{2}\epsilon^{im}q^m(\partial_\zeta-\partial_{\bar{\zeta}})$$

and the identity

$$\epsilon^{ik}\mathbf{q}^2+q^i\epsilon^{km}q^m-q^k\epsilon^{im}q^m=0.$$

Straightforward calculations show that the spectral functions (1.29) satisfy the relations

$$\alpha\omega_c(d_n^\alpha(\omega)-\bar{d}_n^\alpha(\omega))=\omega(d_n^\alpha(\omega)+\bar{d}_n^\alpha(\omega)), \qquad (1.32)$$
$$(\alpha\omega_c)^2\int d\omega e^{-i\omega t}\left[d_n^\alpha(\omega)+\bar{d}_n^\alpha(\omega)\right]=$$
$$=(\alpha\omega_c)^2\int d\omega e^{-i\omega t}\left[d_n^\alpha(0)+\bar{d}_n^\alpha(0)\right]+$$

$$+\int d\omega e^{-i\omega t}\omega^2\left[d_n^\alpha(\omega)+\bar{d}_n^\alpha(\omega)\right]. \tag{1.33}$$

Indeed, the Green function has the following structure

$$\langle m|\mathcal{G}(z)|m\rangle=\frac{a_m}{z-E_m-i\delta}+\frac{b_m}{z-E_m+i\delta}.$$

Integrating, one obtains

$$\begin{aligned}D_{n\alpha}^+(\omega)&=\int\frac{dE}{2\pi}\left(d_n^\alpha(\omega)+\bar{d}_n^\alpha(\omega)\right)=\\&=2i\omega_c\alpha\frac{a_{n+\alpha}b_n-a_nb_{n+\alpha}}{(\omega_c\alpha+i\delta)^2-\omega^2},\end{aligned} \tag{1.34}$$

$$\begin{aligned}D_{n\alpha}^-(\omega)&=\int\frac{dE}{2\pi}\left(d_n^\alpha(\omega)-\bar{d}_n^\alpha(\omega)\right)=\\&=2i\omega\alpha\frac{a_{n+\alpha}b_n-a_nb_{n+\alpha}}{(\omega_c\alpha+i\delta)^2-\omega^2},\end{aligned} \tag{1.35}$$

and hence (1.32). Further, in order to obtain the correlator, one encounters the integral over the frequencies ω:

$$\int\frac{d\omega}{2\pi}e^{-i\omega t}\frac{\omega^2}{(\omega^2-(\alpha\omega_c)^2)}=\int\frac{d\omega}{2\pi}e^{-i\omega t}\frac{\omega^2}{\left(\omega^2-(\alpha\omega_c)^2\right)}+\int\frac{d\omega}{2\pi}e^{-i\omega t}$$

which gives (1.33).

Integrating, we use the limiting procedure

$$\begin{aligned}&\int\frac{dE}{2\pi i}\frac{1}{(E-E_m-i\delta)}\cdot\frac{1}{(E-E_m+i\delta)}=\\&=\int\frac{dE\,e^{iE0^+}}{2\pi i}\frac{1}{(E-E_m-i\delta)}\cdot\frac{1}{(E-E_m+i\delta)}=\\&=\frac{1}{E_m-E_n+i\delta},\end{aligned}$$

and

$$\int\frac{dE}{2\pi i}\frac{1}{(E-E_m\pm i\delta)}\cdot\frac{1}{(E-E_m\pm i\delta)}=0.$$

Equations (1.32) and (1.33) permit to write (1.30) and (1.31) in the following form:

$$\begin{aligned}&[Y_i(\mathbf{q})\sigma_{n\alpha}]_{\mathbf{q}=\mathbf{q}'}=\frac{1}{2m}q_i\frac{\omega}{\omega_c}\left[\kappa^{\alpha-1}\Phi_n^\alpha(\kappa)\Phi_n^\alpha(\kappa)\right]D_{n\alpha}^+(\omega)+\\&+\frac{i}{2m}\epsilon_{ij}q_j\left[\kappa^\alpha\frac{d\Phi_n^\alpha(\kappa)}{d\kappa}\Phi_n^\alpha(\kappa)+\frac{1}{2}\alpha\kappa^{\alpha-1}\Phi_n^\alpha(\kappa)\Phi_n^\alpha(\kappa)\right]D_{n\alpha}^+(\omega),\\&\left[Y_i(\mathbf{q})Y_k^\dagger(\mathbf{q}')\sigma_{n\alpha}\right]_{\mathbf{q}=\mathbf{q}'}=\frac{1}{2m^2\ell^2}\delta_{ik}\frac{\omega^2}{\omega_c^2}\left[\kappa^{\alpha-1}\Phi_n^\alpha(\kappa)\Phi_n^\alpha(\kappa)\right]D_{n\alpha}^+(\omega)+\end{aligned}$$

$$+\frac{i}{m^2\ell^2}\epsilon_{ik}\frac{\omega}{\omega_c}\left[\frac{1}{2}\alpha\kappa^{\alpha-1}\Phi_n^\alpha(\kappa)\Phi_n^\alpha(\kappa)+\kappa^\alpha\frac{d\Phi_n^\alpha(\kappa)}{d\kappa}\Phi_n^\alpha(\kappa)\right]D_{n\alpha}^+(\omega)+$$
$$+\frac{1}{m^2}\left(\delta_{ik}\mathbf{q}^2-q_iq_k\right)\left[\frac{d\Phi_n^\alpha(\kappa)}{d\kappa}\frac{d\Phi_n^\alpha(\kappa)}{d\kappa}+\alpha\kappa^{\alpha-1}\frac{d\Phi_n^\alpha(\kappa)}{d\kappa}\Phi_n^\alpha(\kappa)\right]D_{n\alpha}^+(\omega)+$$
$$+\frac{1}{2m^2\ell^2}\delta_{ik}\alpha^2\left[\kappa^{\alpha-1}\Phi_n^\alpha(\kappa)\Phi_n^\alpha(\kappa)\right]D_{n\alpha}^+(0).$$

Introduce the function

$$S(\kappa,\kappa',\omega)\equiv\frac{i}{2\pi\ell^2}\sum_{n=0}^{\infty}\sum_{\alpha=0}^{\infty}\left(1-\frac{\delta_{\alpha0}}{2}\right)\frac{n!}{(n+\alpha)!}s_{n\alpha}(\kappa,\kappa',\omega)D_{n\alpha}^+(\omega)=$$
$$=\frac{i}{2\pi\ell^2}\sum_{n=0}^{\infty}\sum_{\alpha=0}^{\infty}\left(1-\frac{\delta_{\alpha0}}{2}\right)\frac{n!}{(n+\alpha)!}\times$$
$$\times\frac{1}{2}(\kappa^\alpha+\kappa^{'\alpha})\Phi_n^\alpha(\kappa)\Phi_n^\alpha(\kappa')D_{n\alpha}^+(\omega). \qquad (1.36)$$

Direct calculations show that the structure functions Π_0, Π_1 and Π_2 can be expressed as

$$\Pi_0(\omega,\mathbf{q})=\frac{\ell^2}{2}\kappa^{-1}S(\kappa,\kappa,\omega),$$
$$\Pi_1(\omega,\mathbf{q})=\frac{1}{m}\frac{\partial}{\partial\kappa}S(\kappa,\kappa',\omega)\Big|_{\kappa'=\kappa},$$
$$\Pi_2(\omega,\mathbf{q})=\frac{1}{m^2}\frac{\partial}{\partial\kappa}\frac{\partial}{\partial\kappa'}S(\kappa,\kappa',\omega)\Big|_{\kappa'=\kappa}.$$

For the structure function Π_3, we get

$$\Pi_3(\omega,\mathbf{q})=\frac{i}{4\pi m^2\ell^4}\sum_{n\alpha}\left(1-\frac{1}{2}\delta_{\alpha0}\right)\frac{n!}{(n+\alpha)!}\alpha^2\kappa^{\alpha-1}\left[\Phi_n^\alpha(\kappa)\right]^2D_{n\alpha}^+(0).$$

As we see, the three different structure functions entering into the correlator can be expressed as derivatives of the same scalar function (1.36) [41], [40]. This function is expressed in terms of infinite series, from which one can extract physical results with a needed accuracy. Remind that in the principal papers on the calculation of current correlators (e.g., [120], [110], [111], [24], [76]), this objects are obtained in the lowest orders and for special configurations of the gauge field after rather lengthy and complicated calculations.

As a further step, let us find the frequency-dependent part of the function (1.36).

Using the explicit form of the Laguerre polynomials, one finds that zero-momentum values of structure functions are given by

$$\Pi_0(\omega,0)=\frac{i}{4\pi}\sum_{n=0}^{\infty}(n+1)D_{n1}^+(\omega),$$

$$\Pi_1(\omega,0)=\frac{i}{2\pi m\ell^2}\sum_{n=0}^{\infty}\left[\frac{1}{2}(n+1)D^+_{n1}(\omega)-\frac{1}{2}\Big(n+\frac{1}{2}\Big)D^+_{n0}\right],$$

$$\Pi_2(\omega,0)=\frac{i}{2\pi m\ell^2}\sum_{n=0}^{\infty}\left[-\frac{1}{2}(n+1)^2D^+_{n1}(\omega)+\frac{1}{2}\Big(n+\frac{1}{2}\Big)^2D^+_{n0}\right].$$

Suppose that the levels $n=0,1,\dots,N-1$ are filled up and the higher levels with $n\geq N$ are empty. This corresponds to the Green function with

$$a_m=\theta(N-1-m),\quad b_m=\theta(m-N),\quad (\theta(0)=1).$$

From (1.34), we get

$$D^+_{n\alpha}=-i\delta_{n,N-1}\delta_{\alpha 1}\frac{2\omega_c}{\omega_c^2-\omega^2}. \tag{1.37}$$

It is not difficult to extract the zero momentum values of the structure functions:

$$\Pi_0(\omega,0)=\frac{1}{2}\frac{N}{\pi}\frac{\omega_c}{\omega_c^2-\omega^2}, \tag{1.38}$$

$$\Pi_1(\omega,0)=\frac{1}{2}\frac{N}{\pi}\frac{\omega_c^2}{\omega_c^2-\omega^2}, \tag{1.39}$$

$$\Pi_2(\omega,0)=-\frac{1}{2}\frac{N^2}{\pi m}\frac{\omega_c^2}{\omega_c^2-\omega^2}, \tag{1.40}$$

$$\Pi_3(\omega,0)=\frac{1}{2m\ell^2}\frac{N}{\pi}. \tag{1.41}$$

The threshold values ($\omega\sim 0,\mathbf{q}\sim 0$) of the structure functions are given by

$$\begin{aligned}\Pi_0(0)&=\frac{m\ell^2N}{2\pi},\\ \Pi_1(0)&=\frac{N}{2\pi},\\ \Pi_2(0)&=-\frac{N^2}{2m\pi}.\end{aligned} \tag{1.42}$$

6. Current Correlators at $T\neq 0$

At a non-zero temperature, one can use the methods of grand canonical ensembles and the imaginary time (Matsubara) technics [52], [89].

The corresponding expressions in the Matsubara formalism can be obtained performing Wick's rotation and the well known substitutions

$$\begin{aligned}E\to i\xi_s=\frac{2\pi i}{\beta}(s+\frac{1}{2}),\quad -\int\frac{dE}{2\pi}\to\frac{1}{\beta}\sum_s,\\ \omega\to i\omega_r=\frac{2\pi i}{\beta}r,\quad -\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\to\frac{1}{\beta}\sum_r e^{-i\omega_r(t-t')}.\end{aligned} \tag{1.43}$$

The thermal Green function is defined as a quantum statistical average

$$G_\beta(\mathbf{r},\tau;\mathbf{r}',\tau') = -\frac{1}{\mathrm{Z}} Tr\{\rho T_\tau[\psi(\mathbf{r},\tau)\psi^\dagger(\mathbf{r}',\tau')]\}.$$

where $\rho = exp[-\beta(H-\mu N)]$ is the grand canonical distribution and $\mathrm{Z} = \mathrm{Tr}\rho$. Matsubara fields are defined as follows

$$\psi(\mathbf{r},\tau) = \sum_n^\infty \int dp e^{-E'_n\tau}\psi_{np}(\mathbf{r})a_{np},$$

$$\psi^\dagger(\mathbf{r},\tau) = \sum_n^\infty \int dp e^{E'_n\tau}\psi^\dagger_{np}(\mathbf{r})a^\dagger_{np}.$$

Here $E'_n = E_n - \mu$.

Due to the antiperiodicity, this thermal Green function can be Fourier-expanded over the half-integer frequencies:

$$G_\beta(\mathbf{r},\tau;\mathbf{r}',\tau) = \frac{1}{\beta}\sum_{s=-\infty}^{+\infty} e^{-i\xi_s(\tau-\tau')}\langle\mathbf{r}|\mathcal{G}_\beta(i\xi_s)|\mathbf{r}'\rangle,$$

$$\xi_s = \frac{2\pi}{\beta}(s+1/2).$$

The kernel

$$\mathcal{G}_\beta(i\xi_s) = \sum_{n=0}^\infty \frac{|n\rangle\langle n|}{i\xi_s - E'_n}$$

can be obtained from the zero temperature kernel by the formal substitution $E \to i\xi_s, E_n \to E'_n = E_n - \mu$. The current correlators are defined by the statistical averages

$$\Gamma^\beta_{\mu\nu}(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) = -\frac{1}{Z}\mathrm{Tr}\ [j_\mu(\mathbf{r}_1,\tau_1)j_\nu(\mathbf{r}_2,\tau_2)]$$

and can be calculated in the same way as the zero-temperature correlators. It is not difficult to see that the main changes concern the frequency dependent parts $D^{\pm}{}_{n\alpha}(\omega)$, which are changed to the temperature dependent quantities

$$\Delta^\pm_{n\alpha}(\omega_r) = -\frac{1}{\beta}\sum_s \left[\langle n+\alpha|\mathcal{G}^\beta(i\xi_s)|n+\alpha\rangle\langle n|\mathcal{G}^\beta(i\xi_s - i\omega_r)|n\rangle \pm (\omega_r \to -\omega_r)\right].$$

The typical sum in these expressions is given by

$$I_\beta(\omega_r) = \frac{1}{\beta}\sum_{s=-\infty}^{+\infty}\frac{1}{i\xi_s - E'_n}\cdot\frac{1}{i\xi_s - E'_{n+\alpha}}.$$

This sum can be converted to the contour integral

$$I_\beta(\omega_r) = -\frac{\beta}{4\pi^2}\int_\Gamma \frac{dz}{\cot\pi z}\frac{1}{\left(z - \frac{\beta}{2\pi i}E'_n\right)}\cdot\frac{1}{\left(z - \frac{\beta}{2\pi i}E'_{n+\alpha} - \frac{\beta}{2\pi}\omega_r\right)}.$$

The complex integration path passes below and above the real axis and is closed at $\pm\infty$'s. Deforming this contour into two paths in the upper and lower half-planes, one gets:

$$I_\beta(\omega_r) = \frac{1}{E'_{n+\alpha} - E'_n + i\omega_r}\left[\frac{1}{1+e^{\beta E'_n}} - \frac{1}{1+e^{\beta E'_{n+\alpha}}}\right].$$

If $\alpha = 0$, there is a double pole at $\omega_r = 0$ in the contour integral, and

$$I_\beta(0) = \frac{\beta}{2\pi}\left(\frac{1-e^{\beta E'_n}}{1+e^{\beta E'_n}}\right).$$

As in the zero temperature case, one can show that the following relations are valid:

$$\alpha\omega_c\Delta^+_{n\alpha}(\omega_r) = i\omega_r\Delta^+_{n\alpha}(\omega_r)$$

and

$$\frac{1}{\beta}\sum_r e^{-i\omega_r\tau}\frac{\omega_r^2}{\omega_r^2+\alpha^2\omega_c^2} = \frac{1}{\beta}\sum_r e^{-i\omega_r\tau}\frac{-\alpha^2\omega_c^2}{\omega_r^2+\alpha^2\omega_c^2} + \frac{1}{\beta}\sum_r e^{-i\omega_r\tau}.$$

The last term here is the periodic delta-function $\delta_\beta(\tau)$.

Further steps repeat the zero-temperature case, where

$$\Delta^+_{n\alpha}(\omega_r) = \frac{2\omega_c}{\omega_c^2+\omega_r^2}\left[\frac{1}{1+e^{\beta E'_n}} - \frac{1}{1+e^{\beta E'_{n+\alpha}}}\right]$$

must be used instead of $D^+_{n\alpha}(\omega)$. Now the zero-momentum structure functions are given by the sums

$$\Pi_0^\beta(\omega_r,0) = \frac{1}{2\pi}\frac{\omega_c}{\omega_c^2+\omega_r^2}\sum_{n=0}^{\infty}\frac{1}{1+e^{\beta E'_n}},$$

$$\Pi_1^\beta(\omega_r,0) = \frac{1}{2\pi}\frac{\omega_c^2}{\omega_c^2+\omega_r^2}\sum_{n=0}^{\infty}\frac{1}{1+e^{\beta E'_n}},$$

$$\Pi_2^\beta(\omega_r,0) = -\frac{1}{2\pi m}\frac{\omega_c^2}{\omega_c^2+\omega_r^2}\sum_{n=0}^{\infty}\frac{2n+1}{1+e^{\beta E'_n}}.$$

7. Effective Action and Response Function

Consider the action

$$\mathcal{S}[A,\phi] = \int dxdy\left[-\frac{1}{2}A(x)K(x,y)A(y) + L(A,\phi)\right].$$

In this action, by $A(x)$ we denote the set of gauge fields, such as the dynamical Maxwell field $A_\mu(x)$, or the Chern–Simons field a_μ and the external electromagnetic potential $\mathcal{A}_\mu(x)$. The block-diagonal matrix $K(x,y)$ represents the kernel of the kinetic term and can include the gauge fixing terms for dynamical gauge fields. The kinetic terms corresponding to the external

fields formally are assumed to be infinitely large. By $\phi(x)$ we denote matter fields.

Construct the Schwinger's functional and the effective action for the gauge fields, integrating out the matter. In the Heisenberg representation, the Green functions generating functional is given by

$$Z[J] = \langle \,\mathrm{out}\,| e^{i\int dx J(x) A(x)(x)} |\,\mathrm{in}\,\rangle = e^{iW[J]}.$$

The corresponding effective action up to the second order is given by

$$S_{eff}[A_c] = -\frac{1}{2}\int dxdy\,(A_c(x) - A_0(x))\,\mathcal{D}^{-1}(x,y)\,(A_c(y) - A_0(y)) + \mathcal{O}(A^3).$$

Here

$$\mathcal{D}(x,y) = i\langle\,\mathrm{out}\,|T\,(A(x)A(y)))\,|\,\mathrm{in}\,\rangle_c$$

is a full propagator and $A_0(x) = \langle out|A(x)|in\rangle$ is the field vacuum average. In what follows, we assume that $A_0(x) = 0$. Those components of the classical field $A_c(x)$ which correspond to external electromagnetic fields are fixed, and are equal to $\mathcal{A}_\mu(x)$.

In the interaction representation, the free propagator is defined by

$$D_c(x,y) = i\langle 0|T\,(A(x)A(y))\,|0\rangle = \int \frac{d^dq}{(2\pi)^d} e^{-iq(x-y)} D_c(q),$$

where

$$D_c(q) = \int \frac{d^dx}{(2\pi)^d} e^{iqx} K^{-1}(x,0) \equiv K^{-1}(q)$$

and the interacting Green function is given by

$$\mathcal{D}(x,y) = i\langle 0|T\left(A(x)A(y)e^{i\int dxL}\right)|0\rangle.$$

For further needs, we give some expressions illustrating the above discussions.

In the Coulomb gauge, the propagators must be extracted from the free Maxwell end Chern–Simons Lagrangians

$$\mathcal{L}_{\mathrm{EM}} = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \frac{1}{2\xi}(\partial_k A_k(x))^2,$$
$$\mathcal{L}_{\mathrm{CS}} = \frac{1}{2}\kappa\varepsilon^{\mu\nu\lambda}a_\mu(x)\partial_\nu a_\lambda(x) + \frac{1}{2\alpha}(\partial_k a_k(x))^2.$$

The Maxwell field propagator $D_{\mu\nu}(q)$ is given by the matrix

$$\tilde{K}^{-1}_{EM}(q) = \frac{1}{\mathbf{q}^2}\begin{pmatrix} 1+\xi\frac{\omega^2}{\mathbf{q}^2} & -\xi\frac{\omega q^i}{\mathbf{q}^2} \\ -\xi\frac{\omega q^i}{\mathbf{q}^2} & [\delta^{ik} - \frac{q^iq^k}{\mathbf{q}^2}] + \xi\frac{q^iq^k}{(\mathbf{q}^2} \end{pmatrix}.$$

The Chern–Simons propagator $\Delta_{\mu\nu}(q)$ is given by the matrix

$$\tilde{K}^{-1}_{CS}(q) = \frac{1}{\kappa \mathbf{q}^2} \begin{pmatrix} \alpha\omega^2 & i\frac{\epsilon^{im}q^m}{\kappa\ q^2} - \alpha\frac{\omega q^i}{(\mathbf{q}^2)^2} \\ i\frac{\epsilon^{im}q^m}{\kappa \mathbf{q}^2} - \alpha\frac{\omega q^i}{(\mathbf{q}^2)^2} & -\alpha\frac{q^i q^k}{\kappa^2 \mathbf{q}^2} \end{pmatrix}.$$

In some cases, it is more convenient to take as a free Chern–Simons Lagrangian the expression (see the Lagrangian (1.6)))

$$\mathcal{L}_{CS} = \frac{\kappa}{2}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - \frac{e^2\rho}{2m}a_k a_k + \frac{\alpha}{2}(\partial_i a_i). \tag{1.44}$$

The corresponding propagator in the limit $\alpha \to \infty$ looks like

$$\Delta_{\mu\nu}(x) = \int \frac{dq}{(2\pi)^3} e^{-iqx}\Delta_{\mu\nu}(q) = i\langle T\{a_\mu(x)a_\nu(0)\}\rangle, \tag{1.45}$$

where

$$\Delta_{\mu\nu}(q) = \frac{1}{\kappa}\frac{1}{\mathbf{q}^2} \begin{pmatrix} -\frac{g^2\rho}{m\kappa} & iq_y & -iq_x \\ -iq_y & 0 & 0 \\ iq_x & 0 & 0 \end{pmatrix}. \tag{1.46}$$

The relativistic free action is

$$\mathcal{L}_{\rm EM} = \int dx \left\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\xi}\ (\partial^\mu A_\mu)^2 \right\}$$

for the Maxwell field and

$$\mathcal{L}_{\rm CS} = \int dx \left\{ \frac{\kappa}{2}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \frac{1}{2\alpha}(\partial^\mu a_\mu)^2 \right\} d^3x$$

for the Chern–Simons field.

The bare propagators of the Maxwell and the Chern–Simons fields in that cases are given by

$$D^{\mu\nu}(q) = \frac{1}{q^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{(q^2)^2}\right) - \frac{q^\mu q^\nu}{k^2}\xi,$$

$$\Delta^{\mu\nu}(q) = -\frac{i}{\kappa}\frac{\varepsilon^{\mu\nu\lambda}q_\lambda}{q^2} - \frac{q^\mu q^\nu}{(q^2)^2}\alpha.$$

The full propagator satisfies the Dyson-Schwinger equation

$$\mathcal{D}(x,y) = D_c(x,y) + \int dudv D_c(x,u)\mathcal{P}(u,v)\mathcal{D}(v,y),$$

where $\mathcal{P}(x,y)$ is a polarization operator. The corresponding expression in the momentum space is

$$\mathcal{D}(q) = D_c(q) + D_c(q)\mathcal{P}(q)\mathcal{D}(q).$$

The equivalent form of this equation is given by

$$\mathcal{D}^{-1}(x,y) = D_c^{-1}(x,y) - \mathcal{P}(x,y)$$

and

$$\mathcal{D}^{-1}(q) = D_c^{-1}(q) - \mathcal{P}(q).$$

The lowest order perturbative calculations show that

$$\begin{aligned}\mathcal{P}(x,y) = \langle 0|T\left(\frac{\partial L}{\partial A(x)}\frac{\partial L}{\partial A(y)}\right)_{A=0}|0\rangle_c + \\ + 2\langle 0|T\left(\frac{\partial^2 L}{\partial A(x)\partial A(y)}\right)_{A=0}|0\rangle_c + \cdots \equiv \\ \equiv P(x,y) + \cdots \ .\end{aligned}$$

Taking into account the above consideration, the effective action is expressed as

$$\begin{aligned}S_{eff}[A_c] = -\frac{1}{2}\int dxdy A_c(x)\mathcal{D}^{-1}(x,y)A_c(y) = \\ = -\frac{1}{2}\int dxdy A_c(x)D_0^{-1}(x,y)A_c(y) + \\ + \frac{1}{2}\int dxdy A_c(x)\mathcal{P}(x,y)A_c(y).\end{aligned}$$

For the case of the external field $A_{ext}(x)$ and a dynamical one denoted by $A(x)$, the effective action looks like

$$\begin{aligned}S_{eff}[A, A_{ext}] = -\frac{1}{2}\int dxdy A(x)K(x,y)A(y) + \\ + \frac{1}{2}\int dxdy\left(A(x) + A_{ext}(x)\right)\mathcal{P}(x,y)\left(A(y) + A_{ext}\right).\end{aligned}$$

As a next step, we can integrate out the dynamical fields, i.e. substitute the solution of the extremal equation

$$A(x) = \int dudy\Pi(x,u)\mathcal{P}(u,y)A_{ext}(y),$$

where the operator $\Pi(x,y) = K(x,y) - \mathcal{P}(x,y)$. In that way, one gets the action

$$\mathcal{S}_{eff}[A_{ext}] = \frac{1}{2}\int dxdy A_{ext}(x)\Sigma(x,y)A_{ext}(y).$$

The Fourier representation of the response function Σ is given by

$$\Sigma(x,y) = \int \frac{d^d q}{(2\pi)^d} e^{-iq(x-y)}\Sigma(q),$$

where

$$\Sigma(q) = \mathcal{P}(q) + \mathcal{P}(q)\Pi^{-1}(q)\mathcal{P}(q).$$

The operators under consideration satisfy the equations ($\Delta(q) \equiv K^{-1}(q)$)

$$\Pi^{-1}(q) = \Delta(q) + \Pi^{-1}(q)\mathcal{P}(q)\Delta(q),$$

and, as a consequence,

$$\Sigma(q) = \mathcal{P}(q) + \mathcal{P}(q)\Delta(q)\Sigma(q).$$

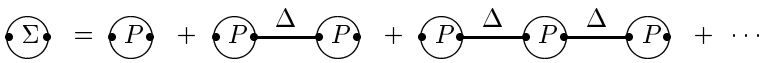
Let us now apply this formalism to the case of matter interacting with a Chern–Simons gauge field in the presence of an external electromagnetic field.

Suppose that the external field consists of background and small fluctuating parts and that the background generates a uniform magnetic field B. In its own turn, represent the Chern–Simons field as $\bar{a}_\mu(x) + a_\mu(x)$, where $\bar{b} = \epsilon_{ik}\bar{a}^k = -\frac{e}{\kappa}\rho$ is a Chern–Simons magnetic field. As a nonperturbed system, take matter interacting with the effective magnetic field $\mathcal{B}_{eff} = \bar{B} + \bar{b}$.

In order to calculate Σ, we need the polarization operator. In the lowest order approximation it is given by eq.(1.15)

$$P_{\mu\nu}(x,y) = e^2\Gamma_{\mu\nu} - \frac{e^2}{m} < 0|j_0(x)|0 > \delta(x-y)\delta_{\mu k}\delta_{\nu l}\delta_{kl}.$$

We see that the last term is exactly cancelled by the Schwinger term presented in the current correlator. As a result for the Fourier-transformed polarization operator, we can use (1.28) with $\Pi_3 = 0$. The diagrammatic representation of $\Sigma(q)$ is given by



The final result looks as follows

$$\begin{aligned}
D\Sigma_{00} &= \mathbf{q}^2\Pi_0, \\
D\Sigma_{0i} &= -q^i\omega\Pi_0 + i\epsilon_{ik}q^k\left[\Pi_1\left(1+\frac{1}{\kappa}\Pi_1\right) - \frac{1}{\kappa}\Pi_0 M\right], \\
D\Sigma_{ik} &= \delta_{ik}\omega^2\Pi_0 + \\
&\quad + \left[\delta_{ik}\mathbf{q}^2 - q_iq_k\right]\left[\Pi_2 + \frac{n_e}{m\kappa^2\mathbf{q}^2}\left(\Pi_0 M - \Pi_1^2\right)\right] + \\
&\quad + i\epsilon_{ik}\omega\left[\Pi_1\left(1+\frac{1}{\kappa}\Pi_1\right) + \frac{\pi}{N}\Pi_0 M\right].
\end{aligned}$$

Here

$$M = \omega^2\Pi_0 + \mathbf{q}^2\Pi_2,$$

and

$$D(q) = e^2\left(1+\frac{1}{\kappa}\Pi_1\right)^2 - \frac{e^2}{\kappa^2}\Pi_0\left(\omega^2\Pi_0 + \mathbf{q}^2\Pi_2\right). \qquad (1.47)$$

CHAPTER 2
SIMILARITY TRANSFORMATION AND INTERPLAY BETWEEN INTEGRAL AND FRACTIONAL QUANTUM HALL EFFECTS

1. Introduction

Experimentally the Quantum Hall Effect (QHE) is observed in two-dimensional electron systems at low temperatures and in strong magnetic fields. The low temperature is needed to freeze (quantum mechanically) the degree of freedom for motion in the perpendicular direction, and strong magnetic field forces the electrons to fill, from bottom to up, the Landau levels (for a review, see [118], [126]).

The Ohm's law for a free electron gas confined on a plane in the presence of a perpendicular magnetic field B is given by

$$J^i = \sigma_{ik} E^k,$$

where $\mathbf{J}$ is the electric current density, $\mathbf{E}$ is the electric field, and σ_{ik} is the conductivity matrix.

Classically, the Hall conductivity

$$\sigma_{xy} = -\sigma_{yx} = -\frac{n_e c e}{B}, \quad \sigma_{xx} = \sigma_{yy} = 0.$$

Here n_e is the electron density. The resistance matrix ρ_{ik} is the inverse of the conductivity matrix $\rho_{ik} = \sigma_{ik}^{-1}$, and its components are

$$\rho_{xx} = \rho_{yy} = 0, \qquad \rho_{xy} = \frac{B}{n_e c e}.$$

The last expression indicates that for an ideal electron gas, the transverse resistance linearly depends on the ratio B/n_e.

Elementary quantum mechanical consideration of a gas of free electrons moving in perpendicular magnetic and electric fields gives for the Hall conductivity the following result

$$\sigma_H \equiv \sigma_{xy} = \frac{e^2}{h}\nu,$$

where the filling fraction

$$\nu = \frac{\#\text{of electrons}}{\#\text{of Landau sites}} = \frac{n_e}{n_B}$$

($n_B = \frac{eB}{2\pi}$ is the density of quantum states per level). In physical terms, ν represents the number of filled Landau levels, and ν^{-1} is the magnetic flux per particle.

In the experimental setup, the charge density or the external magnetic field can be varied. A striking feature of experimental data was that at very low temperatures ($T < 1°\mathrm{K}$) and very strong magnetic field ($B > 10\,\mathrm{Tesla}$),

the Hall conductivity, plotted as a function of the filling fraction develops a series of plateaux at the values

$$\sigma_{xy} = \frac{e^2}{h}\nu, \quad \nu = \begin{cases} 1, 2, 3, \ldots (\pm 10^{-8}) & \text{IQHE} \\ \frac{1}{5}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{2}{3}, \ldots (\pm 10^{-5}) & \text{FQHE} \end{cases}.$$

More generally $\nu = p/q$, p and q being relatively prime integers with q odd. The purer the sample, the greater is the number of observed fractions, and there is more or less a definite order in which new fractions appear as the sample quality is improved.

The effect is not actually restricted to the lowest Landau level ($\nu < 1$) and the cases $\nu = 4/3, 7/3, 7/5, 8/5$, and so on, have been observed. Moreover, plateaux with an even denominator have been observed for $\nu = 5/2$. However, this is exceptional, and is attributed to a spin-unpolarized state in distinction to the ordinary spin-polarized fractionally quantized states.

At the same time, the longitudinal conductivity σ_{xx}, appeared to be essentially zero when σ_{xx} was nearly constant, and conversely when σ_{xy} varies, σ_{xx} is non zero.

The high precision (up to 10^{-8}) of the observed quantization of Hall conductance is deeply related to the fundamental principle of gauge invariance [101]. The arguments used are based on the idea of spectral flow: the adiabatic change of the Hamiltonian by its gauge equivalent does not change the spectrum as a whole, but can produce a charge transfer of an integer number of the charge carriers from one edge to the other. If, during this process, N elementary charge carriers of the charge fe move from one to another edge, the Hall conductivity will be

$$\sigma_{xy} = \frac{j_x}{E_y} = \frac{I_x}{V_y} = \frac{e^2}{h} Nf,$$

i.e.,

$$\nu = Nf$$

and the charge carrier carry the charge $\frac{\nu}{N}e$. In the case of IQHE, $N = \nu$ and $f = 1$, i.e., one electron is transferred from one edge to another for each fully occupied Landau level. In the case of FQHE, for the most exploited Laughlin series $\nu = \frac{1}{(2p+1)}$, $N = 1$, and elementary charged excitations are vortices with a fractional charge $e/(2p+1)$.

As it will be discussed below, FQHE can be interpreted as IQHE for the system of so called composite fermions, consisting of electrons carrying an even number of elementary magnetic flux quanta. This picture, proposed by Jain [86], can be realized in the framework of the Chern–Simons gauge theory, where the additional magnetic flux is produced by a statistical gauge field.

In the present chapter, we will consider an alternative version of Jain's approach, which exhibits an algebraic similarity between integral and fractional QHE's.

In Section 2, we get the conductivity tensor for a system of electrons interacting with the Maxwell and Chern–Simons fields. In Section 3, we review the properties of the Hall fluid. In Section 4, we formulate the similarity criterion and introduce the corresponding transformation. In Section 5, we develop the picture and discuss the composite fermion approach and FQHE wave functions. In our presentation, we follow the papers [42], [43], [44].

2. Hall Conductivity and Response Function

In the preceding chapter, we have obtained the expression for the effective action for the electromagnetic field interacting with matter. In its own turn, matter is supposed to interact with the Chern–Simons gauge field. In the mean field approximation, the initial Hamiltonian corresponds to particles moving in an effective magnetic field created by external currents as well as by the Chern–Simons background magnetic field.

The effective action is given by

$$S_{\text{eff}}[\mathcal{A}_\mu] = \frac{1}{2}\int d^3x d^3y \mathcal{A}_\mu(x)\Sigma^{\mu\nu}(x,y)\mathcal{A}_\nu(y).$$

The electromagnetic current

$$J^\mu(x) = -\frac{\delta S_{\text{eff}}}{\delta \mathcal{A}_\mu(x)} = -\int d^3y \Sigma^{\mu\nu}(x,y)\mathcal{A}_\nu(y).$$

In the momentum space, one has

$$J^\mu(q) = -\Sigma^{\mu\nu}(-q)\mathcal{A}_\nu(q).$$

The electromagnetic response is defined by

$$\begin{aligned}
\Sigma^{i0}(-q) =& \omega q^i \Sigma_0(-\omega,\mathbf{q}^2) - i\epsilon^{im}q^m\Sigma_1(-\omega,\mathbf{q}^2),\\
\Sigma^{ik}(-q) =& \delta^{ik}\omega^2\Sigma_0(-\omega,\mathbf{q}^2) - i\epsilon^{ik}\omega\Sigma_1(-\omega,\mathbf{q}^2) + \left[\delta^{ik}\mathbf{q}^2 - q^iq^k\right]\Sigma_2(-\omega,\mathbf{q}^2).
\end{aligned}$$

The Fourier representation of the electric field is

$$E^k(q) = i\left(q^m\mathcal{A}_0(q) - \omega\mathcal{A}^m(q)\right).$$

As a result,

$$\begin{aligned}
J^i(q) = i\omega\Sigma_0(-q)E^i(q) + \epsilon^{ik}\Sigma_1(-q)E^k(q) -\\
- \Sigma_2(-q)\left[\delta^{ik}\mathbf{q}^2 - q^iq^k\right]\mathcal{A}_k(q).
\end{aligned}$$

The conductivity tensor is given by (see, e.g., [94])

$$\sigma_{xx} = \lim_{\omega\to 0}\left(i\omega\Sigma_0(-\omega,0)\right),$$

$$\sigma_{xy} = \lim_{\omega \to 0} \Sigma_1(-\omega, 0).$$

From the previous chapter, we know that in the low energy limit (see (7)),

$$\Sigma_0(0) = \frac{e^2 \Pi_0(0)}{(1 + \frac{1}{\kappa}\Pi_1(0))^2}, \quad \Sigma_1(0) = \frac{e^2 \Pi_1(0)}{1 + \frac{1}{\kappa}\Pi_1(0)}.$$

In the mean field approximation, when the ground state corresponds to N totally filled Landau levels, the threshold values of the structure functions is given by

$$\Pi_1(0) = \frac{N}{2\pi}.$$

In the sequel, it will be shown that $\kappa = \frac{1}{4\pi p}$ (p is an integer). As a result,

$$\sigma_{xx} = 0, \quad \sigma_{xy} = \frac{e^2}{2\pi} \frac{N}{2Np+1}.$$

Consequently, for the filling fraction one gets (in units, where $\hbar = 1$)

$$\nu = \frac{N}{2Np+1}.$$

These fractions correspond to the generalized Laughlin series.

3. Noncompressible Fluid and $\mathcal{W}$ Symmetry

The IQHE can be understood using the picture of noninteracting electrons filling from bottom to up exactly ν Landau levels. In the picture of free electrons exactly filling up the Landau levels, there must exist an energy gap in the charged particle spectrum. Indeed, the energy to create a separated particle-hole pair should be just the energy to excite a fermion into the lowest empty Landau level, *viz.*$E_{pair} = eB/m$. This predicts that the many-body ground state at the plateaux have a uniform density and gap for a density waves.

The FQHE ground state cannot be obtained from non-interacting electrons by continuously turning on interaction and must be a new type of many body condensate. The most important common feature of FQH and IQH states turns out to be that both of them are incompressible fluid states. In the case of IQHE, incompressibility can be understood in terms of completely filled Landau levels and Fermi statistics.

However, at fractional values of ν, which corresponds to only partially filled Landau levels, the incompressibility is a non-trivial property that originates from the interelectron interactions.

To the many-body system corresponds energy operator, which takes into account the inter-electron Coulomb interactions as well as the presence of a perpendicular strong magnetic field:

$$H(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \frac{1}{2m} \sum_{I=1}^{N} \left(\mathbf{p}_I - \frac{e}{c\hbar} \mathbf{A}(\mathbf{r}_I) \right)^2 - \frac{1}{2} \sum_{I<J} \frac{e^2}{|\mathbf{r}_I - \mathbf{r}_J|^2}.$$

In this expression, $\mathbf{A}(\mathbf{r})$ is a two - dimensional electromagnetic potential creating the magnetic field $B = \epsilon_{ik}\partial_k A^k$. The second term is the static interaction. In two dimensions, the Coulomb potential is logarithmic. However, in the above two-dimensional systems, the ordinary $3D$ Coulomb potential can be used because $2D$ electron systems have a small but finite thickness. Only their motion at low energies is two-dimensional.

When the fraction is $\nu = \frac{1}{2p+1}$, the ground state can be described very accurately by the variational wave function proposed by Laughlin [99]

$$\Psi_p^e = \mathcal{N}_p \prod_{I<J} (z_I - z_J)^{2p+1} e^{-\frac{eB}{4\hbar c}\sum_I |z_I|^2}, \tag{2.1}$$

where z_i is the complex coordinate for the i-th electron and $\mathcal{N}_p$ is a normalization factor. Mathematical features and physical meaning of LWF are the following:

- The prefactor $f(z_1, \ldots, z_N) = \prod_{i<j}(z_i - z_j)^{2p+1}$ is purely analytic, which means that all particles are in the lowest Landau level.
- The prefactor of Jastrow form: it has a zero of order $2p+1$ at coincident points, showing that electrons tend very strongly to repel each other. Each electron sees $2p+1$ zeros bound to the positions of other electrons, as if each particle carries $2p+1$ flux quanta.
- The total angular momentum

$$J = \sum_{k=1}^{N} j_k = (2p+1)\frac{1}{2}N(N-1)$$

  is a good quantum number, which is proportional to the area occupied by the system.

The interpretation of the Laughlin wave function as a FQHE ground state is based on the plasma analogy and is justified by the very high accuracy overlap with the numerically calculated wave functions.

Despite the above mentioned differences between integral and fractional QHE, they can be viewed from unified position. As it will be argued in the present chapter, IQHE and FQHE can be considered as non-unitary equivalent realizations of the same underlying symmetry – infinite dimensional $W_{1+\infty}$.

The main idea permitting to treat IQHE and FQHE at the same footing is related to the physical picture of incompressible quantum Hall fluid. In an incompressible fluid, the fluctuations in the bulk induce fluctuations at the boundary. The relevant degrees of freedom are given by small fluctuations which are localized at the edge – the so called edge density waves [72], propagating around the edges of the sample. Furthermore, it was shown that the spectrum of edge excitations converges towards the one of the theory of relativistic chiral fermions in 1+1 dimensions. (Wen has shown [133] that the spectrum of edge states is determined by level-one $U(1)$ Kac-Moody algebra).

At the same time, all possible configurations of a droplet of uniform density can be obtained by deformation which preserves the area, and the mathematical description of the deformations of an incompressible liquid droplet can be given in terms of a group of area preserving diffeomorphisms [20], [23], [81]. The corresponding algebraic structures are generalizations of the well-known Virasoro algebra, with additional generators induced by extended conformal symmetries. W_N is a conformal algebra which contains the generators of integral conformal spins $2, 3, \ldots N$. ($W_2 = V$ is the Virasoro algebra). W_∞ can be viewed as the $N \to \infty$ limits of finite-N W_N algebras, and contains an infinite number of generating currents of conformal spin $3, 4, \ldots \infty$, in addition to the spin-2 stress tensor of Virasoro [8], [124].

In what follows, we exploit the following particular limit known as w_∞

$$[v_m^i, v_n^j] = [(j+1)m - (i+1)n]v_{m+n}^{i+j}.$$

This algebra can be enlarged to $w_{1+\infty}$ with conformal spins $s = i + 2 \geq 1$, simply by allowing the indices i and j to take the value -1 as well as the non negative integers.

The resulting algebra admits a geometrical interpretation as the algebra of area-preserving diffeomorphisms.

The quantum version of $w_{1+\infty}$ is known as $W_{1+\infty}$. It reads

$$[V_n^i, V_m^j] = (jn - im)V_{n+m}^{i+j-1} + q(i.j, m, n)V_{n+m}^{i+j-3} + \cdots + c^i(n)\delta^{i,j}\delta_{n+m,0}.$$

Here $i + 1 = h \geq 1$ represents the conformal spin of the generator V_n^i, while $-\infty < n < +\infty$ is the angular momentum. The first term on the right hand side reproduces the classical w_∞ algebra by the correspondence $v_{i-n}^i \to V_n^i$ and identifies $W_{1+\infty}$ as the algebra of "quantum area-preserving diffeomorphisms". The additional terms are quantum operator corrections with polynomial coefficients $q(i, j, n, m)$ due to the algebra of higher derivatives. The c-number term $c^i(n)$ is the quantum anomaly.

4. Infinite Symmetry and Similarity Transformation

The current understanding of the quantum Hall effect is essentially based on the Laughlin's picture of the incompressible two-dimensional quantum fluid which exhibits an energy gap [99], [14], [56], [57].

As we have already noted, the notion of incompressibility can be related to an infinite symmetry, which on the classical level is represented by the group of area preserving diffeomorphisms [20], [21], [81].

As an outcome, a two-dimensional quantum fluid can be characterized by the unitary irreducible highest weight representations of the $W_{1+\infty}$ algebra [22].

The derivation of this basic conclusion is straightforward for the IQHE, when liquid is formed by the non-interacting planar electrons in the lowest

Landau level ($\nu = 1$). For the clarity and to fix the notations, we will reproduce some essential points.

In the appropriately chosen system of units ($c = \hbar = m = 1, e = 2, B = 1$), and symmetric gauge $\mathbf{A} = \frac{B}{2}(-y, x)$, the quantum-mechanical Hamiltonian and angular momentum of N electrons in the orthogonal uniform magnetic field $B = \epsilon_{ik}\partial_i A^k$ ($i, k = 1, 2$) can be written in terms of harmonic oscillator operators

$$\hat{H} = \frac{1}{2m}\sum_{I=1}^{N}\left[\mathbf{p}_I - e\mathbf{A}(\mathbf{r}_I)\right]^2 = \sum_{I=1}^{N}\left(a_I a_I^+ + a_I^+ a_I\right),$$
$$\hat{J} = \sum_{I=1}^{N}\left(z_i\partial_I - \bar{z}_I\bar{\partial}_i\right) = \sum_{I=1}^{N}\left(b_i^+ b_I - a_I^+ a_I\right).$$

In the complex notations

$$z = x + iy, \quad \partial = \frac{1}{2}(\partial_x - i\partial_y),$$

these operators are given by

$$a_I = \frac{z_I}{2} + \bar{\partial}_I \qquad a_k^+ = \frac{\bar{z}_I}{2} - \partial_I,$$
$$b_I = \frac{\bar{z}_I}{2} + \partial_I \qquad b_I^+ = \frac{z_I}{2} - \bar{\partial}_I,$$
$$\left[a_I, a_J^+\right] = \left[b_I, b_J^+\right] = \delta_{IJ}.$$

The $W_{1+\infty}$ is generated by the operators

$$v_n^i = -\sum_{I=1}^{N}(b_I^+)^{n+i}(b_I)^i, \quad i \geq 0, \quad n + i \geq 0,$$

which commute with the Hamiltonian and satisfy the commutation relations

$$\left[v_n^i, v_m^j\right] = (jn - im)v_{n+m}^{i+j-1} + \cdots, \tag{2.2}$$

where ellipses correspond to the quantum deformations [20].

The $\nu = 1$ ground state is given by the wave function

$$\Psi_0(z_1, \ldots, z_N) = \prod_{1 \leq I < K \leq N}(z_I - z_K)e^{-1/2\sum_J |z_J|^2}, \tag{2.3}$$
$$\hat{H}\Psi_0 = E_0\Psi_0 = N\Psi_0,$$
$$\hat{J}\Psi_0 = \frac{N(N-1)}{2}\Psi_0.$$

The action of generators v_n^i on this state can be easily calculated (especially if one uses the second quantization formalism). The basic results are as

follows [22]:

$$\begin{aligned} &a) \quad v_n^i \Psi_0 = 0 \\ &\qquad\qquad for \qquad\qquad -i \le n < 0, i \ge 1 \\ &b) \quad v_0^i \Psi_0 = const \cdot \Psi_0 \\ &c) \quad v_n^i \Psi_0 = \Phi_n^i(z_1, \ldots, z_N) \cdot \Psi_0 \qquad\qquad for \qquad\qquad n \ge 0, i \ge 1. \end{aligned} \tag{2.4}$$

Here $\Phi_n^i(z_1, \ldots, z_N)$ is some symmetric polynomial.

The equality $a)$ in (2.4) is the highest weight condition which is a mathematical transcription of the incompressibility. $b)$ and $c)$ characterize the excitation spectrum.

The situation is drastically changed in the case of fractional fillings. Now the ground state (for $\nu = \frac{1}{2p+1}$, p-integer) is given by the Laughlin wave function

$$\Psi_p(z_1, \ldots, z_n) = \prod_{1 \le K < L \le N} (z_k - z_l)^{2p+1} e^{-1/2 \sum_I |z_I|^2} \tag{2.5}$$

and is believed to describe the incompressible state of interacting electrons.

Now if one wants to construct an algebraic classification of a quantum fluid, the ground state (2.5) must be subjected to the action of the symmetry generators. In order to carry out these calculations, in the paper [54] the authors have changed the definition of operators b_k by introducing an interaction term

$$b_K \Longrightarrow B_K = b_K - 2p \sum_{I \ne K} \frac{1}{z_K - z_I}. \tag{2.6}$$

Note that b_K^+ is not changed

$$b_K^+ \Longrightarrow B_K^+ = b_K^+. \tag{2.7}$$

The infinite symmetry is generated by the operators

$$V_n^i = -\sum_{K=1}^{N} (B_K^+)^{n+i} (B_K)^i$$

which satisfy the same algebra as v_n^i in (2.2) up to the terms involving delta-functions. These terms can be ignored, because the wave functions vanish as $z_K \to z_I$. As a result, it can be shown that V_n^i acts on Ψ_p as on the highest weight state.

The operators B_K and B_K^+ are not Hermitian conjugate. This will be improved if one introduces a new integration measure in the configuration space, i.e.,

$$dz_1 \cdots dz_N \Longrightarrow dz_1 \cdots dz_N \mu(z, \bar{z}),$$

where [130]

$$\mu(z,\bar{z}) = \prod_{I<J} |z_I - z_J|^{-4p},$$

and simultaneously changes the definition of operators a_K and a_K^+ in the following way:

$$a_K \Longrightarrow A_K = a_K,$$
$$a_K^+ \Longrightarrow A_K^+ = a_K^+ + 2p \sum_{L\neq K} \frac{1}{z_K - z_L}.$$

Note that the newly introduced operators B_K act on the ground state of interacting electrons Ψ_p in a way analogous to the action of b_K's on Ψ_1:

$$b_K \Psi_1 = \sum_{K\neq J} \frac{1}{z_K - z_J} \Psi_0,$$
$$B_K \Psi_p = \sum_{K\neq J} \frac{1}{z_K - z_J} \Psi_p.$$

It seems that this circumstance has initiated the *Ansatz*-type substitutions (2.6)–(2.7), which in turn leads to the introduction of the measure $\mu(z,\bar{z})$ and operators A_K and A_K^+.

Now we can make a simple observation, which perhaps clarifies the meaning of this procedure [42]: wave functions and algebra generating operators for the fractional ($\nu = \frac{1}{2p+1}$) and integer ($\nu = 1$) filling fractions are related by the following similarity transformation

$$\Psi_p(z_1,\ldots,z_N) = S_p(z_1,\ldots,z_N)\Psi_0(z_1,\ldots,z_N), \tag{2.8}$$
$$\hat{O}_p = S_p(z_1,\ldots,z_N)\hat{O}_0 S_p^{-1}(z_1,\ldots,z_N), \tag{2.9}$$

where

$$S_p(z_1,\ldots,z_N) = \prod_{k<l} (z_k - z_l)^{2p}. \tag{2.10}$$

(2.8) is evident (S_p is a mapping operator T_{n-m} between the ground states corresponding to different filling fractions [3]), and (2.9) can be easily verified by the direct calculations, letting $\hat{O}_0 = \{b_K, b_K^+, a_K, a_K^+, v_n^i\}$ and $\hat{O}_p = \{B_K, B_K^+, A_K, A_k^+, V_n^i\}$, respectively.

