ON MATRIX NEAR-RINGS

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Abstract: This paper extends the work on matrix near-rings $\mathcal{M}_n(R)$, the near-rings of $n \times n$ matrices over right near-rings with identity [3]. Our main aim is to investigate matrix near-rings constructed over right near-rings, not necessarily with identity. We show many similarities to the ring case. It is of interest that one can find some striking contrasts as well. For example, unlike the ring case, not all ideals of $\mathcal{M}_n(R)$ are full (Theorem 2.13), which solves a problem posed in [3].

Introduction

Since the construction of matrix near-rings over arbitrary near-rings by using a functional view of matrices [3], a number of very satisfying structural results have been obtained ([1], [4], [5], [6], [7]). This encourages one to believe that matrix near-rings will play a very important role in the theory of near-rings similar to the role played by matrix rings in ring theory.

This work is divided into two sections. In Section 1 we deal with

matrix near-rings constructed over near-rings which need not have an identity element.

It turns out that the existence of an identity element in the base near-ring R is an important condition (Theorem 1.19). Thus, like other researchers in matrix near-ring theory, in Section 2 we will be studying matrix near-rings $\mathcal{M}_n(R)$, where R is a near-ring with identity.

1. In Meldrum and Van der Walt [3], matrix near-rings $\mathcal{M}_n(R)$ are defined over near-rings with identity, and without identity, separately. We use their first definition for arbitrary near-rings, not necessarily with identity.

Let R be a right near-ring and $n \in \mathbb{N}$, the set of all natural numbers. The direct sum of n copies of the group $(R, +)$ is denoted by R^n. The elements of R^n are thought of as column vectors, but for typographical reasons we write then in transposed form with pointed brackets. We define elementary matrices

$$f_{ij}^r : R^n \to R^n$$

by

$$f_{ij}^r = \iota_i f^r \pi_j, \text{ for } r \in R, 1 \le i, j \le n$$

where ι_j and π_j are jth coordinate injection and projection functions and

$$f^r(s) = rs \text{ for all } s \in R.$$

For typographical reasons, we use the symbol $[r; i, j]$ for f_{ij}^r.

Definition 1.1. The *near-ring of $n \times n$ matrices over R*, denoted by $\mathcal{M}_n(R)$ is the subnear-ring of $M(R^n)$, the near-ring of all maps from R^n to itself, generated by the set $\{[r; i, j] : r \in R, 1 \le i, j \le n\}$.

We emphasise that R need not have an identity in this definition. We wish to carry over the additive laws of $\mathcal{M}_n(R)$ to R.

Definition 1.2. [6] An R-module G is called a *connected R-module* if for any g_1, g_2 in G, there are g in G and x, y in R such that $g_1 = xg$ and $g_2 = yg$.

Lemma 1.3. *Let G be a connected R-module. If $(R, +) \in V$, a variety of additive groups, then $G \in V$.*

Proof. Let $w(x_1, \ldots, x_p)$ be a law of V. If $g_1, \ldots, g_p \in G$ then there exists g in G and $x_1, \ldots, x_p$ in R such that $g_1 = r_1 g, \ldots, g_p = r_p g$, by 3.2 of [6]. Now $w(g_1, \ldots, g_p) = 0_G$ by 12.9 of [2], the hypothesis and

2.12 of [2]. So the law $w(x_1, \ldots, x_p)$ holds in G. This is true for all the laws of V. Hence $G \in V$. $\Diamond$

Theorem 1.4. *Let $(R, +)$ be a connected R-module. Then $(R, +) \in V$ if and only if $(\mathcal{M}_n(R), +) \in V$.*

Proof. The necessary condition follows by Lemma 8 of [1] and the converse follows from 3.3 of [6] and Lemma 1.3. $\Diamond$

The following are some immediate consequences of this result.

Corollary 1.5. *Let $(R, +)$ be a connected R-module. Then $(R, +)$ is in V if and only if $(\mathcal{M}_n(R), +)$ is in V, where V is one of abelian, nilpotent or soluble.* $\Diamond$

We have analogous results to Lemma 1.3 and to Corollary 1.5 for a monogenic R-module as every monogenic R-module is connected.

To extend Theorem 9 of [1], we first state a rewording of 12.9, [2].

Lemma 1.6. *Let $w(v_1, \ldots, v_p)$ be a word in p variables $v_1, \ldots, v_p$. Then $w(x_1, \ldots, x_p)\alpha = w(x_1\alpha, \ldots, x_p\alpha)$ where $x_1, \ldots, x_p \in \mathcal{M}_n(R)$ and $\alpha \in R^n$.* $\Diamond$

We remind the reader here that if I is an ideal of R, then I^+ is the ideal of $\mathcal{M}_n(R)$ generated by $\{[a; i, j]; \ a \in I, \ 1 \le i, \ j \le n\}$ and $I^* := \{X \in \mathcal{M}_n(R); \ X\alpha \in I \text{ for all } x \in R^n\}$ is also an ideal of $\mathcal{M}_n(R)$. Also, if J is an ideal of $\mathcal{M}_n(R)$, then $J_* := \{a \in R; \ a = \pi_j X\alpha, \text{ for some } j, 1 \le j \le n, \ X \in J, \ \alpha \in R^n\}$ is an ideal of R. These results and definitions come from [3].

