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ON QUASI-ANTIORDER IN SEMIGROUPS

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Abstract. Partially ordered semigroups with apartness under an antiorder are investigated from the point of view of Bishop's constructive mathematics. We analyze quasi-antiorder relations on ordered semigroups under an antiorder. The connection between two quasi-antiorders on a semigroup is presented.

1. Introduction

The main goal of this paper is to provide a constructive definition of quasi-antiorder for an arbitrary ordered semigroup with apartness under an antiorder. Our setting is Bishop's constructive mathematics [2–4, 6, 12, 13], mathematics developed with constructive logic (or intuitionistic logic [23])—logic without the Law of the Excluded Middle $P \vee \neg P$. We have to note that 'the crazy axiom' $\neg P \implies (P \implies Q)$ is included in constructive logic. (Precisely in constructive logic, the 'Double Negation Law' $P \iff \neg\neg P$ does not hold, but the following implication $P \implies \neg\neg P$ holds even in minimal logic. In constructive logic, the Weak Law of the Excluded Middle $\neg P \vee \neg\neg P$ does not hold. It is interesting, in constructive logic the following deduction principle $A \vee B, \neg A \vdash B$ holds, but this is impossible to prove without 'the crazy axiom'). One advantage of working in this manner is that proofs and results have more interpretations. On the one hand, Bishop's constructive mathematics is consistent with traditional mathematics. On the other hand, the results can be interpreted recursively or intuitively ([2, 6, 23]). If we are working constructively, the first problem is to obtain appropriate substitutes for the classical definitions. The classical theory of partially ordered sets is based on the negative concept of partial order. Unlike the classical case, an affirmative concept, introduced in the author's papers [14, 17, 19–21] and similar to von Plato's [16] and Baroni's [1] excess relation, will be used as a primary relation.

This investigation is in Bishop's constructive algebra in the sense of [9, 12–14, 17–20, 22] and [23] (Chapter 8: Algebra). Let $(S, =, \neq)$ be a constructive set in

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the sense of Mines [12], Mulvey [13], Ruitenburg [22], and Troelstra and van Dalen [23]. The relation $\neq$ is a binary relation on S with the following properties:

$$\neg(x \neq x), \quad x \neq y \Longrightarrow y \neq x, \quad x \neq z \Longrightarrow x \neq y \vee y \neq z,$$
$$x \neq y \wedge y = z \Longrightarrow x \neq z.$$

It is called *apartness* (Heyting). Let S and T be two sets with apartness, then the relation $\neq$ on $S \times T$ is defined by

$$(x, y) \neq (u, v) \Longleftrightarrow (x \neq u \vee y \neq v)$$

for any $x, u \in S$ and any $y, v \in T$.

A relation q on S is a *coequality relation* on S if and only if it is consistent with the apartness, symmetric and cotransitive [14, 17–19]:

$$q \subseteq \neq, \quad q^{-1} = q, \quad (\forall x, y, z \in S)((x, z) \in q \Longrightarrow (x, y) \in q \vee (y, z) \in q).$$

Let $(S, =, \neq, \cdot)$ be a semigroup with an apartness. Here the semigroup operation '$\cdot$' has to be extensional and strongly extensional in the following sense

$$(\forall x, y, u \in S)((x = y \Longrightarrow (xu = yu \wedge ux = uy)),$$
$$(\forall x, y, u \in S)((xu \neq yu \vee ux \neq uy) \Longrightarrow x \neq y).$$

As in [19], a relation q on S is an *anti-congruence* ('cocongruence' in [14, 17] if and only if it is a coequality relation on S compatible with the semigroup operation in the following sense:

$$(\forall x, y, z \in S)(((xz, yz) \in q \Longrightarrow (x, y) \in q) \wedge ((zx, zy) \in q \Longrightarrow (x, y) \in q)).$$

A. We will briefly recall the constructive definition of linear order and we will use a generalization of von Plato's [16] and Baroni's [1] excess relation for the definition of a partially ordered set. Let S be a nonempty set. A binary relation $<$ (less than) on S is called a *linear order* if the following axioms are satisfied for all elements x and y:

$$\neg(x < y \wedge y < x),$$
$$x < y \Longrightarrow (\forall z \in S)(x < z \vee z < y).$$