Following the scheme of algebraic classification [22] all the essential information about a Hall fluid for the fractional filling is encoded in the action of symmetry generators V_n^i on the highest weight state Ψ_p, which due to (2.8) - (2.10) can be simply deduced from (2.4):

$$V_n^i \Psi_p = S_p \cdot v_n^i \Psi_0.$$

In particular, one automatically obtains the highest weight condition

$$V_n^i \Psi_p = 0, \quad for \quad -i \leq n < 0, i \geq 1.$$

Transformation (2.10) becomes singular as $z_K \to z_L$ but it acts in the space of functions which vanish in that limit. What seems to be more important, is that it is not an unitary transformation:

$$S_p^\dagger S_p = \prod_{K<L} |z_K - z_L|^{4p} = \mu(z,\bar{z})^{-1}.$$

The last equality is not accidental. In the Hilbert space where the operators $\hat{O}_0$ and $\hat{O}_0^\dagger$ act. the Hermitian conjugation is defined by the scalar product

$$\langle \Psi | \hat{O}_0^\dagger | \Phi \rangle = \overline{\langle \Phi | \hat{O}_0 | \Psi \rangle}. \tag{2.11}$$

It is evident, that the operators $\hat{O}_p$ and $\hat{O}_p^\dagger = (S_p^{-1})^\dagger O_0^\dagger S_p^\dagger$ are not Hermitian conjugate in the sense of (2.11). Introduce a metric operator $\hat{\eta}$ and define a new scalar product

$$\langle \Psi | \hat{\eta} \hat{O}_p^\dagger | \Phi \rangle = \overline{\langle \Phi | \hat{\eta} \hat{O}_p | \Psi \rangle}.$$

This operator is given by

$$\hat{\eta} = (S_p^{-1})^\dagger \cdot S_p^{-1} = \mu(z,\bar{z}).$$

The transformations (2.8)–(2.10) interconnect the ground state vectors and spectrum generating quantum operators corresponding to two different physical phenomena: IQHE can be understood using a picture of non-interacting electrons, while FQHE is essentially a manifestation of inter-electron interactions. On the other hand, one can say that from the point of view of algebraic classification in the sense of [22], the IQHE and FQHE are non-unitary equivalent realizations of the same underlying symmetry.

Evidently Ψ_p is an eigenfunction of the transformed Hamiltonian

$$\hat{H}_p = \hat{H} + 4p \sum_{I \neq K} \frac{1}{z_I - z_K} a_I + 2p\pi \sum_{I \neq K} \delta(z_I - z_K)$$

which must be considered as a Hamiltonian of an interacting electron system.

Similar consideration relates the $\nu = m$ states and operators to the representation of $W_{1+\infty}$ at $\nu = \frac{m}{2mp+1}$. The corresponding similarity transformation is given by

$$S_{p,m} = \prod_{A<B} \prod_{I<J} (z_I^A - z_J^B)^{K_{A,B}} \prod_A \prod_{I<J} (z_I^A - z_J^B)^{K_{A,A}-1},$$

where the $m \times m$ matrix K_{AB} is defined by [57]

$$K = \begin{vmatrix} 2p+1 & 2p & \ldots & 2p \\ 2p & 2p+1 & \ldots & 2p \\ . & . & . & . \\ 2p & \ldots & 2p & 2p+1 \end{vmatrix}.$$

It is interesting to note that non-unitary similarity transformations recently have been considered in the context of the quantum gravity, where they are related to the temporal evolution between unstable quantum backgrounds, indicating a deep connection between the string quantum gravity and incompressible Hall fluid [48].

Note that the non-unitary transformations (2.9) induce non-canonical, complex transformations of the phase space variables – complex coordinates $z_K = x_K + iy_K$ and conjugated momenta $\bar{p}_K = \frac{1}{2}(p_{Kx} - ip_{Ky})$:

$$z_k \to S_p z_k S_p^{-1} = z_k, \quad \bar{p}_k \to S_p \bar{p}_k S_p^{-1} = \bar{p}_k + i2p \sum_{l \neq k} \frac{1}{z_k - z_l},$$
$$\bar{z}_K \to S_p \bar{z}_K S_p^{-1} = \bar{z}_K, \quad p_K \to S_p p_K S_p^{-1} = p_K.$$

The substitutions $p \to p - \frac{e}{2}f$, $\bar{p} \to \bar{p} - \frac{e}{2}\bar{f}$ can be interpreted as the introduction of complex, non-local vector potentials

$$f_K(\mathbf{r}_1, \ldots, \mathbf{r}_N) \equiv f_{Kx} + if_{Ky} = 0, \tag{2.12}$$
$$\bar{f}_k(\mathbf{r}_1, \ldots, \mathbf{r}_N) \equiv f_{kx} - if_{ky} = -i2p \sum_{l \neq k} \frac{1}{z_k - z_l}, \tag{2.13}$$

which depend on the positions of all N particles.

The magnetic field associated to these potentials which acts on the k - th particle, is given by the curl

$$\mathcal{B}_K = i\left(\bar{\partial}_K \bar{f}_K - \partial_K f_K\right) = 2p\pi \sum_{K \neq L} \delta(z_K - z_L),$$

i.e., each particle sees the $N-1$ others as vortices carrying $2p$ elementary flux quanta.

In the chosen system of units, flux quantum $\phi_0 = \pi$, and the density of Landau states $n_B = \frac{eB}{2\pi} = 1/\pi$. Hence the filling fraction

$$\nu = \frac{N}{(\Phi/\phi_0)} = \frac{1}{2p+1},$$

where the total flux $\Phi = \pi N(2p+1)$.

Using mean-field arguments, one can say that electrons move in the average magnetic field $2p+1$, in accordance with Jain's hierarchical construction [86]. However, the additional magnetic field $\mathcal{B} = 2p$ is generated now by complex gauge potentials, in contrast to the composite fermion approach [87], where magnetic fluxes attached to point particles are produced by the real singular vector potentials

$$\vec{\mathcal{A}}_K = p\vec{\nabla}_K \sum_{L \neq K} \varphi_{KL}, \quad \varphi_{KL} = -i \arg(z_K - z_L),$$

or, in complex notation,

$$\mathcal{A}_K = ip \sum_{L \neq K} \frac{1}{\bar{z}_K - \bar{z}_L}, \quad \bar{\mathcal{A}}_K = -ip \sum_{L \neq K} \frac{1}{z_K - z_L}. \tag{2.14}$$

The corresponding ground state is known to be [87], [100]

$$\Phi_p = \prod_{K<L} \frac{(z_K - z_L)^{2p}}{|z_K - z_L|^{2p}} \Psi_0(z_1, \ldots, z_N) \equiv$$
$$\equiv U_p(\mathbf{r}_1, \ldots, \mathbf{r}_N) \Psi_1(z_1, \ldots, z_N)$$

which contains one particle states from the higher Landau orbitals and its energy is higher than the FQHE ground state energy.

Note that the gauge potentials (2.14) can be introduced as a singular unitary transformation

$$U_p p_K U_p^\dagger = p_K - \frac{e}{2} \mathcal{A}_K,$$
$$U_p \bar{p}_K U_p^\dagger = \bar{p}_K - \frac{e}{2} \bar{\mathcal{A}}_K.$$

The last remark can be related to the equivalence between a system of electrons bounded to the even number of magnetic flux quanta and the same system without these fluxes [108]. From our consideration it follows that these two theories can be related by a unitary operator U_p as well as by a similarity transformation S_p. In the former case, the equivalence assertion given in [108] is in fact a quantum-mechanical unitary equivalence. At the same time, the non-unitary character of S_p is a loophole which enables to evade the consequences of the equivalence statement, and reduces the study of FQHE of the mutually interacting electrons to the IQHE of non-interacting composite particles.

The potentials (2.12), (2.13) and (2.14) have a form typical for a statistical interaction with a parameter $\theta = 4p\pi$ (see, e.g., [105]) and naturally can be incorporated into the framework of the Chern–Simons theories.

Introduce the particle density at the point $\mathbf{r}$:

$$\varrho(\mathbf{r}) = \sum_{l=1}^{N} \delta(\mathbf{r} - \mathbf{r}_l),$$

and vector potentials satisfying

$$\epsilon_{ik} \partial_i f^k(\mathbf{r}) = 2p\pi \varrho(\mathbf{r}) \tag{2.15}$$

and

$$\partial_i f^i(\mathbf{r}) = -i 2\pi p \varrho(\mathbf{r}). \tag{2.16}$$

The solutions to (2.15)-(2.16) can be easily found:

$$f^i(\mathbf{r}) = -ip\partial_i \int d\mathbf{r}' \ln(z-z')\varrho(\mathbf{r}'). \tag{2.17}$$

Here $G_c(\mathbf{r}) = \frac{1}{2\pi}\ln r$ is a Green's function, and $\varphi(z) = -i\arg z$.

Substituting the particle density into (2.17) and letting $\mathbf{r} = \mathbf{r}_k$, we immediately recover (2.12) and (2.13).

Note that the potential (2.14) can be written in an analogous form

$$\mathcal{A}^i(\mathbf{r}) = -2p\pi\epsilon_{ik}\partial_k \int d\mathbf{r}' G_c(\mathbf{r}-\mathbf{r}')\varrho(\mathbf{r}') = p\partial_i \int d\mathbf{r}'\varphi(z-z')\varrho(\mathbf{r}'). \tag{2.18}$$

At the same time, (2.15) is a field equation for the Chern-Simons Lagrangian

$$\begin{aligned} &\mathcal{L} = i\psi^\dagger(\partial_0 + iea_0)\psi - \frac{1}{2}(D_k\psi^\dagger)(D_k\psi) - \frac{e^2}{8\pi p}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda, \\ &D_i\psi = (\partial_i + ieA_i + iea_i)\psi, \quad D_i\psi^\dagger = (\partial_i - ieA_i - iea_i)\psi^\dagger. \end{aligned} \tag{2.19}$$

The solutions given by (2.17) and (2.18), produce a magnetic field

$$b(\mathbf{r}) = 2p\pi\sum_{l=1}^{N}\delta(\mathbf{r}-\mathbf{r}_l),$$

and are related by the complex gauge transformation

$$f^k(\mathbf{r}) = \mathcal{A}^k(\mathbf{r}) - ip\partial_k \int d\mathbf{r}' \ln|z-z'|\varrho(\mathbf{r}').$$

In conclusion, we can say the following. According to [22] the quantum states of an incompressible fluid can be exhaustively classified by unitary irreducible highest weight representations of the algebra $W_{1+\infty}$. Applying to the representation at ν =integer the similarity transformation (2.8) or (2.10), one automatically (at least in principle) obtains the corresponding classification for the fractional values of filling fraction. This transformation seems to be equivalent to the introduction of a complex abelian C-S gauge potentials in terms of which a field-theoretic description of FQHE can be given.

5. On the Composite Fermion Approach in the FQHE

Practically all the essential information about the quantum Hall effect can be encoded analytically in the form of the Laughlin wave function (2.1), which describes the incompressible ground state of N_e spin-polarized planar electrons moving in the orthogonal magnetic field B. At the same time, two-dimensional Hall fluid can be classified by the unitary irreducible representations of the infinite dimensional algebra $W_{1+\infty}$, where Lauglin state is a highest weight vector [22]. The analytic and algebraic aspects of QHE are

supplemented in the physically transparent way by Jain's composite electron picture [86], [87], [88]. In this picture, FQHE is related to a system of non-interacting composite particles consisting from electrons bound to the magnetic fluxes $2p\phi_0$ ($\phi_0 = \frac{ch}{e} = \frac{2\pi}{e}$).

These fictitious magnetic fluxes can be associated with the singular gauge potential $\mathbf{a}(\mathbf{r})$ such that the magnetic field

$$b(\mathbf{r}) = \epsilon_{ik}\partial_i a^k(\mathbf{r}) = 2p\phi_0\varrho(\mathbf{r}), \tag{2.20}$$

where $\varrho(\mathbf{r})$ is the particle density.

The statistical gauge field

$$a^i(\mathbf{r}) = -i\frac{p\phi_0}{\pi}\partial_i \int d\mathbf{r}' \arg(z - z')\rho(\mathbf{r}') \tag{2.21}$$

provides the required amount of magnetic flux [137], but the corresponding ground state turns out to be compressible and can not be related to FQHE [100].

As an alternative, there exists another gauge potential satisfying (2.20), the Knizhnik-Zamolodchikov connection [96]:

$$a^i(\mathbf{r}) = -i\frac{p\phi_0}{\pi}\partial_i \int d\mathbf{r}' \ln(z - z')\varrho(\mathbf{r}'). \tag{2.22}$$

The first-quantized form of covariant derivatives with this connection

$$D_I = \frac{\partial}{\partial z_I} - 2p\sum_{J\neq I}\frac{1}{z_I - z_J}, \quad \bar{D}_I = \frac{\partial}{\partial \bar{z}_I} \tag{2.23}$$

has been used in [130] to study the scattering problem for particles obeying braid statistics, and in [54] to construct the Hamiltonian and $W_{1+\infty}$ algebra generators for the Laughlin function as a highest weight state.

It is not difficult to notice that these covariant derivatives can be presented in the form of the similarity transformation [42], [43],

$$D_I = S_p(z_1, \cdots, z_{N_e})\frac{\partial}{\partial z_I}S_p^{-1}(z_1, \cdots, z_{N_e}),$$
$$\bar{D}_I = S_p(z_1, \cdots, z_{N_e})\frac{\partial}{\partial \bar{z}_I}S_p^{-1}(z_1, \cdots, z_{N_e}),$$

where

$$S_p(z_1, \cdots, z_N) = \prod_{K<L}(z_K - z_L)^{2p}$$

is a singular non-unitary operator. This observation is helpful in expressing the wave function and quantum operators for the fractional value of the filling factor as a similarity transformation of the corresponding quantities for the non-interacting quasi-particle system.

Note that analogous transformations earlier have been introduced as a mapping operator between ground states with different filling factors [3], [91].

In this section, we will consider the second-quantized version of this transformation. Introduce the fermion Hamiltonian

$$H = \frac{1}{2m}\int d\mathbf{r}\,(\partial_k - ieA_k(\mathbf{r}))\,\chi^\dagger(\mathbf{r})\,(\partial_k + ieA_k(\mathbf{r}))\,\chi(\mathbf{r}) \qquad (2.24)$$

and define the transformed fields

$$\psi(\mathbf{r}) = S_p\chi(\mathbf{r})S_p^{-1} = e^{2p\int d\mathbf{r}'\ln(z-z')\varrho(\mathbf{r}')}\chi(\mathbf{r}),$$
$$\psi^\star(\mathbf{r}) = S_p\chi^\dagger(\mathbf{r})S_p^{-1} = \chi^\dagger(\mathbf{r})e^{-2p\int d\mathbf{r}'\ln(z-z')\varrho(\mathbf{r}')}.$$

The operator S_p can be represented as $S_p = e^{G_p}$, where the generator G_p is a singular quadratic functional of the density operator (see Appendix B)

$$\varrho(x) = \psi^\star(x)\psi(x) = \chi^\dagger(x)\chi(x).$$

It is not difficult to show that the transformed fields obey the normal Fermi statistics

$$\chi(\mathbf{r}_1)\chi(\mathbf{r}_2) = (-1)e^{\omega(z_2-z_1)-\omega(z_1-z_2)}\chi(\mathbf{r}_2)\chi(\mathbf{r}_1) = (-1)e^{-2ip\pi}\chi(\mathbf{r}_2)\chi(\mathbf{r}_1),$$
$$\chi(\mathbf{r}_1)\chi^\dagger(\mathbf{r}_2) = \delta(\mathbf{r}_1-\mathbf{r}_2) - e^{\omega(z_1-z_2)-\omega(z_2-z_1)}\chi^\dagger(\mathbf{r}_2)\chi(\mathbf{r}_1),$$
$$= \delta(\mathbf{r}_1-\mathbf{r}_2) - e^{2ip\pi}\chi_c^\star(\mathbf{r}_2)\chi_c(\mathbf{r}_1).$$

Here

$$\omega(z_1-z_2) = -2p\ln(z_1-z_2).$$

Following [86], we will interprete $\psi(\mathbf{r})$ as an electron field, and associate $\chi(\mathbf{r})$ with composite particles carrying an even $(2p)$ number of magnetic flux quanta. In terms of the electron field, the Hamiltonian (2.24) reads as

$$H = \frac{1}{2m}\int d\mathbf{r}\,(\partial_k\psi^\star - ie\psi^\star A_k - ie\psi^\star a_k)\,(\partial_k\psi + ieA_k\psi + iea_k\psi)\,,$$

where

$$a_k(\mathbf{r}) = i\frac{2p}{e}\partial_k\int d\mathbf{r}'\ln(z-z')\varrho(\mathbf{r}').$$

In the Fock space where the operators $\chi(\mathbf{r})$ and $\chi^\dagger(\mathbf{r})$ act, the Hermitian conjugation is defined by the scalar product

$$\langle\Phi_1|\hat{O}^\dagger|\Phi_2\rangle = \overline{\langle\Phi_2|\hat{O}|\Phi_1\rangle}. \qquad (2.25)$$

It is evident that the operators $\psi(\mathbf{r})$ and $\psi^\star(\mathbf{r})$ are not Hermitian conjugate in the sense of (2.25). Introduce the metric operator $\hat{\eta}$ and define a new scalar product

$$\langle\Phi_1|\hat{\eta}\psi^\star(\mathbf{r})|\Phi_2\rangle = \overline{\langle\Phi_2|\hat{\eta}\psi(\mathbf{r})|\Phi_1\rangle}.$$

This operator will be given by

$$\hat{\eta} = (S_p^{-1})^\dagger \cdot S_p^{-1},$$

and its matrix element in the N-particle subspace coincides with the integration measure (4) introduced in [130], [54]. (These aspects will be discussed in the next chapter.)

The coordinate representation bra- and ket vectors are generated by the action of the physical electron field $\psi(\mathbf{r})$ on the vacuum (which is not changed under the action of S_p), and are defined by the expressions

$$\langle z_1, \ldots, z_N| = \langle 0|\psi(1)\psi(2)\cdots\psi(N) =$$
$$= \langle 0|e^{2p\sum_{K=1}^{N}\int dr \ln(z-z_K)\varrho(r)} \prod_{J<L} e^{2p\ln(z_J-z_L)}\chi(1)\chi(2)\cdots\chi(N) =$$
$$= \prod_{K<L}(z_K - z_L)^{2p}\langle 0|\chi(1)\cdots\chi(N)$$

and

$$|z_1, \ldots, z_N\rangle = \psi^\star(N)\cdots\psi^\star(2)\cdot\psi^\star(1)|0\rangle =$$
$$= \chi^\dagger(N)\cdots\chi^\dagger(2)\chi^\dagger(1)\prod_{J<L} e^{-2p\ln(z_J-z_L)}e^{-2p\int dr[\sum_{K=1}^{N}\ln(z-z_K)]\varrho(\mathbf{r}')}|0\rangle =$$
$$= \prod_{K<L}(z_K - z_L)^{-2p}\chi^\dagger(N)\cdots\chi^\dagger(1)|0\rangle.$$

The quasi-particle field satisfy the Schrödinger equation for the fermion in the uniform magnetic field. Expand $\chi(\mathbf{r})$ into modes

$$\chi(\mathbf{r}) = \chi_0(\mathbf{r}) + \tilde{\chi}(\mathbf{r}),$$

where

$$\chi_0(\mathbf{r}) = \sum_{j=0}^{N_B-1} f_j u_j(\mathbf{r})$$

contains only the lowest Landau level wave functions. For the disk geometry and the symmetric gauge $\mathbf{A} = \frac{B}{2}(-x, y)$, they are angular momentum eigenfunctions

$$u_j(\mathbf{r}) \sim z^j e^{-\frac{eB}{4}|z|^2}. \tag{2.26}$$

The Fock space operators satisfy usual fermionic anticommutation relations

$$\{f_j, f_l^+\} = \delta_{jl}.$$

The modes corresponding to the lowest Landau level satisfy

$$\Big(\frac{\partial}{\partial\bar{z}} + \frac{eB}{4}z\Big)\chi_0(\mathbf{r}) = 0,$$

and the similarity transformation does not cause the level mixing

$$\Big(\frac{\partial}{\partial\bar{z}}+\frac{eB}{4}z\Big)S_p\chi_0(\mathbf{r})S_p^{-1}=0.$$

The ground state of the Hamiltonian H is extremely degenerate. All the N-particle states of the form

$$|N\rangle=f_{j_1}^+f_{j_2}^+\cdots f_{j_N}^+|0\rangle$$

have the same energy. One can select a particular ground state applying Bogolubov's concept of quasi-averages [16]. Following this method, modify the Hamiltonian by the infinitesimal perturbation which lifts the degeneracy, and find the unique ground state. After performing necessary calculations and taking the thermodynamic limit, the perturbation is switched off, leaving the results marked by this particular ground state.

In the case under consideration, such a degeneracy lifting naturally arises due to an external confining potential which keeps particles together. This circumstance selects the ground state as the state with a minimal angular momentum. It has been shown that the transition to the states with a higher angular momentum costs energy [21], promoting the incompressible state of non-interacting quasi-particles as a unique candidate for the ground state.

Consider the Hamiltonian eigenstate

$$|\Omega;N\rangle=\prod_{j=0}^{N-1}f_j^+|0\rangle.$$

Case $N=N_B$ corresponds to the complete filling of the lowest Landau level. The Laughlin ground state is given by

$$\langle z_1,\ldots,z_{N_e}|\Omega;N_e\rangle=\prod_{K<L}(z_K-z_L)^{2p}\langle 0|\chi(1)\cdots\chi(N_e)|\Omega;N_e\rangle,$$

where the last factor

$$\langle 0|\chi(1)\cdots\chi(N_e)f_0^+\cdots f_{N_e-1}^+|0\rangle=\prod_{1\le K<L\le N_e}(z_K-z_L)e^{-\frac{eB_{ext}}{4}\sum_{I=1}^{N_e}|z_I|^2}$$

is the Slater determinant of one-particle states (2.26).

Consequently, the Laughlin function can be defined as a similarity transformation of an incompressible state of N_e non-interacting composite particles.

The spectrum generating quantum operators are related by the S_p transformation to the corresponding quantities of the non-interacting quasi-particle theory. In particular, $W_{1+\infty}$ is generated by the operators

$$V_n^i=-\int d\mathbf{r}\chi^\dagger(\mathbf{r})(B_0^\dagger)^{n+i}(B_0)^i\chi(\mathbf{r})=-\int d\mathbf{r}\psi^\star(\mathbf{r})(B_p^\dagger)^{n+i}(B_p)^i\psi(\mathbf{r}),$$

where

$$B_p = \frac{\partial}{\partial z} + \frac{eB}{4}\bar{z} - 2p\int d\mathbf{r}' \frac{\varrho(\mathbf{r}')}{z-z'},$$
$$B_p^\dagger = -\frac{\partial}{\partial \bar{z}} + \frac{eB}{4}z.$$

Unlike the conformal field theory, the generators V_n^i are bounded from below ($n + i \geq 0$), i.e., they form so called "wedge" $W_\Lambda = \{V_n^i, |n| > i\}$, plus the positive modes $n > i$ [21].

At the same time, the operators

$$W_n^i = -\int d\mathbf{r}\psi^\star(\mathbf{r}) \exp\{nB_p^\dagger\}(B_p)^i\psi(\mathbf{r})$$

satisfy the W-algebra commutation relations for any integer $n \in \mathbb{Z}$, $i \geq 0$, i.e., they correspond to the full $W_{1+\infty}$.

When $p \neq 0$, only $N_e = \frac{N_B}{1+2p}$ states in the lowest Landau level are occupied. The remaining $N_h = N_B - N_e = N_B(1 - \frac{1}{1+2p})$ states are empty, i.e., they are described by the hole wave function [64]

$$\Psi_p^h(N_e+1,\ldots,N_e+N_h) = \langle\Omega; N_B|S_p^{-1}\psi^\star(N_e+N_h)\cdots\psi^\star(N_e+1)|\Omega; N_e\rangle =$$
$$= \int\cdots\int\prod_{K=1}^{N_h}[d\mathbf{r}_K]\langle\Omega; N_B|S_p^{-1}\psi^\star(N_e+N_h)\cdots\psi^\star(N_e+1)\psi^\star(N_e)\cdots\psi^\star(1)|0\rangle\times$$
$$\times\langle 0|\psi(1)\cdots\psi(N_e)|\Omega; N_e\rangle =$$
$$= \int\cdots\int\prod_{K=1}^{N_e}[d\mathbf{r}_K]\overline{\Psi_0^e(1,\cdots,N_B)}\times\Psi_p^e(1,\cdots,N_e).$$

The same wave function reappears while considering a particle-hole conjugate system or, equivalently, a system of electrons in a magnetic field $-B < 0$. Consider a Hamiltonian

$$H_c = \frac{1}{2m}\int d\mathbf{r}\,(\partial_k + ieA_k)\,\chi_c^\dagger(\mathbf{r})(\partial_k - ieA_k)\chi_c(\mathbf{r}),$$

where χ_c is a composite hole field.

The physical holes are introduced by the transformations

$$\psi_c(\mathbf{r}) = \bar{S}_p\chi_c(\mathbf{r})\bar{S}_p^{-1} = e^{2p\int d\mathbf{r}'\ln(\bar{z}-\bar{z}')\varrho_c(\mathbf{r}')}\chi_c(\mathbf{r}),$$
$$\psi_c^\star(\mathbf{r}) = \bar{S}_p\chi_c^\dagger(\mathbf{r})\bar{S}_p^{-1} = \chi_c^\dagger(\mathbf{r})e^{-2p\int d\mathbf{r}'\ln(\bar{z}-\bar{z}')\varrho_c(\mathbf{r}')},$$
$$\varrho_c(x) = \psi_c^\star(x)\psi_c(x) = \chi_c^\dagger(x)\chi_c(x).$$

Express the quasi-hole Hamiltonian in terms of physical fields

$$H_c = \frac{1}{2m}\int d\mathbf{r}\,(\partial_k\psi_c^\star + ie\psi_c^\star A_k + ie\psi_c^\star\bar{a}_k)\,(\partial_k\psi_c - ieA_k\psi - ie\bar{a}_k\psi_c),$$

where

$$\bar{a}_k(\mathbf{r}) = -i\frac{2p}{e}\partial_k \int d\mathbf{r}' \ln(\bar{z}-\bar{z}')\varrho_c(\mathbf{r}')$$

is charge-conjugate to the connection (2.22).

Charge-conjugate operators can be used to construct the state vectors for filling factors others then $\nu = 1/(2p+1)$.

Expand $\chi_c(\mathbf{r})$ into the modes

$$\chi_c(\mathbf{r}) = \sum_{j=0}^{N_B-1} f_{cj}u_{cj}(\mathbf{r}) + \tilde{\chi}_c(\mathbf{r}), \quad (u_{cj}(\mathbf{r}) = \bar{u}_j(\mathbf{r}))$$

and define the vacuum $|0_c\rangle$ (completely filled Fermi-Dirac sea):

$$\chi_c(\mathbf{r})|0_c\rangle = 0.$$

Introduce an incompressible quasi-hole state

$$|\Omega_c; N\rangle = \prod_{j=0}^{N-1} f_{cj}^+|0_c\rangle,$$

and the state representing N_B holes in the lowest Landau level

$$\bar{S}_p|\Omega_c; N_B\rangle. \tag{2.27}$$

Note that (2.27) is not an eigenvector of the Hamiltonian H_c. Then

$$\begin{aligned}
&\langle\Omega_c; N_B|\bar{S}_p^{-1}\psi_c^\star(1)\cdots\psi_c^\star(N_e)|\Omega_c; N_h\rangle = \\
&= \int\cdots\int \prod_{K=N_e+1}^{N_B}[d\mathbf{r}_K]\langle\Omega_c; N_B|\bar{S}_p^{-1}\psi_c^\star(1)\cdots\psi_c^\star(N_B)|0_c\rangle \times \\
&\quad \times\langle 0_c|\psi_c(N_B)\cdots\psi_c(N_e+1)|\Omega_c; N_h\rangle = \\
&= \int \prod_{K=N_e+1}^{N_B}[d\mathbf{r}_K]\overline{\Psi_p^e(N_e+1,\cdots,N_B)}\times\Psi_0^e(1,\cdots,N_B)
\end{aligned} \tag{2.28}$$

will be the electron wave function for the filling factor

$$\nu = 1 - \frac{1}{1+2p}.$$

So the "chiral" partners of (2.23)

$$D_I = \frac{\partial}{\partial z_I}, \quad \bar{D}_I = \frac{\partial}{\partial \bar{z}_I} - 2p\sum_{J\neq I}\frac{1}{\bar{z}_I - \bar{z}_J}$$

are engaged in the charge conjugate sector of the theory.

Concluding this chapter, note that the similarity transformation relating the integral and fractional QHE represents an isometric transformation. Such isometries can be used to study physically equivalent theories or to

reduce the problem to a more tractable form. Naturally, this transformation needs the introduction of a special measure in the Hilbert space [4].

CHAPTER 3
ON THE HOLOMORPHIC GAUGE QUANTIZATION

1. Introduction

In chapter 2 we have described some aspects of the FQHE in terms of similarity transformation and complex Chern–Simons gauge potentials. In this chapter, we consider the problem from the point of view of canonical quantization.

One of the characteristic features of two-dimensional systems is that the Green functions (correlators) can be factorized into the product of holomorphic and antiholomorphic parts, and corresponding gauge connections can take complex values.

A known example of a non-real gauge potential is provided by an integrable connection over the configuration space arising from the Yang-Baxter equations. This connection is represented by the one-form [95]

$$\omega = const \cdot \sum_{I \neq J} T_I^a \otimes T_J^a \, d \ln(z_I - z_J), \tag{3.1}$$

where $z_I = x_I + iy_I$ are the complex coordinates and T_I^a are the generators of the symmetry group for the I^{th} particle. This connection governs the monodromy behavior of conformal blocks in (1+1) dimensional current algebra and enters into the Knizhnik - Zamolodchikov (KZ) [96] equation

$$\left(\frac{\partial}{\partial z_I} - \frac{1}{k+c} \sum_{I \neq J} \frac{T_I^a \otimes T_J^a}{z_I - z_J} \right) \Psi(z_1, \ldots, z_N) = 0, \tag{3.2}$$

$$\frac{\partial}{\partial \bar{z}_I} \Psi(z_1, \ldots, z_N) = 0.$$

The KZ connection plays an essential role in the physics of particles obeying the braid statistics and in the theory of the quantum Hall effect (see, e.g., [126]). In the latter case, the holomorphic part of the Laughlin wave function satisfies (3.2) and could be expressed as an N-point correlation function in a certain conformal field theory [114].

The gauge potential (3.1) can be incorporated into the framework of the Chern-Simons (CS) gauge theory in 2+1 dimensions. Formally, the problem reduces to the quantization of the theory describing the matter interacting with the CS fields in the holomorphic gauge, where the corresponding gauge condition is expressed by a complex matrix equation

$$\mathcal{A}_x + i\mathcal{A}_y = 0 \tag{3.3}$$

($\mathcal{A}_\mu$ is a Lie-algebra-valued gauge connection).

Remind that this type of gauge has been presented as a solution of a Gauß law constraint in discussions of quantum holonomies [68], and BRST quantization of non-abelian CS gauge theories [103].

Note that the holomorphic gauge quantization as considered in, e.g., [103] leads to a non-Hermitean Hamilton operator, and for consistency one has to introduce in the Hilbert space a compensating integration measure, respectively to which the Hamiltonian is self-adjoint [130].

It must be emphasized that as well as the complex gauge condition is imposed in the CS theory with a compact gauge group and real gauge fields, equation (3.3) must be understood in the sense of some analytic continuation.

It is worth pointing out at this point that in the paper [140], Witten had considered the theory with a non-compact (complex) gauge transformation group and complex CS gauge fields.

It was shown in this paper that the quantization of self-interacting CS gauge fields can be performed as precisely as for compact groups, using standard tools and without any specific difficulties (see also [18], [47]).

In the present item we consider the same scheme as in [140], enlarging the system by the matter fields. The point of departure is the observation that in a holomorphic gauge, in order to have a real Lagrangian (i.e., a unitary theory), the matter fields as well as the gauge degrees of freedom must be accompanied by their complex conjugate counterparts. In the quantization procedure, we follow Dirac's classical method [29].

As a physical application, we will try to give some convincing arguments, that the models with complex gauge groups can provide a consistent description of a variety of QHE wave functions.

In Section 2, we consider holomorphic polarization and corresponding wave functionals for the Chern–Simons field interacting with matter. In Section 3, we define the action and Euler-Lagrange equations for complex non-Abelian CS gauge fields interacting with non-relativistic fermions. Imposing the holomorphic gauge we perform the Dirac quantization.

In Section 4, we introduce the non-unitary similarity transformation and reduce the Hamiltonian to a (quasi)free form. Diagonalization is complete in the Abelian case. In Section 5, we consider a system of planar electrons in an external magnetic field. As an output, we give the construction of the relevant wave functions for a quantum Hall fluid with both Abelian and non-Abelian CS gauge interactions.

Together with the conventional cartesian coordinates $\mathbf{r} = x^k = (x, y)$, it is convenient to use the complex notation

$$z = x + iy, \quad \partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y),$$
$$\bar{z} = x - iy, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$$

for particle coordinates and the corresponding Cauchy-Riemann operators. The vector fields $\mathbf{A}(r) = (A_x, A_y)$ will be represented by their holomorphic

and antiholomorphic components

$$A(\mathbf{r}) = A_x + iA_y, \qquad \bar{A}(\mathbf{r}) = A_x - iA_y.$$

A non-Abelian matrix-valued vector potential can be decomposed with respect to a basis of the real Lie algebra of a compact gauge group G:

$$\mathcal{A}_\mu(x) = \sum_a A^a_\mu(x) \cdot t^a, \qquad a = 1, 2, \ldots, r = \dim G. \tag{3.4}$$

The group generators are anti-Hermitean, traceless matrices t^a obeying the Lie algebra

$$t^{a\,\dagger} = -t^a, \quad [t^a, t^b] = f_{abc}t^c \tag{3.5}$$

with f^{abc} totally antisymmetric, real structure constants. In the case of an abelian group, (3.4) is replaced by $\mathcal{A}_\mu(x) = iA_\mu(x)$.

We will abbreviate the spatial coordinates of I^{th} particle $\mathbf{r}_I$ to I, when this will not be ambiguous.

2. Complex Chern–Simons Gauge Fields and Wave Functionals

The Lagrangian describing a Chern–Simons field coupled to matter can be written in the Hamiltonian form

$$\mathcal{L} = i\tilde{\psi}\dot{\psi} - \frac{1}{2m}D_k\tilde{\psi}D_k\psi - \frac{1}{2}\kappa\epsilon^{ik}f_i\dot{f}_k - f_0(x)[\tilde{\psi}\psi - \kappa\epsilon^{ik}\partial_i f_k].$$

Canonical momenta are defined as usual

$$\pi(x) = i\tilde{\psi}(x), \qquad \Pi^k(x) = \frac{1}{2}\kappa\epsilon^{km}a_m(x),$$

and the canonical Hamiltonian density is given by

$$\begin{aligned} H_c &= \pi\dot{\psi} + \Pi^k\dot{f}_k - \mathcal{L} = \\ &= \frac{1}{2m}D_k\tilde{\psi}(x)D_k\psi + f_0(x)[\tilde{\psi}(x)\psi(x) - \kappa\epsilon^{ik}\partial_i f_k(x)]. \end{aligned}$$

The system is singular. The primary constraint

$$\phi_1(x) = \Pi^0(x) \approx 0$$

reflects the absence of canonical momentum conjugate to time component of the gauge field, and the secondary constraint

$$\phi_2(x) = \tilde{\psi}(x)\psi(x) - \kappa\epsilon^{ik}\partial_i f_k(x)$$

is the Gauß law.

Canonical Poisson brackets are given by

$$\left\{\psi(x), \tilde{\psi}(y)\right\}_{x_0=y_0} = -i\delta(\mathbf{x} - \mathbf{y})$$

and

$$\{f_i(x), f_k(y)\}_{x_0=y_0} = \frac{2}{\kappa}\epsilon^{ik}\delta(\mathbf{x}-y).$$

Following Dirac's [29] treatment of constraint systems, the dynamics is governed by the extended Hamiltonian

$$H' = H_c + \lambda_1\phi_1 + \lambda_2\phi_2,$$

where $\lambda_{1,2}$ are the Lagrange multipliers which must be determined by the gauge choice.

First consider the Coulomb gauge [60] determined by the conditions

$$\chi_1(x) = a_0 = 0, \qquad \chi_2(x) = \frac{1}{2}\Delta^{-1}\partial_m a_m(x).$$

In this gauge, one gets

$$f_i(x) = \frac{e}{\kappa}\epsilon^{ik}\partial_k^{-1}\psi^\dagger(x)\psi(x).$$

Here the non-local operator ∂_k^{-1} is defined with the help of the Green function

$$\partial_k^{-1} f(\mathbf{r}) = \partial_k \int G_c(\mathbf{r}-\mathbf{r}')f(\mathbf{r}'),$$

where

$$G_c(\mathbf{r}-\mathbf{r}') = \Delta^{-1} = \frac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r}'|$$

is the Green function in the Coulomb gauge.

Another reasonable gauge choice is the axial one [7]:

$$\chi_1(x) = a_0(x) = 0 \qquad \chi_2(x) = a_y(x) = 0.$$

The Gauß law can be solved in an explicit way:

$$f_x = -\frac{e}{\kappa}\partial_y^{-1}\psi^\dagger\psi, \quad f_y = 0.$$

The formal solution for ∂_y^{-1} is given by the Green function

$$\partial_y^{-1} f(\mathbf{r}) = \int d\mathbf{r}' G_a(\mathbf{r}-r')f(\mathbf{r}'),$$

where

$$G_a(\mathbf{r}-r') = \frac{1}{2}\delta(x-x')\epsilon(y-y').$$

For the needs of the Chern–Simons description of the quantum Hall effect, in what follows we will use the called holomorphic gauge. In this gauge, the gauge fixing conditions are expressed by the complex equations [68], [103], [42], [46]

$$\chi_1 = a_0 = 0, \quad \chi_2 = a_x + ia_y = 0,$$

and the solution for the gauge field is given by

$$a_\mu(x) = -\frac{i}{2\kappa}\partial_\mu \int d\mathbf{r}'\frac{1}{\pi}\ln(z-z')\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}').$$

To get P better insight into the problem, consider first the complex gauge determined by the conditions

$$\chi_1(x) = a_0 = 0, \quad \chi_2(x) = \partial_m f_m(x) - \eta\tilde{\psi}\psi.$$

Here η is a complex number.

The Lagrange multipliers are determined from the self-consistency conditions

$$\dot{\lambda}_a(x) = \{\lambda(x), H'\} \approx 0.$$

In this gauge, one gets $\lambda_1 = 0$, and λ_2 is determined from the equation

$$\Delta\lambda_2(x) = \frac{1}{2}\eta\partial_t\left(\tilde{\psi}(x)\psi(x)\right)$$

which is reduced to

$$\Delta\lambda_2(x) = -\frac{1}{\kappa}\partial_i\epsilon^{ik}J_k(x) - \eta\frac{i}{2m}\left(\tilde{\psi}(x)D_kD_k\psi(x) - D_kD_k\tilde{\psi}(x)\psi(x)\right) = \\ = \frac{1}{2}\eta\partial_t\left(\tilde{\psi}(x)\psi(x)\right).$$

The Chern–Simons field can be determined from the constraint equation

$$\kappa\epsilon_{ik}\partial_i f_k(x) = \tilde{\psi}(x)\psi(x).$$

Define the holomorphic Green function

$$G(\mathbf{r} - \mathbf{r}') = \frac{1}{\pi}\ln(z - z').$$

The Green functions in different gauges are related by the equation

$$G(\mathbf{r} - \mathbf{r}') = 2G_c(\mathbf{r} - \mathbf{r}') + \frac{i}{\pi}\arg(z - z').$$

The main properties of the introduced functions are summarized below:

$$\Delta G_c(\mathbf{r}) = \delta(\mathbf{r}), \quad \Delta G(\mathbf{r}) = \delta(\mathbf{r}), \quad \Delta\arg(z - z') = 0$$

and

$$\epsilon^{ik}\partial_i\partial_k G_c(\mathbf{r}) = 0, \qquad \epsilon^{ik}\partial_i\partial_k G(\mathbf{r}) = 2i\delta(\mathbf{r}),$$

$$\epsilon^{ik}\partial_i\partial_k \arg(z - z') = 2\pi\delta(\mathbf{r}), \qquad \epsilon^{ik}\partial_k G_c(\mathbf{r}) = -\frac{1}{2\pi}\partial_i \arg z.$$

We see that as a solution, one can take

$$f_k(x) \equiv a_k + ib_k(x),$$

where

$$a_k(x) = -\frac{1}{\kappa}\epsilon^{kn}\partial_n \int d\mathbf{r}' G_c(\mathbf{r} - \mathbf{r}')\tilde{\psi}(\mathbf{r}'),$$

$$b_k(x) = -i\eta\partial_k \int d\mathbf{r}' G_c(\mathbf{r} - \mathbf{r}')\tilde{\psi}(\mathbf{r}')\psi(\mathbf{r}').$$

Consider the particular case

$$\eta = -\frac{i}{\kappa}.$$

Then

$$f_k(x) = -\frac{i}{\kappa}\partial_k \int d\mathbf{r}' \left[G_c(\mathbf{r}-\mathbf{r}') + \frac{i}{2\pi}\arg(z-z')\right] \tilde{\psi}(\mathbf{r}')\psi(\mathbf{r}')$$

or

$$f_k(x) = -\frac{i}{2\kappa}\partial_k \int d\mathbf{r}' G(\mathbf{r}-\mathbf{r}')\tilde{\psi}(\mathbf{r}')\psi(\mathbf{r}').$$

These gauge fields satisfy the equation

$$f_x(x) + i\, f_y(x) = 0$$

which in fact is the holomorphic gauge condition.

The corresponding Lagrange multiplier

$$\lambda_2(x) \equiv f_0(x) = -\frac{i}{2\kappa}\partial_t \int d\mathbf{r}' G(\mathbf{r}-\mathbf{r}')\tilde{\psi}(\mathbf{r}')\psi(\mathbf{r}').$$

Dynamics is governed by the Hamiltonian

$$H' = \frac{1}{2m} D_k\tilde{\psi}(x) D_k\psi(x) + f_0[\tilde{\psi}(x)\psi(x) - \kappa\epsilon^{ik}\partial_i f_k(x)].$$

After calculating all the commutators, the constraint ϕ_2 is set to zero and for the gauge field, we take the solution

$$f_\mu(x) = -\frac{i}{2\kappa}\partial_\mu \int d\mathbf{r}' G(\mathbf{r}-\mathbf{r}')\tilde{\psi}(\mathbf{r}')\psi(\mathbf{r}').$$

Above we have quantized the system after solving the constraints. There is an alternative way: first quantize and afterwards implement the constraints as relations for the quantum states.

The Chern–Simons Lagrangian written in the Hamiltonian form

$$\mathcal{L}_{\mathrm{CS}} = \kappa\epsilon^{ik}\dot{f}_i(x) f_k(x)$$

determines a symplectic structure of the theory, and establishes the canonical commutation relations

$$[f_i(\mathbf{x}), f_k(\mathbf{y})] = \frac{2i}{\kappa}\epsilon^{ik}\delta(\mathbf{x}-\mathbf{y}).$$

As a further step, one must choose a polarization [83], [84], or declare which one of the phase space variables is a canonical coordinate and momentum. One possible choice is the Cartesian polarization

$$\phi(x) = f_1(x), \qquad \pi(x) = -\frac{i\kappa}{2} f_2(x).$$

Another choice is the so called holomorphic polarization which reflects the complex structure of the underlying two-dimensional manifold. In the complex notation canonical commutation relations look like

$$\left[f(\mathbf{r},t),\bar{f}(\mathbf{r}',t)\right] = \frac{4}{\kappa}\delta(\mathbf{r}-\mathbf{r}'). \tag{3.6}$$

Introduce the holomorphic variables

$$a^\star(x) = \frac{\sqrt{\kappa}}{2}\bar{f}(x), \quad a(x) = \frac{\sqrt{\kappa}}{2}f(x),$$
$$[a(\mathbf{r},t),a^\star(\mathbf{r}',t)] = \delta(\mathbf{r}-\mathbf{r}'), \tag{3.7}$$

In the functional space, these operators are realized by

$$a(x) = \frac{\delta}{\delta a^\star(x)}, \qquad a^\star(x) = a^\star(x).$$

The quantum states $|\Psi>$ are represented by the wave functionals $\Psi[a^\star]$. The physical states must satisfy the constraint equation

$$\left(\rho(x) + 2i\sqrt{\kappa}\left[\bar{\partial}a^\star(x) - \partial a(x)\right]\right)|\Psi\rangle = 0.$$

In terms of the wave functionals, this equation reads

$$\left(\rho(x) + i2\sqrt{\kappa}\left[\partial a^\star(x) - \partial\frac{\delta}{\delta a^\star(x)}\right]\right)\Psi[a^\star] = 0. \tag{3.8}$$

The solution to (3.8) can be easily found:

$$\Psi[a^\star] = e^{\Omega[a^\star]}F[\psi,\tilde{\psi}],$$

where

$$\Omega[a^\star] = \int d\mathbf{r}\left[-\frac{i}{2\sqrt{\kappa}}a^\star(\mathbf{r})\partial^{-1}\rho(\mathbf{r}) + \frac{1}{2}a^\star(\mathbf{r})\left(\partial^{-1}\bar{\partial}a^\star\right)(\mathbf{r})\right], \tag{3.9}$$

and $F[\psi]$ depends only on the matter field variables. (Here we use the symbolic notation for integro-differential operators).

One can now fix the gauge. In the holomorphic gauge, the physical states satisfy the functional equation

$$a(x)|\Psi\rangle = 0 \qquad \text{or} \qquad \frac{\delta}{\delta a^\star(x)}\Psi[a^\star] = 0.$$

The solution to this equation is

$$a^\star(x) = \frac{i}{2\sqrt{\kappa}}\bar{\partial}^{-1}\rho(x) \tag{3.10}$$

and the corresponding wave functional is given by

$$\Psi_H[\rho] = \exp\left[\frac{1}{8\kappa}\int d\mathbf{r}\bar{\partial}^{-1}\rho(\mathbf{r})\partial^{-1}\rho(\mathbf{r})\right]F =$$

$$= \exp\left[-\frac{1}{8\kappa}\int d\mathbf{r}\int d\mathbf{r}'\rho(\mathbf{r})G(z-z')\rho(\mathbf{r}')\right]F,$$

where the holomorphic Green function is used.

3. Action and Quantization

Let G be a compact r-dimensional Lie group. The group elements are parametrized by the set of real parameters $g = g(\omega_1, \omega_2, \ldots, \omega_r)$. The irreducible unitary representations of G are denoted by $D_{(\sigma)}(g) \equiv D_{(\sigma)}(\omega_a)$. Matrices $T^a_{(\sigma)}$ are the corresponding group generators. They satisfy the commutation relations

$$\left[T^a_{(\sigma)}, T^b_{(\tau)}\right] = \delta_{\sigma\tau} f_{abc} T^c_{(\sigma)}, \qquad 1 \le a, b, c \le r.$$

The matter fields and quantum states in the representation $D_{(\sigma)}$ are labeled by the weight vectors $\mathbf{w}_\sigma \equiv w^m_\sigma$ $(m = 1, \ldots, R = \operatorname{rank} G)$.

Consider the non-compact group G_c (the complex extension of G), regarding the group parameters ω_a as complex quantities and with a group multiplication law given by a holomorphic function.

Recall some facts about representations of complex groups.

- Associated with any irreducible representation $D_{(\sigma)}$ of a Lie group G one, can define its analytic and antianalytic continuations, $\Delta_{(\sigma)}(g) = D_{(\sigma)}(\omega_a)$ and $\Delta^\star_{(\sigma)}(g) = D_{(\sigma)}(\omega^\star_a)$, respectively.
- For any two representations $D_{(1)}$ and $D_{(2)}$, the tensor product $\Delta_{(1,2)}(g) = \Delta_{(1)}(g) \otimes \Delta^\star_{(2)}(g)$ is an irreducible representation of G_c.

The other irreducible representations of interest are contragradient $\tilde{\Delta}_{(1,2)}$, and complex conjugate representations $\Delta^\star_{(1,2)}$ and $\tilde{\Delta}^\star_{(1,2)}$. Introduce the matter fields. It is convenient to define the doublet field

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \tilde{\psi}^\star(x) \end{pmatrix}, \tag{3.11}$$

transforming under reducible representation $\mathcal{R}(g) = \Delta_{(1,2)} \oplus \tilde{\Delta}^\star_{(1,2)}$. (The complex conjugation * for fermions is defined as an involution operation for Grassmann variables [12].)

The corresponding contravariantly transforming fields

$$\tilde{\Psi}(x) = \left(\tilde{\psi}(x), \psi^\star(x)\right)$$

are unified in the representation $\tilde{\mathcal{R}}(g) = \tilde{\Delta}_{(1,2)} \oplus \Delta^\star_{(1,2)}$. It means that there exists a non-degenerate real bilinear form $< \tilde{\Psi}, \Psi >$ invariant under the group transformations.