Theorem 1.7. *Let R be a near-ring and I an ideal of R. If $(I, +) \in V$, then $(I^*, +) \in V$.*

Proof. Exactly the same method of proof as for Theorem 1.3, and Lemma 1.6 enable us to get this result. $\Diamond$

We shall show in Theorem 2.4. that in the case of near-rings with identity the converse of the above statement also holds.

Lemma 1.8. $w([a_1; i, j]), \ldots, [a_p; i, j]) = [w(a_1, \ldots, a_p); \ i, \ j]$ *where $a_1, \ldots, a_p \in R$ and $1 \le i, \ j \le n$.*

Proof. By using induction on the length q of the word $w(v_1, \ldots, v_p)$, and 3.1 (1) of [3], we get what we want. $\Diamond$

Theorem 1.9. *Let R be a near-ring and I be an ideal of R. If $(I, +) \in \in V$, then $(I^+, +) \in V$.*

Proof. Immediate from Proposition 1 of [7] and Theorem 1.7. $\Diamond$

Some immediate consequences of Theorems 1.7 and 1.9 are as follows.

Corollary 1.10. *Let R be a near-ring and I be an ideal of R. If $(I, +)$*

is in V, then $(I^+, +)$ and $(I^*, +)$ are in V where V is one of abelian, nilpotent or soluble. $\Diamond$

The next result we aim to prove is

Theorem 1.11. *If R is distributive over I, then $\mathcal{M}_n(R)$ is distributive over I^*.*

The proof will be based on the following lemmas.

Lemma 1.12. *If R is distributive over I, then*

$$xa + yb = yb + xa$$

where $x, y \in R$ and $a, b \in I$.

Proof. Expand $(x + y)(a + b)$, first using the hypothesis, then the right distributivity of R and vice versa. $\Diamond$

Lemma 1.13. *If R is distributive over I, then*

$$[x; i, j]\alpha + [y; k, l]\beta = [y; k, l]\beta + [x; i, j]\alpha$$

where $x, y \in R$, $\alpha, \beta \in R^n$ and $1 \leq i, j, k, l \leq n$.

Proof. Follows from 3.1 of [2], [3], Lemma 1.12 and simple calculation. $\Diamond$

Lemma 1.14. *Let R be a zero-symmetric near-ring. If R is distributive over I, then*

$$X\alpha + Y\beta = Y\beta + X\alpha$$

where $X, Y \in \mathcal{M}_n(R)$ and $\alpha, \beta \in I^n$.

Proof. Follows by induction on $w(X) + w(Y)$, Lemma 1.13 and 2.16 of [2], 3.2 of [3] and 4.1 of [3]. $\Diamond$

Lemma 1.15. *If R is distributive over I, then*

$$[x; i, j](\alpha + \beta) = [x; i, j]\alpha + [x; i, j]\beta$$

where $x \in R$, $\alpha, \beta \in I^n$ and $1 \leq i, j \leq n$.

Proof. Simple calculation. $\Diamond$

Lemma 1.16. *If R is a zero-symmetric near-ring and R is distributive over I, then $\mathcal{M}_n(R)$ is distributive over I^n.*

Proof. Let $X \in \mathcal{M}_n(R)$ and $\alpha, \beta \in I^n$. By using induction on the weight $w(X)$ of X, Lemmas 1.15, 1.14, 2.16 of [2] and 4.1 of [3], we can show that

$$X(\alpha + \beta) = X\alpha + X\beta. \ \Diamond$$

We are now able to prove Theorem 1.11.

Proof of Theorem 1.11. Let $X \in \mathcal{M}_n(R)$, $A, B \in I^*$, $\alpha \in R^n$. Then $X(A + B)\alpha = X(A\alpha + B\alpha)$. Now the rest of the proof follows immediately from Lemma 1.16 and the definition of I^*. $\Diamond$

Theorem 1.17. *Let R be a zero-symmetric near-ring. If R is distributive over I, then $\mathcal{M}_n(R)$ is distributive over I.*

Proof. Immediate from Proposition 1 of [7] and Theorem 1.11. $\Diamond$

Corollary 1.18. *If R is distributive then $\mathcal{M}_n(R)$ is distributive.* $\Diamond$

We conclude this section with a result which shows the convenience of considering the case in which R has an identity element. Recall that if R is a near-ring with identity then the map $I \to I^*$ is an injection.

Theorem 1.19. *The map $I \to I^*$ need not be an injection, in general.*

Proof. Let R be a zero-symmetric near-ring without identity and I be a non-trivial proper ideal of R such that $xy \in I$ for all x, y in R; we aim to show that $R^* = I^*$. Let $X \in \mathcal{M}_n(R)$. By using induction on the weight $w(X)$ of X, and 2.16 of [2], 3.2 and 4.1 of [3], we can show that $X \in I^*$. $\Diamond$

2. We start this section with a couple of results which show the similarities to the ring case. R is, henceforth, a near-ring with identity.