An example is the standard strict order relation $<$ on $\mathbf{R}$, as described in [3]. For an axiomatic definition of the real number line as a constructive ordered field, the reader is referred to [3, 4, 6, 12]. A detailed investigation of linear orders in lattices can be found in [6]. The binary relation $\nleqslant$ on S is called an *excess relation* if it satisfies the following axioms:

$$\neg(x \nleqslant x),$$
$$x \nleqslant y \Longrightarrow (\forall z \in S)(x \nleqslant z \vee z \nleqslant y).$$

We say that x exceeds y whenever $x \nleqslant y$. Clearly, each linear order is an excess relation. As shown in [16], we obtain an apartness relation $\neq$ and a partial order $\leqslant$ on X by the following definitions:

$$x \neq y \Longleftrightarrow (x \nleqslant y \vee y \nleqslant x),$$
$$x \leqslant y \Longleftrightarrow \neg(x \nleqslant y).$$

Note that the statement $\neg(x \leqslant y) \Longrightarrow x \nleqslant y$ does not hold in general.

As in [20], we define our notion of an antiorder: a relation α on a semigroup $((S,=,\neq),\cdot)$ is an *anti-order* on S if and only if

$$\alpha \subseteq \neq,$$
$$(\forall x,y,z\in S)((x,z)\in\alpha \Longrightarrow ((x,y)\in\alpha \vee (y,z)\in\alpha)),$$
$$(\forall x,y\in S)(x\neq y \Longrightarrow ((x,y)\in\alpha \vee (y,x)\in\alpha), \text{(linearity) and}$$
$$(\forall x,y,z\in S)(((xz,yz)\in\alpha \Longrightarrow (x,y)\in\alpha) \wedge ((zx,zy)\in\alpha \Longrightarrow (x,y)\in\alpha)).$$

B. Let S be a semigroup with apartness [9, 13, 14]. A relation ρ on S is a *quasi-order* [5, 8] if

$$\Delta_S \subseteq \rho, \quad \rho\circ\rho \subseteq \rho.$$

where the operation '$\circ$' is the standard composition of relations. If a quasi-order ρ is compatible with the semigroup operation on S in the sense that $(a,b)\in\rho$ implies $(ac,bc)\in\rho$ and $(ca,cb)\in\rho$ for each $a,b,c\in S$, then the relation C on S, defined by $C=\rho\cap\rho^{-1}$, is a congruence on S [5, 8]. In [10] and [11] Kehayopulu and Tsingelis developed a theory of pseudo-orders (called a 'quasi-order' in [5] and [8]) in ordered semigroup. The constructive notion of a quasi-antiorder relation is the parallel notion to the classical notion of a quasi-order relation. Let $(S,=,\neq,\cdot)$ be a semigroup with apartness. A relation σ on S is a *quasi-antiorder* [14, 17–21] on S if

$$\sigma \subseteq \neq,$$
$$(\forall x,y,z\in S)((x,z)\in\sigma \Longrightarrow ((x,y)\in\sigma \vee (y,z)\in\sigma)),$$
$$(\forall x,y,z\in S)(((xz,yz)\in\sigma \Longrightarrow (x,y)\in\sigma)\wedge((zx,zy)\in\sigma \Longrightarrow (x,y)\in\sigma)).$$

In this paper and some other papers (for example, in [20, 21]) we try to research the properties of quasi-antiorders.

C. Let x be an element of S and A a subset of S. We write $x\bowtie A$ iff $(\forall a\in A)(x\neq a)$, and $A^C=\{x\in S : x\bowtie A\}$. If σ is a quasi-antiorder on S, then the relation $q=\sigma\cup\sigma^{-1}$ is an anti-congruence on S. As to the first, the relation $q^C=\{(x,y)\in S\times S : (x,y)\bowtie q=\sigma\cup\sigma^{-1}\}$ is a congruence on S compatible with q, in the following sense

$$q^C\circ q \subseteq q \wedge q\circ q^C \subseteq q$$

[19, Theorem 1].

For a homomorphism $f:(S,=,\neq)\longrightarrow(T,=,\neq)$ between two semigroups we say that it is a *strongly extensional homomorphism* if and only if

$$(\forall a,b\in S)(f(a)\neq f(b)\Longrightarrow a\neq b).$$

In this article we give some new characteristics of quasi-antiorder relations on semigroups. The new results in this article are one of the answers to the question: 'What kind of connection exists between two quasi-antiorder relations σ and ϱ if $\sigma \subseteq \varrho$?' These results are given in Theorem 3.1 (on the existence of a quasi-antiorder on a semigroup S/q), Theorem 3.2, Theorem 3.3, Theorem 3.4 (the Decomposition Theorem), and Theorem 3.5 (on the existence of the quasi-antiorder relation σ/ϱ).