Gauging the rigid group G_c, we consider the group parameters as complex functions of the space-time coordinates. The Lie-algebra valued gauge potential $\mathcal{F}_\mu(x) \equiv F_\mu^A(x) \cdot T^A$ transforms as follows

$$\mathcal{F}_\mu(x) \to \mathcal{F}_\mu^\omega(x) = \mathcal{R}(g)\mathcal{F}_\mu(x)\mathcal{R}(g)^{-1} + \partial_\mu \mathcal{R}(g) \cdot \mathcal{R}(g)^{-1}. \quad (3.12)$$

The matrices $T^A\,(A = 1,,\ldots,r,r+1,\ldots 2r)$ are anti-Hermitean Lie-algebra generators in the representation of the matter field :

$$\begin{aligned} T^a &= \begin{pmatrix} T^a_{(1)} \otimes I_{(2)} & 0 \\ 0 & I_{(1)} \otimes T^a_{(2)} \end{pmatrix}, \\ T^{r+a} &= \begin{pmatrix} I_{(1)} \otimes T^a_{(2)} & 0 \\ 0 & T^a_{(1)} \otimes I_{(2)} \end{pmatrix}. \end{aligned} \quad (3.13)$$

These generators are associated with the group parameters ω_a and $\omega_{r+a} \equiv \omega_a^\star$. The defining commutation relations are

$$\left[T^A, T^B\right] = f_{ABC}T^C, \qquad A, B = 1,\ldots,2r.$$

With the help of the gauge fields $F_\mu^a(x)$, $F_\mu^{r+a}(x) \equiv (F_\mu^a(x))^\star$, define the covariant derivatives:

$$D_\mu\Psi(x) = \partial_\mu\Psi(x) - \mathcal{F}_\mu(x)\Psi(x), \quad D_\mu\tilde{\Psi}(x) = \partial_\mu\tilde{\Psi}(x) + \tilde{\Psi}(x)\mathcal{F}_\mu(x) \quad (3.14)$$

These ingredients permit to construct a real Lagrangian invariant under the involution and the group of complex gauge transformations G_c:

$$\begin{aligned} \mathcal{L} = \frac{\kappa}{2}\varepsilon^{\mu\nu\lambda}[F_\mu^A(x)\partial_\nu F_\lambda^A(x) + \frac{1}{3}f_{ABC}F_\mu^A F_\nu^B F_\lambda^C] + \\ + i < \tilde{\Psi}(x), D_0\Psi(x) > -\frac{1}{2m} < D_k\tilde{\Psi}(x), D_k\Psi(x) > \end{aligned} \quad (3.15)$$

The Euler–Lagrange equations for the matter and gauge fields are given by the set

$$\begin{aligned} \frac{1}{\kappa}J^A &= -2i\bar{\partial}F_0^A - i\partial_0 F^A - if_{ABC}F^B F_0^C, \\ \frac{1}{\kappa}\bar{J}^A &= -2i\partial F_0^A + i\partial_0\bar{F}^A + if_{ABC}\bar{F}^B F_0^C, \\ \frac{i}{\kappa}\rho^A &= \bar{\partial}\bar{F}^A - \partial F^A - \frac{i}{4}f_{ABC}(F^B\bar{F}^C - \bar{F}^B F^C), \\ i\partial_t\psi &= -\frac{1}{m}(D\bar{D} + \bar{D}D)\psi + iF_0\psi. \end{aligned} \quad (3.16)$$

Here

$$\begin{aligned} D &= \frac{1}{2}(D_x - iD_y) = \partial + \frac{1}{2}\bar{\mathcal{F}}, \\ \bar{D} &= \frac{1}{2}(D_x + iD_y) = \bar{\partial} + \frac{1}{2}\mathcal{F} \end{aligned}$$

are covariant derivative operators.

The gauge invariant currents

$$J_0^A(x) \equiv \rho^A(x) = i < \tilde{\Psi}, T^A \Psi >,$$

$$J^A(x) = J_x^A(x) + iJ_y^A(x) = \frac{1}{m}[< \tilde{\Psi}, T^A \bar{D}\Psi > - < \bar{D}\tilde{\Psi}, T^A \Psi >],$$

$$\bar{J}^A = J_x^A - iJ_y^A = \frac{1}{m}[< \tilde{\Psi}, T^A D\Psi > - < D\tilde{\Psi}, T^A \Psi >]$$

are covariantly conserved:

$$\partial_t \rho^A + \partial J^A + \bar{\partial}\bar{J}^A - f_{ABC}[F_0^B \rho^C - \frac{1}{2}(F^B \bar{J}^C + \bar{F}^B J^C)] = 0. \quad (3.17)$$

Note that the gauge coupling constant is set to 1. Its actual value can be restored by rescaling the gauge fields and the statistical parameter

$$A_\mu \to gA_\mu, \qquad \kappa \to \frac{\kappa}{g^2}.$$

For the canonical quantization, we will use Dirac's method, and try to adapt it for the case of the complex gauge group.

To begin with, consider the classical theory and discuss the setup of holomorphic gauge quantization. The canonical Hamiltonian is given by the expression

$$\mathcal{H}_c = \int d\mathbf{r}\Big[\frac{1}{m}\Big(< D\tilde{\Psi}(\mathbf{r}), \bar{D}\Psi(\mathbf{r} > + < \bar{D}\tilde{\Psi}(\mathbf{r}), D\Psi(\mathbf{r}) >\Big) + F_0^A(\mathbf{r})\phi^A(\mathbf{r})\Big]. \quad (3.18)$$

The system is constrained by the first class constraints

$$\Pi_0^A = \frac{\partial \mathcal{L}}{\partial \dot{F}_0^A} \approx 0,$$

and

$$\phi^A = \rho^A + i\kappa\Big[\bar{\partial}\bar{F}^A - \partial F^A - \frac{1}{4} f_{ABC}(F^B \bar{F}^C - \bar{F}^B F^C)\Big] \approx 0. \quad (3.19)$$

The canonical variables satisfy the Poisson brackets relations

$$\{\Psi(\mathbf{r}, t), \tilde{\Psi}(\mathbf{r}', t)\}_{\mathrm{PB}} = -i\delta(\mathbf{r} - \mathbf{r}') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.20)$$

$$\{F^A(\mathbf{r}, t), \bar{F}^B(\mathbf{r}', t)\}_{\mathrm{PB}} = -i\frac{4}{\kappa}\delta_{AB}\delta(\mathbf{r} - \mathbf{r}'). \quad (3.21)$$

Due to the presence of quadratic terms, the constraint equations (3.19) are not easy to solve. The obvious way out is to impose the gauge conditions which linearize them. This possibility is realized in the axial type gauge, e.g., $F_y^A = 0$ [7].

As an alternative solution, one can use a holomorphic gauge, with the gauge fixing conditions

$$\eta^A = F_0^A = 0, \qquad \chi^a = F^a = 0, \qquad \chi^{r+a} = \bar{F}^{r+a} = 0. \quad (3.22)$$

In conformity with Dirac's procedure, introduce the total Hamiltonian

$$H_T = \mathcal{H}_c + \int d\mathbf{r}\Lambda^A(\mathbf{r})\phi^A(\mathbf{r}), \tag{3.23}$$

where $\Lambda^A(\mathbf{r})$ are the Lagrange multipliers. These functions must be subjected to the self-consistency conditions

$$\partial_t \eta^A(\mathbf{r}) = \{\eta^A(\mathbf{r}, H_T\} = 0 \quad \text{and} \quad \partial_t \chi^A(\mathbf{r}) = \{\chi^A(\mathbf{r}), H_T\} = 0,$$

and are given by

$$\Lambda^a(\mathbf{r}) = \frac{i}{2\kappa}\int d\mathbf{r}'\mathcal{G}(\mathbf{r}-r')\cdot J^a(\mathbf{r}'), \tag{3.24}$$

$$\Lambda^{r+a}(\mathbf{r}) = -\frac{i}{2\kappa}\int d\mathbf{r}'\bar{\mathcal{G}}(\mathbf{r}-r')\cdot J^{r+a}(\mathbf{r}'). \tag{3.25}$$

Here we have formally introduced the operator $\bar{\partial}^{-1}$ which defines the Green function

$$\bar{\partial}^{-1}J^a(\mathbf{r}) = \int d\mathbf{r}'\mathcal{G}(\mathbf{r}-r')J^a(\mathbf{r}'). \tag{3.26}$$

The Green function $\bar{\partial}^{-1} = \mathcal{G}(\mathbf{r})$ can be represented as a derivative of the holomorphic Green function

$$\mathcal{G}(\mathbf{r}) = \partial G(z) = \frac{1}{\pi z} = \partial\frac{1}{\pi}\ln z. \tag{3.27}$$

We see that (3.27) is an ill-defined multivalued function. In the nonrelativistic case when the particle density is a sum of δ-functions, using appropriate regularization one may ignore this point and consider G(z) as a normal function vanishing at the origin [85], [105].

In an analogous way, one can define the antiholomorphic Green functions

$$\bar{\mathcal{G}}(\mathbf{r}) = \bar{\partial}\bar{G}(z) = \frac{1}{\pi\bar{z}} = \frac{1}{\pi}\bar{\partial}\ln\bar{z}. \tag{3.28}$$

In the holomorphic gauge, the Gauß law constraints (3.19) look like

$$\phi^a = \rho^a + i\kappa\bar{\partial}\bar{F}^a = 0, \quad \phi^{r+a} = \rho^{r+a} - i\kappa\partial F^{r+a} = 0,$$

and can be easily solved

$$\bar{F}^a(\mathbf{r}) = \frac{i}{\kappa}\int d\mathbf{r}'\mathcal{G}(\mathbf{r}-r')\cdot\rho^a(\mathbf{r}'), \tag{3.29}$$

$$F^{r+a}(\mathbf{r}) = -\frac{i}{\kappa}\int d\mathbf{r}'\bar{\mathcal{G}}(\mathbf{r}-r')\cdot\rho^{r+a}(\mathbf{r}'). \tag{3.30}$$

In the chosen gauge, $J^a = 0$, $\bar{J}^{r+a} = 0$. Using the continuity equation (3.17), one can express the Lagrange multipliers (3.24) and (3.25) as time derivatives:

$$\Lambda^a(\mathbf{r}) = -\frac{i}{2\kappa}\partial_t \int d\mathbf{r}' G(\mathbf{r}-r')\rho^a(\mathbf{r}',t), \tag{3.31}$$

$$\Lambda^{r+a}(\mathbf{r}) = \frac{i}{2\kappa}\partial_t \int d\mathbf{r}' \bar{G}(\mathbf{r}-r')\rho^{r+a}(\mathbf{r}',t). \tag{3.32}$$

The last expressions may be unified with (3.29) and (3.30) composing 3-vectors

$$a^a_\mu(x) \equiv (\Lambda^a, F^a_k) = -\frac{i}{2\kappa}\partial_\mu \int d\mathbf{r}' G(\mathbf{r}-r')\rho^a(\mathbf{r}',t), \tag{3.33}$$

$$a^{r+a}_\mu(x) \equiv (\Lambda^{r+a}, F^{r+a}_k) = \frac{i}{2\kappa}\partial_\mu \int d\mathbf{r}' \bar{G}(\mathbf{r}-r')\rho^{r+a}(\mathbf{r}',t) \tag{3.34}$$

which is the solution for the CS gauge fields in the holomorphic gauge.

Up to now, we have been considering the classical canonical formalism. The corresponding second quantized Hamiltonian operator is given by

$$H_T = \int d\mathbf{r}\Big[\frac{1}{m}\Big(< D\tilde{\Psi}(\mathbf{r}), \bar{D}\Psi(\mathbf{r}) > + < \bar{D}\tilde{\Psi}(\mathbf{r}), D\Psi(\mathbf{r}) >\Big) + \Lambda^A(\mathbf{r})\cdot\phi^A(\mathbf{r})\Big].$$

Dynamical equations are defined by the commutator

$$i\partial_t \mathcal{O}(\mathbf{r},t) \approx \big[\mathcal{O}(\mathbf{r},t), H_T\big]. \tag{3.35}$$

The Heisenberg equation of motion for the matter field is given by

$$i\partial_t \Psi(\mathbf{r},t) = \mathcal{H}_S\Psi(\mathbf{r},t) \equiv -\frac{1}{m}(D\bar{D}+\bar{D}D)\Psi(\mathbf{r},t) + ia_0^A T^A \Psi(\mathbf{r}). \tag{3.36}$$

Here the operator $\mathcal{H}_S$ contains the solutions (3.29)–(3.30) for the statistical gauge fields.

It is not difficult to notice that the many-particle wave function

$$\Phi(\mathbf{r}_1,\ldots,\mathbf{r}_N;t) = < 0|\Psi(\mathbf{r}_1,t)\cdots\Psi(\mathbf{r}_N,t)|\Phi >$$

satisfies the Shrödinger equation

$$i\partial_t\Phi(\mathbf{r}_1,\ldots,\mathbf{r}_N;t) = -\frac{1}{m}\sum_{I=1}^{N}\Big[\bar{D}_I D_I + D_I^\star \bar{D}_I^\star\Big]\Phi(\mathbf{r}_1,\ldots,\mathbf{r}_N;t) \tag{3.37}$$

with the derivative operators given by

$$D_I = \partial_I - \frac{1}{2\pi\kappa}\sum_{I\neq J}\frac{T_I^a\otimes T_J^a}{z_I - z_J}, \quad \bar{D}_I = \bar{\partial}_I,$$

$$D_I^\star = \partial_I, \quad \bar{D}_I^\star = \partial_I + \frac{1}{2\pi\kappa}\sum_{I\neq J}\frac{T_I^{r+a}\otimes T_J^{r+a}}{\bar{z}_I - \bar{z}_J}$$

(the matrices T_I^A act on the group variables of I^{th} particle).

As the first application of the framework described above, consider the case where the fermions are in the fundamental ("chiral") representation $T^a_{(1)} = t^a$ and $T^a_{(2)} = 0$. The Hamiltonian takes the form

$$H = -\frac{1}{m}\sum_I \Big[\bar{\mathcal{D}}_I \mathcal{D}_I \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{D}^\star_I \bar{\mathcal{D}}^\star_I \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\Big],$$

where the covariant derivative operators are given by the KZ connections

$$\mathcal{D}_I = \partial_I - \frac{1}{2\pi\kappa}\sum_{I\neq J}\frac{t^a_I \otimes t^a_J}{z_I - z_J}, \quad \bar{\mathcal{D}}_I = \bar{\partial}_I,$$

$$\mathcal{D}^\star_I = \partial_I, \quad \bar{\mathcal{D}}^\star_I = \bar{\partial}_I + \frac{1}{2\pi\kappa}\sum_{I\neq J}\frac{t^a_I \otimes t^a_J}{\bar{z}_I - \bar{z}_J}.$$

Another case of interest is a "symmetric" representation $T^a_{(1)} = T^a_{(2)} = t^a$. Now

$$T^a = T^{r+a} = \begin{pmatrix} t^a & 0 \\ 0 & t^a \end{pmatrix}.$$

The corresponding Hamiltonian is given by

$$H = -\frac{1}{m}\sum_I \Big[\bar{\mathcal{D}}_I \mathcal{D}_I \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{D}^\star_I \bar{\mathcal{D}}^\star_I \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big].$$

Note that in the case of "chiral" representation the Hamiltonian is not Hermitean [130], [103], and conjugation causes the interchange between upper and lower components of the matter field doublet (3.11). At the same time, in the "symmetric" representation, $H^\dagger = H$.

4. Similarity Transformation and Hamiltonian Diagonalization

The matter Hamiltonian

$$H_{\text{Matter}} = \int d\mathbf{r}' \frac{1}{2m} < D_k \tilde{\Psi}(\mathbf{r}), D_k \Psi(\mathbf{r}) > \tag{3.38}$$

contains the gauge connection in the form of a gradient

$$\mathcal{F}_k(\mathbf{r}) = \frac{i}{2\kappa}\partial_k\Big[\int d\mathbf{r}' G(\mathbf{r}-r')\rho^a(\mathbf{r}')\cdot T^a - \int d\mathbf{r}' \bar{G}(\mathbf{r}-r')\rho^{r+a}(\mathbf{r}')\cdot T^{r+a}\Big].$$

The situation is simplified when $G = U(1)$: the generators commute and CS fields can be eliminated by means of a suitably chosen complex gauge transformation, reducing the Hamiltonian to the free (diagonal) form.

Formally the gauge fields can be removed by going to the new field variables

$$X(x) = \begin{pmatrix} \chi(x) \\ \tilde{\chi}^\star(x) \end{pmatrix} \quad \text{and} \quad \tilde{X}(x) = (\tilde{\chi}(x), \chi^\star(x))$$

defined by

$$\Psi(\mathbf{r}) = U(\mathbf{r};\gamma)X(\mathbf{r}), \quad \tilde{\Psi}(\mathbf{r}) = \tilde{X}(\mathbf{r})U^{-1}(\mathbf{r};\gamma), \tag{3.39}$$

where $U(\mathbf{r};\gamma)$ is a holonomy operator (or monodromy matrix) associated to an oriented open path γ in R^2 connecting the points $\mathbf{r}_0$ and $\mathbf{r}$:

$$U(\mathbf{r};\gamma) = P\exp\Big(\int\limits_{\gamma} dx^k\,\mathcal{F}_k(\mathbf{x})\Big). \tag{3.40}$$

(In (3.40), P is a path ordering operation and $\mathbf{r}_0$ is some fixed point.)

Note that due to the non-commutativity of density operators

$$[\rho^A(\mathbf{r}),\rho^B(\mathbf{r}')] = f_{ABC}\rho^C(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}'),$$

the path-ordering is a non-trivial operation. Below we describe a much simpler procedure which in principle permits to get some information on non-Abelian wave functions.

Introduce the operator

$$\Omega(\mathbf{r}) = -\frac{i}{2\kappa}\int d\mathbf{r}' G(\mathbf{r}-\mathbf{r}')\rho^m(\mathbf{r}')\cdot H^m + \frac{i}{2\kappa}\int d\mathbf{r}'\bar{G}(\mathbf{r}-\mathbf{r}')\rho^{r+m}(\mathbf{r}')\cdot H^{r+m},$$

where

$$\rho^M(\mathbf{r}) = i\tilde{\Psi}(\mathbf{r})H^M\Psi(\mathbf{r})$$

are mutually commuting charge densities. The matrices

$$H^m = \begin{pmatrix} H^m_{(1)} & 0 \\ 0 & H^m_{(2)} \end{pmatrix}, \quad H^{r+m} = \begin{pmatrix} H^m_{(2)} & 0 \\ 0 & H^m_{(1)} \end{pmatrix}, \quad m = 1,\dots,R,$$

are the Cartan generators in the representation $\mathcal{R}(g)$. Consider the transformations

$$\Psi(\mathbf{r}) = e^{\Omega(\mathbf{r})}X(\mathbf{r}), \quad \tilde{\Psi}(\mathbf{r}) = \tilde{X}(\mathbf{r})e^{-\Omega(\mathbf{r})}. \tag{3.41}$$

The action of the diagonal Cartan generators on the Ψ-fields

$$H^M\Psi_{\mathbf{w}} = \Upsilon^M_{\mathbf{w}}\cdot\Psi_{\mathbf{w}} \tag{3.42}$$

defines a $2R$ dimensional weight vector $\Upsilon^M_{\mathbf{w}}$ $(M = 1,\dots,R,r+1,\dots,r+R)$

$$\Upsilon^m_{\mathbf{w}} = \begin{pmatrix} w^m_1 & 0 \\ 0 & w^m_2 \end{pmatrix}, \quad \Upsilon^{r+m}_{\mathbf{w}} = \begin{pmatrix} w^m_2 & 0 \\ 0 & w^m_1 \end{pmatrix}.$$

(Remind that $\mathbf{w}_\sigma$'s are the weight vectors of the representation $D_{(\sigma)}$.)

The transformation (3.41) can be written in the component form

$$\Psi_{\mathbf{w}}(\mathbf{r}) = \sum_{\mathbf{w}'}\Big(e^{\Omega(\mathbf{r})}\Big)_{\mathbf{w},\mathbf{w}'}X_{\mathbf{w}'}(\mathbf{r}) = e^{\Omega_{\mathbf{w}}(\mathbf{r})}X_{\mathbf{w}}(\mathbf{r}), \tag{3.43}$$

where the operators

$$\Omega_{\mathbf{w}}(\mathbf{r}) = -\frac{i}{2\kappa}\int d\mathbf{r}' G(\mathbf{r}-r')\rho^m(\mathbf{r}')\cdot\Upsilon^m_{\mathbf{w}} + \\ + \frac{i}{2\kappa}\int d\mathbf{r}'\bar{G}(\mathbf{r}-r')\rho^{r+m}(\mathbf{r}')\cdot\Upsilon^{r+m}_{\mathbf{w}}$$

are labeled by the corresponding weight vectors.

In order to find the (anti)commutation rules obeyed by matter fields, we use the relations

$$[\Omega_{\mathbf{w}}(1), \Psi_{\mathbf{w}'}(2)] = -\frac{1}{2\kappa} G(1-2) < \mathbf{w}, w' > \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Psi_{\mathbf{w}}(2). \qquad (3.44)$$

Suppose that both matter fields $X(x)$ and $\Psi(x)$ correspond to fermions. Straightforward calculations show that the requirement of the fermionic commutation relations together with (3.43) and (3.44) leads to the conditions imposed on the weight vectors of the representations $D_{(1)}$ and $D_{(2)}$:

$$e^{\frac{i}{2\kappa}<\mathbf{w}_1, \mathbf{w'}_1>} = e^{\frac{i}{2\kappa}<\mathbf{w}_2, \mathbf{w'}_2>} = 1. \qquad (3.45)$$

Note that in certain circumstances, it is more suitable to consider $X(x)$ as bosonic fields (e.g., in the Ginzburg–Landau description of QHE). In these cases, instead of (3.45) we have to choose a weight lattice which satisfies the condition

$$e^{\frac{i}{2\kappa}<\mathbf{w}_1, \mathbf{w'}_1>} = e^{\frac{i}{2\kappa}<\mathbf{w}_2, \mathbf{w'}_2>} = -1. \qquad (3.46)$$

Let $|\Phi>$ be the eigenstate of the Hamiltonian H_{matter}. The corresponding N-particle wave function is given by the matrix elements

$$< 0|\Psi_{\mathbf{w}(1)}(\mathbf{r}_1) \cdots \Psi_{\mathbf{w}(N)}(\mathbf{r}_N)|\Phi >=$$

$$= \prod_{I<K} \mathcal{U}(z_I, \mathbf{w}(I); z_J, \mathbf{w}(J)) < 0|e^{\sum \Omega_{\mathbf{w}}(I)(\mathbf{r}_I)} X_{\mathbf{w}(1)}(\mathbf{r}_1) \cdots X_{\mathbf{w}(N)}(\mathbf{r}_N)|\Phi > .$$

Here

$$\mathcal{U}(z_I, \mathbf{w}(I); z_J, \mathbf{w}(J) >) = G(z_I - z_K) \begin{pmatrix} w_1^m(I) \cdot w_1^m(K) & 0 \\ 0 & w_2^m(I) \cdot w_2^m(K) \end{pmatrix} -$$

$$-\bar{G}(z_I - z_K) \begin{pmatrix} w_2^m(I) \cdot w_2^m(K) & 0 \\ 0 & w_1^m(I) \cdot w_1^m(K) \end{pmatrix}.$$

Following (3.45), the weight vectors must belong to a lattice defined by the equations

$$\frac{1}{2\kappa\pi} < \mathbf{w}_1(I), \mathbf{w}_1(K) >= \pm 2 p_{IK},$$

$$\frac{1}{2\kappa\pi} < \mathbf{w}_2(I), \mathbf{w}_2(K) >= \pm 2 q_{IK}$$

with p_{IK} and q_{IK} integers. (In the bosonized theory, r.h.s. of these relations are changed by ± 1 giving odd numbers)

As we see, the wave function is factorized into a "kinematical" prefactor and some dynamical part. The typical term in the prefactor is of the form

$$\begin{pmatrix} (z_I - z_K)^{\pm 2 p_{IK}} (\bar{z}_I - \bar{z}_K)^{\mp 2 q_{IK}} & \\ 0 & (z_I - z_K)^{\pm 2 q_{IK}} (\bar{z}_I - \bar{z}_K)^{\mp 2 p_{IK}} \end{pmatrix}, \qquad (3.47)$$

and depends on representations and quantum numbers carried by particles under consideration.

As a practical application of the proposed scheme, one can indicate the theory of the quantum Hall effect (see, e.g., [126]), where the expressions like (3.47) are used as building blocks for many-particle wave functions.

This and other developments will be considered in the subsequent section.

5. The Quantum Hall Effect and the Laughlin States

In the Abelian case which is obtained by the formal substitutions $H^A \to i$, $\rho^a \to -\rho = -\tilde{\psi}\psi$, $\alpha_A \to i$, the gauge fields can be removed by the non-unitary similarity transformations [91], [42]

$$\psi(\mathbf{r}) = S\chi(\mathbf{r})S^{-1} = e^{-\frac{1}{2\kappa}\int d\mathbf{r}' G(z-z')\bar{\chi}(\mathbf{r}')\chi(\mathbf{r}')}\chi(\mathbf{r}), \tag{3.48}$$

$$\tilde{\psi}(\mathbf{r}) = S\tilde{\chi}(\mathbf{r})S^{-1} = \tilde{\chi}(\mathbf{r})e^{\frac{1}{2\kappa}\int d\mathbf{r}' G(z-z')\bar{\chi}(\mathbf{r}')\chi(\mathbf{r}')}. \tag{3.49}$$

(Here and hereafter we abandon the doublet notation.) The χ fields will satisfy the Fermi–Dirac commutation relations, i.e.,

$$\frac{1}{2\kappa} = \pm 2\pi p \tag{3.50}$$

with a positive integer p.

Note that in the Abelian case, the holomorphic ($A = 0$) and axial ($A_y = 0$) gauges are related by the complex gauge transformation

$$A^k_{axial} = A^k_{hol} - i\frac{1}{\kappa}\partial_k\lambda = -\frac{1}{2\kappa}\delta_{k1}\int d\mathbf{r}'\delta(x-x')\epsilon(y-y')\varrho(\mathbf{r}'), \tag{3.51}$$

where

$$\lambda(\mathbf{r}) = \frac{1}{2\pi}\int d\mathbf{r}' \ln|z-z'|\varrho(\mathbf{r}') + \\ + \frac{i}{2\pi}\int d\mathbf{r}' \tan^{-1}|\frac{y-y'}{x-x'}|\epsilon(x-x')\epsilon(y-y')\varrho(\mathbf{r}').$$

The Hamiltonian

$$H_{matter} = \int d\mathbf{r}'\frac{1}{2m}[D_k\tilde{\psi}D_k\psi + D^\star_k\psi^\star D^\star_k\tilde{\psi}^\star]$$

can be represented in a free form:

$$H_{matter} = \int d\mathbf{r}'\frac{1}{2m}[\partial_k\tilde{\chi}\partial_k\chi + \partial_k\chi^\star\partial_k\tilde{\chi}^\star]. \tag{3.52}$$

Remind that in the theory there are two pairs of canonically conjugate variables: $(\chi, \tilde{\chi})$ and $(\chi^\star, \tilde{\chi}^\star)$. It can be shown (see Appendix C) that in the Abelian case *tilde* -operation can be identified with the Hermitean conjugation, i.e.,

$$\tilde{\chi} = \chi^\dagger, \quad \tilde{\chi}^\star = \chi^{\star\dagger}. \tag{3.53}$$

(This fact will be useful in order to study completeness relations in the corresponding Hilbert space.)

For further needs, it is convenient to use a charge conjugate field $\chi_c \equiv \chi^\star$. In terms of newly introduced fields, the Hamiltonian is expressed as follows

$$H_{matter} = \int d\mathbf{r}' \frac{1}{2m} \partial_k \chi^\dagger(\mathbf{r}') \partial_k \chi(\mathbf{r}') - \\ - \int d\mathbf{r}' \frac{1}{2m} \partial_k \chi_c^\dagger(\mathbf{r}') \partial_k \chi_c(\mathbf{r}') + \Delta_1, \tag{3.54}$$

i.e., it corresponds to the two types of free fermions. Δ_1 is a reordering constant which will be specified below. The basic anticommutators are given by the relations

$$\{\chi(\mathbf{r}), \chi^\dagger(\mathbf{r}')\} = \delta(\mathbf{r} - r'), \quad \{\chi_c(\mathbf{r}), \chi_c^\dagger(\mathbf{r}')\} = \delta(\mathbf{r} - r').$$

The quantum Hall effect is a condensed matter phenomenon, taking place at low temperatures when the planar system is exposed to a strong perpendicular magnetic field $\mathcal{B} = \epsilon_{ik} \partial_i A^k$. Below we consider the standard case of external homogeneous magnetic field generated by the symmetric gauge potential $A_x = -\frac{1}{2}\mathcal{B}y, A_y = \frac{1}{2}\mathcal{B}x$. The corresponding Hamiltonian will be now

$$H_{matter} = \int d\mathbf{r}' \frac{1}{2m} \{\nabla_k \chi^\dagger(\mathbf{r}') \nabla_k \chi(\mathbf{r}') - \nabla_k \chi_c^\dagger(\mathbf{r}') \nabla_k \chi_c(\mathbf{r}')\} + \Delta_1, \tag{3.55}$$

where the covariant derivatives are defined by

$$\nabla_k \chi = (\partial_k - ieA_k)\chi, \quad \nabla_k \chi_c = (\partial_k + ieA_k)\chi_c.$$

For simplicity assume that the system is spin polarized and treat electrons as scalar fermions. The fermion fields can be decomposed into the normal modes

$$\chi(\mathbf{r}, t) = \sum_{n=0}^{\infty} \sum_{j=0}^{N_B - 1} F_{nj} U_{nj}(\mathbf{r}) e^{-iE_n t}, \tag{3.56}$$

$$\chi_c(\mathbf{r}, t) = \sum_{n=0}^{\infty} \sum_{j=0}^{N_B - 1} F_{nj}^c \bar{U}_{nj}(\mathbf{r}) e^{iE_n t}, \tag{3.57}$$

where $U_{nj}(\mathbf{r})$ are solutions of the one-particle Schrödinger equation

$$-\frac{1}{2m} \nabla_k^2 U_{nj}(\mathbf{r}) = E_n U_{nj}(\mathbf{r}),$$

and $E_n = \frac{|e\mathcal{B}|}{m}(n + \frac{1}{2})$ are the energy eigenvalues. The quantity

$$N_B = \frac{|e\mathcal{B}|}{2\pi} \cdot (Area)$$

is the number of quantum states per Landau level. F_{nj} and $F^{\dagger}_{nj}$ are the Fock space lowering and rising Fermi operators and satisfy the usual relations

$$\{F_{nj}, F^{\dagger}_{ml}\} = \delta_{nm}\delta_{jl}.$$

The same is valid for the charge conjugate operators F^{c}_{nj} and F^{c+}_{ml}.

The Hamilton and angular momentum operators are given by

$$H_{matter} = \sum_{nj} E_n F^{+}_{nj} F_{nj} - \sum_{nj} E_n F^{c+}_{nj} F^{c}_{nj} + \Delta_1, \tag{3.58}$$

$$J = \sum_{nj} [j F^{+}_{nj} F_{nj} - j F^{c+}_{nj} F^{c}_{nj}] + \Delta_2. \tag{3.59}$$

In (3.58)–(3.59), we abbreviate

$$\sum_{nj} \equiv \sum_{n=0}^{\infty} \sum_{j=0}^{N_B-1}.$$

The reordering constants

$$\Delta_1 = \sum_{n=0}^{\infty} \sum_{j=0}^{N_B-1} E_n = N_B \sum_{n=0}^{\infty} E_n, \tag{3.60}$$

$$\Delta_2 = \sum_{n=0}^{\infty} \sum_{j=0}^{N_B-1} j = \frac{N_B(N_B-1)}{2} \sum_{n=0}^{\infty} 1 \tag{3.61}$$

are the energy and the angular momentum of a state with totally occupied one-particle excitations.

The eigenstates of the Hamiltonian H_m are represented by the direct products

$$|N> \otimes |M_c> \sim F^{+}_{n_1 j_1} \cdots F^{+}_{n_N j_N} |0> \otimes F^{c+}_{m_1 l_1} \cdots F^{c+}_{m_M l_M} |0_c>, \tag{3.62}$$

where the vacuum states are annihilated by the lowering operators

$$F_{nj}|0> = F^{c}_{ml}|0_c> = 0, \quad n, m = 0, 1, \ldots; \quad j, l = 0, 1, \ldots, N_B - 1. \tag{3.63}$$

The state vector (3.62) corresponds to the energy eigenvalue

$$\frac{|e\mathcal{B}|}{m}[\sum_{i=1}^{N}[(n_i + \frac{1}{2}) - \sum_{k=1}^{M}(m_k + \frac{1}{2})] + \Delta_1.$$

The angular momentum of this state is

$$J = \sum_{i=1}^{N} j_i - \sum_{k=1}^{M} l_k + \Delta_2.$$

Represent the matter Hamiltonian as the sum

$$H_{matter} = H + H_c, \tag{3.64}$$

where

$$H = \sum_{nj} E_n F^+_{nj} F_{nj}, \quad H_c = -\sum_{nj} E_n F^{c+}_{nj} F^c_{nj} + \Delta_1. \tag{3.65}$$

The first term here corresponds to the particle degrees of freedom while the second one to the holes in the charge conjugate sector. In order to justify this assertion, define the states

$$|\Omega >= \prod_{n=0}^{\infty} \prod_{j=0}^{N_B-1} F^+_{nj}|0> \tag{3.66}$$

and

$$|\Omega_c >= \prod_{m=0}^{\infty} \prod_{l=0}^{N_B-1} F^{c+}_{ml}|0_c> . \tag{3.67}$$

Together with the vacua $|0>$ and $|0_c>$, they satisfy the following relations

$$H|0>= H_c|\Omega_c>= 0, \quad H|\Omega>= \Delta_1|\Omega>, \quad H_c|0_c>= \Delta_1|0_c> .$$

The elementary charged excitations with the energy E_n and angular momentum j can be identified with the states

$$F^+_{nj}|0> \quad \text{or} \quad F^c_{nj}|\Omega_c> .$$

In the same time, the states

$$F_{nj}|\Omega> \quad \text{or} \quad F^{c+}_{nj}|0_c>$$

can be interpreted as opposite charge hole excitations with the energy $-E_n$ and the angular momentum $-j$.

The corresponding wave functions are determined by the matrix elements

$$< 0|\chi(\mathbf{r},t)F^+_{nj}|0>=< \Omega_c|\chi^\dagger_c(\mathbf{r},t)F^c_{nj}|0>= U_{nj}(\mathbf{r})e^{-iE_n t},$$
$$< 0_c|\chi_c(\mathbf{r},t)F^{c+}_{nj}|0_c>=< \Omega|\chi^\dagger(\mathbf{r},t)F_{nj}|\Omega>= \bar{U}_{nj}(\mathbf{r})e^{iE_n t}.$$

As basic sets in the Hilbert space, one can use the coordinate representation vectors which satisfy the completeness relations

$$|0><0| + \sum_{N\geq 1} \frac{1}{N!} \int [\prod_{1\leq I\leq N} d\mathbf{r}_I] \chi^\dagger(1)\cdots\chi^\dagger(N)|0> $$
$$< 0|\chi(N)\cdots\chi(1) = \mathbb{1}, \tag{3.68}$$
$$|\Omega><\Omega| + \sum_{N\geq 1} \frac{1}{N!} \int [\prod_{1\leq I\leq N} d\mathbf{r}_I] \chi(1)\cdots\chi(N)|\Omega> $$
$$< \Omega|\chi^\dagger(N)\cdots\chi^\dagger(1) = \mathbb{1}. \tag{3.69}$$

Similar relations hold in the conjugate sector. Note that the validity of these completeness relations is guaranteed by (3.53).

So the multi-particle states can be represented by the wave functions

$$< 0|\chi(1)\cdots\chi(N_e)|\Phi_e > \quad \text{or} \quad < \Omega|\chi^\dagger(1)\cdots\chi^\dagger(N_h)|\Phi_e > \tag{3.70}$$

and multi-hole state by the wave functions

$$< 0_c|\chi_c(1)\cdots\chi_c(N_h)|\Phi_h > \quad \text{or} \quad < \Omega_c|\chi_c^\dagger(1)\cdots\chi_c^\dagger(N_e)|\Phi_h > . \tag{3.71}$$

In the theory of QHE, a distinguished role is played by the states where all the particles are in the lowest Landau level (LLL). For the LLL ($n = 0$) operators and wave functions, we will use the simplified notation

$$F_{0j} \equiv f_j, \qquad U_{0j}(\mathbf{r}) \equiv u_j(\mathbf{r}).$$

Decompose

$$\chi(\mathbf{r}) = \chi_0(\mathbf{r}) + \chi'(\mathbf{r}),$$

where

$$\chi_0(\mathbf{r}) = \sum_{j=0}^{N_B-1} f_j u_j(\mathbf{r})$$

is the lowest level field operator and LLL states are built up by the application of lowering and rising operators satisfying the oscillator algebra:

$$\{f_j, f_l^\dagger\} = \delta_{jl}.$$

Totally filled LLL state is presented by the vector

$$|\omega >= \prod_{0\le j\le N_B-1} f_j^\dagger|0> .$$

The analogous state in the conjugate sector will be given by

$$|\omega_c >= \prod_{0\le j\le N_B-1} f_j^{c\dagger}|0_c> .$$

Instead of the identity resolution (3.69) for the LLL states, one can use the LLL projection operator

$$\Pi = |\omega >< \omega, | + \tag{3.72}$$

$$+ \sum_{1\le N\le N_B} \frac{1}{N!}\int [\prod_{1\le I\le N} d\mathbf{r}_I]\chi(1)\cdots\chi(N)|\omega >< \omega|\chi^\dagger(N)\cdots\chi^\dagger(1)$$

and its conjugate partner

$$\Pi_c = |\omega_c, N_B >< \omega_c, N_B| + \tag{3.73}$$

$$+ \sum_{1\le N\le N_B} \frac{1}{N!}\int [\prod_{1\le I\le N} d\mathbf{r}_I]\chi_c(1)\cdots\chi_c(N)|\omega_c >< \omega_c|\chi_c^\dagger(N)\cdots\chi_c^\dagger(1).$$

The eigenstates of the Hamiltonian (3.58) are expressed in terms of χ quanta excitations. At the same time, the physical observables and wave functions must be expressed in terms of the fields ψ. As we have already

noted, these operators are related by the similarity transformations (3.48)–(3.49). In terms of ψ fields, the completeness relations and the projection operators are given by

$$S|0><0|S^{-1} + \tag{3.74}$$
$$+\sum_{N\geq 1}\frac{1}{N!}\int[\prod_{1\leq i\leq N} d\mathbf{r}_i]\tilde{\psi}(1)\cdots\tilde{\psi}(N)S|0><0|S^{-1}\psi(N)\cdots\psi(1)=\mathbb{1},$$

$$S|0_c><0_c|S^{-1} + \tag{3.75}$$
$$+\sum_{N\geq 1}\frac{1}{N!}\int[\prod_{1\leq i\leq N} d\mathbf{r}_i]\tilde{\psi}^\star(1)\cdots\tilde{\psi}^\star(N)S|0_c><0_c|S^{-1}\psi^\star(N)\cdots\tilde{\psi}^\star(1)=\mathbb{1}_c,$$

$$\Pi = S|\omega><\omega|S^{-1} + \tag{3.76}$$
$$+\sum_{1\leq N\leq N_B}\frac{1}{N!}\int[\prod_{1\leq I\leq N} d\mathbf{r}_I]\psi(1)\cdots\psi(N)S|\omega><\omega|S^{-1}\tilde{\psi}(N)\cdots\tilde{\psi}(1),$$

$$\Pi_c = S|\omega_c><\omega_c|S^{-1} + \tag{3.77}$$
$$+\sum_{1\leq N\leq N_B}\frac{1}{N!}\int[\prod_{1\leq I\leq N} d\mathbf{r}_I]\psi^\star(1)\cdots\psi^\star(N)S|\omega_c><\omega_c|S^{-1}\tilde{\psi}^\star(N)\cdots\tilde{\psi}^\star(1).$$

Equations (3.74)–(3.77) together with the properties of the similarity transformation can be used in order to make a reasonable choice of the Hilbert space basis. Below we list these sets indicating the corresponding coordinate representation bra-vectors.

1. Vacua are invariant under the similarity transformation

$$S|0>=|0>,\ <0|S^{-1}=<0| \quad\rightarrow\quad <0|\psi(1)\cdots\psi(N), \tag{3.78}$$
$$S|0_c>=|0_c>,\ \ <0_c|S^{-1}=<0_c| \quad\rightarrow\quad <0_c|\psi^\star(1)\cdots\psi^\star(N). \tag{3.79}$$

2. The operator S does not lead to the Landau level mixing

$$S\Pi S^{-1}=\Pi \quad\rightarrow\quad <\omega|S^{-1}\tilde{\psi}(1)\cdots\tilde{\psi}(N), \tag{3.80}$$
$$S\Pi_c S^{-1}=\Pi_c \quad\rightarrow\quad <\omega_c|S^{-1}\tilde{\psi}^\star(1)\cdots\tilde{\psi}^\star(N). \tag{3.81}$$

The LLL projected Hamiltonian is

$$H_0=\mathcal{H}+\mathcal{H}_c, \tag{3.82}$$

where

$$\mathcal{H}=E_0\sum_{j=0}^{N_B-1} f_j^+ f_j$$

and

$$\mathcal{H}_c=-E_0\sum_{j=0}^{N_B-1} f_j^{c+} f_j^c + N_B E_0.$$

The corresponding angular momentum operator is given by

$$J_0 = \sum_{j=0}^{N_B-1} j[f_j^+ f_j - f_j^{c+} f_j^c] + \frac{N_B(N_B-1)}{2}.$$

Consider the state

$$|\Phi; N_e >= \prod_{j=0}^{N_e-1} f_j^+ |0> . \tag{3.83}$$

This state corresponds to a system of N_e electrons in LLL with the energy $N_e E_0$ and the minimal total angular momentum $J = \frac{1}{2}N_e(N_e-1)$.

The supplementary state in the conjugate sector

$$|\Phi_c; N_h >= \prod_{j=0}^{N_h-1} f_j^{c+} |0_c> \tag{3.84}$$

describes a system of $N_h = N_B - N_e$ holes with the same total energy and the angular momentum

$$J_c = \frac{1}{2}N_B(N_B-1) - \frac{1}{2}N_h(N_h-1).$$

Consequently,

$$|L; N_e >= |\Phi; N_e > \otimes |\omega_c > \tag{3.85}$$

and

$$|G; N_e >= |0> \otimes |\Phi_c; N_h > \tag{3.86}$$

are degenerate eigenstates of H_0 with the energy $N_e E_0$.

Now it is easy to show that (3.85) describes the Laughlin state [99] with filling fraction

$$\nu = \frac{1}{2p+1}.$$

The corresponding wave function is obtained by applying the projection operator Π:

$$\begin{aligned} \mathbf{1} \otimes \Pi |L; N_e > \to \Psi_\nu^e(1,,,,,N_e) &=< 0|\psi(1)\cdots\psi(N_e)|\Phi, N_e >= \\ &= \prod_{K<L} (z_K - z_L)^{2p} \langle 0|\chi(1)\cdots\chi(N_e)|\Phi; N_e\rangle. \end{aligned} \tag{3.87}$$

The last factor

$$\langle 0|\chi(1)\cdots\chi(N_e) f_0^+ \cdots f_{N_e-1}^+|0\rangle = \prod_{1\le K<L\le N_e} (z_K - z_L) e^{-\frac{eB}{4}\sum_{I=1}^{N_e}|z_I|^2}$$

is the Slater determinant of one-particle LLL states.

As an alternative, one can use the bosonized version of a similarity transformation, take as $\chi(x)$ the Bose fields and fix the statistical parameter by

$$-\frac{1}{2\pi\kappa} = 2p+1.$$

As a ground state, consider

$$|\Phi_b; N_e >= \left(f_0^+\right)^{N_e} |0> . \tag{3.88}$$

The wave function is given by the matrix element

$$\begin{aligned}\Psi^e_\nu(1,,,,,N_e) &=< 0|\psi(1)\cdots\psi(N_e)|\Phi_b, N_e >= \\ &= \prod_{K<L}(z_K - z_L)^{2p+1}\langle 0|\chi(1)\cdots\chi(N_e)|\Phi_b; N_e\rangle. \end{aligned}\tag{3.89}$$

The factor

$$\langle 0|\chi(1)\cdots\chi(N_e)\left(f_0^+\right)^{N_e}|0\rangle = e^{-\frac{eB}{4}\sum_{I=1}^{N_e}|z_I|^2}$$

is a symmetric wave function describing N_e lowest Landau level bosons condensed in the state with the zero angular momentum.

Another state of interest is the Girvin state (3.86). The corresponding wave function is extracted acting by the projection operator Π_c:

$$\begin{aligned}\mathbb{1}\otimes\Pi_c|G; N_e >\to \Psi^e_{\nu_c}(1,\ldots,N_e) &= \langle\omega_c|S^{-1}\tilde\psi^\star(1)\cdots\tilde\psi^\star(N_e)|\Phi_c\rangle = \\ &= \int\cdots\int\prod_{K=N_e}^{N_B}[d\mathbf{r}_K]\langle\omega_c|S^{-1}\tilde\psi^\star(1)\cdots\tilde\psi^\star(N_B)|0_c\rangle\times \\ &\quad\times\langle 0_c|\psi^\star(N_B)\cdots\psi^\star(N_e+1)|\Phi_c >= \\ &= \int\cdots\int\prod_{K=1}^{N_e}[d\mathbf{r}_K]\overline{\Psi^e_0(1,\cdots,N_B)}\times\Psi^e_p(1,\cdots,N_e),\end{aligned}\tag{3.90}$$

where the relation $< \omega_c|S^{-1} =< \omega_c|$ is assumed to be valid. This wave function describes the state with filling fraction $\nu_c = 1-\nu = 2p/2p+1$ [64] (Ψ^e_0 corresponds to the totally filled lowest level).

Another representation of the same state will be given by the matrix element

$$\begin{aligned}&\langle\omega|S^{-1}\tilde\psi(N_e+N_h)\cdots\tilde\psi(N_e+1)|\Phi\rangle = \\ &= \int\cdots\int\prod_{K=1}^{N_h}[d\mathbf{r}_K]\langle\omega|S^{-1}\tilde\psi^(N_e+N_h)\cdots\tilde\psi(N_e+1)\tilde\psi^(N_e)\cdots\tilde\psi^(1)|0\rangle\times \\ &\quad\times\langle 0|\psi(1)\cdots\psi(N_e)|\omega\rangle = \\ &\int\cdots\int\prod_{K=1}^{N_e}[d\mathbf{r}_K]\overline{\Psi^e_0(1,\cdots,N_B)}\times\Psi^e_p(1,\cdots,N_e).\end{aligned}\tag{3.91}$$

The holomorphic factor $\prod(z_I - z_J)^{2p}$ is usually associated to the $2p$ magnetic flux quanta attached to electrons forming what is called Jain's composite particles [86]. In the present discussion, it is a matrix element of a complex gauge transformation relating two different, non-unitary equivalent bases.

In the same way, one can consider wave functions for noncompressible states of fractionally charged quasiparticles.

Although in the non-Abelian case we do not know the exact wave function, one can nevertheless get some sort of kinematical information contained in the form of similarity transformations (3.43). In order to find the N-particle wave function, we need some basis vectors, e.g.,

$$\begin{aligned} &< 0|\psi_{\mathbf{w}(1)}(1)\cdots\psi_{\mathbf{w}(N)}(N) = \\ =&< 0|\exp[\sum \Omega_{\mathbf{w}(I)}(I)]\exp[\frac{1}{2\kappa}\sum_{I>K} G(z_I - z_K) < \mathbf{w}(I), \mathbf{w}(K) >] \\ &< 0|\chi_{\mathbf{w}(1)}(1)\cdots\chi_{\mathbf{w}(N)}(N) = \\ =&\prod_{I<K}(z_I - z_K)^{-\frac{1}{2\pi\kappa}<\mathbf{w}(I),\mathbf{w}(K)>} < 0|\chi_{\mathbf{w}(1)}(1)\cdots\ \chi_{\mathbf{w}(N)}(N). \quad (3.92) \end{aligned}$$

(deriving (3.92) we have used the fact that vacuum is annihilated by the operators $\Omega_{\mathbf{w}(I)}(I)$).

Apply this formula to the case of $SU(2)$ non-Abelian theory. For the ψ's in the fundamental representation, the weight vectors

$$w^\alpha = \pm i$$

correspond to the isospin up $\uparrow$ an down $\downarrow$ components. The corresponding basis vector is given by the expression

$$\begin{aligned} &\prod(z_{I\uparrow} - z_{K\uparrow})^{2p}\prod(z_{R\downarrow} - z_{S\downarrow})^{2p} \times \\ &\times\prod(z_{I\uparrow} - z_{R\downarrow})^{-2p} < 0|\chi_{\mathbf{w}(1)}(1)\cdots\chi_{\mathbf{w}(N)}(N). \quad (3.93) \end{aligned}$$

So we see that the wave function of any Hamiltonian eigenstate in this basis contains an holomorphic prefactor indicating the attraction between different isospins and repulsion between the same ones. It was conjectured that this type of wave functions may be related to the multilayered QHE states [71], [50].

CHAPTER 4
ANYON SUPERCONDUCTIVITY

1. Introduction

The BCS theory of superconductivity contains two essential points: the first one is a temperature dependent gap in the spectrum of charged particles, and the second one is the spontaneous breaking of U(1) gauge symmetry.

The gap provides the rigidity of the wave function. It is given in terms of the order parameter which is a characteristic of a spontaneous symmetry breaking and can be calculated using microscopical equations or the Ginzburg-Landau effective theory.

The Nambu-Goldstone particle associated with the broken symmetry is combined with an electromagnetic potential resulting a massive photon and hence the Meissner effect. The gauge group U(1) is broken to the Z_2 and according to general assertions about the spontaneously broken symmetries a massless excitation is described by a field that transforms under U(1) like the coordinates of the coset space $\mathrm{U}(1)/Z_2$. In other words, under a gauge transformation with a parameter χ, the electron field transforms as

$$\psi(x) \to e^{i\frac{e}{\hbar c}\chi}\psi(x)$$

and the Nambu-Goldstone field undergoes the transformation

$$\Phi(x) \to \Phi + \frac{2e}{\hbar c}\chi.$$

Remind the essence of the Anderson-Higgs-Kibble mechanism in the $U(1)$ gauge theory (see, e.g., [1], [70]).The exact photon propagator can be represented as

$$\mathcal{D}_{\mu\nu}(q) \sim \frac{\eta_{\mu\nu}}{q^2}\frac{1}{1-e^2\Pi(q^2)}.$$

Here the $\Pi(q^2)$ is the invariant structure function determined from the current-current correlator:

$$(q^2\eta_{\mu\nu} - q^\mu q^\nu)\Pi(q^2) \sim \int dx e^{iqx} < 0|T(J_\mu(x)J_\nu(0))|0> .$$

A photon becomes massive if $q^2\Pi(q^2)$ does not vanish as $q_\mu \to 0$, i.e., if there is a pole in the structure function. This pole at $q^2 = 0$ must be associated with the zero mass particles which interact with the electromagnetic current.

In order to make our discussion more transparent, recall some facts about the Meissner effect and London's equation [52]. London's equation for the supercurrent describes a linear response of a superconductive matter to an external electromagnetic field. The supercurrent is defined by

$$\mathbf{j}_s = \frac{\hbar e}{2m}n_s(\nabla\Phi - \frac{2e}{\hbar c}\mathbf{A}), \tag{4.1}$$

where $e = -|e|$ is the electron charge, m is its mass and n_s is the density of the superconducting electrons. $\Phi(x)$ is the phase of the condensate wave function and is defined by the phase of the abnormal electron Green function.

This definition is invariant under the gauge transformation

$$\mathbf{A} \to \mathbf{A} + \nabla\chi(\mathbf{r}),$$
$$\Phi \to \Phi + \frac{2e}{\hbar c}\chi.$$

Equation (4.1) leads to London's equation

$$\nabla^2\mathbf{B} = \lambda_L^2\mathbf{B} \tag{4.2}$$

for the magnetic field inside the superconductor.

The quantity

$$\lambda_L^2 = \frac{mc^2}{4\pi n_s e^2}$$

is known as the London penetration depth. The superconducting density and the penetration depth are temperature dependant quantities.

The solution of (4.2)

$$\mathbf{B} = \mathbf{B}_0 e^{-x/\lambda_L}$$

explains the Meissner effect.