Theorem 2.1. *Let R be a zero-symmetric near-ring. If $n > 1$, then $\mathcal{M}_n(R)$ cannot be integral.*

Proof. Assume the result to be false and choose two non-zero elements, say x and y, of R. The hypothesis and 3.1 (3) of [3] imply that $[x; i, j][y; k, l] = 0$ if $j \neq k$. Therefore $[x; i, j] = 0$ or $[y; k, l] = 0$. Hence $x = 0$ or $y = 0$. This is a contradiction. $\Diamond$

Theorem 2.2. *A sum of distinct matrix units E_{kk}, $1 \leq k \leq n$, is an idempotent in $\mathcal{M}_n(R)$.*

Proof. E_{ii} for $i = 1, 2, \ldots, n$ is an idempotent, by 3.1 (3) of [3]. By right distributivity in $\mathcal{M}_n(R)$ and 3.1 (5) of [3], it can be seen easily that

$$(E_{11} + \ldots + E_{ll})(E_{11} + \ldots + E_{ll}) = E_{11} + \ldots + E_{ll}.$$

This completes the proof. $\Diamond$

Corollary 2.3. *If $n > 1$, then $\mathcal{M}_n(R)$ cannot be a local near-ring.* $\Diamond$

Our next result takes Theorem 1.7 further.

Theorem 2.4. *$(I, +) \in V$ if and only if $(I^+, +) \in V$.*

Proof. We use a technique similar to that of the proof of Lemma 1.3.

Proposition 1 of [7], the definition of I^+, the hypothesis, and Lemma 1.8 give us the desired result. $\Diamond$

Theorem 2.5. $(I,+) \in V$ *if and only if* $(I^*,+) \in V$.

Proof. Only the converse needs a proof which is exactly the same as that of the above result. $\Diamond$

Corollary 2.6. $(I,+)$ *is in* V *if and only if* $(I^*,+)$ *is in* V, *if and only if* $(I^+,+)$ *is in* V, *where* V *is one of abelian, nilpotent or soluble.* $\Diamond$

R is, henceforth, assumed to be a zero-symmetric near-ring.

Theorem 2.7. *If* J *is an ideal of* $\mathcal{M}_n(R)$, *then* R *is distributive over* J_* *if and only if* $\mathcal{M}_n(R)$ *is distributive over* J.

Proof. Let $X \in \mathcal{M}_n(R)$ and $A, B \in J$. By Proposition 3 of [7], 4.6 of [3] and Theorem 1.11, we get $X(A + B) = XA + XB$. To prove the sufficiency, let $x \in R$, $a, b \in J_*$. By 4.5 of [3], 3.1 of [3] and the hypothesis, we get $[x(a + b); 1, 1] = [xa + xb; 1, 1]$. Hence $x(a + b) = xa + xb$. $\Diamond$

Exactly the same method of proof as for the sufficiency of the condition of Theorem 2.7 enables us to show the converse of Theorems 1.11 and 1.17.

Theorem 2.8. R *is distributive over* I *if and only if* $\mathcal{M}_n(R)$ *is distributive over* I^*. $\Diamond$

Theorem 2.9. R *is distributive over* I *if and only if* $\mathcal{M}_n(R)$ *is distributive over* I^+. $\Diamond$

To end, we answer the question posed in [3]: Does, in general, $\mathcal{M}_n(R)$ possess ideals which are not full?

First we establish the following lemmas.

Lemma 2.10. *If* $I^+ = I^*$ *for any ideal* I *of* R *then all ideals of* $\mathcal{M}_n(R)$ *are full.*

Proof. If J is an ideal of $\mathcal{M}_n(R)$, it can be seen easily that $J = (J_*)^*$ (by 4.6 of [3], the hypothesis and Proposition 3 of [7]). $\Diamond$

Furthermore

Lemma 2.11. $I^+ = I^*$ *for each ideal* I *of* R *if and only if all ideals of* $\mathcal{M}_n(R)$ *are full.*

Proof. For sufficiency, take an ideal I^+ of $\mathcal{M}_n(R)$. $I^+ = L^*$ for some ideal L of R. We aim to show that $I = L$. Let $a \in L$, then $[a; 1, 1] \in I^+ = L^*$, therefore $a \in L$. Now Proposition 1 of [7], the hypothesis and Proposition 2 of [7] imply $L \subseteq I$. This completes the proof. $\Diamond$

Lemma 2.12. *If there exists an ideal I of R such that $I^+ \neq I^*$ then not all ideals of $\mathcal{M}_n(R)$ are full.*

Proof. Assume that all ideals of $\mathcal{M}_n(R)$ are full. Then $I^+ = I^*$ for each ideal I of R by Lemma 2.11. This contradicts the hypothesis. $\Diamond$

Theorem 2.13. *In general $\mathcal{M}_n(R)$ possesses ideals which are not full.*

Proof. Immediate from Lemma 2.12 and Example 4 of [7]. $\Diamond$

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