2. Preliminaries

Our first proposition gives us an explanation of what kind of relation is a complement of an antiorder relation.

LEMMA 2.1 *Let α be an anti-order relation on the semigroup $(S, =, \neq, \cdot)$. Then the relation α^C is a partial order relation on $(S, \neg \neq, \neq, \cdot)$. If the apartness $\neq$ is tight, then α^C is a partial order relation on the semigroup S.*

Proof. (1) Let (u, v) be an arbitrary element of α and let x be an element of S. Then, from $(u, x) \in \alpha \vee (x, v) \in \alpha$ it follows that $u \neq x \vee x \neq v$, i.e., $(u, v) \neq (x, x)$. So, the relation α^C is reflexive.

(2) Let $(x, y) \in \alpha^C$ and $(y, x) \in \alpha^C$ and suppose that $x \neq y$. Then by linearity of α, we have $(x, y) \in \alpha$ or $(y, x) \in \alpha$, which is impossible. So, we must have $\neg(x \neq y)$ and $x = y$ if the relation is tight.

(3) Now, we suppose that $(x, y) \in \alpha^C$ and $(y, z) \in \alpha^C$ and let (u, v) be an arbitrary element of α . Then, by cotransitivity of α, from $(u, x) \in \alpha$ or $(x, y) \in \alpha$ or $(y, z) \in \alpha$ or $(z, v) \in \alpha$ we have $(u, x) \in \alpha$ or $(z, v) \in \alpha$ because $(x, y) \in \alpha^C$ and $(y, z) \in \alpha^C$. Therefore, $u \neq x$ or $z \neq v$. So, $(x, z) \neq (u, v) \in \alpha$.

(4) Let a, b, x, y be elements of S and let $(x, y) \in \alpha^C$ and let (u, v) be an arbitrary element of α. Then from $(u, axb) \in \alpha$ or $(axb, ayb) \in \alpha$ or $(ayb, v) \in \alpha$ we conclude $u \neq axb$ or $ayb \neq v$ because from $(axb, ayb) \in \alpha$ we would have $(x, y) \in \alpha$, which is impossible. So, $(axb, ayb) \neq (u, v) \in \alpha$. ■

Similarly, in the next sentences we will try to make clearer the notion of anti-congruence to the reader: let q be an anti-congruence on S. Then the relation q^C—the strong complement of q—is a congruence on S compatible with q [19, Theorem 1] and we can construct the semigroup $S/(q^C, q) = \{aq^C : a \in S\}$, where $aq^C = \{u \in S : (a, u) \in q\}$, with

$$aq^C = bq^C \Longleftrightarrow (a, b) \bowtie q, \quad aq^C \neq bq^C \Longleftrightarrow (a, b) \in q,$$
$$aq^C \cdot bq^C = (ab)q^C$$

([14, Corollary 1.1.; Theorem 2], [19, Theorem 2]) and the semigroup $S/q = \{aq : a \in S\}$, where $aq = \{u \in S : (a, u) \in q\}$, with

$$aq = bq \Longleftrightarrow (a, b) \bowtie q, \quad aq \neq bq \Longleftrightarrow (a, b) \in q,$$
$$aq \cdot bq = (ab)q$$

[19, Theorem 3]. Besides, by Corollary 3.0 in [19], there exists an isomorphism $S/(q^C, q) \cong S/q$. At the end of this comment let us note that $q^C = \neg q$.

It is well known that any epimorphism $f : S \longrightarrow T$ of semigroups—without order—is completely determined by the congruence $\varphi = f \circ f^{-1}$. Two isomorphism theorems of semigroups based on congruences, a homomorphism theorem of semi-groups based on congruences have been given in [5, 8], respectively, and they are frequently used. In the case of ordered semigroups, quasi-orders in the sense of [4] and [8] play the role of congruences. Here we study some theorems from [5] and [8] for anti-ordered semigroups. As mentioned above, if σ is a quasi-antiorder on S, then the relation $q = \sigma \cup \sigma^{-1}$ is an anti-congruence on S, as the first thing. As to the second, the strong complement σ^C of a quasi-antiorder σ has the well known property:

LEMMA 2.2 *If σ is a quasi-antiorder on S, then the relation $\sigma^C = \{(x, y) \in S \times S : (x, y) \bowtie \sigma\}$ is a quasi-order on S.*

Proof. It is clear that σ^C is a reflexive relation.