The phase function Φ is associated with a scalar massless mode – a phason or a Goldstone particle, which couples to the Maxwell field. This coupling is provided by the term

$$\mathcal{L} \sim (\partial_\mu\Phi - \frac{2e}{\hbar c}A_\mu)^2.$$

In $3+1$ dimensions, the presence of zero mass particles is guaranteed by the Goldstone theorem [65], and is a consequence of a spontaneous symmetry breaking. However, in the lower dimensions, there is no spontaneous symmetry breaking (see, e.g., [70]) and the origin of the zero mass pole must be founded somewhere else. Such a mechanism was proposed by the Fetter, Hanna and Laughlin in their paper on the Chern–Simons superconductivity [53].

In $2+1$ dimensions, a conserved current can couple not only the electromagnetic potentials, also the Chern–Simons gauge field. The gauge invariance tells us that the corresponding interaction is described by the Lagrangian

$$\mathcal{L}_{int} \sim j^\mu(x)\left(A_\mu(x) - \eta\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda(x)\right),$$

where j^μ is a gauge invariant current and η-is a constant.

The needed massless field is associated with the three dimensional curl

$$\partial_\mu\Phi \sim \varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda,$$

and hence with the gauge field $a_\mu(x)$ (in $2+1$ dimensions, a massless gauge field has one degree of freedom).

The kinematical part for this field can be constructed from the Chern-Simons term $\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$ which is gauge invariant up to the total divergence. The Maxwell-like term

$$-\frac{1}{4}(\partial_\mu a_\nu(x)-\partial_\nu a_\mu(x))^2$$

contains higher derivatives and seems to be less important in the static limit.

In the absence of electromagnetic interactions, the massless pole must be presented in the current-current correlators. Besides the massless pole, there must be a gap in the fermion spectrum. This gap stabilizes the Goldstone mode against decay into fermion-hole pairs.

In summary, a reasonable criterion for superconductivity is the presence of a massless pole in the current-current correlation function combined with a gap in the fermion spectrum.

These conditions can be realized in a model describing a matter interacting with the Chern–Simons field and the external electromagnetic potential. The magnetic part of the Chern–Simons field can provide a homogeneous background which organizes the electrons into Landau bands with a finite gap. Integrating out the matter fields, one gets an effective Lagrangian of the following form [10]

$$\Lambda[f_{\mu\nu}]=\epsilon e_i^2+\epsilon_i e_i-\chi b^2+\beta b+\kappa\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda.$$

Here

$$b=\varepsilon_{ik}\partial_i a_k,\quad e_i=\partial_0 a_i-\partial_i a_0$$

are the Chern–Simons magnetic and electric fields. The corresponding strength tensor is $f_{\mu\nu}=\partial_\mu a_\nu-\partial_\nu a_\mu$. Quantities ϵ,ϵ_i,χ do not depend on the Chern–Simons fields.

Consider the partition function

$$\mathcal{Z}=\int Da_\mu e^{i\int d^3x\Lambda[a_\mu]}. \tag{4.3}$$

Performing the change of variables, this integral can be rewritten as [110], [31]

$$\mathcal{Z}=\int D\varphi e^{i\int d^3x L_{eff}[\varphi]},$$

where

$$\begin{aligned}L_{eff}[\varphi]=&-\frac{1}{4\epsilon}\epsilon_i^2+\frac{1}{4\chi}\beta^2-\frac{1}{2}(\partial_0\varphi-qA_0)^2-\frac{v^2}{2}(\partial_k\varphi-qA_k)^2-\\&-\sqrt{\frac{\chi}{2\epsilon^2}}\varepsilon_{ik}\epsilon_i(\partial_k\varphi-qA_k)+\frac{1}{\sqrt{2\chi}}\beta(\partial_0\varphi-qA_0)\end{aligned} \tag{4.4}$$

is expressed in terms of a scalar field φ. This field represents the sound wave with characteristic speed of propagation v. The parameters are given by

$$q = \frac{\kappa}{\sqrt{2\chi}}, \quad v^2 = \frac{\chi}{\epsilon}.$$

The effective Lagrangian (4.4) is invariant under the gauge transformation

$$\varphi \to \varphi + q\lambda, \quad A_\mu \to A_\mu + \partial_\mu \lambda$$

(φ transforms as the phase of the complex field carrying the charge q).

In the present chapter, we study the application of the Chern–Simons theory to the description of the anyon superconductor. In Section 2, we consider the problem using the formalism of thermo field dynamics. In Sections 3, we introduce a relativistic model and in Section 4, we calculate the thermodynamical potential. In Section 5, this model is applied to the analysis of the Meissner effect.

2. Anyon Superconductivity in Thermo Field Dynamics

In 1989, Fetter, Hanna and Laughlin [53] showed, that a free gas of anyons, has a massless pole in the electromagnetic linear response function. This result was confirmed by different authors [24], [120], [10], [110], [77], [61], [76], [40] with a hope to realize the anyonic mechanism of the high temperature superconductivity. The central objects of this calculations are current-current correlation functions, in terms of which one can express both the linear response and the effective action or the free energy.

The assertion that the Chern–Simons theory provides an adequate framework for the description of planar superconductors can be easily justified. Consider a matter interacting with a statistical gauge field with a statistical parameter κ. A common approach is based on the expansion around a system of planar fermions in the homogeneous Chern–Simons magnetic field which is generated by the net particle density

$$\bar{b} = \epsilon_{mn}\partial_m \bar{a}^n = -\frac{e}{\kappa}\,\rho.$$

The filling fraction

$$\nu = \frac{\rho}{n_L}$$

is expressed in terms of the electron density ρ and the density of states n_L.

$$n_L = \frac{eb}{2\pi}, \quad \nu = -\frac{2\kappa\pi}{e^2} \quad \text{for} \quad \text{spin} \quad 0,$$
$$n_L = \frac{eb}{\pi}, \quad \nu = -\frac{\kappa\pi}{e^2} \quad \text{for} \quad \text{spin} \quad \frac{1}{2}.$$

For spinless particles the statistical parameter $\theta = \frac{\pi}{\nu}$ ($\kappa = -\frac{e^2\nu}{2\pi}$) and for fermion doublet $\theta = \frac{\pi}{2\nu}$ ($\kappa = -\frac{e^2\nu}{\pi}$).

Consider the situation when the ground state $|\Phi_0\rangle$ corresponds to the N totally filled Landau levels, i.e., $\nu = N$.

The threshold behavior of the linear response can be studied using the corresponding formulae of the preceding chapter. The pole in correlator corresponds to the zero of the function (1.47)

$$D(q) = \left(1 + \frac{1}{\kappa}\Pi_1(q)\right) - \frac{1}{\kappa^2}\left(\omega^2\Pi_0(q) + \mathbf{q}^2\Pi_2(q)\right).$$

For the spinless particles, the threshold values of structure functions are given by (1.42)

$$\Pi_0(0) = \frac{1}{2}\frac{N}{\pi}m\ell^2, \quad \Pi_1(0) = \frac{1}{2}\frac{N}{\pi} \quad \text{and} \quad \Pi_2(0) = -\frac{1}{2}\frac{N^2}{\pi m}.$$

(For spin 1/2 fermions, the same quantities are to be taken with factor 2.)

As we have seen, $\Pi_1(0) = -\kappa$, and the dispersion law for the zero mass excitation is given by

$$\omega^2 = \frac{2\pi\rho}{m^2}\mathbf{q}^2 \quad \text{for} \quad \text{spin} \quad 0,$$
$$\omega^2 = \frac{\pi\rho}{m^2}\mathbf{q}^2 \quad \text{for} \quad \text{spin} \quad \frac{1}{2}.$$

The linear response function for the system under consideration was calculated several times. We will utilize the abbreviations of the paper [24], and quote the result for the kernel

$$\begin{aligned}
K_{\mu\nu}(q) &= \frac{e}{m}\rho\left(\delta_{\mu\nu} - \delta_{\mu 0}\delta_{\nu 0}\right) - ie\Lambda_{\mu\nu}(q), \quad q^\mu = (\omega, \mathbf{q}),\\
\Lambda_{00}(q) &= -\left(\frac{\pi^2\rho}{mN^2}D\right)^{-1} q^2\Sigma_0(q),\\
\Lambda_{0i}(q) &= -\left(\frac{\pi^2\rho}{mN^2}D\right)^{-1}\Big[-q^i\omega\Sigma_0(q) + \varepsilon_{ij}q^j\frac{e^2\rho}{m\kappa}\Xi(q)\Big],\\
\Lambda_{ik}(q) &= -\left(\frac{\pi^2\rho}{mN^2}D\right)^{-1}\Big\{\frac{\pi^2\rho^2}{m^2N^2}\left[\Xi(q) - \Sigma_1(q) + \Sigma_2(q)\right]\delta_{ik} +\\
&\quad + \frac{q^iq^k}{\mathbf{q}^2}\Big[\omega^2\Sigma_0(q) - \frac{\pi^2\rho^2}{m^2N^2}\left(\Xi(q) - \Sigma_1(q) + \Sigma_2(q) + \Sigma_3(q)D\right)\Big] -\\
&\quad - i\varepsilon_{ik}\frac{\pi\rho}{mN}\omega\Xi(q)\Big\}.
\end{aligned}$$

The functions Σ_a, Ξ and D are the following combinations of the structure functions

$$\Sigma_0(q) = -\frac{\pi^2\rho}{mN^2}\Pi_0(q),$$
$$\Sigma_1(q) = -\frac{\pi}{N}\Pi_1(q),$$

$$\Sigma_2(q) = -\frac{m}{\rho}\left[\omega^2\Pi_0(q) + \Pi_3(q) + q^2\Pi_2(q)\right],$$
$$D(q) = [1+\Sigma_1(q)]^2 - \Sigma_0(q)\,[1+\Sigma_2(q)],$$
$$\Xi(q) = -\Sigma_1(q)[1+\Sigma_1(q)] + \Sigma_0(q)[1+\Sigma_2(q)].$$

Note that the existent calculations of this functions are rather tedious and the results are usually given in the form of the low momentum expansion [53], [24], [61], [77].

In the preceding chapter, we have demonstrated a more transparent way to calculate the correlators. It was shown that all invariant functions actually can be expressed with the help of a single function, both for the zero and finite temperatures. At the same time, we do not use the low-momentum expansion.

All the above consideration is valid for the case of the non-zero temperature. The crucial moment is to get information on the finite temperature behavior of the response function and the correlators. This can be done using the Matsubara formalism or some real time statistical field theory. As a first step, we consider the problem in the framework of a real time thermal field theory.

Here we present a brief account of the evaluation of the thermodynamic potential of a non-relativistic fermion matter ψ in 2+1 dimensions, coupled to Chern–Simons and Maxwell fields [41] (for other finite temperature calculations see [120], [76], [77], [61], [90]). The basic Lagrangian is given by

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{e^2\nu}{2\pi}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + en_eA_0 +$$
$$+ i\psi^\dagger i\ D_0\psi - \frac{1}{2m}|D_k\psi|^2 + U\psi^\dagger\psi.$$

Here

$$D_\mu = \partial_\mu + i(eA_\mu + ga_\mu),$$

and n_e is a background neutralizing charge density.

Maxwell and Chern-Simons electric and magnetic fields are defined by the equalities

$$E^i = F^{i0}, \qquad B = \varepsilon_{mn}\partial_m A^n,$$
$$e^i = f^{i0}, \qquad b = \varepsilon_{mn}\partial_m a^n.$$

In the present discussion, we will use the mean-field approximation (MFA): gauge fields are replaced by the average values which are divided into given backgrounds and small fluctuations

$$A_\mu \to < A_\mu > = \bar{A}_\mu + A_\mu, \quad a_\mu \to < a_\mu > = \bar{a}_\mu + a_\mu.$$

Background fields are chosen to correspond to homogeneous magnetic fields $\bar{B}$ and $\bar{b}$. The matter part of the Hamiltonian is given by

$$H_e = \int \left\{ \frac{1}{2m} |\nabla_k \psi|^2 + (e\bar{A}_0 + g\bar{a}_0)\psi^\dagger \psi \right\} \mathrm{d}\mathbf{r} + \\ + \int \{F_0 I_0 - F_k I_k\} \, \mathrm{d}\mathbf{r} = H_0 + H_I. \tag{4.5}$$

Here

$$\nabla_k = \partial_k + i(e\bar{A}_k + g\bar{a}_k),$$
$$F_0 = e\bar{A}_0 + g\bar{a}_0 + \frac{1}{2m}(e\bar{A}_k + g\bar{a}_k)^2, \quad F_k = e\bar{A}_k + g\bar{a}_k,$$
$$I_0 = \psi^\dagger \psi, \quad I_k = \frac{i}{2m} \left[\psi^\dagger \nabla_k \psi - (\nabla \psi)^\dagger \psi \right].$$

The dynamical equations for the gauge fields in MFA are given by

$$\partial_\nu < F^{\nu\mu}(x) > = < J^\mu(x) > - e n_e \delta^{\mu 0},$$
$$- \frac{e^2 \nu}{2\pi} \varepsilon^{\mu\nu\lambda} < f_{\nu\lambda} > = < j^\mu > .$$

Here $< J^\mu >$ and $< j^\mu >$ are the thermal averages

$$< J^\mu > = \left\langle \frac{\delta H_e}{\delta A_\mu(x)} \right\rangle = \frac{\delta \Omega}{\delta < A_\mu(x) >},$$
$$< j^\mu > = < \frac{\delta H_e}{\delta a_\mu(x)} > = \frac{\delta \Omega}{\delta < a_\mu(x) >}.$$

The thermodynamic potential $\Omega\{< A_\mu >, < a_\mu >, \beta\}$ is defined with the help of the grand canonical partition function

$$e^{-\beta\Omega} = \mathrm{Tr} e^{-\beta(H_e - \mu N)},$$

where $\beta = 1/T$ is the inverse temperature, μ is the chemical potential, and N is the particle number operator.

In order to calculate this quantity, we adopt the real-time formalism known as Thermo Field Dynamics (TFD) [129], [102]. TFD was constructed by the requirement to express thermal averages of quantum operators in the form of vacuum expectation values. This idea was achieved by introducing fictitious "tilde" operators $\tilde{A}$ corresponding to each of the operators A describing the system under consideration, and of a thermal vacuum $|0, \beta >$, which is required to satisfy

$$< 0, \beta | A | 0, \beta > = \frac{\mathrm{Tr} A e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}}.$$

For the hamiltonian $H = H_0 + H_I$, the free energy is given by [113]

$$\Omega_e = \Omega_0 + \int_0^1 ds < 0, \beta | \mathrm{T}\{H_I(t_0) e^{-is\int_{-\infty}^{+\infty} dt \hat{H}_I(t)}\} | 0, \beta >,$$

where $|0, \beta >$ is the thermal vacuum in the interaction representation with respect to H_0. The thermal interaction Hamiltonian is

$$\hat{H}_I(t) = H_I(t) - \tilde{H}_I(t).$$

In our case, H_0 and H_I are given by (4.5) (the term with a chemical potential must be added to H_0), and for the free energy, we obtain

$$\begin{aligned}\Omega = \Omega_0 + \int F_\mu(x) < 0, \beta | I^\mu(x) | 0, \beta > \mathrm{d}\mathbf{r} - \\ - \frac{i}{2} \int F_\mu(x) F_\nu(x_1) [< 0, \beta | \mathrm{T}\{I^\mu(x) I^\nu(x_1)\} | 0, \beta > - \\ - < 0, \beta | \mathrm{T}\{I^\mu(x) \tilde{I}^\nu(x_1)\} | 0, \beta >] \mathrm{d}\mathbf{r} \mathrm{d}\mathbf{r}_1 dt_1 + O(e^4),\end{aligned}$$

where

$$\Omega_0 = -\beta^{-1} \ln \mathrm{Tr}\{e^{-\beta H_0}\}$$

does not depend on the gauge field fluctuations.

For our purposes, it is sufficient to consider time independent fluctuations. In that case, the final expression for the free energy up to the desired order in gauge coupling constants will be given by

$$\begin{aligned}\Omega = \Omega_0 + \int \Gamma_0 \Phi_0(\mathbf{r}) \mathrm{d}\mathbf{r} + \qquad (4.6) \\ + \frac{i}{4\pi\ell^2} \int \Phi(\mathbf{r}_1) [S(-\ell^2\Delta_1/2, -\ell^2\Delta_2/2) \delta(\mathbf{r}_1 - \mathbf{r}_2)] \Phi(\mathbf{r}_2) \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2 - \\ - \frac{i\varepsilon(h)}{2\pi m\ell^2} \int \Phi(\mathbf{r}_1) [S'(-\ell^2\Delta_1/2, -\ell^2\Delta_2/2) \delta(\mathbf{r}_1 - \mathbf{r}_2)] H(\mathbf{r}_2) \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2 + \\ + \frac{i}{4\pi m^2\ell^2} \int H(\mathbf{r}_1) [S''(-\ell^2\Delta_1/2, -\ell^2\Delta_2/2) \delta(\mathbf{r}_1 - \mathbf{r}_2)] H(\mathbf{r}_2) \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2.\end{aligned}$$

Here $h = e\bar{B} + g\bar{b}$ is determined by the background ($\ell^2 = 1/|h|$), and

$$\Phi_0 = eA_0 + ga_0, \quad H = eB + gb.$$

The differential operator $S(-\ell^2\Delta_1/2, -\ell^2\Delta_2/2)$ is determined as follows

$$S(x, y) = e^{-(x+y)/2} \sum_{n=0}^{\infty} \sum_{\alpha=0}^{\infty} \frac{n!}{(n+\alpha)!} \frac{x^\alpha + y^\alpha}{2} L_n^\alpha(x) L_n^\alpha(y) \lambda_n^\alpha(\beta), \qquad (4.7)$$

where $L_n^\alpha(x)$ is an adjoint Laguerre polynomial, and the temperature dependence is determined by the functions

$$\lambda_n^0(\beta) = i\,\beta(1 - \sin^2\theta_n) \sin^2\theta_n,$$

$$\lambda_n^\alpha(\beta) = 2\,i\,\frac{\sin^2\theta_n - \sin^2\theta_{n+\alpha}}{E_{n+\alpha} - E_n}.$$

Here

$$\sin^2\theta_n = \frac{1}{1+e^{\beta E_n}}$$

is the Fermi distribution, and E_n is the Landau level energy

$$E_n = \frac{1}{m\ell^2}(n+1/2) - \mu.$$

The quantities S' and S'' are given by

$$S'(x,y) = \frac{\partial}{\partial x}S(x,y),$$
$$S''(x,y) = \frac{\partial^2}{\partial x \partial y}S(x,y)$$

and

$$\Gamma_0 = \frac{|h|}{2\pi}\sum_n \sin^2\theta_n.$$

Corresponding currents are given by

$$\begin{aligned}
&< J^0(\mathbf{r}) > = \frac{\delta\Omega}{\delta A_0(\mathbf{r})} = e\Gamma_0 + \frac{ie}{2\pi\ell^2}S\Phi_0(\mathbf{r}) - \frac{ie\varepsilon(h)}{2\pi m\ell^2}S'H(\mathbf{r}),\\
&< J^k(\mathbf{r}) > = \frac{\delta\Omega}{\delta A_k(\mathbf{r})} =\\
&\qquad = \frac{i}{2\pi m\ell^2}\varepsilon_{kl}\partial_l\Big\{[e'S/2 + e\varepsilon(h)S']\Phi_0(\mathbf{r}) -\\
&\qquad - \frac{\varepsilon(h)}{m}[e'S'/2 + e\varepsilon(h)S'']H(\mathbf{r})\Big\},
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&< j^0(\mathbf{r}) > = \frac{\delta\Omega}{\delta a_0(\mathbf{r})} = g\Gamma_0 + \frac{ig}{2\pi\ell^2}S\Phi_0(\mathbf{r}) - \frac{ig\varepsilon(h)}{2\pi m\ell^2}S'H(\mathbf{r}),\\
&< j^k(\mathbf{r}) > = \frac{\delta\Omega}{\delta a_k(\mathbf{r})} =\\
&\qquad = \frac{i}{2\pi m\ell^2}\varepsilon_{kl}\partial_l\Big\{[g'S/2 + g\varepsilon(h)S']\Phi_0(\mathbf{r}) -\\
&\qquad - \frac{\varepsilon(h)}{m}[g'S'/2 + g\varepsilon(h)S'']H(\mathbf{r})\Big\}.
\end{aligned} \tag{4.9}$$

As a practical application of obtained results, we will reconsider the simplest model already studied in [120], [77], [76].Now, ψ will be a two-component spinor, the Chern–Simons gauge coupling $g = e$. The background $\bar{B} = 0$, $e^2\nu\bar{b} = \pi e n_e$, $\bar{A}_0 = \bar{a}_0 = 0$.

Note that our main result – formulas (4.6) and (4.8)- (4.9) must be slightly modified, taking into account the doubling of fermion degrees of freedom.

It can be shown that at zero temperature,

$$\begin{aligned} S &= -im\ell^2(\nu\ell^2\Delta + 3\nu^2\ell^4\Delta^2/8), \\ S' &= im\ell^2(\nu + 3\nu^2\ell^2\Delta/4), \\ S'' &= -im\ell^2(\nu^2 + \nu^3\ell^2\Delta/2), \\ \Gamma_0 &= n_e. \end{aligned}$$

Hence the fermionic contribution to the free energy is given by

$$\begin{aligned} \Omega = \int \mathrm{d}\mathbf{r}\{en_e(A_0 + a_0) + e^2\varepsilon(h)\frac{\nu}{\pi}(A_0 + a_0)(B + b) + \\ + e^2[-\frac{1}{2}(\frac{\nu}{\pi})^2\frac{m}{n_e}(E_i + e_i)^2 + \frac{1}{2}(\frac{\nu}{\pi})^2\frac{\pi}{m}(B + b)^2] + \\ + e^2[\frac{3}{16}(\frac{\nu}{\pi})^4\frac{\pi m}{n_e^2}\partial_i(E^i + e^i)\cdot\partial_k(E^k + e^k) - \\ -\frac{1}{4}\Big(\frac{\nu}{\pi}\Big)^4\frac{\pi^2}{mn_e}\partial_i(B + b)\cdot\partial_i(B + b) - \frac{3}{4}\varepsilon(h)\Big(\frac{\nu}{\pi}\Big)^3\frac{\pi}{n_e}\partial_i(E^i + e^i)\cdot(B + b)]\}. \end{aligned}$$

From the gauge field equations,

$$\begin{aligned} -\Delta A_0 &= < J^0 > -en_e, \\ \Delta B &= -\varepsilon_{mn}\partial_m < J^n >, \\ e^2\nu\Delta a_0 &= \pi\varepsilon_{mn}\partial_m < j^n >, \\ e^2\nu\Delta b &= \pi\Delta < j^0 >, \end{aligned}$$

and from the equations (4.8)–(4.9) it follows

$$\begin{aligned} \left[e^2 + \frac{3e^2\nu^2}{4\pi n_e}\Delta - \frac{m}{n_e}\Delta - \frac{3m\nu^2}{8\pi n_e^2}\Delta^2\right] B + \\ + \left[\frac{e^2 m\nu}{\pi n_e}\Delta + \frac{3e^2 m\nu^3}{8\pi^2 n_e^2}\Delta^2 - \frac{3\nu}{4n_e}\Delta^2\right] A_0 = 0, \qquad (4.10) \\ \left[\frac{e^2\nu}{m}\Delta - \frac{3\nu}{4n_e}\Delta^2 + \frac{e^2\nu^3}{2m\pi n_e}\Delta^2\right] B + \\ + \left[e^2\Delta + \frac{3e^2\nu^2}{4\pi n_e}\Delta^2 - \Delta^2\right] A_0 = 0. \qquad (4.11) \end{aligned}$$

Adopting numerical estimates used in [76],

$$\frac{e^2}{\pi m} = 1.1\cdot 10^{-5}, \quad \frac{me^2}{\pi^2 n_e} = 24, \quad \frac{\pi n_e}{m^2} = 4.7\cdot 10^{-7}$$

the dominant terms of (4.10) and (4.11) are given by

$$\left[e^2 - \frac{m}{n_e}\Delta\right] B + \frac{e^2 m\nu}{\pi n_e}\Delta A_0 = 0,$$

$$\left[\frac{e^2\nu}{m}\Delta - \frac{3\nu}{4n_e}\Delta^2\right] B + \left[e^2\Delta + \frac{3e^2\nu^2}{4\pi n_e}\Delta^2 - \Delta^2\right] A_0 = 0.$$

The superconducting solution of this coupled system is given by the configurations

$$A_0 = \lambda\tilde{E}e^{-\mathbf{n}\cdot r}, \quad B = \tilde{B}e^{-\mathbf{n}\cdot r}, \quad \mathbf{n}^2 = 1/\lambda^2.$$

The relevant equations are now given by

$$(1-\gamma)\tilde{B} + \frac{e^2\nu}{\pi}\lambda\gamma\tilde{E} = 0,$$
$$\frac{\nu}{\pi}\left(1 - \frac{3}{4}\gamma\right)\frac{1}{\lambda}\tilde{B} + \left[1 - \left(\frac{\pi n_e}{m^2} - \frac{3e^2\nu^2}{4\pi m}\right)\right]\tilde{E} = 0,$$

where $\gamma = \lambda_L^2/\lambda^2$ and $\lambda_L = (m/e^2 n_e)^{1/2}$ is the London penetration depth. For γ we obtain

$$\gamma = \left(1 + \frac{e^2\nu^2}{4\pi m}\right)^{-1} \approx 1 - \frac{e^2\nu^2}{4\pi m}$$

in a good agreement with a result cited in [76].

It can be shown that at high temperatures γ remains positive, so in this model there is no phase transition to the normal phase. (The temperature dependence will be considered in the subsequent section).

In conclusion, note that the proposed scheme of calculations permits one to analyse in principle a large variety of models for arbitrary static gauge field configurations up to any desired order in the derivative expansion.

3. Relativistic Anyon Superconductors

The zero-temperature Meissner effect presented in the 2+1 dimensional anyon matter provoked considerable efforts in order to promote the Chern-Simons gauge theory as a hypothetical candidate for the high T_c superconductivity.

The most important points in that development are the existence of the massless (Goldstone) pole in the current correlators [53], the cancelation of bare and induced C-S terms [10], and detailed calculations of the effective action and the thermodynamical potential for the fermions interacting with the C-S and Maxwell fields [24], [120], [76], 110].

Among the others, it was shown that the Meissner effect is partial, i.e., the magnetic field starts to penetrate into the sample at any non-zero temperature [76].

In the present section, we try to give some complementary insights into these intriguing question.

It will be demonstrated that the Meissner effect exists only if a matter consists of two types of fermions with opposite signs of the magnetic moment interaction. Such a system can be naturally realized considering planar relativistic fermions. Note that different versions of the relativistic anyon

superconductivity have been considered in [110], [31], [128]. Below we follow [45].

The model under consideration describes 2+1 dimensional Dirac fermions coupled to the Maxwell electromagnetic (A_μ) and the Chern-Simons (a_μ) gauge fields.

Metric $\eta^{\mu\nu}$ and Levi-Civita tensors $\varepsilon^{\mu\nu\lambda}$ in 2+1 dimensions are defined by

$$\eta^{\mu\nu} = \mathrm{diag}(+1,-1,-1) \qquad \varepsilon^{012} = +1, \quad \varepsilon^{kl} \equiv \varepsilon^{0kl}. \tag{4.12}$$

The space-time coordinates and the vector fields are defined by

$$x^\mu = (ct,\, \mathbf{r}), \quad A^\mu = (A^0,\, \mathbf{A}) = (c^{-1}\varphi,\, \mathbf{A}), \quad a^\mu = (a^0,\, \mathbf{a}), \tag{4.13}$$

where φ is the Maxwell scalar potential, and the magnetic field is

$$B = \partial_x A_y - \partial_y A_x = -\varepsilon^{kl}\,\partial_k A_l.$$

For the Dirac matrices in 2+1 dimensional space-time we use the representation

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$
$$\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}, \quad [\gamma^\mu,\gamma^\nu] = 2i\varepsilon^{\mu\nu\lambda}\gamma_\lambda. \tag{4.14}$$

The fermionic Lagrangian, including the interaction with gauge fields, is given by

$$\begin{aligned} L_e &= \int \bar\psi\{i\hbar c\gamma^\mu D_\mu - \sigma mc^2\}\psi \mathrm{d}\mathbf{r}, \\ D_\mu &= \partial_\mu + i(e/\hbar)(A_\mu + a_\mu), \end{aligned} \tag{4.15}$$

where $m > 0$ and $\sigma = \pm 1$. In order to clarify the meaning of the parameter σ, consider translation and Lorentz rotation generators, which in the spinor representation are given by

$$P^\mu = -i\hbar\partial^\mu, \quad L^{\mu\nu} = \frac{i}{4}\,[\gamma^\mu,\gamma^\nu] = \frac{1}{2}\,\varepsilon^{\mu\nu\lambda}\gamma_\lambda.$$

Introducing the Pauli-Lubanski scalar $W = (1/2)\varepsilon^{\mu\nu\lambda}L_{\mu\nu}P_\lambda$, we get

$$W = -\frac{1}{2}\,i\hbar\gamma^\mu\partial_\mu.$$

Obviously, W coincides (up to a constant factor) with the kinetic part of the Dirac operator. Further, taking $m > 0$, we have

$$P^\mu P_\mu = \frac{E^2}{c^2} - \mathbf{P}^2 = m^2c^2 \quad \to \quad mc = (P^\mu P_\mu)^{1/2} > 0,$$

and rewrite the 2+1 dimensional free Dirac equation as

$$\left\{ W + s\cdot(P^\mu P_\mu)^{1/2} \right\}\psi = 0.$$

We see that $\sigma = \pm 1$ corresponds to the particles with opposite helicities.

In the non-relativistic limit, the corresponding one-particle Hamiltonian takes the form

$$\mathcal{H}_{\rm NR} = -\frac{\hbar^2}{2m}\,\mathbf{D}^2 - \sigma\,\frac{e\hbar}{2m}\,B + e\varphi.$$

As we see, different values of σ in 2+1 dimensions give rise to different signs of the magnetic moment interaction, which (as we will see later) leads to non-trivial effects in the magnetic and thermal properties of the system.

As it follows from (4.15), spinors carry the dimensionality $[\psi] = length^{-1}$ and consequently, $J^\mu = ec\bar{\psi}\gamma^\mu\psi$ turns out to be the planar density describing the current distribution over the layer of a sample. On the other hand, we consider A_μ as the $\mu = 0,1,2$ part of the real 3+1 dimensional electromagnetic field. The latter should satisfy the Maxwell equations where the sources are spatial densities of fermion current. Spatial densities in a multilayered system can be introduced using the planar ones, averaging them along the interplanar distance, i.e., as $\delta^{-1}J^\mu$ ($\delta \sim 10^{-9}cm$). In this consideration, the Maxwell equations take the form

$$\frac{1}{\mu_0}\,\partial_\nu F^{\mu\nu} \;=\; \frac{1}{\delta}\,ec\bar{\psi}\gamma^\mu\psi,$$

where μ_0 is the universal magnetic constant.

These equations can be derived from (4.15) and the Maxwell Lagrangian which, being adapted to 2+1 dimensions, reads as

$$L_{Maxwell} \;=\; -\,\frac{\delta}{4\mu_0}\int F_{\mu\nu}F^{\mu\nu}\mathrm{d}\mathbf{r},$$

where the field strengths E_x, E_y and $B \equiv B_z$ are usual in 3+1 dimensional quantities.

The total Lagrangian for this system is a sum of matter and gauge field Lagrangians

$$\begin{aligned}\mathcal{L} = &-\frac{\delta}{4\mu_0}\,F_{\mu\nu}F^{\mu\nu} - \frac{c}{\hbar}\,\frac{e^2\nu_0}{2\pi}\,\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + ecn_eA_0 + \\ &+ \bar{\psi}\{i\hbar c\gamma^\mu D_\mu - \sigma mc^2\}\psi, \\ D_\mu = &\,\partial_\mu + i(e/\hbar)(A_\mu + a_\mu), \quad m > 0, \quad \sigma = \pm 1,\end{aligned} \tag{4.16}$$

where en_e is the planar density of the background neutralizing charges.

The Euler-Lagrange equations derived from (4.16) are given by

$$\begin{aligned} ecn_e g^{\mu 0} + (\delta/\mu_0)\partial_\nu F^{\nu\mu} &= ec\bar{\psi}\gamma^\mu\psi, \\ \frac{c}{\hbar}\,\frac{e^2\nu_0}{\pi}\,\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda &= ec\bar{\psi}\gamma^\mu\psi, \\ i\,\hbar c\gamma^\mu D_\mu\psi - \sigma\,mc^2\psi &= 0. \end{aligned}$$

In what follows, we consider the case of an external (non dynamical) Maxwell field. The Chern–Simons field will be considered in the mean

field approximation (MFA), operating with its quantum average. In this consideration, we take classical values for the gauge fields, while the Dirac field will be quantized.

In order to study static properties of this relativistic system, we can take into account only the time-independent configurations of gauge fields ($\partial_0 A_\mu = \partial_0 a_\mu = 0$). Analysis will be performed in a self-consistent field approximation developed in [76]. It means that fermionic currents in the classical equations of motion will be replaced by the corresponding thermal averages which are defined in terms of the grand canonical ensemble

$$\langle \cdots \rangle \, \mathrm{Tr} e^{-(H_e - \mu mc^2 N)/k_B T} = \mathrm{Tr} \left\{ \cdots e^{-(H_e - \mu mc^2 N)/k_B T} \right\}.$$

Here H_e is the quantized fermionic hamiltonian

$$H_e = \int \psi^\dagger(\mathbf{r}) \{ i\hbar c \gamma^0 \gamma_k D_k(\mathbf{r}) + ecA_0(\mathbf{r}) + eca_0(\mathbf{r}) + \sigma mc^2 \gamma^0 \} \psi(\mathbf{r}) \mathrm{d}\mathbf{r}, \quad (4.17)$$

and N is the particle number operator

$$N = \int \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) \mathrm{d}\mathbf{r}$$

while k_B, T and μ are the Boltzmann constant, the temperature and the dimensionless chemical potential, respectively.

The resulting set of dynamical equations is given by

$$-\frac{\delta}{\mu_0} \Delta A_0 + ecn_e = \langle J^0(\mathbf{r}) \rangle, \quad (4.18)$$

$$-\frac{\delta}{\mu_0} \varepsilon^{kn} \partial_n B = \langle J^k(\mathbf{r}) \rangle, \quad (4.19)$$

$$\frac{c}{\hbar} \frac{e^2 \nu_0}{\pi} b = \langle J^0(\mathbf{r}) \rangle, \quad (4.20)$$

$$\frac{c}{\hbar} \frac{e^2 \nu_0}{\pi} \varepsilon^{kn} \partial_n B = \langle J^k(\mathbf{r}) \rangle. \quad (4.21)$$

Introduce the thermodynamic potential

$$\Omega_e(T, \mu, A, a) = -k_B T \ln \mathrm{Tr} \exp \left\{ -\frac{H_e(A, a) - \mu mc^2 N}{k_B T} \right\}.$$

The current operators can be expressed in terms of functional derivatives

$$J^\mu(\mathbf{r}) = \frac{\delta H_e}{\delta A_\mu(\mathbf{r})} = \frac{\delta H_e}{\delta a_\mu(\mathbf{r})}.$$

Using the cyclic property of the trace operation, one gets

$$\langle J^\mu(\mathbf{r}) \rangle = \frac{\delta \Omega_e}{\delta A_\mu(\mathbf{r})} = \frac{\delta \Omega_e}{\delta a_\mu(\mathbf{r})}. \quad (4.22)$$

Present the gauge fields as sums of fixed background and fluctuating parts

$$A_\mu = A^{\rm b}_\mu + A^{\rm f}_\mu, \quad a_\mu = a^{\rm b}_\mu + a^{\rm f}_\mu.$$

Background corresponds to the uniform Maxwell and Chern–Simons magnetic fields $B^b = const$, $b^b = const$.

Separate the fermionic Hamiltonian into the free and interacting parts $H_e = mc^2(H_0 + H_{int})$, where

$$\begin{aligned} H_0 &= \int \psi^\dagger(\mathbf{r})\{i\,\ell_0\gamma^0\gamma_k D^{\rm b}_k(\mathbf{r}) + \sigma\gamma^0\}\psi(\mathbf{r})\mathrm{d}\mathbf{r}, \\ D^{\rm b}_k(\mathbf{r}) &= \partial_k + i(e/\hbar)(A^{\rm b}_k + a^{\rm b}_k), \\ H_{int} &= \frac{1}{mc^2}\int J^\mu(\mathbf{r})\{A^{\rm f}_\mu(\mathbf{r}) + a^{\rm f}_\mu(\mathbf{r})\}\mathrm{d}\mathbf{r} \end{aligned} \tag{4.23}$$

($\ell_0 = \hbar/mc$ is the Compton wave length for the fermions).

Applying the perturbation theory formalism, we get for the thermodynamic potential

$$\Omega_e = \Omega_0 - k_B T \ln \left\langle \mathrm{T}\exp\left\{-\int\limits_0^\beta H_{int}(\tau)d\tau\right\}\right\rangle_0, \quad \beta = \frac{mc^2}{k_B T}. \tag{4.24}$$

Ω_0 is the thermodynamical potential for the system in the uniform magnetic background

$$\Omega_0(T,\,\mu,\,A^{\rm b},\,a^{\rm b}) = -k_B T \ln \mathrm{Tr}\exp\left\{-\beta(H_0(A^{\rm b},\,a^{\rm b}) - \mu N)\right\} \tag{4.25}$$

and $\langle\cdots\rangle_0$ is defined as

$$\langle\cdots\rangle_0 \mathrm{Tr}\left\{e^{-\beta(H_0-\mu N)}\right\} = \mathrm{Tr}\left\{\cdots e^{-\beta(H_0-\mu N)}\right\}.$$

In (4.24), $H_{int}(\tau)$ is the interaction Hamiltonian in the Matsubara representation

$$H_{int}(\tau) = \frac{1}{mc^2}\int \bar\psi(\tau,\,\mathbf{r})\gamma^\mu\psi(\tau,\,\mathbf{r})\{A^{\rm f}_\mu(\mathbf{r}) + a^{\rm f}_\mu(\mathbf{r})\}\mathrm{d}\mathbf{r},$$

and T denotes a τ-ordering. Matsubara fields are given by

$$\psi(\tau,\,\mathbf{r}) = e^{\tau(H_0-\mu N)}\psi(\mathbf{r})e^{-\tau(H_0-\mu N)}, \tag{4.26}$$

$$\bar\psi(\tau,\,\mathbf{r}) = e^{\tau(H_0-\mu N)}\bar\psi(\mathbf{r})e^{-\tau(H_0-\mu N)} = \psi^\dagger(-\tau,\,\mathbf{r})\gamma^0. \tag{4.27}$$

One-particle Hamiltonian describing a fermion in the uniform magnetic background is given by

$$\begin{aligned} \mathcal{H}^{\rm b} &= i\,\ell_0\gamma^0\gamma_k D^{\rm b}_k + \sigma\gamma^0, \\ D^{\rm b}_k &= \partial_k + i(e/\hbar)(A^{\rm b}_k + a^{\rm b}_k), \\ A^{\rm b}_k(\mathbf{r}) + a^{\rm b}_k(\mathbf{r}) &= \frac{1}{2}\varepsilon_{kl}x^l(B^{\rm b} + b^{\rm b}) - \partial_k\xi(\mathbf{r}), \end{aligned}$$

where $\xi(\mathbf{r})$ is the gauge fixing term.

Solving the Schrödinger equation for $\mathcal{H}^{\rm b}(\sigma,\,\varepsilon)$, one gets the following set of the positive and the negative energy eigenvectors (see Appendix A)

$$\begin{aligned}\mathcal{H}^{\rm b}(\sigma,\,\varepsilon)u_n(\sigma,\,\varepsilon) &= +\,\omega_n u_n(\sigma,\,\varepsilon),\\ \mathcal{H}^{\rm b}(\sigma,\,\varepsilon)v_n(\sigma,\,\varepsilon) &= -\,\omega_n v_n(\sigma,\,\varepsilon),\end{aligned} \tag{4.28}$$

where the eigenvalues, being independent of σ and ε, are given by

$$\omega_n = \sqrt{1+2hn}, \quad h = \frac{\ell_0^2}{\ell^2}.$$

Note that except the lowest energy eigenvalues, the solutions of the Dirac equation are always paired. For the lowest energy, there is an asymmetry, i.e., there is no v_0 mode for $\sigma\varepsilon = 1$ and no U_0 mode for $\sigma\varepsilon = -1$.

As a complete set of commuting operators, we take the Hamiltonian $\mathcal{H}^{\rm b}$ and operator

$$p = -i\ell D_1^{\rm b} - \varepsilon\ell^{-1}y. \tag{4.29}$$

The fermion field operators are presented as follows

$$\begin{aligned}\psi &= \sum_{np}\{a_{np}u_{np} + b_{np}^\dagger v_{np}\},\\ \bar\psi &= \sum_{np}\{a_{np}^\dagger \bar u_{np} + b_{np}\bar v_{np}\}.\end{aligned}$$

Fermion creation-annihilation operators satisfy the standard relations

$$\begin{aligned}\{\,a_{np}\,,\,a_{n'p'}^\dagger\,\} &= \{\,b_{np}\,,\,b_{n'p'}^\dagger\,\} = \delta_{nn'}\,\delta(p-p'),\\ u_{np} = u_n\cdot|p>, &\qquad n_{np} = v_n\cdot|p>\,.\end{aligned}$$

The quantized Hamiltonian H_0 and the particle number operator N are defined in the normal ordered form

$$H_0 = \sum_{np}\omega_n a_{np}^\dagger a_{np} + \sum_{np}\omega_n b_{np}^\dagger b_{np}, \tag{4.30}$$

$$N = \sum_{np} a_{np}^\dagger a_{np} - \sum_{np} b_{np}^\dagger b_{np} \tag{4.31}$$

avoiding the problems with the negative energy states.

4. Thermodynamic Potential

In this section we will find an analytic expression for the thermodynamic potential in the second order approximation with respect to the gauge field fluctuations. All the operator expressions like Hamiltonian, currents, *etc.*, are assumed to be normal ordered.

Substituting H_0 into (4.25), we get

$$\Omega_0(A^{\rm b},\, a^{\rm b}) = k_B T \sum_{np} \ln(1-\rho_n^+) + k_B T \sum_{np} \ln(1-\rho_n^-),$$

where ρ_n^+ and ρ_n^- are the Fermi distribution functions for the particles and antiparticles

$$\rho_n^{\pm} = \{1+\exp[\,\beta(\omega_n \mp \mu)]\}^{-1}, \quad \omega_n = \sqrt{1+2hn},$$

while n and p are the quantum numbers labeling the one-particle states. Due to the degeneracy of the Landau levels, summation over p gives the factor $V/2\pi\ell^2$, where V is the area occupied by the sample. We get

$$\frac{\Omega_0(A^{\rm b},\, a^{\rm b})}{V} = \frac{k_B T}{2\pi\ell^2} \sum_{n} \ln(1-\rho_n^+) + \frac{k_B T}{2\pi\ell^2} \sum_{n} \ln(1-\rho_n^-), \tag{4.32}$$

where ℓ is a magnetic length.

Calculating the linear and second order correction to Ω_0, one needs an explicit form of the averages $\langle J^\mu(\tau,\, \mathbf{r})\rangle_0$ and $\langle J^\mu(\tau_1,\, \mathbf{r}_1) J^\nu(\tau_2,\, \mathbf{r}_2)\rangle_0$.

The thermal fields are given by

$$\psi(\tau,\, \mathbf{r}) = e^{\mu\tau} \sum_{np} \{a_{np} u_{np}(\mathbf{r}) e^{-\omega_n \tau} + b_{np}^\dagger v_{np}(\mathbf{r}) e^{\omega_n \tau}\},$$
$$\bar\psi(\tau,\, \mathbf{r}) = e^{-\mu\tau} \sum_{np} \{a_{np}^\dagger \bar u_{np}(\mathbf{r}) e^{\omega_n \tau} + b_{np} \bar v_{np}(\mathbf{r}) e^{-\omega_n \tau}\}.$$

The non-vanishing thermal averages are

$$\langle a_{np}^\dagger a_{n'p'}\rangle_0 = \delta_{nn'}\delta_{pp'}\rho_n^+, \quad \langle b_{np}^\dagger b_{n'p'}\rangle_0 = \delta_{nn'}\delta_{pp'}\rho_n^-,$$

where $\delta_{pp'}$ stands for $\delta(p-p')$. Using normal ordered current operators, we get

$$\langle J^\mu(\tau,\, \mathbf{r})\rangle_0 = \langle J^\mu(0,\, \mathbf{r})\rangle_0 = \langle J^\mu(\mathbf{r})\rangle_0 = ec\Pi^\mu(\mathbf{r}), \tag{4.33}$$
$$\Pi^\mu(\mathbf{r}) = \sum_{np} \{\rho_n^+ \bar u_{np}(\mathbf{r}) \gamma^\mu u_{np}(\mathbf{r}) - \rho_n^- \bar v_{np}(\mathbf{r}) \gamma^\mu v_{np}(\mathbf{r})\}. \tag{4.34}$$

Wick's theorem for non vanishing quartic combinations yields

$$\langle a_{np}^\dagger a_{n'p'} a_{mq}^\dagger a_{m'q'}\rangle_0 = \delta_{nn'}\delta_{mm'}\delta_{pp'}\delta_{qq'}\rho_n^+\rho_m^+ + \delta_{nm'}\delta_{mn'}\delta_{pq'}\delta_{qp'}\rho_n^+(1-\rho_m^+),$$
$$\langle b_{np}^\dagger b_{n'p'} b_{mq}^\dagger b_{m'q'}\rangle_0 = \delta_{nn'}\delta_{mm'}\delta_{pp'}\delta_{qq'}\rho_n^-\rho_m^- + \delta_{nm'}\delta_{mn'}\delta_{pq'}\delta_{qp'}\rho_n^-(1-\rho_m^-).$$

As a result, the current-current correlators are expressed by

$$\langle J^{\mu}(\tau_1,\mathbf{r}_1)J^{\nu}(\tau_2,\mathbf{r}_2)\rangle_0 = \langle J^{\mu}(\mathbf{r}_1)\rangle_0\langle J^{\nu}(\mathbf{r}_2)\rangle_0 + e^2c^2J^{\mu\nu}(\tau_1-\tau_2,\mathbf{r}_1,\mathbf{r}_2). \quad (4.35)$$

The second term in the last expression is the contribution of the connected part

$$J^{\mu\nu}(\tau,\,\mathbf{r}_1,\,\mathbf{r}_2) =$$
$$= \sum\left\{\rho_n^+(1-\rho_m^+)\bar{u}_{np}(\mathbf{r}_1)\gamma^{\mu}u_{mq}(\mathbf{r}_1)\bar{u}_{mq}(\mathbf{r}_2)\gamma^{\nu}u_{np}(\mathbf{r}_2)e^{(\omega_n-\omega_m)\tau}\right\} +$$
$$+\sum\left\{\rho_n^+\rho_m^-\bar{u}_{np}(\mathbf{r}_1)\gamma^{\mu}v_{mq}(\mathbf{r}_1)\bar{v}_{mq}(\mathbf{r}_2)\gamma^{\nu}u_{np}(\mathbf{r}_2)e^{(\omega_n+\omega_m)\tau}\right\} +$$
$$+\sum\left\{(1-\rho_n^-)(1-\rho_m^+)\bar{v}_{np}(\mathbf{r}_1)\gamma^{\mu}u_{mq}(\mathbf{r}_1)\bar{u}_{mq}(\mathbf{r}_2)\gamma^{\nu}v_{np}(\mathbf{r}_2)e^{-(\omega_n+\omega_m)\tau}\right\} +$$
$$+\sum\left\{(1-\rho_n^-)\rho_m^-\bar{v}_{np}(\mathbf{r}_1)\gamma^{\mu}v_{mq}(\mathbf{r}_1)\bar{v}_{mq}(\mathbf{r}_2)\gamma^{\nu}v_{np}(\mathbf{r}_2)e^{-(\omega_n-\omega_m)\tau}\right\},$$

where the summation over n and m should be performed, taking into account the spectral asymmetry of the one-particle Hamiltonian.

Using (4.35) and performing the τ-integrations, we get

$$\int\frac{d\tau_1 d\tau_2}{\beta^2e^2c^2}\langle \mathrm{T}\{J^{\mu}(\tau_1,\mathbf{r}_1)J^{\nu}(\tau_2,\mathbf{r}_2)\}\rangle_0 = \Pi^{\mu}(\mathbf{r}_1)\Pi^{\nu}(\mathbf{r}_2) - \frac{1}{\beta}\Pi^{\mu\nu}(\mathbf{r}_1,\mathbf{r}_2), \quad (4.36)$$

$$\Pi^{\mu\nu}(\mathbf{r}_1,\,\mathbf{r}_2) = \sum_{nm}\sum_{pq}\frac{\rho_n^+-\rho_m^+}{\omega_n-\omega_m}\,\bar{u}_{np}(\mathbf{r}_1)\gamma^{\mu}u_{mq}(\mathbf{r}_1)\bar{u}_{mq}(\mathbf{r}_2)\gamma^{\nu}u_{np}(\mathbf{r}_2) +$$
$$+\sum_{nm}\sum_{pq}\frac{\rho_n^--\rho_m^-}{\omega_n-\omega_m}\,\bar{v}_{np}(\mathbf{r}_1)\gamma^{\mu}v_{mq}(\mathbf{r}_1)\bar{v}_{mq}(\mathbf{r}_2)\gamma^{\nu}v_{np}(\mathbf{r}_2) +$$
$$+\sum_{nm}\sum_{pq}\frac{\rho_n^++\rho_m^-}{\omega_n+\omega_m}\,\bar{u}_{np}(\mathbf{r}_1)\gamma^{\mu}v_{mq}(\mathbf{r}_1)\bar{v}_{mq}(\mathbf{r}_2)\gamma^{\nu}u_{np}(\mathbf{r}_2) +$$
$$+\sum_{nm}\sum_{pq}\frac{\rho_n^-+\rho_m^+}{\omega_n+\omega_m}\,\bar{v}_{np}(\mathbf{r}_1)\gamma^{\mu}u_{mq}(\mathbf{r}_1)\bar{u}_{mq}(\mathbf{r}_2)\gamma^{\nu}v_{np}(\mathbf{r}_2). \quad (4.37)$$

Terms with $n=m$ in the first two lines of the last expression are defined as

$$\left.\frac{\rho_n^{\pm}-\rho_m^{\pm}}{\omega_n-\omega_m}\right|_{m=n} = \lim_{\alpha\to 0}\frac{\rho_{n+\alpha}^{\pm}-\rho_n^{\pm}}{\omega_{n+\alpha}-\omega_n} = -\rho_n^{\pm}(1-\rho_n^{\pm})\beta.$$

In (4.37), we made the zero temperature subtraction at zero chemical potential.

Expanding (4.24) with respect to gauge fluctuations and substituting (4.33) and (4.36), we get the thermodynamic potential in the second order approximation

$$\Omega_e(A,\,a) = \Omega_0(A^{\mathrm{b}},\,a^{\mathrm{b}}) + ec\int\Pi^{\mu}(\mathbf{r})\{A_{\mu}^{\mathrm{f}}(\mathbf{r}) + a_{\mu}^{\mathrm{f}}(\mathbf{r})\}\mathrm{d}\mathbf{r} +$$

$$+\frac{e^2}{2m}\int \Pi^{\mu\nu}(\mathbf{r}_1,\,\mathbf{r}_2)\{A^{\rm f}_\mu(\mathbf{r}_1)+a^{\rm f}_\mu(\mathbf{r}_1)\}\{A^{\rm f}_\nu(\mathbf{r}_2)+a^{\rm f}_\nu(\mathbf{r}_2)\}{\rm d}\mathbf{r}_1{\rm d}\mathbf{r}_2.$$

Expressions for $\Pi^{\mu}(\mathbf{r})$ and $\Pi^{\mu\nu}(\mathbf{r}_1,\,\mathbf{r}_2)$ can be presented in the translation invariant form

$$\begin{aligned}
\Pi^{\mu}(\mathbf{r}) &= \frac{1}{2\pi\ell^2}\sum_n \rho^+_n \bar u_n\gamma^\mu u_n - \frac{1}{2\pi\ell^2}\sum_n \rho^-_n \bar v_n\gamma^\mu v_n = const,\\
\Pi^{\mu\nu}(\mathbf{r}_1,\,\mathbf{r}_2) &= \frac{1}{\ell^4}\int\frac{{\rm d}\mathbf{k}}{8\pi^3}\,e^{i\mathbf{k}(\mathbf{r}_1-\mathbf{r}_2)/\ell}\Pi^{\mu\nu}(\mathbf{k}),
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
\Pi^{\mu\nu}(\mathbf{k}) = &\sum_n\sum_m \frac{\rho^+_n-\rho^+_m}{\omega_n-\omega_m}\,\bar u_n\gamma^\mu Z^\dagger(\mathbf{k})u_m\bar u_m\gamma^\nu Z(\mathbf{k})u_n +\\
&+\sum_n\sum_m \frac{\rho^-_n-\rho^-_m}{\omega_n-\omega_m}\,\bar v_n\gamma^\mu Z^\dagger(\mathbf{k})v_m\bar v_m\gamma^\nu Z(\mathbf{k})v_n +\\
&+\sum_n\sum_m \frac{\rho^+_n+\rho^-_m}{\omega_n+\omega_m}\,\bar u_n\gamma^\mu Z^\dagger(\mathbf{k})v_m\bar v_m\gamma^\nu Z(\mathbf{k})u_n +\\
&+\sum_n\sum_m \frac{\rho^-_n+\rho^+_m}{\omega_n+\omega_m}\,\bar v_n\gamma^\mu Z^\dagger(\mathbf{k})u_m\bar u_m\gamma^\nu Z(\mathbf{k})v_n,
\end{aligned} \tag{4.39}$$

where u_n and v_n are the one-particle states in the occupation number representation, while $Z\equiv Z(\mathbf{k})$ is the coherent state operator.

Due to the spectral asymmetry, summations in the different terms of (4.38) and (4.39) are performed either from 0 to ∞ or from 1 to ∞.

Operator $\Pi^{\mu\nu}(\mathbf{k})$ can be represented in a planar-transverse form. In particular, consider the following identity

$$\sqrt{h}\,k^m\gamma^0\gamma^m Z(\mathbf{k}) = i\,\sqrt{h}\,\varepsilon^{mn}k^m\gamma^n Z(\mathbf{k}) = [\,\mathcal{H}^{\rm b},\,Z(\mathbf{k})\,], \tag{4.40}$$

where $\mathcal{H}^{\rm b}$ is the relativistic Hamiltonian.

Using equations (4.39), (4.40) and (4.28), we get

$$\begin{aligned}
k^l\Pi^{l0}(\mathbf{k}) \propto &\sum_n\sum_m(\rho^+_n-\rho^+_m)u^\dagger_m Z^\dagger(\mathbf{k})u_n u^\dagger_n Z(\mathbf{k})u_m -\\
&-\sum_n\sum_m(\rho^-_n-\rho^-_m)v^\dagger_m Z^\dagger(\mathbf{k})v_n v^\dagger_n Z(\mathbf{k})v_m +\\
&+\sum_n\sum_m(\rho^+_n+\rho^-_m)v^\dagger_m Z^\dagger(\mathbf{k})u_n u^\dagger_n Z(\mathbf{k})v_m -\\
&-\sum_n\sum_m(\rho^-_n+\rho^+_m)u^\dagger_m Z^\dagger(\mathbf{k})v_n v^\dagger_n Z(\mathbf{k})u_m.
\end{aligned}$$

This expression and analogous ones for $\varepsilon^{mn}k^nk^l\Pi^{ml}(\mathbf{k})$ and $k^mk^n\Pi^{mn}(\mathbf{k})$ together with $\{\gamma^0,\,\mathcal{H}^{\rm b}\}=2\sigma$ lead to the identities

$$k^l\Pi^{l0}(\mathbf{k}) = \varepsilon^{mn}k^nk^l\Pi^{ml}(\mathbf{k}) = k^mk^n\Pi^{mn}(\mathbf{k}) = 0. \tag{4.41}$$

Since the vectors $k^m = (k_x, k_y)$ and $\varepsilon^{mn}k^n = (k_y, -k_x)$ define a local orthogonal basis, one can rewrite the polarization operator as

$$\Pi^{00}(\mathbf{k}) = \frac{2\pi}{\hbar}\,\Pi_{\rm E}(\mathbf{k}^2/2), \tag{4.42}$$

$$\Pi^{0k}(\mathbf{k}) = \frac{2\pi}{\sqrt{\hbar}}\,i\varepsilon^{kl}k^l\Pi_{\rm CS}(\mathbf{k}^2/2), \tag{4.43}$$

$$\Pi^{kl}(\mathbf{k}) = 2\pi\varepsilon^{km}\varepsilon^{ln}k^m k^n\Pi_{\rm M}(\mathbf{k}^2/2), \tag{4.44}$$

where $\Pi_{\rm E}$, $\Pi_{\rm CS}$ and $\Pi_{\rm M}$ are the structure functions (the explicit expressions for these functions are given in the Appendix E). These structure functions for the non-relativistic case have been evaluated in [41].

Explicit calculations show that

$$\Pi_0 = \frac{1}{2\pi\ell^2}\sum_n \rho_n^+ - \frac{1}{2\pi\ell^2}\sum_n \rho_n^-, \quad \Pi_k = 0. \tag{4.45}$$

The corresponding expression for the thermodynamic potential is given by

$$\begin{aligned}\Omega_e(A,\,a) = \Omega_0(A^{\rm b},\,a^{\rm b}) + ec\,\Pi_0\int\{A_0^{\rm f}(\mathbf{r}) + a_0^{\rm f}(\mathbf{r})\}{\rm d}\mathbf{r} + \\ + \frac{e^2}{2m}\,\frac{m^2c^2}{\hbar^2}\int\{A_0^{\rm f}(\mathbf{r}) + a_0^{\rm f}(\mathbf{r})\}\hat{\Pi}_{\rm E}\{A_0^{\rm f}(\mathbf{r}) + a_0^{\rm f}(\mathbf{r})\}{\rm d}\mathbf{r} + \\ + \frac{e^2}{2m}\,\frac{2mc}{\hbar^2}\int\{A_0^{\rm f}(\mathbf{r}) + a_0^{\rm f}(\mathbf{r})\}\hat{\Pi}_{\rm CS}\{B^{\rm f}(\mathbf{r}) + b^{\rm f}(\mathbf{r})\}{\rm d}\mathbf{r} + \\ + \frac{e^2}{2m}\int\{B^{\rm f}(\mathbf{r}) + b^{\rm f}(\mathbf{r})\}\hat{\Pi}_{\rm M}\{B^{\rm f}(\mathbf{r}) + b^{\rm f}(\mathbf{r})\}{\rm d}\mathbf{r},\end{aligned} \tag{4.46}$$

where $\hat{\Pi}$ stands for the differential operators $\Pi(-\ell^2\Delta/2)$ (Δ is the Laplace operator).

Now, using (4.22), we get the desired expressions for the current averages. They are represented as linear functions of the gauge field fluctuations

$$\langle\, J^0(\mathbf{r})\,\rangle = ec\,\Pi_0 + \frac{e^2mc^2}{\hbar^2}\,\hat{\Pi}_{\rm E}\left(A_0^{\rm f} + a_0^{\rm f}\right) + \frac{e^2c}{\hbar}\,\hat{\Pi}_{\rm CS}\left(B^{\rm f} + b^{\rm f}\right), \tag{4.47}$$

$$\langle\, J^k(\mathbf{r})\,\rangle = -\,\varepsilon^{kn}\,\partial_n\left\{\frac{e^2c}{\hbar}\,\hat{\Pi}_{\rm CS}\left(A_0^{\rm f} + a_0^{\rm f}\right) + \frac{e^2}{m}\,\hat{\Pi}_{\rm M}\left(B^{\rm f} + b^{\rm f}\right)\right\}. \tag{4.48}$$

5. Uniform Magnetic Field and the Meissner Effect

As a starting point, consider the system in the zeroth order approximation, i.e., $A_\mu^{\rm f} = a_\mu^{\rm f} = 0$. Equations of motion (4.18) and (4.19) are reduced to

$$n_e = \Pi_0, \tag{4.49}$$

$$b^{\rm b} = \frac{\pi\hbar}{e\nu_0}\,n_e, \tag{4.50}$$

where Π_0 is given by (4.45).

Note that n_e is the free fermion density in the sample and (4.49) serves to define the chemical potential $\mu = \mu(T, B^{\rm b}, b^{\rm b}, n_e)$.

We will study a compound system consisting of two sorts of fermions with equal gauge couplings $e_1 = e_2 = e$ and different helicities $\sigma_1 = -\sigma_2$ corresponding to different signs of the magnetic moment interaction, which on its turn can be associated with spin up and spin down fermions. In that case, the r.h.s. of the equation (4.49) gets the contributions, defined by (4.45) from both sorts of particles. Remark that, since the different sorts have the equal charges, corresponding magnetic lengths are also equal. Moreover, as it follows from (4.50),realistic values of $B^{\rm b}$ are small compared with those of $b^{\rm b}$ and can be neglected:

$$\varepsilon = \mathrm{sgn}(eB^{\rm b} + eb^{\rm b}) = \mathrm{sgn}(eb^{\rm b}).$$

In other words, ε can be considered as independent of $B^{\rm b}$, and without loss of generality we can set $\sigma_1\varepsilon_1 = -\sigma_2\varepsilon_2 = 1$. Taking into account the contributions of both types, the equation of motion (4.49) can be rewritten as

$$n_e = \frac{h}{2\pi\ell_0^2}\,(\nu_1 + \nu_2), \tag{4.51}$$

where ν_1 and ν_2 are the corresponding filling fractions

$$\nu_1 \equiv \sum_{n=0} \rho_n^+(\mu_1) - \sum_{n=1} \rho_n^-(\mu_1), \tag{4.52}$$

$$\nu_2 \equiv \sum_{n=1} \rho_n^+(\mu_2) - \sum_{n=0} \rho_n^-(\mu_2), \tag{4.53}$$

and μ_1 and μ_2 are the chemical potentials for the particle types 1 and 2, respectively. In (4.52) and (4.53), we took into the consideration the spectral asymmetry of the one-particle Hamiltonian, which is reflected in the absence of $n = 0$ modes in certain terms.

Introduce the partial contributions to the particle density

$$n_e^{(1,2)} = \frac{h}{2\pi\ell_0^2}\,\nu_{1,2}. \tag{4.54}$$

By means of (4.51), we can express h in terms of the average filling fraction

$$h = \frac{\pi n_e \ell_0^2}{\nu}, \qquad \nu = \frac{\nu_1 + \nu_2}{2}. \tag{4.55}$$

Using (4.54) and (4.55), we express the partial filling fractions in terms of the average one

$$\nu_{1,2} = \frac{2\, n_e^{(1,2)}}{n_e^{(1)} + n_e^{(2)}}\,\nu. \tag{4.56}$$

Substituting h and $\nu_{1,2}$ from (4.55) and (4.54) into (4.52) and (4.53), we see that the chemical potentials depend on the temperature and the average filling fraction ν.

At the same time, the expression for the magnetic length (4.32) together with equations (4.50) and (4.55) yields

$$\frac{1}{\nu} \;=\; \frac{\varepsilon}{\nu_0} \;+\; \frac{\varepsilon e B^{\rm b}}{\pi n_e \hbar}, \tag{4.57}$$

which reflects the one-to-one correspondence between ν and $B^{\rm b}$. Consequently, any quantity depending on $B^{\rm b}$ can be also viewed as a function of ν, and *vice versa*.

The value of the background magnetic field is determined by the external magnetic field $B^{\rm ext}$ and the magnetization M

$$B^{\rm b} = B^{\rm ext} + M(B^{\rm b}), \quad M(B^{\rm b}) = -\,\frac{\mu_0}{\delta}\,\frac{d\mathcal{F}(B^{\rm b})}{dB^{\rm b}}, \tag{4.58}$$

where $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ is the Helmholtz free energy density of the composite system. The individual contributions are

$$\mathcal{F}_{1,2}(B^{\rm b}) \;=\; \frac{1}{Area}\,\{\,\Omega_0(\mu_{1,2}) \;+\; \mu_{1,2}\,mc^2\langle\, N_{1,2}\,\rangle\},$$

where $\langle\, N_{1,2}\,\rangle$ are the thermal averages of the fermion numbers defined by

$$\begin{aligned}
\frac{\langle\, N_1\,\rangle}{Area} \;&=\; \frac{h}{2\pi\ell_0^2}\sum_{n=0}\rho_n^{+}(\mu_1) - \;\frac{h}{2\pi\ell_0^2}\sum_{n=1}\rho_n^{-}(\mu_1) \;=\; n_e^{(1)},\\
\frac{\langle\, N_2\,\rangle}{Area} \;&=\; \frac{h}{2\pi\ell_0^2}\sum_{n=1}\rho_n^{+}(\mu_2) - \;\frac{h}{2\pi\ell_0^2}\sum_{n=0}\rho_n^{-}(\mu_2) \;=\; n_e^{(2)}.
\end{aligned}$$

Here the quantities h, μ_1 and μ_2 are functions of ν. In order to exhibit the global behaviour of $\mathcal{F}(\nu)$, we have used numerical methods. Consider the case of the equal concentrations $n_e^{(1)} = n_e^{(2)} = n_e/2$. As one can see, the free energy density of the composite system has local minima at integer values of the average filling fraction. However, the individual contributions

of each type of particles exhibit no minima (see Fig.1).

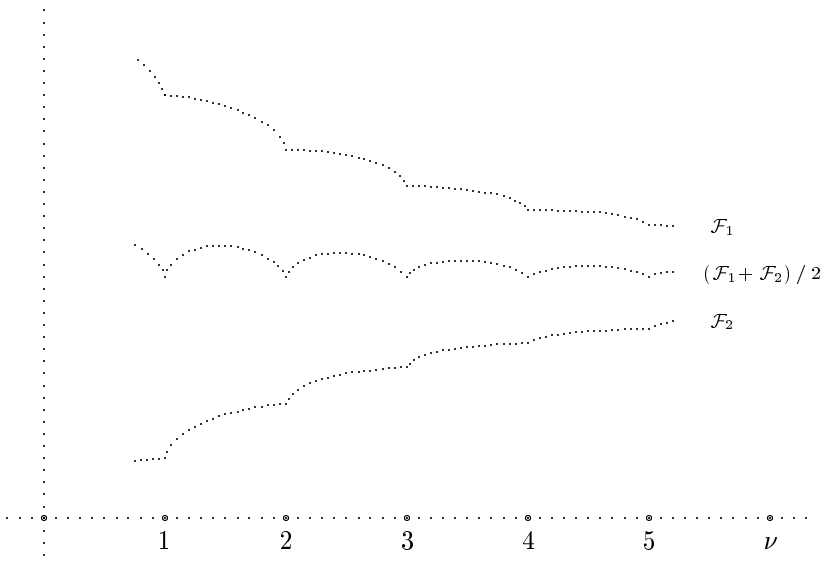


Figure 1. Helmholtz free energy densities versus ν for $\bar{\alpha} = 0$ and $T = 50\,^{\circ}K$.

The cusp-like structure of the free energy is the manifestation of the Meissner effect, when the system tries to expel the magnetic field from inside the sample [76].

As we see, in contrast with the previous calculations, the Meissner effect does not exist in the single fermion system, but only in the composite one, where the diversity in the magnetic moment interaction plays a decisive role.

Magnetization is expressed with a help of the free energy. The lateer contains additive contributions from the one-particle state energies. One-particle Hamiltonian considered in [76] takes into account the magnetic moment interaction only with Maxwell magnetic field. Further simplification is achieved by taking

$$E_n = \frac{\hbar^2}{m\ell^2}\left(n + \frac{1}{2}\right), \tag{4.59}$$

assuming that the magnetic moment interaction vanishes.

In our approximation, the magnetic interaction term contains contributions from both Maxwell and Chern-Simons magnetic fields. The non-

relativistic limit of eru one-particle energy spectrum is given by

$$E_n(\sigma\varepsilon) = \frac{\hbar^2}{m\ell^2}\left(n + \frac{1-\sigma\varepsilon}{2}\right).$$

Note that (4.59) is in fact the half sum of $E_n(+)$ and $E_n(-)$. Separate use of $E_n(+)$ or $E_n(-)$ does not lead to the free energy with localized minima and only their simultaneous contribution has a cusp-like structure.

Equations for the chemical potentials cannot be solved in general. However, one can find an analytic form of $\mu_{1,2}(\nu)$ nearby integer values of ν, where the free energy achieves local minima. This enables to analyze the system in more details near these minima, where the Meissner effect just takes place. Below we present these calculations for asymmetric concentrations, i.e., when $n_e^{(1)} \neq n_e^{(2)}$.

Here we will deal with the minimum corresponding to $\nu = 1$, and for concreteness we will set $e > 0$ and $\varepsilon = 1$. In that case, we have $\nu_0 = 1$ and

$$h = \pi n_e \ell_0^2 (1+\alpha), \quad \alpha = \frac{eB}{\pi n_e \hbar}. \tag{4.60}$$

Note that for the characteristic values of the internal magnetic field ($B < 200\,Gauss$)b we have $\alpha < 10^{-5}$, $(1+\alpha)^{-1} = 1-\alpha$ and represent (4.56) as

$$\begin{aligned}
\nu_1 &= (1+\bar{\alpha})(1-\alpha) \equiv 1-\alpha_1, \\
\nu_2 &= (1-\bar{\alpha})(1-\alpha) \equiv 1-\alpha_2, \\
\bar{\alpha} &= \frac{n_e^{(1)} - n_e^{(2)}}{n_e^{(1)} + n_e^{(2)}}.
\end{aligned}$$

Here $\bar{\alpha}$ measures the asymmetry in the concentrations of different types of fermions.

Further simplifications are due to the fact that in the considered range of temperaturesb we have $\beta > 3 \cdot 10^7$, and consequently for $\mu_{1,2} > 0$

$$\rho_n^-(\mu_{1,2}) = \frac{1}{1+e^{\beta(\omega_n+\mu_{1,2})}} < e^{-10^7},$$

meaning that main contributions to (4.52) and (4.53) come from $\rho_n^+(\mu_{1,2})$, forcing $\nu_{1,2}$ to be positive. With this assumption, we take $\mu_1 > 0$ and represent it as

$$\mu_1 = 1 + \frac{1-w_1}{2}\,h,$$

where w_1 is to be found. Due to (4.60), one has $h \sim 5 \cdot 10^{-7}$. Consequently, for the Landau levels with $2hn << 1$b we can use $\omega_n = 1 + hn$ and write down

$$\beta(\omega_n - \mu_1) = \beta h\left(n - \frac{1}{2} + \frac{w_1}{2}\right).$$

The characteristic values of β and h are such that $\beta h > 15$, allowing to neglect the contributions coming from higher Landau levels. This permits to write (4.52) as $\nu_1 = \rho_0^+(\mu_1) + \rho_1^+(\mu_1)$ or in the equivalent form

$$1 - \alpha_1 = \frac{1}{1 + e^{-\eta} e^{\eta w_1}} + \frac{1}{1 + e^{\eta} e^{\eta w_1}},$$

where $\eta \equiv \pi n_e \ell_0^2 \beta(1+\alpha)/2$. From this equation, we easily find a solution for w_1:

$$e^{\pm \eta w_1} = \frac{1}{1 \mp \alpha_1} \left(\sqrt{1 + \alpha_1^2 \mathrm{sh}^2 \eta} \pm \alpha_1 \mathrm{ch}\eta \right), \quad \alpha_1 = \alpha - \bar{\alpha} + \alpha\bar{\alpha}. \tag{4.61}$$

At the same time, taking

$$\mu_2 = 1 + \frac{3 - w_2}{2} h$$

and performing the same manipulations, we obtain

$$e^{\pm \eta w_2} = \frac{1}{1 \mp \alpha_2} \left(\sqrt{1 + \alpha_2^2 \mathrm{sh}^2 \eta} \pm \alpha_2 \mathrm{ch}\eta \right), \quad \alpha_2 = \alpha + \bar{\alpha} - \alpha\bar{\alpha}. \tag{4.62}$$

So, the leading contributions in the different physical quantities come from the following Fermi distribution functions

$$\begin{aligned} \rho_0^+(\mu_1) &= \frac{1}{1 + e^{-\eta} e^{\eta w_1}}, \quad \rho_1^+(\mu_1) = \frac{1}{1 + e^{\eta} e^{\eta w_1}}, \\ \rho_1^+(\mu_2) &= \frac{1}{1 + e^{-\eta} e^{\eta w_2}}, \quad \rho_2^+(\mu_2) = \frac{1}{1 + e^{\eta} e^{\eta w_2}}, \end{aligned} \tag{4.63}$$

where $e^{\eta w_1}$ and $e^{\eta w_2}$ are defined by (4.61) and (4.62). In this approximation, the system magnetization is $M = M_1 + M_2$ where

$$\begin{aligned} M_1(B^{\mathrm{b}}) &= -\frac{e\mu_0 mc^2}{2\pi\hbar\delta} \sum_{n=0}^{N} \left\{ \frac{1}{\beta} \ln\left[1 - \rho_n^+(\mu_1)\right] + \frac{hn}{\omega_n} \rho_n^+(\mu_1) \right\} \approx \\ &\approx \frac{\pi n_e \ell_0^2}{2} (w_1 - 1) \approx \frac{\varepsilon e \mu_0 n_e \hbar}{4m\delta} \left\{ f(\alpha - \bar{\alpha}N) + 1 \right\}, \\ M_2(B^{\mathrm{b}}) &= -\frac{e\mu_0 mc^2}{2\pi\hbar\delta} \sum_{n=1}^{N+1} \left\{ \frac{1}{\beta} \ln\left[1 - \rho_n^+(\mu_2)\right] + \frac{hn}{\omega_n} \rho_n^+(\mu_2) \right\} \approx \\ &\approx \frac{\pi n_e \ell_0^2}{2} (w_2 + 1) \approx \frac{\varepsilon e \mu_0 n_e \hbar}{4m\delta} \left\{ f(\alpha + \bar{\alpha}N) - 1 \right\}, \end{aligned}$$

while the basic function $f(z)$ which defines all magnetic and thermal properties of the system is given by

$$f(z) = \frac{1}{\xi} \ln \frac{\sqrt{z^2 + 4\mathrm{e}^{-\xi}} - z}{\sqrt{z^2 + 4\mathrm{e}^{-\xi}} + z}, \quad \xi = \frac{1}{N} \frac{\pi n_e \hbar^2}{m k_B T}.$$

Its typical form is depicted in figure 2.

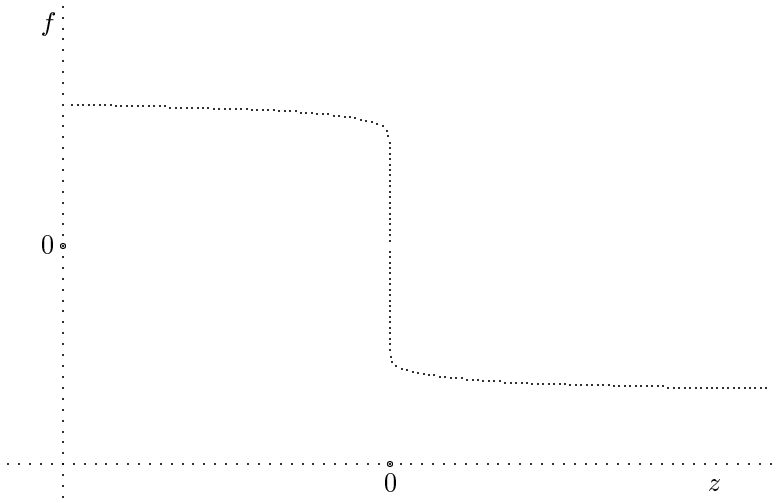


Figure 2. Typical behaviour of the function $f(z)$.

In the case of equal concentrations ($\bar{\alpha} = 0$), the system magnetization becomes independent of ε and of the electric charge sign. It reads as

$$M(T, B^{\rm b}) = \frac{|e| n_e \mu_0 \hbar}{2m\delta} \frac{1}{\xi} \ln \frac{\sqrt{\mathcal{B}^2 + 4\,{\rm e}^{-\xi}} - \mathcal{B}}{\sqrt{\mathcal{B}^2 + 4\,{\rm e}^{-\xi}} + \mathcal{B}}, \quad \mathcal{B} = \frac{\nu_0^2 |e|}{\pi n_e \hbar} B^{\rm b}.$$

Consider the values of $B^{\rm b}$ satisfying $\mathcal{B}^2 << 4{\rm e}^{-\xi}$. Then the magnetization takes the simple form

$$M = -\chi B^{\rm b}, \quad \chi = -\frac{\partial M}{\partial B^{\rm b}} = \frac{N^2 e^2 \mu_0}{2\pi m \delta} \frac{{\rm e}^{\xi/2}}{\xi},$$

where χ is the magnetic susceptibility of the system.

Now the equation (4.58) can be easily solved, and we get

$$B^{\rm b} = (1 + \chi)^{-1} B^{\rm ext}.$$

The magnetic susceptibility of the system becomes exponentially large in the low temperature (large values of ξ) regime. Consequently, the magnetic field is expelled from the sample, what is just the manifestation of the Meissner effect. However, $B^{\rm b}$ does not vanish exactly, but only with exponential precision, i.e., the Meissner effect is not complete. Originally such an observation was made in [76]. Note that the complete Meissner effect is achieved at absolute zero, when the magnetic susceptibility becomes infinite.

Consider now the case $\mathcal{B}^2 >> 4{\rm e}^{-\xi}$. The corresponding expression for M is given by

$$M = -{\rm sgn}(B^{\rm b}) \frac{|e| n_e \mu_0 \hbar}{2m\delta} \left(1 + \frac{2}{\xi} \ln|\mathcal{B}|\right),$$

and for large values of ξ, one can drop the logarithmic term. In that case, one gets

$$B^{\mathrm{b}} \;=\; -\,\mathrm{sgn}(B^{\mathrm{b}})\;\frac{|e|n_e\mu_0\hbar}{2m\delta}\;+\;B^{\mathrm{ext}},$$

and for sufficiently large values of B^{ext}, the first term on the r.h.s. of the last equation can be neglected, meaning that the external magnetic field practically penetrates inside the sample.

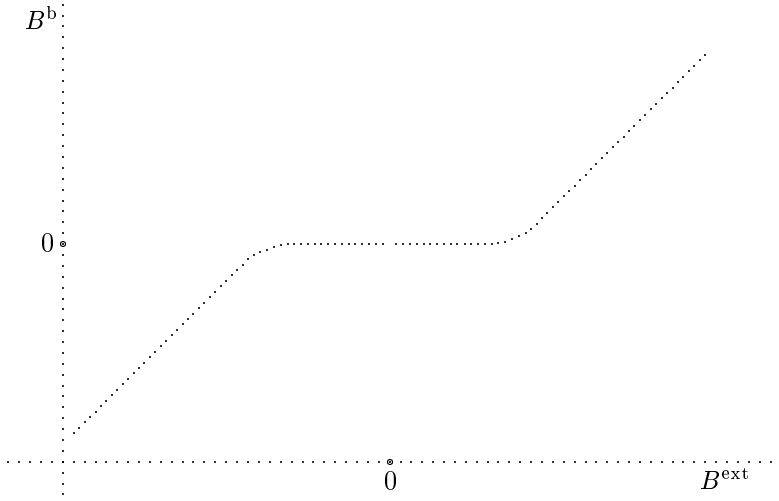


Figure 3. Typical behaviour of function $B^{\mathrm{b}}(B^{\mathrm{ext}})$.

The region of this curve which is almost horizontal corresponds to $\mathcal{B}^2 << 4\mathrm{e}^{-\xi}$, while those with greater slope correspond to $\mathcal{B}^2 >> 4\mathrm{e}^{-\xi}$. As one can see, the magnitude of B^{b} is quite small until B^{ext} reaches some critical value B^{cr}. Above this value, the external magnetic field begins a notable penetration in the sample. The critical value of the external magnetic field is evidently related to a small interval at B^{b} axis, where the magnetization curve drastically changes its direction, i.e., where the curve passes the point of maximal curvature (PMC). The lower is temperature (greater is ξ), more narrow is the interval and it is easier to establish the corresponding critical magnetic field. In order to find the approximated value of B^{cr}, we can use the derivative

$$\frac{\partial M}{\partial B^{\mathrm{b}}} = -\,\frac{\nu_0^2\mu_0 e^2}{\pi m\delta}\;\frac{1}{\xi}\;\left(\alpha^2+4\mathrm{e}^{-\xi}\right)^{-1/2}$$

and consider its behaviour in the low temperature regime.

Until the curve $M(B^{\mathrm{b}})$ reaches the PMC, its slope can be considered to be constant and therefore can be determined by its value at the origin which is exponentially large. On the other hand, the curve becomes practically horizontal after passing the PMC. Obviously, somewhere in the vicinity of the PMC one has $\partial M/\partial B^{\mathrm{b}} = -1$, and using this relation as the definition of the location of PMC, one gets the corresponding value of the internal

magnetic field to be

$$B_0(T) = \left\{ \left(\frac{e n_e \mu_0 \hbar}{m\delta} \, \frac{1}{\xi} \right)^2 - \left(\frac{2\pi n_e \hbar}{\nu_0^2 e} \right)^2 \, \mathrm{e}^{-\xi} \right\}^{1/2} .$$

Substituting $B_0(T)$ into $B^{\mathrm{cr}} = B_0 - M(B_0)$ and keeping the leading terms, one gets

$$B^{\mathrm{cr}}(T) = \frac{|e| n_e \mu_0 \hbar}{2m\delta} + \frac{|e| \mu_0 N k_B T}{\pi \hbar \delta} \ln \frac{e^2 \mu_0 N^3 k_B T}{\pi^2 \hbar^2 n_e \delta}.$$

We observe that starting from the absolute zero and rising the temperature, the value of the critical magnetic field decreases. The area of the maximal curvature becomes smeared for high temperatures and the critical magnetic field becomes ill-defined. On the contrary, the PMC is well localized for $T = 0^\circ K$ and the corresponding value of the critical magnetic field is given by

$$B^{\mathrm{cr}}(0) = \frac{|e| n_e \mu_0 \hbar}{2m\delta}.$$

As a practical realization, we present results for concrete values of the parameters m, e, n_e and δ. In particular, the mass and the electric charge are identified with those of an electron. For the neutralizing background density we take $n_e = 10^{18} m^{-2}$ used in [76] while the value of the interplanar distance in the high T_c superconductors is of order of $\delta = 10^{-9} m$, and we consider the case $N \equiv |\nu_0| = 1$. In the range $|B^{\mathrm{b}}| < 200\, Gauss$ and $T < 200^\circ K$, we have

$$\left| \frac{e\nu_0 B^{\mathrm{b}}}{\pi n_e \hbar} \right| < 5 \cdot 10^{-5} \qquad \pi n_e \ell_0^2 < 4.7 \cdot 10^{-7} \qquad \mathrm{e}^{-\beta \cdot \pi n_e \ell_0^2} < \mathrm{e}^{-30}.$$

As we see, the given values satisfy the conditions (2.52)–(2.54). Figures 4–8 represent the corresponding results for symmetric concentrations of fermion types.

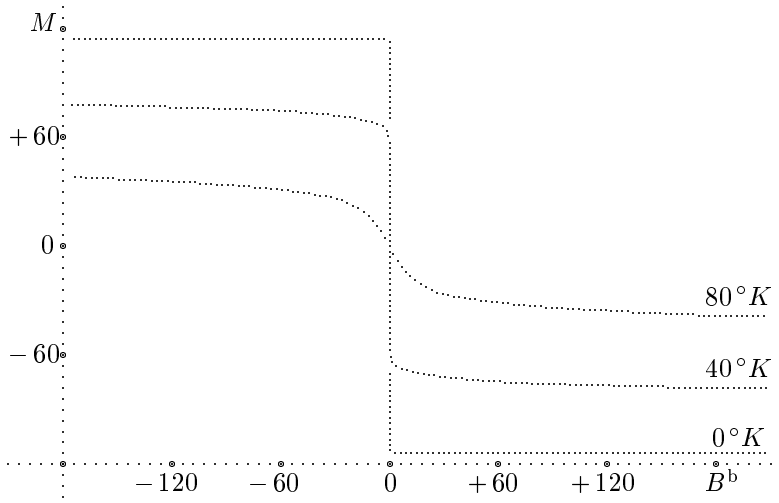


Figure 4. M $[Gauss]$ versus B^{b} $[Gauss]$ for $\bar{\alpha} = 0$.

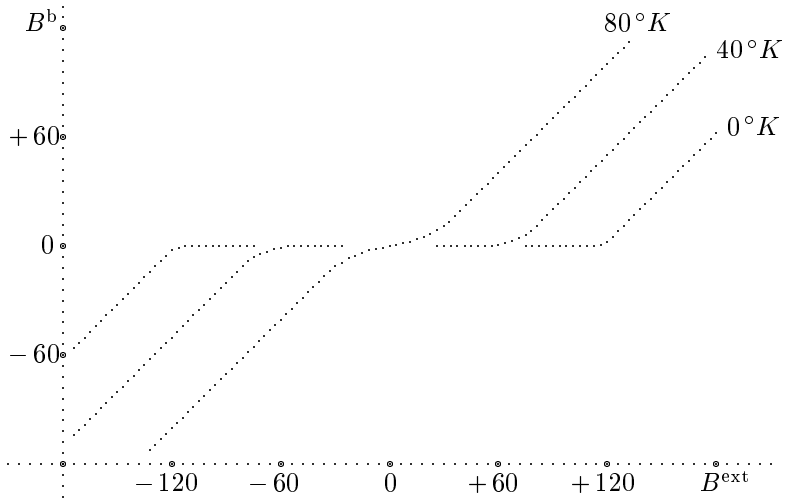


Figure 5. B^{b} $[Gauss]$ versus B^{ext} $[Gauss]$ for $\bar{\alpha} = 0$.

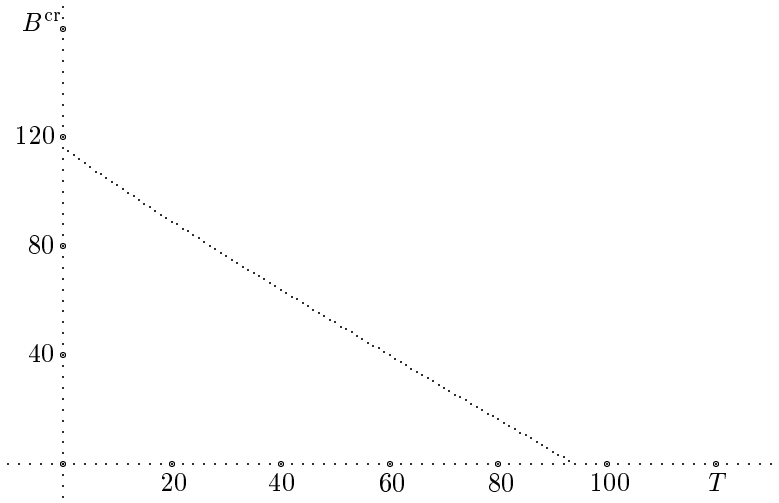


Figure 6. Critical magnetic field $[Gauss]$ versus temperature $[^\circ K]$ for $\bar{\alpha} = 0$.

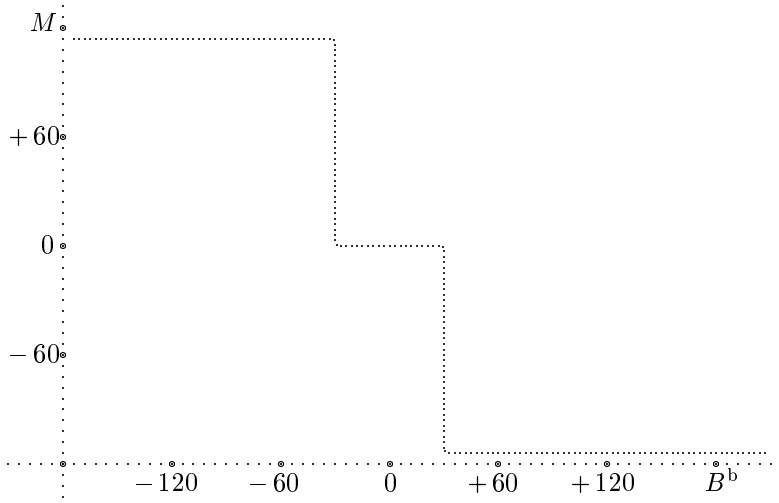


Figure 7. M $[Gauss]$ versus B^{b} $[Gauss]$ for $(\pi n_e \hbar/e) \cdot \bar{\alpha} = 30\, Gauss$ and $T = 0\,^\circ K$.

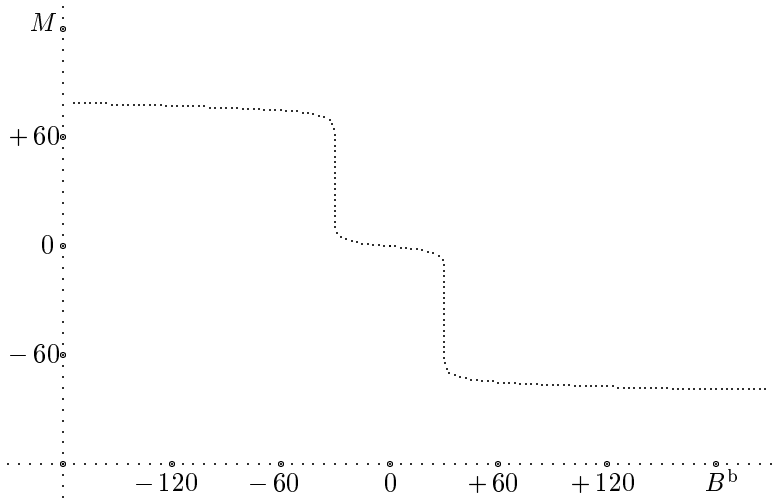


Figure 8. M $[Gauss]$ versus $B^{\rm b}$ $[Gauss]$ for $(\pi n_e \hbar/e)\cdot\bar{\alpha} = 30\,Gauss$ and $T = 40\,^\circ K$.

6. Penetration Depth

Consider now non-vanishing gauge fluctuations. In particular, we study the equations of motion (4.18)–(4.21) for a compound system in the second order approximation, i.e., with the current averages (4.47) and (4.48), where the structure functions take the contributions from both types of fermions.

The equations of motion lead to the following solution for the Chern-Simons field

$$a_0^{\rm f} = -\,\frac{\pi n_e \ell_0 \lambda_{\rm L}^2}{\nu_0}\,B^{\rm f}, \quad b^{\rm f} = -\,\frac{\pi n_e \ell_0 \lambda_{\rm L}^2}{\nu_0}\,\Delta\,A_0^{\rm f},$$

where $\lambda_{\rm L}$ is the London penetration depth given by

$$\lambda_{\rm L}^2 \,=\, \frac{m\delta}{e^2 n_e \mu_0}.$$

Now, the equations of motion get the following effective form

$$\left\{\hat{\Pi}_{\rm E}^{\rm tot} + n_e \ell_0^2 \lambda_{\rm L}^2 \left(1 - \frac{\pi}{\nu_0}\hat{\Pi}_{\rm CS}^{\rm tot}\right)\Delta\right\} A_0^{\rm f} + \left\{\hat{\Pi}_{\rm CS}^{\rm tot} - n_e \lambda_{\rm L}^2\,\frac{\pi}{\nu_0}\hat{\Pi}_{\rm E}^{\rm tot}\right\}\ell_0 B^{\rm f} = 0, \quad (4.64)$$

$$\left\{\hat{\Pi}_{\rm CS}^{\rm tot} - n_e \ell_0^2 \lambda_{\rm L}^2 \frac{\pi}{\nu_0}\hat{\Pi}_{\rm M}^{\rm tot}\,\Delta\right\} A_0^{\rm f} + \left\{\hat{\Pi}_{\rm M}^{\rm tot} + n_e \lambda_{\rm L}^2 \left(1 - \frac{\pi}{\nu_0}\hat{\Pi}_{\rm CS}^{\rm tot}\right)\right\}\ell_0 B^{\rm f} = 0, \quad (4.65)$$

where the superscript "tot" means that both sorts of fermions contribute.

Consider a sample with the geometry of a semi-plane, where the boundary is located at $x = 0$ and the sample occupies the region $x > 0$. The magnetic field inside the sample is $B(r) = B^{\rm b} + B^{\rm f}(r)$ and we look for the fluctuations of the form

$$B^{\rm f}(r) = \left(B^{\rm ext} - B^{\rm b}\right)\mathrm{e}^{-\,x\,/\,\lambda}, \quad A_0^{\rm f}(r) \propto \mathrm{e}^{-\,x\,/\,\lambda} \qquad (4.66)$$

which agree with $B(x \to 0) = B^{\rm ext}$ and $B(x \to \infty) = B^{\rm b}$. Here λ denotes the bare penetration depth which differs from the effective one [76]. Substituting (4.66) into (4.64) and (4.65), we obtain a uniform system of algebraic equations which has a non-trivial solution only if the corresponding determinant vanishes. This condition can be written as

$$\frac{\lambda^2}{n_e^2 \ell_0^2 \lambda_{\rm L}^4} \left\{ \Pi_{\rm CS}^{\rm tot}(z)\, \Pi_{\rm CS}^{\rm tot}(z) - \Pi_{\rm E}^{\rm tot}(z)\, \Pi_{\rm M}^{\rm tot}(z) - n_e \lambda_{\rm L}^2\, \Pi_{\rm E}^{\rm tot}(z) \right\} =$$

$$= \frac{1}{n_e \lambda_{\rm L}^2}\, \Pi_{\rm M}^{\rm tot}(z) - \frac{\pi^2}{\nu_0^2}\, \Pi_{\rm E}^{\rm tot}(z)\, \Pi_{\rm M}^{\rm tot}(z) + \left\{ 1 - \frac{\pi}{\nu_0}\, \Pi_{\rm CS}^{\rm tot}(z) \right\}^2 ,$$

and if solved for $z \equiv -\ell^2 / 2\lambda^2$, sets the bare penetration depth as a function of $B^{\rm ext}$ and the temperature. In figs. 9 and 10, we present the results obtained by numerical methods. (We use the power series expansion of $\Pi^{\rm tot}(z)$ given in Appendix D).

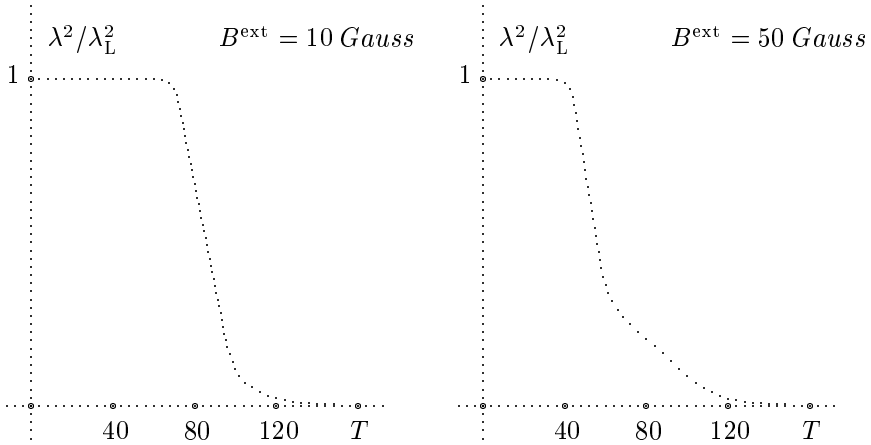


Figure 9. $\lambda^2/\lambda_{\rm L}^2$ versus temperature $[^\circ K]$ for $\bar{\alpha} = 0$.

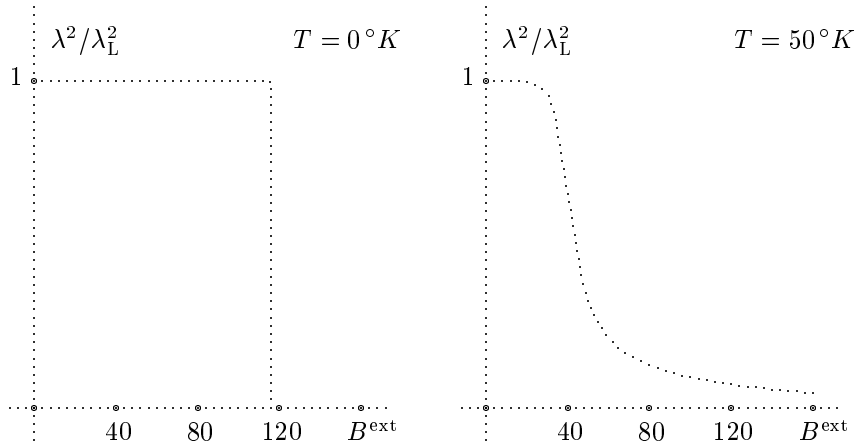


Figure 10. $\lambda^2/\lambda_{\rm L}^2$ versus $B^{\rm ext}$ $[Gauss]$ for $\bar{\alpha} = 0$.

The effective penetration depth is defined as a gradient of the magnetic field inside the sample

$$\frac{1}{\lambda_{\rm eff}(d)} \;=\; -\,\frac{1}{d}\int_0^d \frac{\partial B(x)}{\partial x}\,\frac{dx}{B(x)} \;=\; \frac{1}{d}\,\ln\frac{B(0)}{B(d)}.$$

Obviously, $\lambda_{\rm eff}(d)$ measures how fast the magnetic field changes over the distance d from the boundary towards the bulk of the sample.

In BCS theory, the magnetic field inside the sample is given by $B(x) = B^{\rm ext}{\rm e}^{-\,x/\lambda}$ causing the effective penetration depth to be coincident with the bare one and to be independent of d. Here this is not the case since the Meissner effect is incomplete, i.e., the magnetic field inside the sample is given by

$$B(x, B^{\rm ext}, T) = B^{\rm b}(B^{\rm ext}, T) + \left\{B^{\rm ext} - B^{\rm b}(B^{\rm ext}, T)\right\}{\rm e}^{-\,x/\lambda\,(B^{\rm ext}, T)},$$

and the effective penetration depth takes the form

$$\frac{1}{\lambda_{\rm eff}(d, B^{\rm ext}, T)} \;=\; \frac{1}{d}\,\ln\frac{B^{\rm ext}}{B^{\rm b} + (B^{\rm ext} - B^{\rm b})\,{\rm e}^{-\,d/\lambda}}.$$

It is reduced to λ when the complete Meissner effect ($B^{\rm b} = 0$) takes place. The effective penetration depth as a function of the temperature is represented in figure 11.

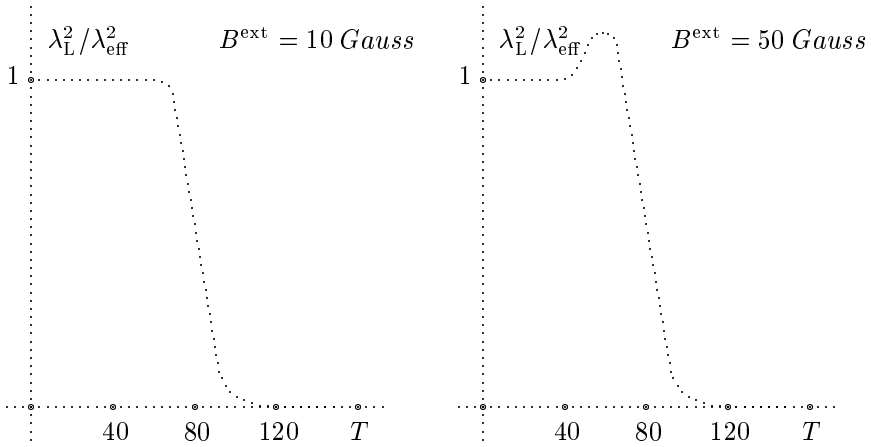


Figure 11. $\lambda_{\mathrm{L}}^2/\lambda_{\mathrm{eff}}^2$ versus temperature $[^\circ K]$ for $\bar{\alpha} = 0$ and $d = \lambda_{\mathrm{L}}$.

CHAPTER 5
ON THE EFFECTIVE POTENTIAL IN THE QUANTUM THEORY

1. Introduction

It is well realized that practically all the physical information contained in the quantum field theory can be encoded in the Green function generating functional (see, e.g., [15]).

$$Z[J] \equiv e^{iW[J]} =< 0|T\exp(i\int d^4x J(x)\phi(x))|0>, \tag{5.1}$$

where the Schwinger functional $W[J]$ generates the connected Green functions:

$$W[J] = -i\ln Z[J] = \sum_n^\infty \frac{1}{n!}\int dx_1\cdots dx_n J(x_1)\cdots J(x_n)G_n^c(x1,\ldots,x_n). \tag{5.2}$$

Closely related objects are the effective action and effective potential, which in principle permit to formulate quantum problems in terms of c-number quantities and variational equations.

Remind that one of the central problems in the quantum field theory is the determination of the ground state (vacuum) which, in its own turn, is characterized by the vacuum expectation value of the quantum field $\phi(x)$. This expectation value should be determined in terms of parameters appearing in the Lagrangian. The classical meaning of this vacuum average is the lowest energy configuration which can be found by minimizing the classical potential. Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x) - U(\phi). \tag{5.3}$$

The energy of the system is given by the functional

$$E[\varphi] = \int\limits_\Omega d^3x\left(\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\varphi)^2 + U(\phi)\right)$$

(Ω is the volume). $U(\phi)$ is the potential energy density and its minimum corresponds to the minimal energy field configuration. In the quantum case, $U(\phi)$ contains the interaction terms and the vacuum expectation values are altered by perturbative corrections and the renormalization procedure must be applied. In any case, it is natural to expect that there must be some quantum object which includes the classical potential and its quantum corrections.