Let $(x, y) \in \sigma^C$ and $(y, z) \in \sigma^C$ and let (u, v) be an arbitrary element of σ. Then

$$(u, x) \in \sigma \lor (x, y) \in \sigma \lor (y, z) \in \sigma \lor (z, v) \in \sigma.$$

Hence, $u \neq x \lor z \neq v$, i.e., $(u, v) \neq (x, z)$. So, $(x, z) \in \sigma^C$.

Let $(a, b) \in \sigma^C$ and $c \in S$, and let (u, v) be an arbitrary element of σ. Then, from

$$(u, ac) \in \sigma \lor (ac, bc) \in \sigma \lor (bc, v) \in \sigma,$$

there follows $u \neq ac$ or $bc \neq v$ because from $(ac, bc) \in \sigma$ there would follow $(a, b) \in \sigma$, which is impossible. So, $(u, v) \neq (ac, bc)$, i.e., $(ac, bc) \in \sigma^C$. Similarly, we have the implication $(a, b) \in \sigma^C \Longrightarrow (ca, cb) \in \sigma^C$. ■

At end of this section let us note that $\sigma^C = \neg q$. If the apartness is tight, then the relation $\neg\sigma$ is a partial order relation (von Platos approach), and, as in article [1], the relation α is an excess relation on S. So, an anti-order is different from an excess relation but it is not more general; its rather vice-versa. Indeed, given a set endowed with an apartness and an equality, an excess relation [16] is an consistent and cotransitive relation. Taking into account the consistency of apartness, it follows that each quasi-antiorder is an excess relation. There is a distinction between of our approach and van Plato's approach. In articles [16] and [1], van Plato and Barony determine excess relation $\nleq$ on set $(S, =)$ firstly and, after that, the apartness on structure $((S, =), \nleq)$ induce by the following way $\neq = \nleq \sqcup \nleq^{-1}$. Besides, the apartness is tight with the equality relation in the following sense $(\forall a, b \in S)(\neg(a \neq b) \Longrightarrow a = b)$. Here, in this article, we proceed from an assumption that a set with apartness $(S, =, \neq)$ is given in advance where the apartness should not be tight with the equality relation. After that, we introduce another relations with the request that these relations must be extensive by the equality relation and strongly extensive by the apartness.

3. Main results

In this part we will give our main results. Let $(S, =, \neq, \cdot)$ be a quasi-ordered semigroup under the quasi-antiorder σ. In Theorem 3.1 we will give the unique solution of the problem of existence of a quasi-antiorder relation on the semigroup S/q, and in Theorems 3.2 and 3.3 we will describe properties of that relation. Theorem 3.4 describes conditions for the existence of a decomposition of a strongly extensional homomorphism between two anti-ordered semigroups. In Theorem 3.5 we give properties of the quasi-antiorder relation σ/ϱ.

THEOREM 3.1. *Let σ be a quasi-antiorder relation on S, $q = \sigma \cup \sigma^{-1}$, and let $\pi(q) : S \longrightarrow S/(q^C, q)$ be the canonical surjective strongly extensional homomorphism of semigroups. Then there exists a unique relation θ on $S/(q^C, q)$ such that $\pi(q)^{-1} \circ \theta \circ \pi(q) = \sigma$, in which case θ is equal to $\pi(q) \circ \sigma \circ \pi(q)^{-1}$.*

Proof. Suppose that such a relation θ satisfying $\pi(q)^{-1} \circ \theta \circ \pi(q) = \sigma$ exists. Since the function $\pi(q)$ is surjective, this relation is unique. Except that we have

$$\sigma = q^C \circ \sigma \circ q^C = \pi(q)^{-1} \circ \pi(q) \circ \sigma \circ \pi(q)^{-1} \circ \pi(q).$$

Indeed, firstly we have $q^C \circ \sigma \circ q^C \subseteq \sigma$, and secondly, by the reflexivity of q^C, we have that $\Delta_S \subseteq q^C$ implies

$$\sigma = \Delta_S \circ \sigma \circ \Delta_S \subseteq q^C \circ \sigma \circ q^C.$$