The functional derivative of $W[J]$ gives the expectation value of ϕ in the presence of varying source:

$$\frac{\delta W[J]}{\delta J(x)} =< \mathrm{vac}|\phi(x)|\mathrm{vac} >_J = \phi_c(x). \tag{5.4}$$

The efective action is defined as a functional Legendre transformation of the Schwinger functional

$$\Gamma[\phi_c] \equiv W[J] - \int d^4y J(y)\phi_c(y). \tag{5.5}$$

If the external source is set to zero, the effective action satisfies the equation

$$\frac{\delta}{\delta\phi_c(x)}\Gamma[\phi_c] = 0.$$

The solutions of this equation are the values of $< \phi(x) >$ in stable quantum states of the theory. For a translation-invariant vacuum state, ϕ_c is independent of x. Sometimes this equation has additional solutions corresponding to localized lumps of field held together by their self-interaction. In this states, called solitons, the solution $\phi_c(x)$ depends on x.

The effective action $\Gamma[\phi_c]$ is an extensive quantity, and for the constant classical fields is proportional to the volume of the space-time region over which the functional integral is taken:

$$\Gamma[\phi_c] = -(VT)\cdot V_{eff}(\phi_c).$$

The function $V_{eff}(\phi_c)$ is called the effective potential [26], [1], [125].

The equilibrium condition reduces to the simple equation

$$\frac{\partial}{\partial\phi_c}V_{eff}(\phi_c) = 0. \tag{5.6}$$

Each solution of the last equation corresponds to a translation-invariant state with $J = 0$, and V_{eff} evaluated at that solution is just the energy density of this quantum state.

The local maxima or saddle points of V_{eff} are unstable configurations. The local minimum corresponds to the metastable vacuum state which can decay to the true vacuum by quantum-mechanical tunneling. The absolute minimum of the effective potential is the state of lowest energy, i.e., the true stable vacuum. A system with a spontaneously broken symmetry will have several minima of V_{eff}, all with the same energy by virtue of the symmetry. The choice of one among these vacua is the spontaneous symmetry breaking. (A spontaneously broken symmetry corresponds to the non-zero values of solutions of (5.6).)

Consequently, the central objects are the vacuum expectation value

$$\phi_c(x) = \frac{< 0|\phi(x)|0 >_J}{< 0|0 >_J}$$

and a functional $\Gamma[\phi_c]$ such that $\phi_c(x)$ is a solution of the classical variational equation

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -J(x).$$

In order to compute the effective action , one must first find $W[J]$ and then invert (5.4) to obtain J as a functional of ϕ_c and replace J in (5.5).

At the tree level, effective action is just the classical action and the full effective action includes all the quantum corrections: the effective action generates proper (connected, truncated, one particle irreducible) n-point functions $\Gamma^{(n)}(x_1, \ldots, x_n)$ via a functional Taylor expansion [19], [82]

$$\Gamma[\phi_c] = \sum_n \frac{1}{n!} \int dx_1 \cdots dx_n \Gamma^{(n)}(x_1, \ldots, x_n)\phi_c(x_1) \cdots \phi_c(x_n).$$

The effective action can be expanded in another way, this time in powers of the momentum, i.e., space-time derivatives of $\phi_c(x)$ [79]

$$\Gamma[\phi_c] = \int dx(-V_{eff}[\phi_c] - \frac{1}{2}Z[\phi_c](\partial_\mu \phi_c)^2 + \cdots).$$

In this expansion, the lowest order term which can be separated out by setting $\phi_c(x) = const$ is the effective potential $V_{eff}(\phi_c)$.

The effective potential can be computed using operator or path integral representations.

In the operator approach, the Schwinger functional $W[J]$ is defined from (5.1). The formal series expansion of the exponential leads to the Taylor expansion of the generating functional, where the coefficients of the expansion are time-ordered products of the field in the vacuum or Green functions. A perturbative expansion of this coefficients leads to the series represented by diagrams, and the Feynman rules are used to evaluate each diagram. If the effective action is expressed as an expansion in powers of momentum, then the lowest order term independent of p is the effective potential. If we expand in powers of $\hbar$, then the term in the expansion of order $\hbar$ is the collection of all 1-loop proper diagrams with vanishing external momenta.

Below we give a well-known expression for the one-loop effective potential for the case of a self-interacting scalar field with a Lagrangian (5.3). Rotating to the Euclidean space, one finds [25], [26]

$$V_{eff}(\phi_c) = U(\phi_c) + \frac{1}{2} \int \frac{d^4 k_E}{(2\pi)^4} \ln \left(1 + \frac{U''(\phi_c)}{k_E^2}\right). \tag{5.7}$$

Integrating the last expression over k_4, one gets

$$V_{eff}(\phi_c) = U(\phi_c) + \int \frac{d^3 k}{(2\pi)^3} \sqrt{\mathbf{k}^2 + U''(\phi_c)}.$$

The second term is a quantum correction to the classical potential and corresponds to fluctuations about a classical field value. In other words, the effective potential is the sum of the classical potential and all zero-point energy fluctuations about ϕ_c.

As it stands, the expression (5.7) is divergent. In a renormalizable theory, it must be possible to absorb the divergence by redefining the theory parameters.

Consider the case of the scalar field with a quartic interaction. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\phi(x)(\partial_\mu \partial^\mu - m_0^2)\phi(x) - \frac{\lambda_0}{4!}\phi^4(x).$$

The effective potential to one loop is

$$V[\phi_c] = \frac{1}{2}m_0^2\phi_c^2 + \frac{\lambda_0}{4!}\phi_c^4 - \frac{i}{2}\int \frac{d^4p}{(2\pi)^4}\ln\left(1 - \frac{\lambda_0\phi_c^2/2}{p^2 - m_0^2 + i\epsilon}\right). \tag{5.8}$$

The renormalized mass and coupling can be defined as follows

$$\frac{d^2}{d\phi_c^2}V[\phi_c]\Big|_{\phi_c=0} = m^2, \quad \frac{d^4}{d\phi_c^4}V[\phi_c]\Big|_{\phi_c=0} = \lambda. \tag{5.9}$$

Since we are only dealing with the effective potential, we can put the wave function renormalization equal to one. The substraction points are arbitrary and we are free to change them using the finite renormalization.

Application of eq.(5.9) yields

$$m^2 = m_0^2 + \frac{1}{2}\lambda_0 \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m_0^2 + i\epsilon}$$

and

$$\lambda = \lambda_0 + \frac{3}{2}i\lambda_0^2 \int \frac{d^4p}{(2\pi)^4}\left(\frac{1}{p^2 - m_0^2 + i\epsilon}\right)^2.$$

Solving these equations for bare mass and coupling (in the one-loop approximation), we can rewrite the effective potential in terms of finite renormalized quantities

$$\begin{aligned} V[\phi_c] = {} & \frac{1}{2}m^2\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 - \frac{i}{2}\int \frac{d^4p}{(2\pi)^4}\ln\left(1 - \frac{(\lambda\phi_c^2/2)}{p^2 - m^2 + i\epsilon)}\right) - \\ & - \frac{1}{4}\lambda\phi_c^2 \int d^4p(2\pi)^4 \frac{i}{p^2 - m^2 + i\epsilon} - \\ & - i\frac{\lambda^2}{16}\phi_c^4 \int \frac{d^4p}{(2\pi)^4}\left(\frac{1}{p^2 - m^2 + i\epsilon}\right)^2. \end{aligned} \tag{5.10}$$

An alternative but equivalent definition of the effective potential can be given using the Schrödinger representation. Here the appropriate method to find the static effective action is by solving the appropriate functional Schrödinger equation, or by variational procedure. Both methods arise because the effective potential is the minimum expected energy per unit volume in the set of states where the expected value of the field is ϕ_c. The vacuum state will minimize this energy density, hence we solve the appropriate Schrödinger equation or use the variational calculation.

In the classical case, the potential for a given constant field configuration $\phi = \phi_c$=const coincides with the energy density

$$V_{cl} = \frac{1}{\Omega} E[\phi_c].$$

The quantum version of this formula will be

$$V(\phi_c) = \frac{1}{\Omega} \min < \psi | H(\phi, \pi) | \psi > . \tag{5.11}$$

Minimization must be carried out under the conditions

$$< \psi | \phi(x) | \psi >= \phi_c, \quad < \psi | \psi >= 1.$$

The static effective action $\Gamma_{static}[\phi_c]$ is defined as minus the minimum of energy

$$\Gamma_{static} = - < \psi_0 | H | \psi_0 > \tag{5.12}$$

in a state $|\psi_0>$ subject to the constraints

$$< \psi_0 | \psi_0 >= 1, \qquad < \psi_0 | \phi(x) | \psi_0 >= \phi_c(x).$$

In order to solve this problem we have to introduce the Lagrange multipliers E and $J(x)$ and find the unconstrained minimum of

$$< \psi_0 | \int d^3x (\mathcal{H} - J(x)\varphi(x)) - E | \psi_0 >; \tag{5.13}$$

$J(x)$ here is a time independent external source and $\phi_c(x)$ is the field vacuum average in the presence of the source. Thus minimizing eq.(5.12) is equivalent to solving the functional Schrödinger equation for the vacuum state functional $|\psi_0>$ of the Hamiltonian H'

$$\left(H - \int d^3x J(x)\varphi(x)\right) | \psi_0 >= E_0 | \psi_0 > .$$

The energy eigenvalue $E_0[J]$ is a source-dependent functional. The application of the Feynmann-Hellmann theorem yields

$$\frac{\delta E_0}{\delta J(x)} =< \psi_0 | \frac{\delta}{\delta J(x)} H' | \psi_0 >= - < \psi_0 | \varphi(x) | \psi_0 >= -\phi_c(x).$$

The last equation implies that the static effective action is the Legendre transform of the vacuum energy

$$\Gamma_{static}[\phi_c] = -E_0[J] - \int d^3x J(x)\phi_c(x),$$

where $J(x)$ is solved in terms of ϕ_c.

In the path integral representation, the generating functional is given by

$$Z[J] = \mathcal{N} \int \mathcal{D}\phi \exp(iS[\phi, J]). \tag{5.14}$$

It is well known that the loop expansion is equivalent to the stationary phase approximation. In order to develop this approximation, we Taylor expand the classical action

$$S[\phi, J] = \int d^4x \left(\mathcal{L}(x) + J(x)\phi(x)\right)$$

about the solution of the Euler-Lagrange equation

$$\frac{\delta S}{\delta \phi(x)} = 0, \quad \phi(x) = \phi_c(x).$$

This results in a Gaussian approximation to the path integral (5.14) which we can compute exactly.

Using this approximation, $Z[J]$ becomes

$$Z[J] \approx \mathcal{N} \int \mathcal{D}\phi \exp\left(iS[\phi_c, J] + \frac{i}{2}\frac{\delta^2 S}{\delta\phi^2}\Big|_{\phi=\phi_c}(\phi - \phi_c)^2\right).$$

After shifting integration variables, $\phi \to \phi - \phi_c$, the Gaussian integration yields

$$Z[J] \approx \mathcal{N} \exp(iS[\phi_c, J]) \det{}^{-\frac{1}{2}}\left(\frac{\delta^2 S}{\delta\phi^2}\Big|_{\phi=\phi_c}\right).$$

Here

$$\frac{\delta^2 S[\phi, J]}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi_c} = -\left(\partial^2 + m_0^2 + \frac{\lambda_0}{2}\phi_c^2\right)\delta^4(x-y).$$

In the absence of the spontaneously broken symmetry, the classical field $\phi_c = 0$ when $J = 0$. Consequently the normalization factor

$$\mathcal{N} = \det{}^{\frac{1}{2}}((\partial^2 + m_0^2)\delta^4(x-y)).$$

Therefore

$$Z[J] \approx \exp(iS[\phi_c, J]) \det{}^{-\frac{1}{2}}\Big(1 + \frac{\lambda_0}{2}G_2^{(0)}(x-y)\phi_c^2(y)\Big),$$

where $G_2^{(0)}$ is the free-field Green function. Using the formula

$$\det M = \exp(\mathrm{tr}\ln M),$$

one can extract the functional $W[J]$

$$W[J] \approx S[\phi_c, J] + \frac{i}{2}\mathrm{tr}\ln\Big(1 + \frac{\lambda_0}{2}G_2^{(2)}(x-y)\phi_c^2(y)\Big)$$

and the effective potential

$$\Gamma[\phi_c] \approx S[\phi_c, 0] + \frac{i}{2}\hbar\mathrm{tr}\ln\Big(1 + \frac{\lambda_0}{2}G_2^{(0)}(x-y)\phi_c^2(y)\Big) + \mathcal{O}(\hbar^2).$$

Setting $\phi_c = \mathrm{const}$, the matrix becomes diagonal in the momentum space, and one obtains theexpression identical to (5.8).

After this lengthy introduction, we can turn to the problem which will be discussed below.

The effective potential (5.8) can be written in the Euclidean variables

$$V[\phi_c] = \frac{1}{2}m_0^2\phi_c^2 + \frac{\lambda_0}{4!}\phi_c^4 + \frac{i}{2}\int \frac{d^4p_E}{(2\pi)^4}\ln\left(1 - \frac{\lambda_0\phi_c^2/2}{p_E^2 + m_0^2}\right). \qquad (5.15)$$

The last expression can be obtained directly if one considers the Euclidean, or Wick rotated Green functions generating functional

$$Z_E[J] = \mathcal{N}\int \mathcal{D}\phi_E \exp(-S_E[\phi_E, J]) = e^{W_E[J]} \equiv$$
$$\equiv \int d\mu(\phi_E) e^{\int d^4x_E J(x)\phi_E(x)}. \qquad (5.16)$$

Here $\phi_E(\tau, \mathbf{x}) = \phi(-i\tau, \mathbf{x})$ is the Wick rotated field variable and S_E is the corresponding Euclidean action:

$$S_E[\phi_E, J] = \int d^4x_E[\mathcal{L}_E - J(x_E)\phi_E(x_E)].$$

The functional $\mathcal{L}_E$ has the form of energy: it is bounded from below and becomes a large when the field ϕ has large amplitude.

With the positively defined integration measure $d\mu$, the Schwinger functional $W_[J]$ satisfies the Hölder inequality [127], [79]

$$e^{W_E[\alpha J_1 + (1-\alpha)J_2]} \leq e^{\alpha W_E[J_1]} e^{(1-\alpha)W_E[J_2]},$$

which is equivalent to say that $W_E[J]$ is a convex functional. The Euclidean effective action is a Legendre transformation of $W_E[J]$. It is well known, that Legendre transform of a convex function is also convex. Now, the effective potential is a value of the effective action for a constant field multiplied by a 4-volume. Consequently, the effective potential given by (5.15) must be a convex function of the classical constant field variable.

Our main objective will be the study of the effective potential for theories with degenerate ground states. As it will be discussed in Section 2, in such cases there is some contradiction between the definition of the effective potential and actual computations: as we have seen, the effective potential must be a convex function while in the theories with vacuum degeneracy, the potential is a non-convex. In the subsequent Section 3, we propose a procedure which permits to redefine generating functionals in such a way that the resulting effective potential is a convex function. In Section 4, we study the same problem from the point of view of path integral representation and show how convexity is related to the correct definition of the boundary conditions.

2. Spontaneous Symmetry Breaking and Convex Effective Potential

Up to now, in our discussion we have tacitly assumed that the system possesses unique vacuum state. At the same time, practically all the physically significant theories are built up using the ideology of degenerate ground states and spontaneous symmetry breaking.

Consider the theory, where the classical potential corresponds to a spontaneously broken symmetry, i.e., there are different minima with equal energy. The simplest example is given by the potential

$$V_{cl}[\varphi] = -\frac{1}{2}m_0^2 + \frac{\lambda_0}{4!}. \tag{5.17}$$

For this potential, $\phi_c = 0$ is an unstable local extremum. There are now two stable minima occurring at

$$\phi_c = \pm\Big(\frac{6m_0^2}{\lambda_0}\Big)^{1/2}.$$

The potential (5.17) is symmetric under the transformation $\phi_c \to -\phi_c$, and the symmetry operation takes us from one vacuum to the other, thus the particular vacuum does not respect or reflect the symmetry present in the action.

The corresponding loop expansion for the effective potential can be obtained if one naively performs the analytic continuation term by term in m_0^2 from $m_0^2 > 0$ to $m_0^2 < 0$ in (5.10).

Introducing the cutoff and integrating over the momentum variables, one obtains (see, e.g., [125])

$$V[\phi_c] = \frac{1}{2}m^2\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 + \frac{1}{64\pi^2}M^2(\phi_c)\ln\frac{M^2(\phi_c)}{m^2} + \\ - \frac{1}{128\pi^2}\lambda\phi_c^2(m^2 + \frac{3}{4}\lambda\phi_c^2),$$

where

$$M^2(\phi_c) = m^2 + \frac{\lambda}{2}\phi_c^2.$$

In the case of spontaneously broken symmetry, we have $m^2 < 0$. From the two last expressions it follows that the one-loop effective potential is complex within the points of inflexion of the classical potential, i.e., those ϕ_c for which $M^2(\phi_c) \le 0$. This complexity is genuinely disturbing, because from the functional integral representation of the Euclidean generating functional (5.16) it follows that both $W_E[J]$ and its Legendre transform must be real functionals. Furthermore, the resulting effective potential is not convex in the same domain of field variable.

The same conclusion must be valid if one uses the definition of the effective potential given by (5.11). This definition becomes problematic if one allows states $|\Psi\rangle$ which are not localized in ϕ. If one includes nonlocalized

states, the set of $|\Psi_0>$'s is too general and the effective potential so defined is always convex. If, however, we restrict ourselves to localized $|\Psi_0>$'s, V_{eff} can be nonconvex, and even complex in the region where it is nonconvex (the complex part is related to an instability for the localized state to decay into a superposition of states [132]).

So one can conclude, that in the theories where the classical potential possesses several minima, the corresponding effective potential in general is neither real nor convex.

3. Quasiaverages and Interpolated Loop Expansion

As we have seen in the previous section, there is some kind of discrepancy in the definition of the effective potential. Namely, in the cases of the theories with spontaneously broken symmetries, the naively applied loop expansion leads to the expression which contradicts the requirement of convexity and reality.

Since the most interesting field theory models are essentially based on the ideas of broken symmetries (for a review, see [70]), the aforementioned contradiction created a definite interest to clarify the situation (see, e.g., [11], [59], [119]).

Below we follow the presentation given in the papers [33], [34], [35], [36].

A general approach to the discussion of the systems with degenerate ground states was proposed by N.Bogolubov and is known as the method of quasiaverages [16].

The essence of this method is to add a small symmetry-breaking perturbation νH_1 to the symmetric Hamiltonian H which removes degeneracy. At first one must calculate the expectation values for the volume Ω:

$$< \mathcal{O} >_{\nu,\Omega} = \frac{\mathrm{tr}_\Omega \mathcal{O} e^{-\beta(H+\nu H_1)}}{\mathrm{tr}_\Omega e^{-\beta(H+\nu H_1)}},$$

where tr_Ω denotes the trace over the Hilbert space of the system in volume Ω. Then one must take the limit of infinite volume, and finally let $\nu \to 0$, obtaining

$$\prec \mathcal{O} \succ = \lim_{\nu \to 0} \lim_{\Omega \to \infty} < A >_{\nu,\Omega} . \tag{5.18}$$

It is characteristic of the broken symmetry that the order of these limits is not reversible and actually the condition $\prec \mathcal{O} \succ \neq 0$ may be taken as a definition of a broken symmetry.

In the case of zero temperatures, the interaction νH_1 breaks the degeneracy of the ground states and in the limiting case of infinite volume, a single member of the set of degenerate states is selected.

The role played by the correct limiting procedure can be illustrated by condensed matter system exhibiting these phenomenon in a physically transparent way – quantum ferromagnetics.

Ferromagnetic systems have a spontaneous magnetization at low temperatures, i.e., below the Curie temperature. When a spontaneous magnetization is present, the Hamiltonian of the system is invariant under transformations that change the sign of the magnetization; however, the state of the system, i.e., probability distribution, is not invariant under this transformation. This possibility can be realized only in the infinite volume limiting case and when this happens, we say that the symmetry is spontaneously broken.

Consider the magnetization density and the free energy as a function of H in the infinite volume limiting case. The symmetry arguments tells us that $f(H) = f(-H)$ and

$$m(H) = -\frac{\partial f(H)}{\partial H} = -m(-H),$$

i.e., $f(H)$ and $m(H)$ are even and odd functions of H, respectively. We have two possibilities: either $m(h)$ is continuous at $H = 0$ and $m(0)$ is consequently zero, or $m(H)$ is discontinuous at $H = 0$ and $m(0)$ is not well defined; this second case corresponds to a spontaneous symmetry breaking. More precisely, these two possibilities correspond to

$$\begin{aligned} &\imath) && f(H) = f(0) + \mathcal{O}(H^\alpha) \qquad \alpha > 1, \\ &\imath\imath) && f(H) = f(0) - m_s|H| + \mathcal{O}(H^\alpha). \end{aligned}$$

In the second case, we have

$$m_+ \equiv \lim_{H\to 0^+} m(H) = m_s \neq m_- \equiv \lim_{H\to 0^-} m(H) = -m_s.$$

The symmetry relation implies only $m_+ = -m_- = m_s$.

The non-differentiability of the free energy with respect to a parameter and the existence of two or more equilibrium states is the distinctive feature of a phase transitions. The nonanalytic nature of the free energy in the infinite volume limiting case can be demonstrated by the following toy model. Consider a system of the volume Ω which can stay in only two states (of equal energy at zero magnetic field) having total magnetization $\pm\Omega$, respectively. The statistical sum as a function of the magnetic field is given by

$$Z(H) = e^{\beta\Omega H} + e^{-\beta\Omega H} = 2\cosh(\beta\Omega H).$$

The corresponding free energy and magnetization are

$$f(H) = -\frac{1}{\beta\Omega}\ln[2\cosh(\beta\Omega H)], \qquad m(H) = \tanh(\beta\Omega H).$$

Spontaneous magnetization in the limit $\Omega \to \infty$ will be

$$m_+ = \lim_{H\to 0^+}\lim_{\Omega\to\infty} = +1, \qquad m_- = \lim_{H\to 0^-}\lim_{\Omega\to\infty} = -1.$$

It means that in order to have a mathematical discontinuity, we must first go to the infinite volume limit and only afterwards switch off the external field.

Let us apply the quasiaverages approach to the calculations of the generating functional and the effective potential. In other words, we will take into account the effect of the limiting procedure $\Omega \to \infty$.

The generating functional in D-dimensions can be written in the following form

$$Z[J_a] = N \int \prod_a \mathcal{D}\phi_a e^{-\int d^D x[\mathcal{L}_E(\phi) - J_a \phi_a]} = e^{W[J_a]}.$$

Here $1 \leq a \leq n$, where n is a number of scalar fields, $\mathcal{L}_E$ – the invariant Euclidean Lagrangian, J_a – external currents, and N is a normalization constant. Integration is performed over the infinite D-dimensional volume.

Following the ideology of quasiaverages, preliminary the system must be considered in the finite D-dimensional volume Ω and under a small external perturbation $J_a \phi_a$, which breaks the symmetry. Define the Green functions generating functional

$$Z_\Omega[J_a] = N_\Omega \int \prod_a \mathcal{D}\phi_a e^{-\frac{1}{\hbar}\int_\Omega d^D x[\mathcal{L}_E(\phi) - J_a \phi_a]} = e^{W_\Omega[J_a]},$$

where we introduced the parameter of quasiclassical expansion $\hbar$.

The object of our interest is the functional $W_\Omega[J_a]$ for small values of currents and large volumes. The quasiaverages are defined as follows

$$\prec \phi_a \succ = \lim_{J \to 0} \lim_{\Omega} \frac{\delta W_\Omega[J]}{\delta J_a}.$$

Remind that the order of the limiting procedure is essential: initially one has to go to the infinite volume limit, and afterwards put $J_a = 0$. The last limit must be taken in the definite way, i.e., the result depends on the direction how the perturbation vanishes.

In the limiting case of the infinite volume, the functional $W_\Omega[J]$ coincides with the generating functional $W[J]$. In what follows, we will consider the domain of small external sources since this region determines the convexity property of the effective potential.

The functional integral can be calculated using the interpolated loop expansion [11], [119], i.e., by the Laplace method. In the one-loop approximation, we have

$$Z_\Omega[J_a] = N \sum_{\bar{\phi}_a} \left[\frac{2\pi\hbar}{\det(-\frac{\delta^2 A_\Omega}{\delta\phi_a \delta\phi_b})} \right]^{\frac{1}{2}}_{\bar{\phi}_a} \exp\left[-1/\hbar \Big(\mathcal{A}_\Omega[\bar{\phi}] - \int_\Omega d^D x J_a \bar{\phi}_a \Big) \right].$$

Here the summation is performed over the set of all the minima of the action functional

$$S_\Omega[J] = \int\limits_\Omega d^D x[\mathcal{L}_E - J_a\phi_a] \equiv \mathcal{A}_\Omega[\phi] - \int\limits_\Omega d^D x J_a\phi_a\,.$$

The fields $\bar{\phi}_a$ are solutions of the extremal equations

$$\frac{\delta\mathcal{A}_\Omega}{\delta\phi_a}|_{\bar{\phi}_a} = J_a.$$

They correspond to the minimum, i.e.,

$$\frac{\delta^2\mathcal{A}_\Omega}{\delta\phi_a\delta\phi_b}|_{\bar{\phi}_a} > 0.$$

Consequently,

$$Z_\Omega[J] = \mathcal{N}\sum_{\bar{\phi}}\sqrt{2\pi\hbar}e^{-1/\hbar[\mathcal{A}_\Omega[\bar{\phi}]-\int_\Omega d^D x J_a\bar{\phi}_a]}\times e^{-1/2\mathrm{tr}\ln(\frac{\delta^2\mathcal{A}_\Omega}{\delta\phi_a\delta\phi_b})|_{\bar{\phi}_a}} =$$
$$= e^{1/\hbar W_\Omega[J_a]}.$$

The effective potential is determined as a functional Legendre transformation of $W_\Omega[J]$, when the sources J_a are x-independent constants:

$$V_\Omega(\phi_a)[-\int\limits_\Omega d^D x] = [W_\Omega[J_a] - \int\limits_\Omega d^D x J_a\phi_a], \qquad \phi_a = \frac{\delta W_\Omega}{\delta J_a},$$

i.e., it is an ordinary Legendre transformation

$$V_\Omega = [J_a\phi_a - w_\Omega(J_a)], \qquad \phi_a = \frac{\partial w_\Omega}{\partial J_a},$$

where

$$W_\Omega[J_a] = w_\Omega(J_a)\int\limits_\Omega d^D x.$$

It is important to note that such a definition of the Legendre transformation is valid only for smooth one-side convex functions. In that case,

$$\frac{\partial^2 V}{\partial\phi_a\partial\phi_b}\cdot\frac{\partial^2 w}{\partial J_b\partial J_a} = \delta_{ac}.$$

In what follows, we will use a more general definition of the Legendre transformation, which is valid for arbitrary (non-convex or non-differentiable) functions [5]

$$J_a, w(J_a) \to \phi_a, V(\phi_a),$$
$$V(\phi_a) = \sup_{J_a}[J_a\phi_a - w(J_a)].$$

As an illustrating model, consider $D = 0$ dimensional case, when the generating functional is given by the ordinary integral

$$z_\Omega(j_a) = \mathcal{N} \int \cdots \int \prod d\phi_a e^{1/\hbar\, \Omega[J_a\phi_a - v(\phi_a)]}.$$

In order to maximally maintain the resemblance with the realistic theory here we formally have introduced the "volume" Ω. The one-loop integration gives

$$z_\Omega(J_a) = \sum_{\bar\phi_a} \mathcal{N}\sqrt{2\pi\hbar} e^{\Omega/\hbar[J_a\bar\phi_a - v(\bar\phi_a)]} e^{-\frac{\Omega}{2}\frac{1}{\Omega}\mathrm{tr}\ln v''(\bar\phi_a)}.$$

The summation is performed over the maxima of the function

$$s(\phi, J) = J_a\phi_a - v(\phi_a),$$

i.e., $\bar\phi_a$ satisfies the equation

$$\frac{\partial v}{\partial \phi_a} = J_a \tag{5.19}$$

under the conditions

$$v''_{ab}(\bar\phi) > 0.$$

Equation (5.19) constitute a system of nonlinear (as usually cubic) equations, which can be solved using the perturbation theory, expanding the currents J_a into series over the small quantities. As we know, the domain in the vicinity of vanishing currents is essential for the convexity and spontaneous symmetry breaking.

After this formal introduction, let us go to the concrete case and consider the simplest two-minima potential

$$v = \frac{1}{4}g(\phi^2 - a^2)^2.$$

For small values of J,($J \le \frac{4}{27}g^2a^2$), the maxima of the function

$$S(\phi, J) = J\phi - v(\phi)$$

are determined by solutions of the cubic equation

$$\frac{\partial v}{\partial \phi} = g\phi^3 - ga^2\phi = J.$$

These solutions are given by

$$\phi_\pm(J) = \pm a + \frac{1}{2}\frac{J}{ga^2} \mp \frac{3}{8}\frac{J^2}{g^2a^5} + \frac{1}{2}\frac{J^3}{g^3a^8} \mp \frac{105}{128}\frac{J^4}{g^4a^{11}} + \cdots .$$

Here

$$S(\phi_+) = a^4 g[j + \frac{1}{4}j^2 - \frac{1}{8}j^3 + \frac{1}{8}j^4 + \cdots],$$

$$S(\phi_-) = a^4 g[-j + \frac{1}{4}j^2 + \frac{1}{8}j^3 + \frac{1}{8}j^4 + \cdots],$$

where

$$j \equiv \frac{J}{a^3 g}.$$

In the tree approximation, one gets

$$e^{\frac{1}{\hbar}\Omega w_\Omega(J)} = e^{\frac{1}{\hbar}\Omega a^4 g[j + \frac{1}{4}j^2 - \frac{1}{8}j^3 + \frac{1}{8}j^4 + \cdots]} + \\ + e^{\frac{1}{\hbar}\Omega a^4 g[-j + \frac{1}{4}j^2 + \frac{1}{8}j^3 + \frac{1}{8}j^4 + \cdots]}.$$

For large $\Omega >> 1$ and small positive $j > 0$, the main contribution is due to the first term, and for negative values of the current $j < 0$, it dominates the second one. Consequently, for small external sources, one can construct the interpolating expression

$$e^{\frac{1}{\hbar}\Omega w_\Omega(J)} = e^{\frac{1}{\hbar}\Omega a^4 g[|j| + \frac{1}{4}j^2 - \frac{1}{8}j^2|j| + \frac{1}{8}j^4 + \cdots]}.$$

In this way, we get the first term of the quasiclassical expansion

$$w(J) = a[|J| + \frac{1}{4}\frac{J^2}{a^2 g} - \frac{1}{8}\frac{J^2|J|}{a^5 g^2} + \frac{1}{8}\frac{J^4}{a^8 g^3} + \cdots .$$

The quasiaverage

$$\prec \phi \succ = \lim_{J\to 0}\left(\frac{\partial w}{\partial J}\right) = \lim_{J\to 0} a[\epsilon(J) + \frac{1}{2}\frac{J}{a^2 g} + \cdots]$$

depends on the current vanishing procedure

$$\prec \phi \succ_{J=0^\pm} = \pm a.$$

It must be noted that the function $w(j)$ is non-analytic: the terms like $|J|$, $|J|J^2$ *etc.* make the function non differentiable at the origin. Here the first derivative is discontinues.

The effective potential will be calculated using the generalized Legendre transformation

$$V(\phi) = \sup_J[\phi J - W(J)].$$

The simple graphical construction shows that the formula above defines a convex function. The convexity is not strict: in the interval $-a \le \phi \le +a$ there is a plateau – a region where the potential achieves its minimal value.

It is not difficult to get the desired expression for the effective potential:

$$V(\phi) = a^4 g \sup_j[\hat{\phi} j - (|j| + \frac{1}{4}j^2 - \frac{1}{8}|j|j^2 + \frac{1}{8}j^4 + \cdots],$$
$$\hat{\phi} \equiv \frac{1}{a}\phi, \qquad j \equiv \frac{1}{a^3 g}J.$$

The source function is given by

$$J(\phi) = a^3 g\theta(|\phi| - a)\epsilon(\phi)[2(|\hat{\phi}| - 1) + 3(|\hat{\phi}| - 1)^2 +$$

$$+ (|\hat{\phi}-1)^3 - \frac{105}{4}(|\hat{\phi}-1)^4 + \cdots].$$

The final result looks as [33]

$$V(\phi) = a^2 g\theta(|\phi|-a)[(|\phi|-a)^2 + \frac{1}{2}(|\phi|-a)^3 + \frac{1}{4a^2}(|\phi|-a)^4 + \cdots].$$

The potential $V(\phi)$ and its derivative remain everywhere continuous but the second derivative

$$V^{''}(\phi) = 2a^2 g\theta(|\phi|-a) + \cdots$$

changes discontinuously at the points of inflexion of the classical potential.

Concluding this item, let us briefly discuss the loop corrections. It is not difficult to see that the one-loop result is given by

$$\begin{aligned} e^{\frac{1}{\hbar}W_\Omega(J)} &= \sum_{\bar{\phi}=\phi_\pm} e^{\frac{\Omega}{\hbar}[S(\bar{\phi}) - \frac{\hbar}{2\Omega}\ln\frac{1}{2ga^2}V^{''}(\bar{\phi})]} = \\ &= \exp\Big[\frac{1}{2\hbar}\Omega(Q_+ + Q_-) + \ln 2\cosh\frac{\Omega}{2\hbar}(Q_+ - Q_-)\Big] = \\ &= \exp\frac{\Omega}{\hbar}\Big[\frac{1}{2}(Q_+ + Q_-) + \frac{1}{2}|Q_+ + Q_-| + \frac{\hbar}{\Omega}\ln(1 + e^{-\frac{\Omega}{\hbar}|Q_+ - Q_-|})\Big]. \end{aligned}$$

Here we have introduced the notation

$$\begin{aligned} Q_\pm &= [S(\phi_\pm) - \frac{\hbar}{2\Omega}\ln\frac{1}{2ga^2}V^{''}(\phi_\pm)] = \\ &= a^4 g[\pm j + \frac{1}{4}j^2 \mp \frac{1}{8}j^3 + \frac{1}{8}j^4 \cdots] - \\ &- \frac{\hbar}{2\Omega}[1 + \frac{3}{2}(\pm j - \frac{1}{2}j^2 \pm \frac{5}{8}j^3 - j^4 \cdots)]. \end{aligned}$$

Consequently, for small values of the currents j, for the generating functional we get

$$\begin{aligned} W_\Omega(J) = a^4 g\{&(1 - \frac{\hbar 3}{\Omega 4a^4 g})|j| + \frac{1}{4}(1 + \frac{\hbar 15}{\Omega 4a^4 g})j^2 - \\ &- \frac{1}{8}(1 + \frac{\hbar 51}{\Omega 4a^4 g})|j|j^2 + \frac{1}{8}(1 + \frac{\hbar 393}{\Omega 16a^4 g})j^4 \cdots\} + \\ &+ \frac{\hbar}{\Omega}\ln[1 + \exp\{-\frac{\Omega a^4 g}{\hbar}[(1 - \frac{\hbar 3}{\Omega 4a^4 g})|j| - \frac{1}{8}(1 + \frac{\hbar 51}{\Omega 4a^4 g}|j|j^2 \cdots]\}]. \end{aligned}$$

It is evident that the factors

$$\frac{\hbar}{\Omega}\frac{1}{a^4 g}$$

correspond to the one-loop contributions which in their turn also exhibit a non-analytic structure.

4. A Model with Two Scalar Fields

Without any doubt, it is interesting to consider the theories where the scalar interactions have a more complicated character, e.g., there are several multiplets of Higgs fields. In that case, the physical content of the theory becomes richer and the problem of minimization is more complex [106]. Consider a $D = 0$ dimensional toy-model with two scalar fields [33]. The classical potential is

$$V_{cl} = -\mu\sigma^2 - \nu\phi^2 + \alpha\sigma^2\phi^2 + \beta\phi^4 + \gamma\sigma^4.$$

The form of this potential can be easily established studying its extrema. Consider the case where the parameters satisfy the conditions

$$\alpha\nu - 2\gamma\mu > 0, \quad \alpha\mu - 2\beta\nu > 0, \quad \alpha^2 - 2\beta\gamma > 0.$$

It is not difficult to find out that this potential possesses

- A local maximum at the point $(\sigma = 0, \phi = 0)$

$$V_a = V_{cl}(0, 0).$$

- A pair of degenerate local minima at the points $(\sigma = 0, \phi = \pm\sqrt{\frac{\nu}{2\gamma}})$

$$V_b = V_{cl}\left(\sigma = 0, \phi = \pm\sqrt{\frac{\nu}{2\gamma}}\right) = -\frac{\nu^2}{4\gamma}.$$

- A pair of degenerate local minima in the points $(\sigma = \pm\sqrt{\frac{\mu}{2\beta}}, \phi = 0)$

$$V_c = V_{cl}\left(\sigma = \pm\sqrt{\frac{\mu}{2\beta}}, \phi = 0\right) = -\frac{\mu^2}{4\beta}.$$

- Four saddle points with the coordinates

$$\sigma^2 = \frac{\alpha\nu - 2\gamma\mu}{\alpha^2 - 4\beta\gamma}, \qquad \phi^2 = \frac{\alpha\mu - 2\beta\nu}{\alpha^2 - 4\beta\gamma}.$$

The classical potential is invariant under the transformations: $\sigma \to -\sigma, \phi \to -\phi$. Our goal is to construct the corresponding convex effective potential and to study the mechanism of the spontaneous symmetry breaking.

The generating functional is given by

$$Z_\Omega(J, K) = \mathcal{N} \int\limits_{-\infty}^{+\infty} d\sigma \int\limits_{-\infty}^{+\infty} d\phi e^{\frac{\Omega}{\hbar}[J\sigma + K\Phi - V_{cl}(\sigma,\phi)]}.$$

In order to calculate this integral using the Laplace method, preliminary we have to find out the maxima of the function

$$S(\sigma, \phi; J, K) = J\sigma + K\phi - V_{cl}(\sigma, \phi).$$

The extrema equations

$$\frac{\partial V_{cl}}{\partial \sigma} = J, \quad \frac{\partial V_{cl}}{\partial \phi} = K$$

will be solved using the perturbation theory for small values of external currents by expanding over the powers of sources.

The tree approximation result reads as follows:

$$Z_\Omega(J,K) = e^{\frac{\Omega}{\hbar} W_\Omega(J,K)} = \mathcal{N} \sum_A e^{\frac{1}{\hbar} S_A(\sigma_a, \phi_a; J, K)},$$

where $\sigma_A = \sigma_A(J,K)$ and $\phi_A = \phi_A(J,K)$ are solutions of the extrema equations at the points of maxima. S_A are the corresponding values of classical action.

For the classical potential under consideration, we have four solutions corresponding to the maxima of the effective potential.

The corresponding contributions up to the forth order terms are

$$S_{\pm<\phi>} = \frac{\nu^2}{4\gamma} \pm <\phi> K + \frac{1}{2}(pJ^2 + \frac{1}{4\nu}K^2) \pm$$

$$\pm <\phi> [\frac{\alpha p^2}{2\nu} J^2 K + \frac{\gamma}{16\nu^3} K^3]$$

$$+ \frac{\alpha^2 - 4\beta\gamma}{4\gamma} p^4 J^4 + \frac{\alpha p^2}{8\nu^2}(1 + \frac{2\alpha p \nu}{\gamma}) J^2 K^2 + \frac{\gamma}{32\nu^4} K^4,$$

$$S_{\pm<\sigma>} = \frac{\mu^2}{4\beta} \pm <\sigma> J + \frac{1}{2}(qK^2 + \frac{1}{4\mu} J^2) \pm$$

$$\pm <\sigma> [\frac{\alpha q^2}{2\mu} J K^2 + \frac{\beta}{16\mu^3} J^3]$$

$$+ \frac{\alpha^2 - 4\beta\gamma}{4\beta} q^4 K^4 + \frac{\alpha q^2}{8\mu^2}(1 + \frac{2\alpha q \nu}{\beta}) J^2 K^2 + \frac{\beta}{32\mu^4} J^4.$$

Here we have introduced the notation

$$<\phi> = \sqrt{\frac{\nu}{2\gamma}}, \quad <\sigma> = \sqrt{\frac{\mu}{2\beta}},$$

$$p = \frac{\gamma}{\beta\nu - 2\gamma\mu}, \quad q = \frac{\beta}{\alpha\mu - 2\beta\nu}.$$

Hence

$$e^{\frac{\Omega}{\hbar barr} W_\Omega(J,K)} = e^{\frac{\Omega}{\hbar} S_{+<\phi>}} + e^{\frac{\Omega}{\hbar} S_{-<\phi>}} + e^{\frac{\Omega}{\hbar} S_{+<\sigma>}} + e^{\frac{\Omega}{\hbar} S_{-<\sigma>}}.$$

The quasiclassical result reads as follows

$$e^{\frac{\Omega}{\hbar} W_\Omega(J,K)} = \exp \frac{\Omega}{\hbar} \{ \frac{\nu^2}{4\gamma} + <\phi> |K| + \frac{1}{2}(pJ^2 + \frac{1}{4\nu} K^2) -$$

$$- <\phi> [\frac{\alpha p^2}{2\nu} J^2 + \frac{\gamma}{16\nu^3} K^2] |K| +$$

$$+\frac{\alpha^2-4\beta\gamma}{4\gamma}p^4J^4+\frac{\alpha p^2}{8\nu^2}(1+\frac{2\alpha p\nu}{\gamma})J^2K^2+\frac{\gamma}{32\nu^4}K^4\}+$$
$$+\exp\frac{\Omega}{\hbar}\{\frac{\mu^2}{4\beta}+<\sigma>|J|+\frac{1}{2}(qK^2+\frac{1}{4\mu}J^2)-$$
$$-<\sigma>[\frac{\alpha q^2}{2\mu}K^2+\frac{\beta}{16\mu^3}J^2]|J|+$$
$$+\frac{\alpha^2-4\beta\gamma}{4\beta}q^4K^4+\frac{\alpha q^2}{8\mu^2}(1+\frac{2\alpha q\nu}{\beta})J^2K^2+\frac{\beta}{32\mu^4}J^4\}=$$
$$=e^{\frac{\Omega}{\hbar}Q_+}+e^{\frac{\Omega}{\hbar}Q_-}.$$

We see that $W(J,K)$ exhibits a nonanalytic dependence on the external currents. Performing further simplifications, one gets

$$W=\frac{1}{2}(Q_++Q_-)+\frac{1}{2}|Q_+-Q_-|+\frac{\hbar}{\Omega}\ln[1+\exp(-\frac{\Omega}{\hbar}|Q_+-Q_-|].$$

For the small values of currents, we have

$$Q_+-Q_-=\frac{\nu^2}{4\gamma}-\frac{mu^2}{4\beta}+<\phi>|K|-<\sigma>|J|,$$

and for the nonvanishing

$$\Delta=\frac{\nu^2}{4\gamma}-\frac{\mu^2}{4\beta},$$

we have

$$W(J,K)\simeq\theta(\Delta)Q_++\theta(-\Delta)Q_-.$$

Let $\Delta>0$. Then

$$W(J,K)\simeq\frac{\nu^2}{4\gamma}+<\phi>|K|+\frac{1}{2}(pJ^2+\frac{1}{4\nu}K^2).$$

The corresponding quasiaverages are given by

$$\prec\sigma\succ=(\frac{\partial W}{\partial K})_{K\to0}=0,$$
$$\prec\phi\succ=(\frac{\partial W}{\partial J})_{J\to0}=\pm<\phi>.$$

We see that the symmetry $\phi\to-\phi$ is spontaneously broken while the reflexion of σ does not break the symmetry.

The effective potential

$$V(\sigma,\phi)=\sup_{J,K}[\sigma J+\phi K-W(J,K)]$$

is a quasi-convex function with a shape of hull.

5. Boundary Conditions

The conventional method to calculate the effective potential is a loop expansion which to each order maintains the symmetries of the input Lagrangian. In this section, we will show that a more transparent understanding of the convexity problem can be achieved if the boundary conditions in the path integral representation of the Green functions generating functional are properly specified. As we have already noted, the loop expansion corresponds to the integration by the Laplace method, and actually it is an expansion around a solution of the classical equation

$$\frac{\partial}{\partial\phi}[U(\phi) - J\phi] = 0. \tag{5.20}$$

For the convex classical potential which possesses a single minimum, the equation (5.20) has a unique solution which provides a minimum of the function $U(\phi) - J\phi$. As a result $w(J)$, is a well defined smooth and convex function, and its Legendre transformation, i.e., the effective potential $V(\phi)$ satisfies all the desired properties.

Let us now turn to the case of non-convex classical potentials, and as an example, consider the double-well potential

$$U(\phi) = -\frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4.$$

This potential possesses two degenerate minima at $\phi = \pm\frac{m}{\sqrt{\lambda}}$ and a local maximum at $\phi = 0$.

Consequently, (5.20) in general has three different solutions denoted here by

$$\phi_+(J), \qquad \phi_- \qquad \text{and} \qquad \phi_0(J).$$

One can consider the function $\bar{\phi}(J)$ – an analytic continuation of this three different solutions which is a smooth multivalued function. The corresponding Schwinger functional $W(J)$ is neither convex nor single-valued and everywhere real. We can distinguish three regions: two physical ones which correspond to the global minima, and a unphysical one, which corresponds to the local maximum. Using the analogy with a ferromagnetic system, one can say that in the last region, the magnetic suspectibility is negative. The corresponding effective potential in the tree approximation coincides with the classical potential.

The boundary values of $\phi_\alpha(J)$'s at $J = 0$ are given by the extrema of V_{cl}. Consequently, one may say that in the standard loop expansion, the functional integral is dominated by the paths $\phi_\alpha(J)$ which satisfy the different boundary conditions as $J \to 0$. Now we must remark that the very definition of the path integral implies the specification of the boundary conditions in the functional space, i.e., the precise form of the transition

amplitude is given by the functional integral

$$\langle \phi_2, \tau_2 | \phi_1, \tau_1 \rangle^J = \mathcal{N} \int_{\phi(\tau_1)}^{\phi(\tau_2)} \mathcal{D}\phi \exp[- \int d^4x (L(\phi) - J\phi]. \tag{5.21}$$

The text-book derivation of the generating functionals proceeds as follows [1]: assuming that the external source $J(\tau)$ vanishes outside a sufficiently large time interval $[T', T'']$, the l.h.s. of (5.21) can be written in the form of the double sum over the eigenstates of the Hamiltonian H:

$$\begin{aligned} &\langle \phi_2, \tau_2 | \phi_1, \tau_1 \rangle^J = \\ &= \int d\sigma'' \int d\sigma' \{ \sum_{n,m} < \phi_2 | n > < n | \sigma'', 0 > e^{-E_n(\tau_2 - T'')} \} \\ &< \sigma'', T'' | \sigma', T' >^J < \sigma', 0 | m > < m | \phi_1 > e^{-E_m(T' - \tau_1)}. \end{aligned} \tag{5.22}$$

The demanded expression for $W[J]$ is obtained in the limit

$$T'', \tau_2 \to +\infty, \quad T', \tau_1 \to -\infty.$$

Then the r.h.s. of (5.22) receives its dominant contribution from the lowest energy state $|0>$. Consequently,

$$\langle \phi_2, \tau_2 | \phi_1, \tau_1 \rangle^J = \Psi_0(\phi_2) \Psi_0^\star(\phi_1) e^{-E_0(\tau_2 - \tau_1)} < 0, T'' | 0, T' >^J, \tag{5.23}$$

where $\Psi_0(\phi) \equiv < \phi | 0 >$ is the ground state wave functional, and

$$< 0, T'' | 0, T' >^J = \int d\sigma'' \int d\sigma' \Psi_0^\star(\sigma'', T'') < \sigma'', T'' | \sigma', T' >^J \Psi_0(\sigma', T'),$$

$$\Psi_0(\sigma, T) = e^{-E_0 T} \Psi_0(\sigma)$$

is the vacuum to vacuum transition amplitude. Recalling the definition (5.20), we obtain for the Green functions generating functional the following expression:

$$\begin{aligned} e^{W[J]} &= \frac{< 0, +\infty | 0, -\infty >^J}{< 0, +\infty | 0, -\infty >} = \frac{< \phi_2, +\infty | \phi_1, -\infty >^J}{< \phi_2, +\infty | \phi_1, -\infty >} = \\ &= \frac{\int_{\phi_1}^{\phi_2} \mathcal{D}\phi \exp[- \int d^4x (L(\phi) - J\phi]}{\int_{\phi_1}^{\phi_2} \mathcal{D}\phi \exp[- \int d^4x (L(\phi)]} \end{aligned} \tag{5.24}$$

which due to (5.23) does not depend on any particular choice of the boundary conditions, i.e., on ϕ_1 and ϕ_2.

Note, however, that if we want to compute the r.h.s. of (5.24) using the steepest descent method, we have to specify boundary conditions in such a way that a dominant path would satisfy them.

Now, it is important to emphasize that (5.23) and (5.24) are valid if the Hamiltonian does not have degenerate lowest energy states, i.e., the classical potential has a single minimum, say at $\phi = v$. Then the steepest descent

path $\bar{\phi}(J)$ satisfies the boundary condition $\bar{\phi}(0) = v$, and it is natural that the functional integral

$$e^{W[J]} = \frac{\int_v^v \mathcal{D}\phi \exp[-\int d^4x(L(\phi) - J\phi]}{\int_v^v \mathcal{D}\phi \exp[-\int d^4x(L(\phi)]}$$

calculated within the loop expansion leads to the correct result.

Consider now a situation, when H possesses a set of degenerate ground states, i.e.,

$$H|0_\alpha\rangle = E_0|0_\alpha\rangle.$$

Instead of (5.23), the equation (5.22) now leads to the following expansion:

$$\lim_{\tau_1\to\infty}\lim_{\tau_2\to-\infty}\langle\phi_2,\tau_2|\phi_1\tau_1\rangle^J =$$
$$= \sum_{\alpha\beta}\Psi_{0,\beta}(\phi_2)\,\Psi^\star_{0,\alpha}(\phi_1)e^{E_0(\tau_2-\tau_1)}\langle 0_\beta,+\infty|0_\alpha,-\infty\rangle^J.$$

Let us suppose that there is no tunneling between different vacua, i.e.,

$$\langle 0_\beta,+\infty|0_\alpha,-\infty\rangle^J \sim \delta_{\alpha\beta}$$

and that the ground state wave functional $\Psi_{0\alpha}$ is essentially a Gaussian centered around $\phi \sim v_\alpha$.

Define the Green functions generating functional corresponding to the particular choice of boundary conditions

$$Z_\alpha[J] = e^{W_\alpha[J]} = \frac{\langle 0_\alpha,+\infty|0_\beta,-\infty\rangle^J}{\langle 0_\alpha,+\infty|0_\beta,-\infty\rangle},$$

where

$$\langle 0_\alpha,+\infty|0_\beta,-\infty\rangle^J =$$
$$= \lim_{\tau_2\to\infty}\lim_{\tau_1\to-\infty}\int d\phi_2\int d\phi_1\,\Psi^\star_{0,\alpha}(\phi_2)\Psi_{0,\alpha}(\phi_1)e^{E_0(\tau_2-\tau_1)}\langle\phi_2,\tau_2|\phi_1,\tau_1\rangle.$$

Then

$$\langle 0_\alpha,+\infty|0_\beta,-\infty\rangle^J \sim \lim_{\tau_1\to\infty}\lim_{\tau_2\to-\infty} e^{E_0(\tau_2-\tau_1)}\langle v_\alpha,\tau_2|v_\beta,\tau_1\rangle^J,$$

and we finally get

$$e^{W_\alpha[J]} = \frac{\langle v_\alpha,+\infty|v_\alpha,-\infty\rangle^J}{\langle v_\alpha,+\infty|v_\alpha,-\infty\rangle} =$$
$$= \frac{\int_{v_\alpha}^{v_\alpha}\mathcal{D}\phi\exp[-\int d^4x(L(\phi)-J\phi]}{\int_{v_\alpha}^{v_\alpha}\mathcal{D}\phi\exp[-\int d^4x(L(\phi)]}. \qquad (5.25)$$

The r.h.s of this equation can be calculated by the steepest descent method. In this method, the integral is dominated by the minimal Euclidean action path which satisfies the boundary conditions

$$\bar{\phi}(\tau\to\pm\infty) = v_\alpha.$$

Consequently, we have to find a solution $\phi_\alpha(J)$ of the extremal equation

$$\frac{\partial U(\phi)}{\partial \phi} = J, \quad \phi = \phi_\alpha(J)$$

such that it provides the minimum of

$$U(\phi) - J\phi, \quad U''(\phi_\alpha(J)) > 0$$

and satisfies the condition $\phi_\alpha(0) = v_\alpha$.

It is not difficult to guess that within quasiclassical approximation, v_α is one of the minima of the classical potential $V_{cl}(\phi)$, so that we can quote the final result [34], [35], [36]

$$W_\alpha[J] = \int d^4x\{J\phi_\alpha(J) - U(\phi_\alpha(J)) - \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\ln[q^2 + U''(\phi_\alpha(J))] +$$
$$+ U(v_\alpha) + \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\ln[q^2 + U''(v_\alpha)]. \tag{5.26}$$

It is essential to note that (5.26) is only valid while $\phi_\alpha(J)$ provides the minimum of $U(\phi) - J\phi$, i.e., for a definite domain of variations of the source J. In this region, $W_\alpha[J]$ is a convex and real function. It is a matter of simple deduction to realize that convexity can be proven only for functional integrals with well defined boundary conditions (like (5.25)) while the standard loop expansion suffers the lack of this prescription.

The convex effective potential can be defined as a generalized Legendre transformation

$$V(\phi) = \sup_J[J\phi - V(J)] \tag{5.27}$$

of the interpolated generating functional

$$W[J] \equiv \int d^4x\omega(J) = \max_\alpha W_\alpha[J]. \tag{5.28}$$

In the tree approximation, $V(\phi)$ coincides with the convex hull of the classical potential $V_{cl}(\phi)$.

The definitions (5.27) and (5.28) can be justified if one appeals to the classical analogue, where effective potential is the minimal energy of the system with a given configuration ϕ, and $-w(j)$ is the minimal energy density of the system interacting with a given classical external source J.

The function $w(J)$ is not analytic and its derivative changes discontinuously. For the double-well potential,

$$\left(\frac{dw(J)}{dJ}\right)_{J\to 0} = \pm\frac{m}{\sqrt{\lambda}} + \mathcal{O}(\hbar).$$

This is a natural manifestation of the spontaneous breaking of the discrete symmetry $\phi \to -\phi$.

Above we have tried to point out that the convexity property is satisfied by the functional integrals with well defined boundary conditions which can be represented in the following form:

$$\left(\frac{d\omega}{dJ}\right)_{J=0} = \phi_\alpha(J=0) = v_\alpha.$$

For the Schwinger functional $W[J]$ given by the standard loop expansion, the derivative $\frac{d\omega}{dJ}$ is ill-defined as $J \to 0$, so one would not expect that the convexity will be maintained.

What kind of physical interpretation can be given to the all this consideration? Let us take for concreteness the double-well potential. The Schwinger functionals $W_\pm[J]$ will give the Green's functions for two different vacua: $|0_\pm>$. The functional $W_0[J]$ corresponds to the quantization near the unstable vacuum $|0_0>$, and is given by the path integral

$$\frac{\int_0^0 \mathcal{D}\phi \exp[-\int d^4x(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 - J\phi]}{\int_0^0 \mathcal{D}\phi \exp[-\int d^4x(L(\phi)]}.$$

For $m^2 < 0$, the Gaussian integration will lead to a divergent expression, but noticing that due to the negative ϕ^4 terms, the integrals must be convergent, we can calculate them in the usual way first for $m^2 > 0$ and afterwards perform an analytic continuation in m^2 from $m^2 > 0$ to $m^2 < 0$.

In this manner, we can obtain a multivalued non-convex Schwinger functional $W[J]$ composed from the three different branches with three different vacua and Green functions. It is just the standard loop expansion result which claims that double-well potential will lead to a theory with two degenerate ground states and one unstable vacuum.

A quite different picture arises when one considers the convex $W[J]$ and $V(\phi)$, and its adequate interpretation is usually associated with the Maxwell construction in thermodynamics.

Here we have two separate Hilbert spaces built up on the vacuum states $|0_->$ and $|0_+>$ with the properties that $<0_|A|0_+>=0$ for any observable A, and $<0_\pm|\phi|0_\pm>=\pm v = \pm a + \mathcal{O}(\hbar)$. Furthermore, one can construct a superposition

$$|\Theta>=\alpha|0_+> + \beta|0_->, \qquad |\alpha|^2 + |\beta|^2 = 1$$

which is also a Hamiltonian eigenstate with the ground state energy

$$<\Theta|H|\Theta>=<0_+|H|0_+>=<0_-|H|0_->,$$

and where the field average takes a value

$$<\Theta|\phi|\Theta>=(|\alpha|^2 - |\beta|^2)v, \quad -v \le <\Theta|\phi|\Theta> \le v.$$

Thus one obtains the state with the vacuum energy and a quantum field average anywhere between the minima of the classical potential. Remark

that only the states $\alpha = 0$ or $\beta = 0$ are accessible on removing the external source, and this is the fact that determines the symmetry of the theory.

Consider once more the procedure of the calculation of the effective potential. For concreteness, take a theory with a single scalar field and Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - U(\phi),$$

where $U(\phi)$ is a polynomial containing the mass term and interactions. The loop expansion is not affected by the division of $\mathcal{L}$ into the free and interacting parts [26], and the functional $\Gamma[\phi_c]$ which generates the one particle irreducible proper diagrams does not depend on the choice of the free part in the Lagrangian. It can be written as follows

$$\Gamma[\phi] = \sum \frac{1}{n!} \int dx_1 \cdots \int dx_n \Gamma_v^n(x_1, \ldots, x_n)(\phi(x_1) - v(x_1)) \cdots (\phi(x_n) - v(x_n)),$$

where $v(x)$ is some given configuration. By the construction, the derivatives

$$\frac{\delta}{\delta\phi(x_1)} \cdots \frac{\delta}{\delta\phi(x_n)} \Gamma[\phi]|_{\phi=v}$$

are the sum of 1PI diagrams representing the average values of T-ordered Heisenberg field products. The essentialis that the averages are taken with respect to the ground state $|\Phi_v>$ such that

$$< \Phi_v^{out} | \phi(x) | \Phi_v^{in} > = \phi_v(x)$$

(in the translation invariant theories, $\phi_v(x) = v = const$).

In another terms, the same generating functional $\Gamma[\phi]$ can generate the Green functions for different phases of the theory. Now, it is well-known that any connected Green function generated by Schwinger's functional $W[J]$ can be written as the sum of three graphs in a theory whose "effective action" is given by $\Gamma[\phi]$ [19]. Consequently, the connected Green functions also must be labeled by the phase index v.

$$W[J] =$$

$$= \sum \frac{1}{n!} \int dx_1 \cdots dx_n W_v^n(x_1, \ldots, x_n)(J(x_1) - J_v(x_1)) \cdots (J(x_n) - J_v(x_n)).$$

It may happen that $W[J]$ is non-differentiable for some values of J. As a simplest example consider again the case of the double-well potential $U(\phi) = \frac{\lambda}{4}(\phi^2(x) - a^2)^2$.

Taking in the effective action $\phi_v = \pm a + \mathcal{O}(\hbar)$, one reconstructs the Green functions for the two stable phases. The corresponding Schwinger's functional is non-differentiable

$$\Big(\frac{\delta W[J]}{\delta J(x)}\Big)_{J=0^\pm} = \pm a + \mathcal{O}(\hbar).$$

We have already seen that this can be interpreted as the fact of existence of two different branches of the Schwinger's functional$W_{\pm}[J]$. The third branch $W_0[J]$ corresponds to the unstable vacuum $\phi_0 = 0$:

$$\Big(\frac{\delta W_0[J]}{\delta J(x)}\Big)_{J=0} = 0.$$

Let us translate this observation into the language of path integrals. As we have noted, different branches correspond to different choice of boundary conditions in the definitions of generating functionals (see also [115], [[27]). Constructive definition of path integrals can be given in terms of a positively defined normalizable integration measure. For the theories with a unique vacuum state, such a definition can be given in terms of Gaussian integrals (in Euclide rotated version). When the free part describes the physical particles, the boundary conditions in the path integrals correspond to the *in* and *out* asymptotic solutions of the Heisenberg equations.

In the case of the unstable vacuum, the quadratic part of the Lagrangian corresponds to the tachions: imaginary mass particles. The corresponding Gaussian measure is not normalizable. But due to the ϕ^4 term, the path integral is convergent. In other words, the integration measure does not correspond to the free asymptotic states. In the practical calculations, one uses the Green functions of the normal theory and then performs the analytic continuations $m^2 \to -m^2$ term by term.

CHAPTER 6
ON THE TEMPORARY SYMMETRY BREAKING AT HIGH TEMPERATURES

1. Introduction

The finite temperature effective action can be interpreted as the Gibbs free energy of a system in statistical mechanics. Therefore $V_{eff}(T)$ is useful for determining the equilibrium state of the system since it gives a picture of the preferred thermodynamic state and includes all effects of thermal fluctuations. This aspect of temperature-dependent generating functionals plays a major role in the applications of the quantum field theory to the problems of symmetry breaking phase transitions in particle physics [107].

To study the field theory at finite temperature, we imagine the system to be placed in contact with a thermal bath at temperature T. Then the background in which we study a particular scattering process is no longer an empty vacuum, but rather a thermal distribution of particles. The finite temperature Green functions are defined in terms of ensemble averages:

$$G_\beta(x_1,\dots,x_N) = \frac{\mathrm{Tr}\left[e^{-\beta H}\phi(x_1)\cdots\phi(x_N)\right]}{\mathrm{Tr}e^{-\beta H}}.$$

In analogy with the zero temperature case, one can define a generating functional

$$Z_\beta[J] = \frac{\mathrm{Tr}\left[e^{-\beta H}e^{i\int dxJ(x)\phi(x)}\right]}{\mathrm{Tr}e^{-\beta H}}$$

from which the temperature Green functions are obtained by varying $Z_\beta[J]$ with respect to $J(x)$ in the usual way. The functional $Z_\beta[J]$ can be written in terms of a Euclidean path integral over scalar fields satisfying periodic (antiperiodic in the case of fermions) boundary conditions in Euclidean time $\beta\hbar$:

$$Z_\beta[J] = \mathcal{N}\int \mathcal{D}_\beta\phi e^{-\int_0^\beta d\tau\int d^3x(\mathcal{L}_E - J\phi)},$$

where the measure is restricted to periodic paths $\phi(0,\mathbf{x}) = \phi(\beta,\mathbf{x})$ and $\mathcal{L}_E$ is the Euclidean Lagrangian.

Let us briefly discuss the finite temperature calculation of the effective potential, and focus attention on the Green functions generating functional

$$Z^T[J] = \frac{\tilde{Z}^T[J]}{\tilde{Z}^T[0]} = \frac{\mathrm{Tr}\,\exp\{-\frac{\beta}{\hbar}(H - \int d^3xJ\phi)\}}{\mathrm{Tr}\,\exp\{-\frac{\beta}{\hbar}H\}} =$$

$$= \frac{\int_{-\infty}^{+\infty}d\phi\int_\phi^\phi D_T\eta\exp\{-\frac{1}{\hbar}\int_0^\beta d\tau\int d^3x[\mathcal{L}(\eta) - J\eta]\}}{\int_{-\infty}^{+\infty}d\phi\int_\phi^\phi D_T\eta\exp\{-\frac{1}{\hbar}\int_0^\beta d\tau\int d^3x\mathcal{L}(\eta)\}}, \qquad (6.1)$$

$$\beta = \frac{1}{T}.$$

In (6.1), the functional integral variables satisfy periodic boundary conditions: $\eta(0) = \eta(\beta) = \phi$. Now we have

$$\tilde{Z}^T[J] = \mathcal{N} \int_{-\infty}^{+\infty} d\phi \int_0^0 D_T\eta \exp\{-\frac{1}{\hbar}\int_0^\beta d\tau \int d^3x[\mathcal{L}(\phi+\eta\sqrt{\hbar}) - J(\phi+\eta\sqrt{\hbar})]\},$$

and in the one-loop approximation may write down

$$\tilde{Z}^T[J] = \mathcal{N} \int_{-\infty}^{+\infty} d\phi \int_0^0 \exp\{-\frac{1}{\hbar}\int_0^\beta d\tau \int d^3x[U(\phi) - J\phi + \\ + \frac{(U'(\phi)-J)^2}{2U''(\phi)} + V_1^T(\phi)]\},$$

where $V_1^T(\phi)$ is the well-known one-loop correction [30]

$$V_1^T(\phi) = V_1^0(\phi) + \Delta V_1^T(\phi) = \frac{\hbar}{2}\int \frac{d^4q}{(2\pi)^4} \ln[q^2 + U''(\phi)] + \\ + \frac{\hbar}{2\pi^2\beta^4}\int_0^\infty dx x^2 \ln[1 - \exp\{-(x^2 + \beta^2 U''(\phi))^{\frac{1}{2}}\}].$$

The last integration in (6.1) will be performed using the steepest descent method. Keeping in the mind that we are treating problems in the one-loop approximation, we can state the result:

$$\tilde{Z}^T[J] = \mathcal{N}\sum_\alpha \exp\{-\frac{1}{\hbar}\int_0^\beta d\tau \int d^3x[U(\phi_\alpha^T(J)) - J\phi_\alpha^T + V_1^T(\phi_\alpha^T)],$$

where $\phi_\alpha^T(J)$'s are the minima of the function $U(\phi) - J\phi + V_1^T(\phi)$.

In the scope of the one-loop approximation, it can be shown that

$$\phi_\alpha^T(J) = \phi_\alpha(J) + \phi_{1\alpha}^T(J),$$

where $\phi_\alpha(J)$ are the extrema of $U(\phi) - J\phi$, and

$$\phi_{1\alpha}^T(J) = -\frac{1}{2U''(\phi_\alpha)}\left(\frac{dv_1^T}{d\phi}\right)_{\phi_\alpha(J)}.$$

Note that there is a crucial difference between the zero and finite temperature cases: at $T = 0$, only the minima of the classical potential contribute to the functional integral, whilst at finite temperatures, it is essential to take into account the minima of the temperature-dependent potential $U(\phi) + V_1^T(\phi)$. Above some critical temperature T_c, the minimum of $U(\phi) + V_1^T(\phi)$ coincides with the maximum of the classical potential $U(\phi)$.

Let us define the generating functionals

$$Z_\alpha^T[J] = \frac{\exp\{-\frac{1}{\hbar}\int_0^\beta d\tau \int d^3x[U(\phi_\alpha^T(J)) + V_1^T(\phi_\alpha^T(J)) - J(\phi_\alpha^T(J))]\}}{\exp\{-\frac{1}{\hbar}\int_0^\beta d\tau \int d^3x[U(\phi_\alpha^T(J)) + V_1^T(\phi_\alpha^T(J))\}} =$$
$$= \exp\{\frac{1}{\hbar}W_\alpha^T[J]\}. \qquad (6.2)$$

Then

$$W^T[J] = \max_\alpha W_\alpha^T[J] \sim \int_0^\beta d\tau \int dx w(J)$$

will be a convex function, and the finite temperature effective potential

$$V_{eff}^T(\phi) = \sup_J[J\phi - w(J)]$$

appears to be the convex hull of the function

$$U^T(\phi) = U(\phi) + V_1^T(\phi).$$

Usually $U^T(\phi)$ is treated as a finite temperature effective potential.

One can calculate the averages

$$< \phi^T(J) >= \frac{dw^T(J)}{dJ},$$

and in the case of the double-well potential obtain

$$\big(\frac{dw^T(J)}{dJ}\big)_{J\to 0^\pm} = \pm\frac{\mu}{\sqrt{\lambda}}\big(1 - \hbar\frac{\lambda}{16\mu^2}T^2\big) + \mathcal{O}(\hbar^2)$$

for

$$T^2 < T_c^2 = \frac{16\mu^2}{\hbar\lambda},$$

and

$$\Big(\frac{dw^T(J)}{dJ}\Big)_{J\to 0^\pm} = 0$$

for $T^2 > T_c^2$ in accordance with the scenario of the high temperature symmetry restoration.

In the present chapter, we consider simple models, where the symmetry behaviour is in some sense opposite to the normal one. In Section 2, we expose the main characteristic features of such model and in Sections 3 and 4, we find its supersymmetric versions.

2. Two Higgs Model

The standard scheme of spontaneous symmetry breaking predicts the symmetry restoration above some critical temperature [107], [93]. This well known phenomenon of the symmetry restoration is presented in most cosmological applications of the gauge theories. On the other hand, it is possible that at $T = 0$ the symmetry is exact, but at finite temperatures it becomes broken spontaneously, i.e., thermal fluctuations lead to a reduction of symmetry. It should be emphasized that a possibility of such a behaviour was pointed out by S. Weinberg [131] for a model with a global $\mathrm{O}(n) \times \mathrm{O}(n)$ symmetry. Following the tradition to illustrate finite temperature effects using analogies with the condensed matter physics, we note that a typical phenomenon characterized by the symmetry restoration is, e.g., the vanishing magnetization of ferromagnetics above the Curie point; on the other hand, the spontaneous polarization of ferroelectrics at finite temperatures corresponds to the symmetry lowering.

In the case of complicated scalar interactions heating may lead to the increase of symmetry as well as to the reduction or to the symmetry breaking in a definite interval of temperature.

The choice of the symmetry temperature beheviour depends on the structure of the Lagrangian under the consideration, and a careful tuning of parameters is needed to realize a concrete one.

In this item, we will consider a simplest scalar model which exhibits some general properties of theories which are characterized by the temporary symmetry breaking at high temperatures [39]. By the temporary breaking we mean the that symmetry is lowered only in some definite temperature interval, being unbroken at the zero tempreture as well as at high ones.

This model reproduces some essential points of the so called two Higgs theories of electro-weak interactions studied by different authors with the goal to find out the possible scenarios for the high temperature phase transitions in the early Universe.

Consider a model with two real scalar fields σ and ϕ. The corresponding Lagrangian is given by

$$\mathcal{L}(\sigma, \phi) = \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\sigma\phi).$$

A tree potential is given by

$$V(\sigma, \phi) = -\frac{\mu^2}{2}\sigma^2 - \frac{m^2}{2}\phi^2 + \frac{\gamma^2}{4}\sigma^4 + \frac{f^2}{4}\phi^4 + g\sigma^2\phi^2. \tag{6.3}$$

This potential is invariant under two symmetries: discrete transformations (reflections)

$$\sigma \to -\sigma, \qquad \phi \to -\phi. \tag{6.4}$$

In what follows, we will concentrate our attention on the high temperature behaviour of the symmetry $\phi \to -\phi$. The invariance $\sigma \to -\sigma$ plays an

auxiliary role. (In the physical applications, one can think of these fields as a standard and additional Higgs's.)

In order to determine the symmetry properties, one has to study the extrema of the high-temperature effective potential [107], [30], [131], [13]

$$V_T(\sigma,\phi) = V_(\sigma,\phi) + \Delta_T V, \tag{6.5}$$

where

$$\Delta_T V = \frac{T^2}{24}\mathrm{Tr}\, M^2 - \frac{\pi^2}{90}NT^4.$$

The mass matrix is defined by the second derivatives of the potential and is given by

$$M^2 = \begin{vmatrix} -\mu^2 + 3\gamma^2\sigma^2 + 2g\phi^2 & 4g\sigma\phi \\ 4g\sigma\phi & -m^2 + 3f^2\phi^2 + 2g\sigma^2 \end{vmatrix}. \tag{6.6}$$

Define the zero temperature stationary points of the potential (6.3). The extrema equations are given by

$$\frac{\partial V}{\partial \sigma} = 2\sigma[-\frac{1}{2}\mu^2 + \frac{1}{2}\gamma^2\sigma^2 + g\phi^2] = 0, \tag{6.7}$$

$$\frac{\partial V}{\partial \phi} = 2\phi[-\frac{1}{2}m^2 + \frac{1}{2}f^2\phi^2 + g\sigma^2] = 0. \tag{6.8}$$

Let us enumerate the solutions for the equations (6.7)-(6.8). The kind of the corresponding extremum (minimum, maximum or saddle point) is determined by the value of the mass matrix (6.6).

A simple analysis leads to the following set of stationary points:

0°. The local maximum

$$(\sigma = 0, \phi = 0), \quad \text{where} \quad V_0 = V(0,0) = 0. \tag{6.9}$$

1°. The extremum

$$\sigma^2 = \frac{\mu^2}{\gamma^2}, \quad \phi = 0,$$

where

$$V_1 = V(\sigma^2 = \frac{\mu^2}{\gamma^2}, \ \ \phi = 0) = -\frac{\mu^2}{4\gamma^2}. \tag{6.10}$$

Supposing that

$$-m^2 + 2\frac{g}{\gamma^2}\mu^2 > 0, \tag{6.11}$$

we conclude that this solution corresponds to a local minimum. In the opposite case, one gets a saddle point;

2°. The extremum

$$\sigma = 0, \phi = \pm\sqrt{\frac{m^2}{f^2}}. \tag{6.12}$$

The corresponding value of the potential is

$$V_2 = V(\sigma = 0, \phi = \pm\frac{m^2}{f^2}) = -\frac{m^2}{4f^2}. \tag{6.13}$$

If

$$-\mu^2 + 2\frac{g}{f^2}m^2 > 0 \tag{6.14}$$

this solution provides a local minimum. As in the previous case, the violation of (6.14) leads to a saddle point.

3° Under the condition

$$4g^2 - f^2\gamma^2 > 0, \tag{6.15}$$

there are non-zero solutions for the extrema equations (6.7)–(6.8). This extrema correspond to stationary saddle points.

The conditions (6.11), (6.14) and (6.15) determine the shape of the potential which has one local maximum and four local minima (pairwise degenerated).

In order to define the global minimum, one has to compare the potential values (6.10) and (6.13).

Supposing that

$$\frac{\mu^2}{\gamma^2} > \frac{m^2}{f^2}, \tag{6.16}$$

we arrive to a pair of degenerate global minima 1° at $\phi = 0, \sigma = \pm\mu/\gamma$.

So we concertize the model supposing that the model parameters satisfy the conditions (6.11), (6.14), (6.15) and (6.16). In other terms, we arrange the vacuum expectation values to lie purely in the direction σ.

If the parameters g, μ^2 and f^2 are of the same order, we do not expect that loop corrections will significant change the extrema distribution picture.

So, at the zero temperature the system is in the ground state with a nonbroken symmetry $\phi \to -\phi$. In this phase, the condensates $< \phi >= 0$ and $< \sigma > \neq 0$, and the scalar fields get the masses

$$\begin{aligned} m_\sigma^2 &= 2\mu^2 = 2\gamma^2 < \sigma >^2, \\ m_\phi^2 &= 2g < \sigma >^2 - m^2. \end{aligned} \tag{6.17}$$

At high temperatures ($T^2 \gg m_\sigma^2, m_\phi^2$), the symmetry properties are determined by the minima of the effective potential (6.5):

$$\begin{aligned} V_T = &-\frac{1}{2}\mu^2(T)\sigma^2 - \frac{1}{2}m^2(T)\phi^2 + \\ &+ \frac{1}{4}\gamma^2\sigma^4 + \frac{1}{4}f^2\phi^4 + g\sigma^2\phi^2 - \frac{\pi^2}{90}NT^4. \end{aligned} \tag{6.18}$$

Here

$$\mu^2(T) = \mu^2(1 - \frac{T^2}{T_1^2}), \quad T_1^2 = \frac{6\mu^2}{g + \frac{3}{2}\gamma^2}, \tag{6.19}$$

$$m^2(T) = m^2(1 - \frac{T^2}{T_2^2}), \quad T_2^2 = \frac{6m^2}{g + \frac{3}{2}f^2}. \tag{6.20}$$

The extrema of the temperature-dependent potential (6.18) are determined as in the case of the zero temperature with the self-evident substitutions $m^2 \to m^2(T), \mu^2 \to \mu^2(T)$.

Taking into account this remark, we can recompute the positions of the extrema, this time as a function of temperature.

0°. The solution $\sigma = \phi = 0$, where $V_0^T = 0$. For $T < \min(T_1, T_2)$, this solution corresponds to a local maximum. As the temperature increases in the range

$$\min(T_1, T_2) < T < \max(T_1, T_2)$$

this extremum evolves into the saddle point, and for $T > \max(T_1, T_2)$, the solution determines potential minimum.

In what follows, we will suppose that $T_1 < T_2$.

1°. For $T < T_1$, there is a solution

$$\phi(T) = 0, \quad \sigma^2(T) = \frac{\mu^2(T)}{\gamma^2}. \tag{6.21}$$

This solution corresponds to a minimum. The necessary condition is expressed by

$$m^2(T) < \frac{2g}{\gamma^2}\mu^2(T). \tag{6.22}$$

The last inequality defines the corresponding temperature range:

$$T^2 \le \Theta_1^2 = \frac{2\frac{g}{\gamma^2}\mu^2 - m^2}{2\frac{g}{\gamma^2}\frac{\mu^2}{T_1^2} - \frac{m^2}{T_2^2}}. \tag{6.23}$$

2°. For $T \le T_2$, there is a solution

$$\sigma(T) = 0, \quad \phi^2(T) = \frac{m^2(T)}{f^2} \tag{6.24}$$

which provides a minimum of the temperature

$$T^2 \le \Theta_2^2 = \frac{2\frac{g}{f^2}m^2 - \mu^2}{2\frac{g}{f^2}\frac{\mu^2}{T_2^2} - \frac{\mu^2}{T_1^2}}. \tag{6.25}$$

3°. The non-zero solutions $\phi(T)\sigma(T) \neq 0$ correspond to saddle points and we do not consider them further.

Now we can compare the extremal values of the effective potential given by (6.21) and (6.24)

$$\begin{aligned} V_1(T) &= -\frac{\mu^4}{4\gamma^2}\left(1-\frac{T^2}{T_1^2}\right)+\mathcal{O}(T^4), \\ V_2(T) &= -\frac{m^4}{4f^2}\left(1-\frac{T^2}{T_2^2}\right)+\mathcal{O}(T^4). \end{aligned} \tag{6.26}$$

Remember that we consider the case where $T_1 < T_2$, one can conclude that there exists a critical temperature Θ_c such that

$$V_1(T) < V_2(T) \qquad \text{for} \qquad T < \Theta_c$$

and

$$V_1(T) > V_2(T) \qquad \text{for} \qquad T < \Theta_c.$$

Now we see that in some temperature interval, the global minimum corresponds to a solution with a non-zero condensate $\langle\phi\rangle$, i.e., a spontaneously broken symmetry $\phi \to -\phi$.

The critical temperature Θ_c can be obtained from the phase equilibrium condition $V_1(\Theta_c) = V_2(\Theta_c)$

$$\Theta_c^2 = \frac{f\mu^2 - \gamma m^2}{f\frac{\mu^2}{T_1^2} - \gamma\frac{m^2}{T_2^2}}.$$

Consequently, up to the temperature Θ_c, the symmetry $\phi \to -\phi$ is unbroken. For the temperature interval (Θ_c, T_2), the global minimum is provided by a spontaneously broken solution. For higher temperatures ($T > Y_2$), a symmetric solution restores symmetry.

Note in conclusion that the theories with a nonstandard symmetry behaviour at high temperatures require the complications in the scalar sector and necessitate a very accurate adjustment of the model parameters [98], [97].

3. SUSY Model

In the previous section, we considered the toy model mimicking some possible properties of the so called two Higgs models [69]. Since supersymmetry requires two (or more) Higgs doublets, it is natural to realize the mechanism of temporary symmetry breaking in SUSY model [112], [32].

Note that in that case, such a temperature behaviour does not mean the parameter fine tuning because of the SUSY constraints. Below we will consider the simplest SUSY model which exhibits most characteristic features of the global symmetry breaking over the finite range of high temperatures [38]. The case of gauge symmetry will be considered in the subsequent section.

The model contains a pair of chiral superfields (for SUSY see [51], [135])

$$\Phi(x,\theta) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x) = \frac{1}{\sqrt{2}}(A - iB) + \cdots ,$$
$$\Sigma(x,\theta) = \sigma(x) + \sqrt{2}\theta\eta(x) + \theta^2 H(x) = \frac{1}{\sqrt{2}}(a - ib) + \cdots . \tag{6.27}$$

The superpotential

$$W(\Phi,\Sigma) = -\lambda\Sigma + \frac{1}{3!}\gamma\Sigma^3 + \frac{1}{2}\gamma\Sigma\Phi^2 \tag{6.28}$$

leads to a theory invariant under the discrete transformations

$$\Phi(x,\theta) \to -\Phi(x,\theta)\Sigma(x,\theta) \to -\Sigma(x,\theta). \tag{6.29}$$

Suppose that SUSY is softly broken by the term [63]

$$V_{SB} = \omega^2 Re\sigma^2 + \kappa^2 Re\phi^2. \tag{6.30}$$

The corresponding scalar potential is given by

$$V = V_{SUSY} + V_{SB} = |-\lambda + \frac{1}{2}\gamma\sigma^2 + \frac{1}{2}f\phi^2|^2 + f^2|\sigma|^2|\Phi|^2 +$$
$$+ \omega^2 Re\sigma^2 + \kappa^2 Re\phi^2. \tag{6.31}$$

While solving the extrema equations

$$\frac{\partial V}{\partial z_a} = 0, \qquad \frac{\partial V}{\partial \bar{z}_a} = 0 \qquad (z_a = \sigma, \phi), \tag{6.32}$$

one can take into account only a P-invariant manifold of the real scalar fields

$$< Imz_a >=< B >=< b >= 0.$$

It is not difficult to find the local maximum

$$V_0(\phi = 0, \sigma = 0) = \lambda^2 \tag{6.33}$$

and 4-pairwize degenerate minima

$$V_{1,2} \equiv V\left(\phi = 0, \sigma = \pm\sqrt{2\tfrac{\lambda\gamma - \omega^2}{\gamma^2}}\right) = \lambda^2 - \left(\lambda - \tfrac{\omega^2}{\gamma^2}\right)^2, \tag{6.34}$$

$$V_{3,4} \equiv V\left(\phi = \pm\sqrt{2\tfrac{\lambda f - \kappa^2}{f^2}}, \sigma = 0\right) = \lambda^2 - \left(\lambda - \tfrac{\kappa^2}{f^2}\right)^2. \tag{6.35}$$

These minima exist under the conditions

$$\frac{1}{2}\frac{f^2}{f^2 + \frac{1}{2}\gamma f} < \frac{\lambda f - \kappa^2}{\lambda\gamma - \omega^2} < 2\frac{f^2 + \frac{1}{2}\gamma f}{\gamma^2}. \tag{6.36}$$

Note that the non-zero solutions for equations (5.7), i.e., such that $\phi \cdot \sigma \neq 0$ do not satisfy the minimum condition and correspond to saddle points. In other words, there are potential barriers between minima locations (6.34) and (6.35).

Letting $\omega = 0$, $\kappa^2 > 0$, we get

$$V_{1,2} < V_{3,4},$$

i.e., (6.34) correspond to the global minima, while (6.35) to the local ones.

Consequently, in the model there are several (quasi)stable states w exhibiting different symmetry. The quantum vacuum corresponds to the global minima ((6.34)) with unbroken symmetry $\Phi \to -\Phi$. It is evident, that the concrete choice of the ground state is provided by the SUSY breaking ($\omega = 0, \kappa^2 > 0$). At the same time, the additional symmetry $\Sigma(x,\theta) \to -\Sigma(x,\theta)$ is spontaneously broken.

These conclusions are reached at the tree level, but are valid for higher corrections. This is guaranteed by the SUSY and the nature of the soft breaking.

Now we have to establish the changes caused by the temperature. Symmetry properties are determined by the finite temperature effective potential [62]

$$V^T = \lambda^2 - \frac{1}{2}\mu^2(T)\sigma^2 - \frac{1}{2}m^2(T)\phi^2 + \frac{1}{4}\gamma^2\sigma^4 + \frac{1}{4}f^2\phi^2 + \\ + \left(f^2 + \frac{1}{2}\gamma f^2\right)\sigma^2\phi^2 - \frac{\pi^2}{90}NT^4. \tag{6.37}$$

The temperature dependent masses are given by

$$\mu^2(T) = 2\lambda\gamma\left(1 - \frac{T^2}{T_1^2}\right), \quad T_1^2 = \frac{8\lambda\gamma}{\gamma^2 + f^2} \\ m^2(T) = 2(\lambda f - \kappa^2)\left(1 - \frac{T^2}{T_2^2}\right), \quad T_2^2 = \frac{4(\lambda f - \kappa^2)}{f^2}. \tag{6.38}$$

Further we assume that $T_1 < T_2$.

It is not difficult to find the solutions for the extrema equations

$$\frac{\partial V^T}{\partial \sigma} = 0, \quad \frac{\partial V^T}{\partial \phi} = 0.$$

a) A temperature independent solution: $\sigma(T) = \phi(T) = 0$ with the corresponding energy

$$V_0(T) = \lambda^2 - \frac{\pi^2}{90}NT^4; \tag{6.39}$$

b) For the temperatures $T < T_1$, there is a solution

$$\sigma(T) = \pm\sqrt{\frac{2\lambda}{\gamma}\left(1 - \frac{T^2}{T_1^2}\right)}, \qquad \phi(T) = 0, \tag{6.40}$$

$$V_{1,2} = \lambda^2 - \lambda^2\left(1 - \frac{T^2}{T_1^2}\right)^2 - \frac{\pi^2}{90}NT^4; \tag{6.41}$$

c) If $T < T_2$, the corresponding solution is given by

$$\sigma(T) = 0, \qquad \phi(T) = \pm\sqrt{2\frac{\lambda f - \kappa^2}{f^2}} \cdot \left(1 - \frac{T^2}{T_2^2}\right), \tag{6.42}$$

$$V_{3,4} = \lambda^2 - \left(\lambda - \frac{\kappa^2}{f}\right)^2 \left(1 - \frac{T^2}{T_2^2}\right)^2 - \frac{\pi^2}{90} N T^4. \tag{6.43}$$

Analyzing the minima existence conditions, we see that for the temperature range $T > T_2$, there is one unique minimum (6.39). In the temperature interval $T_1 < T < T_2$, there is a pair of degenerate minima (6.43). At lower temperatures $T < T_1$, the situation becomes more complicated. The extrema (6.39) and (6.42) are augmented by the plus stationary points (6.40). All these solutions correspond to minima up to the temperature $\Theta_1 < T_1$, where Θ_1 is determined by the condition

$$\frac{\lambda f - \kappa^2}{\lambda\gamma} = 2\frac{f^2 + \frac{1}{2}\gamma f}{\gamma^2} \cdot \frac{1 - \frac{\Theta_1^2}{T_1^2}}{1 - \frac{\Theta_1^2}{T_2^2}}.$$

Taking into account that at the temperature

$$\Theta_c = \sqrt{\frac{8\gamma\kappa^2}{f(\gamma - f)^2}} < \Theta_1 < T_1,$$

the depth of the two minima (6.40) and (6.42) becomes equal to

$$V_{1,2}(\Theta_c) = V_{3,4}(\Theta_c).$$

It must be noted that

$$V_{1,2}(T < \Theta_c) < V_{3,4}(T < \Theta_c),$$
$$V_{1,2}(T < \Theta_c) > V_{3,4}(T > \Theta_c).$$

Now we can restore the picture of how the global minimum depends on the temperature, and trace the temperature evolution of the symmetry.

At $T = 0$, the system possesses a global minimum (6.34) with unbroken symmetry $\Phi \to -\Phi$, and local minima (6.35). There are also a local maximum (6.33) and saddle points.

While the temperature rises, the difference between the minima decreases, and starting from the temperature Θ_c, the solutions with the broken symmetry (6.35) give location of the true vacuum. For higher temperatures ($T > T_2$), there remains a single minimum ($\sigma = \phi = 0$) and the symmetry is restored.

As we see, the heating results in the competing change of the global minimum. In some temperature range, $T \sim \Theta_c$, the symmetric ground state transforms into the metastable false vacuum and must decay into a new degenerate ground state.

The size of the temperature interval where symmetry is temporarily broken is determined by the SUSY breaking parameters.

4. Symmetry Antirestoration in SUSY QED

In the previous sections, we have considered a class of models, where the effect of finite temperature is to lower the symmetry, in contrast to the standard scenario of symmetry restoration. Up to now, we presented the cases of some global symmetry. The SUSY model with global U(1) invariance was studied in [112], where the phenomenon of temporary symmetry breaking is referred as symmetry anti-restoration.

In this section, we present a model of SUSY electrodynamics, where the symmetry under consideration is local U(1) [32], [37]. It will be demonstrated that symmetry anti-restoration still occur when the theory is gauged (see also [109]).

Consider the SUSY model with a U(1) gauge symmetry. The theory contains the gauge superfield $V(A_\mu, \chi)$ and three chiral superfields $\Phi_A(z_A, \psi_A)$ (two of them carrying opposite charges and one neutral)

$$\Phi_a = z_a + \sqrt{2}\Theta\psi_a + \Theta^2 f_a, \quad a = 1, 2, \tag{6.44}$$

$$\Phi_0 = z_0 + \sqrt{2}\Theta\psi_0 + \Theta^2 f_0. \tag{6.45}$$

Supergauge transformations are written as usually [51], [135]

$$\begin{aligned} \delta V &= i(\Lambda - \Lambda^+), \\ \delta\Phi_a &= ie\Lambda\varepsilon_{ab}\Phi_b, \quad \varepsilon_{12} = 1, \\ \delta\Phi_0 &= 0. \end{aligned} \tag{6.46}$$

The superpotential – a gauge-invariant function of the neutral and charged superfields looks like

$$W(\Phi_A) = -\lambda\Phi_0 + \frac{1}{3!}\gamma\Phi_\cdot^2 + \frac{1}{2}h\Phi_0(\Phi_1^2 + \Phi_2^2). \tag{6.47}$$

As in the previous section, supersymmetry is assumed to be softly broken. This will be achieved if one introduces in the Lagrangian the SUSY non invariant term

$$\mathcal{L}_{SB} = -V_{SB} = -[\omega^2 Re\, z_0^2 + \kappa^2(Re\, z_1^2 + Re\, z_2^2)]. \tag{6.48}$$

The SUSY Lagrangian contains gauge couplings and mass terms, qubic and quadratic interactions for the particles of scalar multiplets. $\mathcal{L}_{\rm SB}$ has a simple SUSY transformation property: it transforms like rmA - term of the chiral superfield. It is the simplest explicit breaking among those considered in [63], and what is essential, it does not generate divergences, at least in the field-dependent part of the effective potential. Furthermore, it

is known that explicit breaking of such a type can be generated by the spontaneous breaking of the supersymmetry without appearance of the physical goldstinos.

Thr symmetry is determined by the scalar fields vacuum expectation values. They are given by the minimizing solutions to the extremal equations:

$$\frac{\partial V}{\partial z_A} = 0, \quad \frac{\partial V}{\partial \bar{z}_A} = 0 \quad (z_a = \sigma, \phi). \tag{6.49}$$

Here V^T is the high temperature effective potential. In the high temperature approximation, it is given by [62]

$$V^T(z_A, \bar{z}_A) = V(Z_A, \bar{z}_A) + \Delta_T V(z_A, \bar{z}_A). \tag{6.50}$$

Here

$$V(z_A, \bar{z}_A) = V_{SUSY} + V_{SB} =$$
$$= \sum_A \left|\frac{\partial W}{\partial z_A}\right|^2 + \frac{1}{2}e^2[\bar{z}_a \varepsilon_{ab} z_b]^2 + +\omega^2 Re\, z_0^2 + \kappa^2(Re\, z_1^2 + Re\, z_2^2) \tag{6.51}$$

is the zero temperature part, and

$$\Delta_T V(z_A, \bar{z}_A) = \frac{T^2}{8}\{\sum_{A,B} \frac{\partial^2 W}{\partial z_A \partial z_B}|^2 + 4e^2 \bar{z}_a z_A\} -$$
$$- \frac{\pi^2}{90}(N_B + \frac{7}{8}N_F)T^4$$

is the temperature dependent one-loop correction.

Assuming the theory is invariant with respect to the parity operation, one can consider the vacua manifold with $< Im\, z_A >= 0$. With this remark, our effective potential takes the form

$$V^T(\sigma, \rho) = \lambda^2 - \frac{1}{2}\mu^2(T)\sigma^2 - \frac{1}{2}m^2\rho^2 +$$
$$+ \frac{1}{4}\gamma^2\sigma^4 + \frac{1}{4}h^2\rho^4 + (h^2 + \frac{1}{2}\gamma h)\sigma^2\rho^2. \tag{6.52}$$

In this expression, we use the notation $\sigma = Re\, z_0$ and $\rho^2 = Re\, z^2$

$$\mu^2(T) = 2(\lambda\gamma - \omega^2)\left(1 - \frac{T}{T_1^2}\right), \quad m^2(T) = 2(\lambda h - \kappa^2)\left(1 - \frac{T}{T_2^2}\right),$$
$$T_1^2 = \frac{2\mu^2(0)}{h^2 + \frac{1}{2}\gamma^2}, \quad T_2^2 = \frac{2m^2(0)}{h^2 + 2e^2}. \tag{6.53}$$

For concreteness in what follows we will assume that $T_1 < T_2$. Find the potential extema location and for this purpose solve the equations

$$\begin{aligned} \frac{\partial V^T}{\partial \sigma} &= 2\sigma \frac{\partial V^T}{\partial \sigma^2}, \\ \frac{\partial V^T}{\partial \rho} &= 2\rho \frac{\partial V^T}{\partial \rho^2}. \end{aligned} \tag{6.54}$$

Enumerate the corresponding solutions a) $\sigma(T) = \phi(T) = 0$. Depending on the temperature, T this solution corresponds to:

a1) $T < T_1$ – local maximum,

a2) $T_1 < T < T_2$ – saddle point,

a3) $T > T_2$ – minimum.

b) In the temperature range $0 < T < T_1$, there are the solutions

$$\sigma^2(T) = \pm \left[\frac{2}{\gamma^2}(\lambda\gamma - \omega^2) \cdot \left(1 - \frac{T}{T_1^2}\right) \right]^{1/2}, \quad \rho(T) = 0 \tag{6.55}$$

corresponding to the degenerate local minima with unbroken U(1). Here the potential takes the value

$$\begin{aligned} V_{1,2}^T &= V^T(\sigma = \sigma(T), \rho = 0) = \\ &= \lambda^2 - \left(\frac{\lambda\gamma - \omega^2}{\gamma}\right)^2 \cdot \left(1 - \frac{T}{T_1^2}^2\right) - \frac{\pi^2}{90}\left(N_B + \frac{7}{8}N_F\right)T^4. \end{aligned} \tag{6.56}$$

c) When the temperature $T < T_2$, there are solutions with the broken gauge symmetry:

$$\sigma = 0, \qquad \rho(T) = \pm \left[\frac{2}{h^2}(\lambda h - \kappa^2) \left(1 - \frac{T}{T_2^2}\right)^2 \right]^{1/2}. \tag{6.57}$$

The value of the corresponding local minimum is given by

$$\begin{aligned} V_{3,4}^T &= V^T(\sigma = 0, \rho = \rho(T)) = \\ &= \lambda^2 - \left(\frac{\lambda h - \kappa^2}{h}\right)^2 \cdot \left(1 - \frac{T}{T_2^2}^2\right) - \frac{\pi^2}{90}\left(N_B + \frac{7}{8}N_F\right)T^4. \end{aligned} \tag{6.58}$$

d) Nonzero solutions $\sigma(T)\rho(T) \neq 0$ correspond to saddle points.

The concrete location of the effective potential extrema depends on the model parameters. If the solutions a) and b) are minima, one has the

conditions

$$\begin{gathered}\frac{1/2\gamma^2}{h^2+1/2\gamma h}<\frac{\lambda\gamma-\omega^2}{\lambda h-\kappa}<\frac{h^2+1/2\gamma h}{1/2h^2},\\ \frac{\lambda\gamma-\omega^2}{\lambda h-\kappa}>\frac{\gamma}{h}\\ \frac{1}{2}\frac{h^2}{h^2+\frac{1}{2}\gamma h}<\frac{h^2+2e^2}{h^2+\frac{1}{2}\gamma^2}<2\frac{h^2+\frac{1}{2}\gamma h}{\gamma^2},\\ \gamma(h^2+2e^2)<h(h^2+\frac{1}{2}\gamma^2).\end{gathered}\tag{6.59}$$

Assuming the validity of the inequalities (6.59), we get the following picture.

1. At $T=0$, the effective potential possesses:
   - global minima with exact U(1) symmetry,
   - local minima with the spontaneously broken gauge invariance,
   - a local maximum $\sigma=\rho=0$,
   - a saddle point $\sigma=\rho\neq 0$, corresponding to the potential barriers between the minima points.
2. When the temperature increases up to the $T=\Theta_c$ with

$$\Theta_c^2=4\frac{\kappa^2-h/\gamma\omega^2}{(h^2+1/2\gamma^2)h/\gamma-(h^2+2e^2)},$$

the minimum with unbroken U(1) lies below the minimum with the non-invariant ground state : $V_{1,2}(T<\Theta_c)<V_{3,4}(T<\Theta_c)$.

3. At $T=\Theta_c$, the minima with exact and broken U(1) become equal to $V_{1,2}(\theta_c)=V_{3,4}(\Theta_c)$. For higher temperatures $\Theta_c<T<T_2$, the symmetry is broken, i.e., the symmetry anti restoration takes place.

4. At high temperatures $T>T_2$, the symmetry is restored and the system undergoes the phase transition to the normal symmetric phase.

In conclusion, one can say that in the model under consideration the gauge symmetry can be broken for some finite temperatures.

Thus, depending on the SUSY breaking, one can realize different patterns of the gauge symmetry breaking at finite temperatures, i.e., SUSY breaking drives the internal symmetry.

In this connection, we have to mention a theory [98] where the magnetic monopole production is suppressed by the breaking chain

$$\mathrm{SU}(5)\to H_1\to\cdots\to H_n\to\mathrm{SU}(3)_c\times\mathrm{U}(1)_{\mathrm{em}}$$

occurring at temperatures $T_1>T_2>\cdots>T_c$. In this scheme, the intermediate symmetry group H_i does not contain a U(1) factor, i.e., at $T>T_c$ the gauge U(1) symmetry is broken, just what we have in the model considered above.

APPENDIX

A. Fermions in Magnetic Field

A particle (charge e mass m) constrained to move in a plane perpendicular to a constant magnetic field B is described by a stationary wave function $\psi(x)$ satisfying the Schrödinger equation

$$\frac{1}{2m}(\mathbf{P} - e\mathbf{A})^2\psi(x) = E\psi(x), \tag{A.1}$$

where $\mathbf{A}(x)$ is a vector potential such that

$$\varepsilon_{ik}\partial_i A^k(x) = B.$$

In what follows, we will ignore the electron spins, assuming that they are aligned along the field since the energy required for the reversal of a spin is comparable with the spacing of the Landau levels.

Different solutions of the equation (A.1) are related by gauge transformations. The usual choices are symmetric by

$$A_x = -\frac{1}{2}By, \qquad A_y = \frac{1}{2}Bx,$$

and asymetric (Landau) gauges

$$A_x = -By, \quad A_y = 0.$$

These gauges are applied in the cases of the disk and rectangular geometry, respectively. Consider first the case of an asymmetric gauge.

It is suitable to introduce new pairs of canonical variables

$$Q = -\frac{i}{\sqrt{\omega_c}}D_1, \quad P = -\frac{i\varepsilon}{\sqrt{\omega_c}}D_2, \quad [Q,P] = i,$$
$$q = \sqrt{\omega_c}x + P, \quad p = -\varepsilon\sqrt{\omega_c}y + Q, \quad [q,p] = i,\ \varepsilon = \operatorname{sign}(eB).$$

Here $\omega_c = |eB| = \frac{1}{\ell^2}$, ℓ is the magnetic length. Operators q and p are known as guiding canter coordinates. In fact, they are the magnetic translation generators for the matter field

$$P_x = \frac{1}{\ell}p \qquad P_y = \frac{1}{\ell}q.$$

Note that they do not commute among themselves.

Rotations are generated by the operator

$$J = \frac{1}{2}(q^2 + p^2) - \frac{1}{2}(Q^2 + P^2).$$

It is natural to define the oscillator operators

$$a = \frac{Q + iP}{\sqrt{2}}, \qquad a^\dagger = \frac{Q - iP}{\sqrt{2}},$$

$$b = \frac{q + ip}{\sqrt{2}}, \qquad b^\dagger = \frac{q - ip}{\sqrt{2}}$$

in terms of which the Hamiltonian and the angular momentum operators are given by

$$H = \frac{\omega_c}{m}(a^\dagger a + 1/2), \quad J = b^\dagger b - a^\dagger a.$$

As a commuting set, one can choose H and one of the momentum operators. Another commonly used set is formed by the Hamiltonian and the angular momentum.

In what follows, we will work in the basis $|n, p\rangle = |n\rangle \otimes |p\rangle$. Here $|n\rangle$ is the occupation number basis and $|p\rangle$ is the basis in which the translation generator P_x is diagonal. In the coordinate representation, the wave function is given by

$$\langle \mathbf{r}|n, p\rangle = \frac{1}{\sqrt{2\pi\ell}} e^{i(\varepsilon xy - \varphi(x,y))} e^{ip_x \cdot x} \Psi_n(\sqrt{\omega_c} y + \ell p_x). \tag{A.2}$$

Here $\Psi_n(Q)$ is a ordinary oscillatory wave function and the function $\varphi(x, y)$ is determined by the detailed form of the background gauge field. In the assymetric gauge $\varphi = \varepsilon\omega xy$. Classically the particle moves around a circle in the x,y plane. The conserved quantity $y_0 = -\ell p_x$ corresponds to the y coordinate of the canter of the classical orbit. To the coordinate x of the centre there corresponds the operator ℓq.

The assymetric gauge has the feature that the wave functions (A.2) are extended in the direction x but sharply peaked about $y = y_0 = -\frac{l}{\sqrt{\omega_c}}$, so the parameter y_0 represents the site of the Landau orbital. Notice that the Landau and symmetric gauges are related by the gauge transformation

$$\mathbf{A}^{Landau} = \mathbf{A}^{sym} + \nabla\lambda$$

with the gauge function $\lambda = \frac{B}{2}xy$.

Usually non-degenerate wave functions are simply related by the phase factor $e^{i\lambda}$. However, due to the degeneracy of the Landau levels, the gauge invariance only implies that $e^{i\lambda}\psi^j_{nsymm}$ is some linear combination of the $\psi_{n,k\,asymm}$ with the same energy. The particular form of this linear combination is gauge dependent, and the conversion of the density from one gauge to another is not such a straightforward matter.

The Landau site y_0 depends only on p_x. So each level is highly degenerate. If the system is of the finite size (with the area L_xL_y), then the allowed values of p_x are discrete: $\Delta p_x = 2\pi/L_x$. Moreover, the condition implies $0 \le |p_x| \le \ell^{-2}L_y$, so the number of allowed values p_x is finite and given by

$$\frac{L_y}{2\pi\ell^2 p_x} = |eB|L_xL_y = e\Phi,$$

and the degeneracy or the number of allowed quantum states per unit area is

$$n_L = \frac{|eB|}{2\pi}. \tag{A.3}$$

(For $B = 1$ Tesla, there are 2.7×10^{14} states per m^2.) The quantity $\omega_c = \frac{eB}{mc}$ is the classical cyclotron frequency. The corresponding harmonic oscillator energy $\hbar\omega_c$ sets the scale for the physics. Taking m to be the free electron mass, we find, for $B = 1$ Tesla, that the corresponding energy is about 10^{-4} eV, or ≈ 1.2 degrees Kelvin. The system should be at lower temperature than this to observe any effect of energy quantization.

In the case of the disk geometry, it is convenient to use the symmetric gauge $\mathbf{A}(x) = \frac{B}{2}(-y, x)$ and as the unit of length the quantity $\sqrt{2}\ell$. This gauge has the property that it is well adapted to the use of complex variables $z = x + iy$, $\partial = \frac{1}{2}(\partial_x - i\partial_y)$. Oscillator operators are given by

$$a = \frac{z}{2} + \bar{\partial}, \quad a^+ = \frac{\bar{z}}{2} - \partial,$$
$$b = \frac{\bar{z}}{2} + \partial, \quad b^+ = \frac{z}{2} - \bar{\partial}.$$

With these notations, the one-particle Hamiltonian reads

$$H = \frac{1}{m\ell^2}[a^+ a + \frac{1}{2}].$$

The operators a and a^+ are lowering and raising operators between different Landau levels characterized by the quantum number n. The energy levels are

$$E_n = \frac{\hbar eB}{m}(n + 1/2),$$

the operators b and b^+ commute with H, and generate infinitely many degenerate states

The properly normalized one-body Landau wave functions are

$$\psi_n^j(z, \bar{z}) = \sqrt{\frac{n!}{\pi 2^{j+1}(n+j)!}} z^j L_n^j(|z|^2) exp(-\frac{|z|^2}{2}),$$

where $n \geq 0$ and $j \geq -n$ are integers and L_n^j are the adjoint Laguerre polynomials. They satisfy

$$H\psi_n^j = E_n \psi_n^j$$

and

$$J\psi_n^j = (z\partial - \bar{z}\bar{\partial})\psi_n^j = j\psi_n^j.$$

The states of the lowest Landau level have $n = 0$, and are annihilated by the operator a. The general solution of the equation

$$a\psi = \bar{\partial}\psi + \frac{z}{2}\psi = 0$$

is of the form

$$\psi(z,\bar{z}) = \pi^{-1/2} exp(-\frac{z\bar{z}}{2})\phi(z),$$

where ϕ is a function only of z, i.e., an analytic (entire) function. The complete orthonormal basis is given by the monomials

$$\phi_j(z) = \frac{z^j}{\sqrt{l!}}, j = 0, 1, 2, \ldots .$$

The operator J becomes $z\partial$, giving the degree of homogeneity and hence the orbital angular momentum around a fixed origin. The fact that these eigenvalues are bounded from below arises from the preferred chirality introduced in the plane by the magnetic field: it requires energy to overcome chirality (for $B < 0$ we must replace z by $\bar{z}$).

The probability density of the j-th state is concentrated in a ring centered around $r \sim \sqrt{j}$ since

$$< j|\bar{z}z|j >= j + 1.$$

In the second quantized formalism, the field operator will be expanded over a set of energy eigenfunctions:

$$\psi(\mathbf{r}, t) = \sum_{n \geq 0} \int dp_x e^{-iE_n t} \Psi_{np_x}(\mathbf{r}) a_{np_x}.$$

Here $E_n = \frac{\omega_c}{M}(n + 1/2) - \mu$ is the energy of the Landau level. The operator a_{np_x} and its Hermitean conjugate must satisfy the canonical anticommutation relations

$$\{a_{np_x}, a^\dagger_{n'p'_x}\} = \delta_{nn'}\delta(p_x - p'_x).$$

As a ground state, consider

$$|\Phi_0\rangle = \prod_{p_x} \prod_{n=0}^{\nu-1} a^\dagger_{np_x}|0\rangle.$$

This ground state corresponds to the totally filled $N = \nu$ Landau levels. Obviously

$$N = \frac{\varrho}{n_L}.$$

Let us calculate the energy density

$$E = \langle\Phi_0|H(\mathbf{r})|\Phi_0\rangle = n_L \sum_{n=0}^{N-1} \frac{eB}{m}(n + 1/2) = \frac{\pi}{m}\varrho^2. \qquad (A.4)$$

So the energy density does not depend on N.

Now consider the case of the partial filling, which takes place when the filling factor

$$\nu = \frac{2\pi\varrho}{|eB|}$$

is not integer. Let us perform a small variation of the magnetic field:

$$B \to B' = B + \delta B.$$

It is not difficult to show that the energy density is given by the expression

$$E' = \frac{\pi\varrho^2}{m}[1 + |\hat{B}| - (N^2 - N\mathrm{sign}\,\hat{B})\hat{B}^2],$$

where

$$\hat{B} = \frac{\delta(eB)}{2\pi\varrho}.$$

So we see that the switching on of the additional magnetic field increases the energy – energetically it is favorable to remain in the state with the integer filling factor. The energy density as a function of B has cusps for those values of the magnetic field which correspond to the integer values of the filling factor.

Now we can introduce the Green functions for the fermion field. At the zero temperature, it is defined as the ground state average:

$$\begin{aligned} G(x,x') &= i\langle\Phi_0|T(\psi(x)\psi^\dagger(x'))|\Phi_0\rangle = \langle\mathbf{r}|G(t-t')|\mathbf{r}'\rangle = \\ &= -\int\frac{dE}{2\pi}e^{-iE(t-t')}\langle\mathbf{r}|\mathcal{G}(E)|\mathbf{r}'\rangle, \end{aligned} \tag{A.5}$$

where the kernel $\mathcal{G}(E)$ is given by

$$\begin{aligned} \mathcal{G}(E) &= \sum_{n=0}^{N-1}\frac{|n\rangle\langle n|}{E-E_n-i\delta} + \sum_{n=N}^{\infty}\frac{|n\rangle\langle n|}{E-E_n+i\delta} = \\ &= -\int\limits_{C_N}\frac{dE}{2\pi}e^{-iE(t-t')}\sum_{n}^{\infty}\frac{|n\rangle\langle n|}{E-E_n}. \end{aligned} \tag{A.6}$$

At finite temperatures, one can use imaginary time(Matsubara) or real time formalisms. In the imaginary time formulation, the finite temperature Green function is defined as a quantum statistical average

$$G_\beta(\mathbf{r},\tau;\mathbf{r}',\tau') = -\frac{1}{\mathrm{Z}}Tr\{\rho T_\tau[\psi(\mathbf{r},\tau)\psi^\dagger(\mathbf{r}',\tau')]\}, \tag{A.7}$$

where $\rho = exp[-\beta(H-\mu N)]$ is the grand canonical distribution and $\mathrm{Z} = Tr\rho$. The Matsubara fields are defined as follows

$$\begin{aligned} \psi(\mathbf{r},\tau) &= \sum_{n}^{\infty}\int dp e^{-E_n\tau}\psi_{np}(\mathbf{r})a_{np}, \\ \psi^\dagger(\mathbf{r},\tau) &= \sum_{n}^{\infty}\int dp e^{E_n\tau}\psi^\dagger_{np}(\mathbf{r})a_{np}. \end{aligned}$$

Due to the antiperiodicity, this thermal Green function can be Fourier expanded over the half-integer frequencies:

$$G_\beta(\mathbf{r},\tau;\mathbf{r}',\tau) = \frac{1}{\beta}\sum_{s=-\infty}^{+\infty} e^{-i\xi_s(\tau-\tau')}\langle\mathbf{r}|\mathcal{G}(i\xi_s)|\mathbf{r}'\rangle,$$
$$\xi_s = \frac{2\pi}{\beta}(s+1/2).$$

The kernel

$$G(i\xi_s) = \sum_{n=0}^{\infty}\frac{|n\rangle\langle n|}{i\xi_s - E_n} \tag{A.8}$$

can be obtained from zero temperature kernel by the formal substitution $E \to i\xi_s, E_n \to E_n = E_n - \mu$ (μ is a chemical potential).

In the thermo field dynamics, the Green functions are defined in a close analogy with the standard field theory. They are given by a 2×2 matrix

$$G_{ab}(x,x') = i\langle 0(\beta)|T(\psi_a(x)\tilde{\psi}_b^\dagger(x'))|0(\beta)\rangle =$$
$$= \langle\mathbf{r}| - \int\frac{dE}{2\pi}e^{-iE(t-t')}G_{ab}(E)|\mathbf{r}'\rangle, \qquad a,b=1,2. \tag{A.9}$$

The kernels $G_{ab}(E)$ are given by

$$G_{11}(E) = \sum_n[\frac{\cos^2\theta_n(\beta)}{E-E_n+i\delta} + \frac{\sin^2\theta_n(\beta)}{E-E_n-i\delta}]|n\rangle\langle n|,$$
$$G_{12}(E) = \sum_n[\frac{1}{E-E_n+i\delta} - \frac{1}{E-E_n-i\delta}]\sin\theta_n(\beta)\cos\theta_n(\beta)|n\rangle\langle n| =$$
$$= -G_{21}(E), \tag{A.10}$$
$$G_{22}(E) = \sum_n[\frac{\sin^2\theta_n(\beta)}{E-E_n+i\delta} + \frac{\cos^2\theta_n(\beta)}{E-E_n-i\delta}]|n\rangle\langle n|,$$

where

$$\sin^2\theta_n(\beta) = \frac{1}{1+e^{\beta E_n}}.$$

One particle Hamiltonian describing a relativistic planar electron in the uniform magnetic background is given by

$$\mathcal{H}^{\mathrm{b}} = i\,\ell_0\gamma^0\gamma_k D_k^{\mathrm{b}} + \sigma\gamma^0,$$
$$D_k^{\mathrm{b}} = \partial_k + i(e/\hbar)(A_k^{\mathrm{b}} + a_k^{\mathrm{b}}),$$
$$A_k^{\mathrm{b}}(\mathbf{r}) + a_k^{\mathrm{b}}(\mathbf{r}) = \frac{1}{2}\varepsilon_{kl}x^l(B^{\mathrm{b}} + b^{\mathrm{b}}) - \partial_k\xi(\mathbf{r}),$$

where $\xi(\mathbf{r})$ is the gauge fixing term. Introduce the canonical transformation $(x, y, -i\partial_x, -i\partial_y) \to (Q, q, P, p)$

$$Q = -i\ell D_1^{\mathrm{b}}, \quad P = -i\varepsilon\ell D_2^{\mathrm{b}},$$

$$q = -i\varepsilon\ell D_2^{\rm b} + \ell^{-1}x, \quad p = -i\ell D_1^{\rm b} - \varepsilon\ell^{-1}y,$$

where $\varepsilon \equiv \mathrm{sgn}(eB^{\rm b} + eb^{\rm b})$, and the magnetic length ℓ is defined by

$$\frac{1}{\ell^2} = \frac{1}{\hbar}\,|\,eB^{\rm b} + eb^{\rm b}|.$$

One can easily verify that the non-vanishing commutators are only $[Q,\,P] = [q,\,p] = i$.

Rewritten in terms of new variables, the Hamiltonian takes the form

$$\mathcal{H}^{\rm b}(\sigma,\,\varepsilon) = \begin{pmatrix} \sigma & \sqrt{h}\,(Q - i\,\varepsilon P) \\ \sqrt{h}\,(Q + i\,\varepsilon P) & -\sigma \end{pmatrix},$$

where $h = \ell_0^2/\ell^2$ and $\ell_0 = \hbar/mc$.

Now we introduce the lowering and rising operators

$$Q + iP = \sqrt{2}\,c, \quad Q - iP = \sqrt{2}\,c^\dagger$$

and use the occupation number representation

$$c\,|n\rangle = \sqrt{n}\,|n-1\rangle, \qquad c^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle.$$

Solving the Schrödinger equation for $\mathcal{H}^{\rm b}(\sigma,\,\varepsilon)$, one finds the following set of the positive and the negative energy eigenvectors

$$\begin{aligned} \mathcal{H}^{\rm b}(\sigma,\,\varepsilon)u_n(\sigma,\,\varepsilon) &= +\,\omega_n u_n(\sigma,\,\varepsilon), \\ \mathcal{H}^{\rm b}(\sigma,\,\varepsilon)v_n(\sigma,\,\varepsilon) &= -\,\omega_n v_n(\sigma,\,\varepsilon), \end{aligned}$$

where the eigenvalues, being independent of σ and ε, are given by

$$\omega_n = \sqrt{1 + 2hn}.$$

Explicit expressions for the eigenvectors are ($\sigma = \varepsilon = 1$)

$$u_n(+,\,+) = \frac{1}{\sqrt{2\omega_n}} \begin{pmatrix} \sqrt{\omega_n + 1} & |n\rangle \\ \sqrt{\omega_n - 1} & |n-1\rangle \end{pmatrix}, \quad n = 0, 1, 2, \ldots,$$

$$v_n(+,\,+) = \frac{1}{\sqrt{2\omega_n}} \begin{pmatrix} \sqrt{\omega_n - 1} & |n\rangle \\ -\sqrt{\omega_n + 1} & |n-1\rangle \end{pmatrix}, \quad n = 1, 2, 3, \ldots,$$

and as one can see, each u_n except u_0 has the partner v_n with the opposite energy eigenvalue. Consequently, there exists some asymmetry between the positive and the negative energy spectra.

Eigenfunctions are normalized as follows

$$u_m^\dagger u_n = v_m^\dagger v_n = \omega_m \bar{u}_m u_n = -\omega_m \bar{v}_m v_n = \delta_{mn}, \quad u_m^\dagger v_n = 0.$$

The completeness condition and the eigenbasis expansion of $\mathcal{H}^{\mathrm{b}}(+,+)$ are

$$\sum_{n=0} u_n u_n^\dagger + \sum_{n=1} v_n v_n^\dagger = 1,$$
$$\sum_{n=0} \omega_n u_n u_n^\dagger - \sum_{n=1} \omega_n v_n v_n^\dagger = \mathcal{H}^{\mathrm{b}}.$$

These relations established for the case $\sigma = \varepsilon = 1$ can be easily generalized to the different values of σ and ε. In particular, the Dirac Hamiltonian has the following properties

$$\mathcal{H}^{\mathrm{b}}(-\sigma, \varepsilon) = -\gamma_0 \mathcal{H}^{\mathrm{b}}(\sigma, \varepsilon)\gamma^0,$$
$$\mathcal{H}^{\mathrm{b}}(\sigma, -\varepsilon) = -\gamma_1 \mathcal{H}^{\mathrm{b}}(\sigma, \varepsilon)\gamma^1$$

which immediately yield

$$u_n(-\sigma, \varepsilon) = \gamma^0 v_n(\sigma, \varepsilon),$$
$$v_n(\sigma, -\varepsilon) = \gamma^1 u_n(\sigma, \varepsilon).$$

Asymmetry between the positive and the negative energy eigenstates is generalized as follows: there is no v_0 mode for $\sigma\varepsilon = +1$, and no u_0 mode for $\sigma\varepsilon = -1$.

Choosing $(\mathcal{H}^{\mathrm{b}}, p)$ as a full set of commuting operators and denoting $u_{np} = u_n \cdot |p\rangle$ and $v_{np} = v_n \cdot |p\rangle$, we write the quantum spinor fields as

$$\psi = \sum_{np} \{a_{np} u_{np} + b_{np}^\dagger v_{np}\},$$
$$\bar{\psi} = \sum_{np} \{a_{np}^\dagger \bar{u}_{np} + b_{np} \bar{v}_{np}\},$$

where $a_{np}^\dagger$ and $b_{np}^\dagger$ are the operators creating the particles and antiparticles, respectively, while a_{np} and b_{np} are the corresponding annihilating ones. These operators satisfy the ordinary anticommuting relations

$$\{\, a_{np}\,,\, a_{n'p'}^\dagger \,\} = \{\, b_{np}\,,\, b_{n'p'}^\dagger \,\} = \delta_{nn'}\,\delta(p - p').$$

For completeness, we give the formulae for (Q, q) and (x, y) representations

$$\langle Q|n\rangle = (\sqrt{\pi}\, 2^n n!)^{-1/2} e^{-Q^2/2} H_n(Q) \equiv \Psi_n(Q),$$
$$\langle q|p\rangle = (2\pi)^{-1/2} e^{ipq},$$
$$\langle x, y|Q, q\rangle = \frac{1}{2\pi\ell} \exp\{i\ell^{-1}(Qx + \varepsilon qy) - (i/2)\varepsilon\ell^{-2}xy - iQq + i(e/\hbar)\xi\},$$
$$\langle x, y|n, p\rangle = \frac{1}{\ell\sqrt{2\pi}}\, \Psi_n(p + \varepsilon\ell^{-1}y) \exp\{i\ell^{-1}px + (i/2)\varepsilon\ell^{-2}xy + i(e/\hbar)\xi\},$$

where $H_n(Q)$ are the Hermite polynomials.

The quantized Hamiltonian H_0 and the particle number operator N are defined in the normal ordered form

$$H_0 = \sum_{np} \omega_n a^\dagger_{np} a_{np} + \sum_{np} \omega_n b^\dagger_{np} b_{np},$$
$$N = \sum_{np} a^\dagger_{np} a_{np} - \sum_{np} b^\dagger_{np} b_{np}$$

allowing to avoid problems with negative energy states.

For the Fourier transformation, one can use the relation

$$e^{-i\mathbf{k}\hat{r}} = e^{i\ell(k_x\hat{q} - \varepsilon k_y \hat{p})} Z(\mathbf{k}),$$

where

$$Z(\mathbf{k}) = e^{\varepsilon \ell^2 \varepsilon^{mn} k^m D_n} = e^{\ell[k_+ a^\dagger - k_- a]}$$

is a coherent state operator $k_\pm = 1/\sqrt{2}(k_x \pm i\varepsilon k_y)$. It is not difficult to show, that the following relations are valid:

$$\{D_m, Z(k)\} = -\frac{2\varepsilon}{\ell^2} \varepsilon^{mn} \frac{\partial}{\partial k^n} Z(\mathbf{k}),$$

$$\int d\mathbf{r} e^{-i\mathbf{k}r} \langle \mathbf{r}|A(a^\dagger, a)|\mathbf{r}\rangle = \frac{2\pi}{\ell^2} tr[Z(\mathbf{k})A],$$
$$\int d\mathbf{r} \int d\mathbf{r}' e^{-i\mathbf{k}r} e^{i\mathbf{k}'r'} \langle \mathbf{r}|A(a^\dagger, a)|\mathbf{r}'\rangle \langle \mathbf{r}'|B(a^\dagger, a)|\mathbf{r}\rangle =$$
$$= \frac{2\pi}{\ell^2} \delta(\mathbf{k} - \mathbf{k}') tr[Z(\mathbf{k}) A Z^\dagger(\mathbf{k}') B].$$

B. Similarity Transformation

The generator of similarity transformation must satisfy the commutation relation

$$[G_p, \chi(\mathbf{r})] = 2p\pi \int d\mathbf{r}' G(z - z') \varrho(\mathbf{r}') \cdot \chi(\mathbf{r}).$$

Represent G_p as a bilinear functional of the density operator:

$$G_p = \int d\mathbf{r}' \int d\mathbf{r}'' \varrho(\mathbf{r}') \Lambda_p(\mathbf{r}', \mathbf{r}'') \varrho(\mathbf{r}'') + \int dr L_p(\mathbf{r}) \varrho(\mathbf{r}).$$

Using the commutator

$$[\varrho(\mathbf{r}'), \chi(\mathbf{r})] = -\delta(\mathbf{r} - \mathbf{r}') \chi(\mathbf{r}),$$

we get

$$\int dr' [\Lambda_p(\mathbf{r}, \mathbf{r}') + \Lambda_p(\mathbf{r}', \mathbf{r})] \varrho(\mathbf{r}') = -2p\pi \int d\mathbf{r}' G(z - z') \varrho(\mathbf{r}')$$

and

$$L_p(\mathbf{r}) = -\Lambda_p(\mathbf{r}, \mathbf{r}).$$

Consequently

$$\Lambda_p(\mathbf{r},\mathbf{r}') = -p\pi G(z-z') + \frac{i\pi}{2}p,$$

and

$$L_p(\mathbf{r}) = -\lim_{\mathbf{r}'\to\mathbf{r}} \Lambda_p(\mathbf{r},\mathbf{r}') = i\frac{\pi}{2}p.$$

In the last expression, we have used the regularized Green function, satisfying the condition $G(0) = 0$. As an example of such a regularization, one can try to use the function

$$G(z) = \lim_{\epsilon\to 0} G_\epsilon(z), \qquad G_\epsilon = \frac{1}{\pi}\ln z \cdot e^{-\epsilon/|z|^2}.$$

C. Toy Model

Consider a toy model describing the couple of fermion oscillators. The basic anticommutators are

$$\{f, f^\dagger\} = 1, \quad \{f_c, f_c^\dagger\} = 1.$$

The Hamiltonian is

$$\hat{H} = f^\dagger f - f_c^\dagger f_c.$$

The explicit realization of basic operators and Hilbert space can be given in terms of Grassmann variables ξ and ξ^*:

$$f = \frac{\partial}{\partial\xi}, \quad f^\dagger = \xi, \quad f_c = \frac{\partial}{\partial\xi^*}, \quad f_c^\dagger = \xi^*.$$

The Hamiltonian takes the form

$$\hat{H} = \xi\frac{\partial}{\partial\xi} - \xi^*\frac{\partial}{\partial\xi^*}.$$

The state vectors are represented by their expansion

$$\Psi = \psi_{00} + \psi_{10}\xi + \psi_{01}\xi^* + \psi_{11}\xi\xi^*.$$

Introduce the dual vector

$$\Psi^\# = -\bar{\psi}_{00}\xi\xi^* + \bar{\psi}_{01}\xi^* - \bar{\psi}_{10}\xi - \bar{\psi}_{11}.$$

The scalar product is defined by the Berezin integral over the Grassmann numbers

$$(\Phi,\Psi) = \int \mathrm{d}\xi\mathrm{d}\xi^*\Phi^\#\cdot\Psi = \bar{\phi}_{00}\psi_{00} + \bar{\phi}_{10}\psi_{10} + \bar{\phi}_{01}\psi_{01} + \bar{\phi}_{11}\psi_{11}.$$

We see that the pairs of Hermitian conjugate operators are given by

$$(\xi,\ \xi^\dagger = \frac{\partial}{\partial\xi}) \quad \text{and} \quad (\xi^\star,\ \xi^{*\dagger} = \frac{\partial}{\partial\xi^*}).$$

So our Hamiltonian is Hermitian and invariant under the involution operation.

D. Structure functions

Below we present the structure functions for the relativistic fermion current correlators

$$\Pi_{\rm E}(x) = -\frac{h}{2\pi}\ I(x) - \frac{1}{4\pi}\ e^{-x}\, S_3(x,x) + \frac{h}{2\pi}\ e^{-x}\,\frac{\partial}{\partial x}\, x\,\frac{\partial}{\partial x}\, S_0(x,y)\Big|_{y=x},$$

$$\Pi_{\rm CS}(x) = \frac{1}{4\pi}\ e^{-x}\left[\frac{\partial}{\partial x}\ S_1(x,y) - \frac{\partial}{\partial x}\ S_2(x,y) + S_2(x,y)\right]\Big|_{y=x},$$

$$\Pi_{\rm M}(x) = -\frac{1}{2\pi}\ e^{-x}\ \frac{\partial^2}{\partial x \partial y}\ S_0(x,y)\Big|_{y=x},$$

where $I(x)$ and $S_a(x,y)$ are given by

$$I(x) = e^{-x}\sum_{n=0}^{\infty}\sum_{\alpha=1}^{\infty}\frac{n!}{(n+\alpha)!}\ \alpha^2\ \Theta_0(n,\alpha)\ x^{\alpha-1} L_n^\alpha(x)\, L_n^\alpha(x),$$

$$S_a(x,y) = \sum_{n=0}^{\infty}\Theta_a(n) L_n(x) L_n(y) +$$

$$+\sum_{n=0}^{\infty}\sum_{\alpha=1}^{\infty}\frac{n!}{(n+\alpha)!}\ \Theta_a(n,\alpha)\,(x^\alpha + y^\alpha)\, L_n^\alpha(x)\, L_n^\alpha(y).$$

Here $L_n^\alpha(x)$ are the adjoint Laguerre polynomials, while Θ_a are temperature-dependent quantities

$$\alpha\,\Theta_a(n,\alpha) = \theta_a(n) - \theta_a(n+\alpha), \quad \Theta_a(n) = \lim_{\alpha\to 0}\Theta_a(n,\alpha),$$

$$\theta_0(n) = \frac{\rho_n}{\omega_n}, \quad \theta_1(n) = \sigma\,\frac{\rho_n}{\omega_n} + \varepsilon\,\bar\rho_n, \quad \theta_2(n) = \sigma\,\frac{\rho_{n+1}}{\omega_{n+1}} - \varepsilon\,\bar\rho_{n+1},$$

$$\theta_3(n) = \frac{1}{2}\left(\omega_n + \frac{1}{\omega_n}\right)\rho_n + \frac{1}{2}\left(\omega_{n+1} + \frac{1}{\omega_{n+1}}\right)\rho_{n+1} + \sigma\varepsilon\,(\bar\rho_n - \bar\rho_{n+1}),$$

$$\rho_n \equiv \rho_n^+(\mu) + \rho_n^-(\mu), \qquad \bar\rho_n \equiv \rho_n^+(\mu) - \rho_n^-(\mu).$$

Structure functions which include the contributions from both sorts of fermions are denoted as $\Pi_{\rm E}^{\rm tot}(x)$, $\Pi_{\rm CS}^{\rm tot}(x)$ and $\Pi_{\rm M}^{\rm tot}(x)$. For our purposes, we consider the case of $T = 0$ and $\bar\alpha = 0$. If the applied magnetic field $B^{\rm ext}$ does not exceed the critical one, then at $T = 0$ it is completely expelled from the sample ($B^{\rm b} = 0$), and according to (4.55) and (4.56), we have $\nu_1 = \nu_2 = \nu = \nu_0 = N + 1$ where N is any non-negative integer. Consequently, the only non-vanishing terms in (4.51) and (4.52) in the zero temperature limiting case ($\beta = \infty$) are

$$\rho_0^+(\mu_1) = \cdots = \rho_{N-1}^+(\mu_1) = \rho_N^+(\mu_1) = 1,$$

$$\rho_1^+(\mu_2) = \cdots = \rho_N^+(\mu_2) = \rho_{N+1}^+(\mu_2) = 1$$

while all others, including the antiparticle contributions vanish exactly. These values of the Fermi distribution functions lead to the following zero temperature expressions

$$\Pi_{\rm E}^{\rm tot}(x) = \frac{1}{2\pi}\sum_{n=0}^{N}\left\{-4x + 3x^2(2n+1) - \frac{x^3}{9}(30n^2+30n+11)\right\}, \quad \text{(D.1)}$$

$$\Pi_{\rm CS}^{\rm tot}(x) = \frac{1}{\pi}\sum_{n=0}^{N}\left\{1 - \frac{3x}{2}(2n+1) + \frac{x^2}{12}(30n^2+30n+11)\right\} + \\ + \frac{x^3}{72\pi}\sum_{n=0}^{N}(70n^3+105n^2+85n+25), \quad \text{(D.2)}$$

$$\Pi_{\rm M}^{\rm tot}(x) = \frac{1}{\pi}\sum_{n=0}^{N}\left\{2n+1 - \frac{3x}{2}(2n^2+2n+1)\right\} + \\ + \frac{x^2}{12\pi}\sum_{n=0}^{N}(30n^3+30n^2+32n+11) + \\ + \frac{x^3}{72\pi}\sum_{n=0}^{N}(35n^4+70n^3+120n^2+85n+25), \quad \text{(D.3)}$$

where the structure functions are expanded up to x^3, i.e., up to $\mathbf{k}^6$ terms.

References

1. E. S. Abers and B. W. Lee, The gauge theories. *Phys. Reports* **9**(1973), 1.

2. Y.Aharonov and D.Bohm, Significance of electromagnetic potential in the quantum theory. *Phys. Rev.*. **115**(1959), 485.

3. N. d'Ambrumenil and R. Morf, Hierarchical classification of fractional quantum Hall states. *Phys. Rev.* **B 40**(1989), 6108.

4. A. Anderson, Quantum-canonical transformations. Physical equivalence of quantum theories. *Phys. Lett.* **B 305**(1993), 67.

5. V. I. Arnold, Mathematical methods of classical mechanics. *Springer-Verlag, Berlin,* 1978.

6. D. Arovas, J. R. Schriefer and F. Wilczek, Statistical mechanics of anyons. *Phys. Rev. Lett.* **53**(1984), 722.

7. D. Bak, R. Jackiw, and S.-Y. Pi, Non-Abelian Chern-Simons particles and their quantization. *Phys. Rev.* **B 49**(1994), 6778.

8. I. Bakas, The large N-limit of extended conformal symmetry. *Phys. Lett.* **B 228**(1989), 57.

9. A. P. Balachandran, E. Ercolessi, G. Morandi, and A. M. Srivastava, The Hubbard model and anyon superconductivity. *World Scientific, Singapore*, 1990.

10. T. Banks and J. D. Lykken, Landau-Ginsburg description of anyonic superconductors. *Nucl. Phys.* **B 336**(1990), 500.

11. C. M. Bender and F. Cooper, Failure of the naive loop expansion for the effective potential in ϕ^4 field theory when there is "broken symmetry. *Nucl. Phys.* **B 224**(1983), 403.

12. F.A.Berezin, The method of second quantization. *Academic Press, New York,* 1966.

13. C. Bernard, Feynman rules for gauge theories at finite temperature. *Phys. Rev.* **D9**(1974), 3312.

14. B. Blok and X. G. Wen, Many-body systems with non-abelian sratistics. *Nucl. Phys.* **B 374**(1992), 625.

15. N. N. Bogolubov and D. V. Shirkov, Introduction to the theory of quantized fields. *New-York–London, Intersci. Publ.,* 1959.

16. N. N. Bogolubov, Quasiaverages in the problems of statistical mechanics. *JINR preprint* D-781, 1961.

17. N. N. Bogolubov, B. N. Struminsky, and A. N. Tavkhelidze, To the problem of composite models in the theory of elementary particles. *JINR preprint* D-1968, 1965.

18. M. Bos and V. P. Nair, Coherent state quantization of Chern-Simons theory. *Int. Journ. of Mod. Phys.* **A 5**(1990), 959.

19. R.H.Brandenberger, Quantum field theory methods and inflationary universe models. *Rev. Mod. Phys.* . **57**(1985), 1.

20. A. Cappelli, C. A. Trugenberger, and G. R. Zemba, Infinite symmetry in the quantum Hall effect. *Nucl. Phys.* **B 396**(1993), 465.

21. A. Cappelli, C. A. Trugenberger, G. V. Dunne, and G. R. Zemba, Conformal symmetry and universal properties of quantum Hall states. *Nucl. Phys.* **B 398**(1993), 531.

22. A. Cappelli, C. A. Trugenberger, and G. R. Zemba, Classification of quantum Hall universality classes by $\mathcal{W}_{1+\infty}$ symmetry. *Phys. Rev. Lett.* **72**(1994), 1902.

23. A. Cappelli, C. A. Trugenberger, and G. R. Zemba, $\mathcal{W}_{1+\infty}$ dynamics of edge excitations in the quantum Hall effect. *preprint* MPI-PH/93-75 (1993).

24. Y. H. Chen, F. Wilczek, E. Witten, and B. I. Halperin, On anyon superconductivity. *Int. Journ. of Mod. Phys.* **B 3**(1989), 1001.

25. S. Coleman and E. Weinberg, Radiative corrections as the origin of spontaneous symmetry breaking. *Phys. Rev.* **D 7**(1973), 1888.

26. S.Coleman, The secret symmetry. *In: "Laws of Hadronic Matter", Erice,* 1973, *ed. A. Zichichi, Academic Press, New York,* 1975.

27. A. Dannenberg, Dysfunctional methods and the effective potential. *Phys. Lett.* **B 202**(1988), 110.

28. P. A. M. Dirac, Quantized singularities in the electromagnetic field. *Proc. R. Soc. London*, **A 133**(1931), 60.

29. P. Dirac, Lectures on quantum mechanics. *Belfer Graduate School of Science, Yeshiva University, New York,* 1964.

30. L. Dolan and R. Jackiw, Symmetry behavior at finite temperature. *Phys. Rev.* **D9**(1974), 3320.

31. N. Dorey and N. E. Mavromatos, QED_3 and two-dimensional superconductivity without parity violation. *Nucl. Phys.* **B 389**(1992), 614.

32. M. Eliashvili, On the breaking of gauge invariance at finite temperatures in SUSY models. *preprint* LAPP-TH-105/84, (1984).

33. M. Eliashvili, On the convexity property of the effective potential. *In: Proc. Int. Seminar "Quarks'86", VNU Sci. Press, Utrecht,* 1987.

34. M. Eliashvili, On the construction of convex effective potential. *preprint* LAPP-TH-205/87, (1987).

35. M. Eliashvili, On the construction of convex effective potential in the quantum field theory with a spontaneously broken symmetry. *In: Proc. Int. Conf. on The Modern Problems in Mathematical Physics, Tbilisi,* 1987.

36. M. Eliashvili, On the construction of convex effective potential at zero and finite temperatures. *In: Proc. Int. Seminar "Quarks'88", World Scientific, Singapore,* 1989.

37. M. Eliashvili, On the temporary symmetry breaking in the supersymmetric theory. *Bull. Georg. Acad. Sci.* **126**(1987), 521.

38. M. Eliashvili, On the breaking of gauge invariance at finite temperatures in the model of supersymmetric electrodynamics. *Bull. Georg. Acad. Sci.* **126**(1987), 62.

39. M. Eliashvili, On symmetry breaking at finite temperatures. *Bull. Georg. Acad. Sci.* **129**(1988), 530.

40. M. Eliashvili and G. Tsitsishvili, On the current correlators for planar fermions in magnetic field. *Mod. Phys. Lett.* **A 40**(1993), 3807.

41. M. Eliashvili and G. Tsitsishvili, On the anyon superconductivity in thermo field dynamics. *Bull. Georg. Acad. Sci.* **147**(1993), 24.

42. M. Eliashvili, The infinite symmetry and interplay between integer and fractional quantum Hall effect. *preprint* ENSLAPP-A-462/94, hep-th/9404070, (1994).

43. M. Eliashvili, $\mathcal{W}_{1+\infty}$, similarity transformation and interplay between integer and fractional quantum Hall effect. *preprint* ENSLAPP-A-466/94, hep-th/9404090, (1994).

44. M. Eliashvili, On the composite fermion approach in the QHE. *preprint* ENSLAPP-A-478/94, HEP-TH/9407126 (1994).

45. M. Eliashvili and G. Tsitsishvili, On the Meissner effect in the relativistic anyon superconductors. *preprint* DTP-TMI-007/95, hep-th/9510214 (1995).

46. M. Eliashvili, On the holomorphic gauge quantization of the Chern-Simons gauge theory and Laughlin wave functions. *preprint* ZU-TH-15/96, hep-th/9605160 (1996).

47. S. Elitzur, G. Moore, A. Schwimmer, and N. Seiberg, Remarks on the canonical quantization of the Chern-Simons-Witten theory. *Nucl. Phys.* **B 326**(1989), 108.

48. J. Ellis, N. E. Mavromatos, and D. V. Nanopoulos, A non-critical string approach to black-holes, time and quantum dynamics. *Preprints* ENSLAPP-A-463/94 and CERN-TH.7195/94 (1994).

49. Z. F. Ezawa, M. Hotta, and A. Iwazaki, Field theory of anyons and fractional quantum Hall effect. *Preprint* TU-376, (1991).

50. Z. F. Ezawa and A. Iwazaki, Landau-Ginsburg theory of anyon superconductivity and the fractional quantum Hall effect. *Phys. Rev.* **B 47**(1993), 7295.

51. P. Fayet and S. Ferrara, Supersymmetry. *Phys. Reports* **C32**(1977).

52. A. L. Fetter and J. D. Walecka, Quantum theory of many particle systems. *McGraw-Hill,* 1971.

53. A. L. Fetter, C. B. Hanna, and R. B. Laughlin, Random-phase approximation in the fractional-statistics gas. *Phys. Rev.* **B 39**(1989), 9679.

54. M. Flohr and R. Varnhagen, Infinite symmetry in the fractional quantum Hall effect. *Journ. of Phys.* **A 27**(1994), 3999.

55. S. Forte, Quantum mechanics and field theory with fractional spin and statistics. *Rev. Mod. Phys.* . **64**(1992), 193.

56. J. Fröhlich and T. Kerler, Universality in quantum Hall system. *Nucl. Phys.* **B 354**(1991), 369.

57. J. Fröhlich and A. Zee, Large scale physics of the quantum Hall fluid. *Nucl. Phys.* **B 364**(1991), 517.

58. S. Fubini, Vertex operators and quantum Hall effect. *Mod. Phys. Lett.* **A A6**(1991), 347.

59. Y. Fujimoto, L. O'Raifeartaigh, and G. Parravicini, Effective potential for non-convex potentials. *Nucl. Phys.* **B 212**(1983), 268.

60. P. de Sousa Gerbert, Anyons and Chern–Simons Lagrangians. *Phys. Rev.* **D 42**(1990), 543.

61. Y. Georgelin, M. Knecht, Y. Leblanc, and J. C. Wallet, Finite temperature linear response theory of anyonic superconductivity. *Mod. Phys. Lett.* **A B5**(1995), 211.

62. L. Girardello, M. Grisaru, and P. Salomonsen, Temperature and supersymmetry. *Nucl. Phys.* **B 178**(1981), 331.

63. L. Girardello and M. Grisaru, Soft breaking of supersymmetry. *Nucl. Phys.* **B 194**(1983), 65.

64. S. M. Girvin, Particle-hole symmetry in the anomalous quantum Hall effect. *Phys. Rev.* **B 29**(1984), 6012.

65. J. Goldstone, Field theories with "superconductor" solutions. *Nuovo Cim.* **19**(1961), 154.

66. O. W. Greenberg, Coupling of internal and space-time symmetries. *Phys. Rev.* **135B**(1964), 1447.

67. E. Guadanini, The universal link-polynomials. *preprint* CERN-TH-5926/90 (1990).

68. E. Guadagnini, M. Martellini, and M. Mintchev, Braids and quantum group symmetry in Chern–Simons theory. *Nucl. Phys.* **B 336**(1990), 581.

69. J. F. Gunion, H. E. Haber, G. L. Kane, and S. Dawson, The Higgs hunters guide. *Addison-Wesley, Menlo Park,* 1989.

70. G. S. Guralnik, C. R. Hagen, and T. W. Kibble, Broken symmetries and the Goldstone theorem. *In: Advances in Particle Physics* (*ed. R. Cool and R. Marshak, v.* 2*, New York; Interscience,* 1968).

71. B. I. Halperin, Double layer systems. *Helv. Phys. Acta.* **56**(1983), 75.

72. B. I. Halperin, Quantized Hall conductance, current currying edge states and the existence of extended states in a two-dimensional disordered potential. *Phys. Rev.* **B 25**(1982), 2185.

73. B. I. Halperin, Statistics of quasiparticles and the hierarchy of fractional quintized Hall states. *Phys. Rev. Lett.* **52**(1982), 1583.

74. B. I. Halperin, P. A. Lee, and N. Read, Theory of the half filled Landau level. *Phys. Rev.* **B 47**(1993), 7312.

75. M. Han and Y. Nambu, Three triplet model with a double SU(3) symmetry. *Phys. Rev.* **139B**(1965), 1006.

76. J. E. Hetrick, Y. Hosotani and B. H. Lee, Magnetic and thermal properties of the anyon superconductor. *Ann.Phys.* (*N.Y*) **209**(1991), 151.

77. Y. Hosotani and S. Chakravarty, Superconductivity in the anyon model. *Phys. Rev.* **B 42**(1990), 342.

78. Y. Hosotani, Neutral and charged anyon fluid. *preprint* UMN-TH-1106/92, cond-matt/9302002 (1993).

79. J. Iliopoulos, C. Itzikson, and A. Martin, Functional methods and perturbation theory. *Rev. Mod. Phys.* **47**(1973), 165.

80. K. Ishikawa, Axial anomaly in three dimensions and planar fermions. *Phys. Rev.* **D 31**(1985), 1432.

81. S. Iso, D. Karabali, and B. Sakita, Fermions in the lowest Landau level, bosonization, $\mathcal{W}_\infty$ algebra, droplets, chiral bosons. *Phys. Lett.* **B 296**(1992), 143.

82. R. Jackiw, Functional evaluation of the effective potential. *Phys. Rev.* **D 9**(1974), 1686.

83. R. Jackiw, Analysis of infinite-dimensional manyfolds - Schrödinger representation for quantized fields. *preprint* CTP# 1852 , (1989).

84. R. Jackiw, Topics in planar physics. em preprint CTP# 1852, (1989).

85. R. Jackiw and S. Y. Pi, Classical and quantal non-relativistic Chern–Simons theory. *Phys. Rev.* **D 42**(1990), 3500.

86. J. K. Jain, Composite fermion approach to the quantum Hall effect. *Phys. Rev. Lett.* **63**(1989), 199.

87. J. K. Jain, Incompressible quantum Hall states. *Phys. Rev.* **B 40**(1989), 8079.

88. J. K. Jain and V. J. Goldman, Hierarchy of states in the fractional quantum Hall effect. *Phys. Rev.* **B 45**(1992), 1225.

89. J. Kapusta, Finite temperature field theory. *Cambridge Univ. Press, Cambridge,* 1988.

90. J. Kapusta, M. E. Carrington, B. Bayman, D. Seibert, and C. S. Song, Superconducting phase transition in $2D$ Chern–Simons theory. *Phys. Rev.* **B 44**(1991), 7519.

91. D. Karabali, Algebraic aspects of the fractional quantum Hall effect. *Nucl. Phys.* **B 419**(1994), 437.

92. D. Karabali, $\mathcal{W}_\infty$ algebras in the fractional quantum Hall effect. *Nucl. Phys.* **B 419**(1994), 437.

93. D. A. Kirzhnitz and A. D. Linde, Symmetry behaviour in gauge theories. *Ann.Phys.* (*N.Y*) **101**(1976), 195.

94. S. Kivelson, D. H. Lee and S. C. Zhang, Global phase diagram in the quantum Hall effect. *Phys. Rev.* **B 46**(1992), 2223.

95. T. Kohno, Monodromy representations of braid groups and Yang–Baxter equations. *Ann. Inst. Fourier Grenoble* **37. 4**(1987), 139.

96. V. G. Knizhnik and A. B. Zamolodchikov, Current algebra and Wess-Zumino model in two dimensions. *Nucl. Phys.* **B 247**(1984), 83.

97. V. A. Kuzmin, M. E. Shaposhnikov, and I. I. Tkachev, Baryon generation and unusual symmetry behaviour at high temperatures. *Nucl. Phys.* **B 196**(1982), 29.

98. P. Langacker and S.-Y. Pi, Magnetic monopoles in grand unified theories. *Phys. Rev. Lett.* , **45**(1980), 1.

99. R. B. Laughlin, Anomalous quantum Hall effect: An incompressible quantum fluid with fractionally charged excitations. *Phys. Rev. Lett.* **50**(1983), 1395.

100. R. B. Laughlin, Superconducting ground state of noninteracting particles obeying fractional statistics. *Phys. Rev. Lett.* **60**(1988), 2677.

101. R. B. Laughlin, Elementary theory: incompressible quantum fluid. *In: R. A. Prange and S. M. Givrin, "The Quantum Hall Effect", Springer Verlag, Berlin,* 1990.

102. N. P. Landsman and Ch. G. van Weert, Real-and imaginary-time field theory at finite temperature and density. *Phys. Reports* **145**(1987). Nos. 3&4, 141.

103. T. Lee and P. Oh, Non-abelian Chern-Simons quantum mechanics and non-abelian Aharonov-Bohm effect. *Ann.Phys. (N.Y)* **235**(1994), 413.

104. J. M. Leinaas and J. Mirheim, On the theory of identical particles. *Nuovo Cim.* **37B**(1977), 1.

105. A. Lerda, Anyons. *Springer-Verlag, Berlin,* 1992.

106. L.-F. Li, Group theory of the spontaneously broken gauge symmetries. *Phys. Rev.* **D 9**(1974), 1723.

107. A. D. Linde, Phase transitions in gauge theories. *Rep. Prog. Phys.* **42**(1979), 389.

108. A. Lopez and E. Fradkin, Fractional quantum Hall effect and Chern–Simons gauge theories. *Phys. Rev.* **B 44**(1991), 5246.

109. A. Love and S. J. Stow, Symmetry restoration and anti-restoration in supersymmetric theories at finite temperature and chemical potential. *Nucl. Phys.* **B 257**(1984), 271.

110. J. D. Lykken, J. Sonnenschien, and N. Weiss, Field theoretic analysis of anyonic superconductivity, *Phys. Rev.* **B 42**(1990), 2161.

111. J. D. Lykken, J. Sonnenschien, and N. Weiss, The theory of anyonic superconductivity. *Int. Journ. of Mod. Phys.* **A 6**(1991), 1335.

112. A. Masiero, D. Nanopoulos, and T. Yanagida, Symmetry anti-restoration in supersymmetric theory. *Phys. Lett.* **B 338**(1984), 91.

113. H. Matsumoto, Y. Nakano, and H. Umezawa, Free energy in thermo field dynamics. *Phys. Rev.* **D 31**(1985), 1495.

114. G. Moore and N. Read, Nonabelions in the fractional quantum Hall effect. *Nucl. Phys.* **B 360**(1991), 362.

115. S. Norimatsu, K. Yamamoto and A. Tanaka, What is the effective potential. *Phys. Rev.* **D 35**(1987), 2009.

116. P. K. Panigrahi, R. Ray, and B. Sakita, Effective Lagrangian for a system of nonrelativistic fermions in 2+1 dimensions coupled to an electromagnetic field: application to anyonic superconductivity. *Phys. Rev.* **B 42**(1990), 4036.

117. W. Pauli, On the connection between spin and statistics. *Phys. Rev.* **58**(1940), 716.

118. R. A. Prange and S. M. Givrin, The quantum Hall effect. *Springer Verlag, Berlin,* 1990.

119. L. O'Raifeartaigh, A. Wipf, and H. Yoneyama, The constrained effective potential. *Nucl. Phys.* **B B271**(1986), 653.

120. S. Randjbar-Daemi, A. Salam, and J. Strathdee, Chern-Simons superconductivity at finite temperatures. *Nucl. Phys.* **B 340**(1990), 403.

121. N. Read, Order parameter and Ginzburg-Landau theory for the fractional quantum Hall effect. *Phys. Rev. Lett.* **62**(1989), 86.

122. R. J. Rivers, Effective potential convexity and finite temperature phase transition. *Z. Phys. C - Particles and Fields* **22**(1987), 137.

123. R. T. Rockafellar, Convex analysis. *Princeton Univ. Press, Princeton,* 1974.

124. X. Shen, $\mathcal{W}$ infinity and string theory. *Int. Journ. of Mod. Phys.* **A**, **28**(1992), 6953.

125. M. Sher, Electroweak Higgs potentials and vacuum stability. Phys. Reports, Nos. 5 & 6, 273 (1989). ???

126. M.Stone ed., The quantum Hall effect. *World Scientific, Singapore,* 1992.

127. K. Symanzik, Renormalizable models with simple symmetry breaking. *Comm. Math. Phys.* **16**(1970), 48.

128. G. Tsitsishvili and M. Eliashvili, On the thermal properties of relativistic anyon superconductor. *In: Proc. Int. Seminar "Quarks'94", World Scientific, Singapore,* 1995.

129. H. Umezawa, H. Matsumoto, and M. Tachiki, Thermo field dynamics and condensed states. *North-Holland Publ. Comp. Amsterdam,* 1982.

130. E. Verlinde, A note on braid statistics and the non-abelian Aharonov–Bohm effect. *In: Modern Quantum Field Theory, ed. S.Das at al., World Scientific, Singapore,* 1991.

131. S. Weinberg, Gauge and global symmetries at high temperature. *Phys. Rev.* **D9**(1974), 3357.

132. E. Weinberg and A. Wu, Understanding complex perturbative effective potential. *Phys. Rev.* **D 36**(1987), 2474.

133. X.-G. Wen, Theory of the edge states in fractional quantum Hall effect. *Int. Journ. of Mod. Phys.* **B 10**(1992), 1711.

134. D. Wesolowski, Y. Hosotani, and S. Chakravarty, The energy density in the Maxwell–Chern–Simons theory. *preprint* UMN-TH-1245/94 hep-th/9406169 (1994).

135. J. Wess and J. Bagger, Supersymmetry and supergravity. *Princeton Univ. Press, Princeton, NJ,* 1983.

136. U. A. Wiedemann, Non-differentiability of the effective potential. *Nucl. Phys.* **B 406**(1993), 808.

137. F. Wilczek, Magnetic flux, angular momentum, and Statistics. *Phys. Rev. Lett.* **48**(1982), 1144.

138. F. Wilczek, Quantum mechanics of fractional-spin particles. *Phys. Rev. Lett.* **48**(1982), 1144.

139. F. Wilczek, ed. Fractional statistics and anyon superconductivity. *World Scientific, Singapore,* 1990.

140. E. Witten, Quantization of Chern-Simons gauge theory with complex gauge group. *Journ. Math. Phys.* **137**(1991), 29.

141. S. C. Zhang, T. Hansson, and S. Kivelson, Effective-field-theory model for the fractional quantum Hall effect. *Phys. Rev. Lett.* **62**(1989), 82.

142. S. C. Zhang, The Chern-Simons-Landau-Ginzburg theory of the fractional quantum Hall effect, *Int. Journ. of Mod. Phys.* **B 6**(1992), 25.

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