Therefore, we have $\sigma = q^C \circ \sigma \circ q^C$. So, if we put

$$\theta = \pi(q) \circ \sigma \circ \pi(q)^{-1},$$

we have that

$$\sigma = \pi(q)^{-1} \circ \theta \circ \pi(q). \quad \blacksquare$$

In the next proposition we will give an explanation of what kind of relation is the relation θ in Theorem 3.1:

THEOREM 3.2. *Let $(S, =, \neq, \cdot)$ be a semigroup with apartness and σ be a quasi-antiorder relation on S. The relation θ on S/q, where $q = \sigma \cup \sigma^{-1}$, defined by $(aq, bq) \in \theta \iff (a, b) \in \sigma$, is a consistent, cotransitive and linear relation on S/q compatible with the semigroup operation on S/q.*

Proof. (i) Let $(aq, bq) \in \theta$, that is $(a, b) \in \sigma$. According to $\sigma \subseteq q$, we have $(a, b) \in q$. So, $aq \neq bq$.

(ii) Let $(aq, cq) \in \theta$, that is $(a, c) \in \sigma$. Thus, $(a, b) \in \sigma$ or $(b, c) \in \sigma$. Finally, we have $(aq, bq) \in \theta$ or $(bq, cq) \in \theta$ which is what it means that θ is a cotransitive relation.

(iii) Let $((axb)q, (ayb)q) \in \theta$, that is $(axb, ayb) \in \sigma$. Hence $(x, y) \in \sigma$, because the relation σ is compatible with the semigroup operation in S. Therefore, $(xq, yq) \in \theta$.

(iv) Let $aq \neq bq$, that is $(a, b) \in q = \sigma \cup \sigma^{-1}$. Then $(a, b) \in \sigma$ or $(b, a) \in \sigma$, i.e., then $(aq, bq) \in \theta$ or $(bq, aq) \in \theta$. Hence θ is linear. $\blacksquare$

The following theorem will give a converse assertion to the above theorem.

THEOREM 3.3. *If $(S, =, \neq, \cdot)$ and $(T, =, \neq, \cdot)$ are semigroups, ϱ a quasi-antiorder on T, and $\varphi : S \longrightarrow T$ a strongly extensional homomorphism, then the relation $\varphi^{-1}(\varrho) = \{(a, b) \in S \times S : (\varphi(a), \varphi(b)) \in \varrho\}$ is a quasi-antiorder on S, the* *relation* $Coker\varphi = \{(a,b) \in S \times S : \varphi(a) \neq \varphi(b)\}$ *is an anti-congruence on* S *compatible with the congruence* $Ker\varphi = \varphi \circ \varphi^{-1}$, *and* $Coker\varphi \supseteq \varphi^{-1}(\varrho) \cup (\varphi^{-1}(\varrho))^{-1}$ *holds. Also, if the relation* ϱ *is linear we have* $Coker\varphi = \varphi^{-1}(\varrho) \cup (\varphi^{-1}(\varrho))^{-1}$.

Proof.

(i) $(a,b) \in \varphi^{-1}(\varrho) \Longleftrightarrow (\varphi(a), \varphi(b)) \in \varrho \subseteq \neq$
$\Longrightarrow \varphi(a) \neq \varphi(b)$
$\Longrightarrow a \neq b$;

(ii) $(a,c) \in \varphi^{-1}(\varrho) \Longleftrightarrow (\varphi(a), \varphi(c)) \in \varrho$
$\Longrightarrow (\forall b \in S)((\varphi(a), \varphi(b)) \in \varrho \vee (\varphi(b), \varphi(c)) \in \varrho)$
$\Longrightarrow (\forall b \in S)((a,b) \in \varphi^{-1}(\varrho) \vee (b,c) \in \varphi^{-1}(\varrho))$;

(iii) $(xay, xby) \in \varphi^{-1}(\varrho) \Longleftrightarrow (\varphi(xay), \varphi(xby)) \in \varrho$
$\Longrightarrow (\varphi(x)\varphi(a)\varphi(y), \varphi(x)\varphi(b)\varphi(y)) \in \varrho$
$\Longrightarrow (\varphi(a), \varphi(b)) \in \varrho$
$\Longleftrightarrow (a,b) \in \varphi^{-1}(\varrho)$;

(iv) Suppose that the relation ϱ is linear. Then we will have
$(a,b) \in Coker\varphi \Longleftrightarrow \varphi(a) \neq \varphi(b)$
$\Longrightarrow (\varphi(a), \varphi(b)) \in \varrho \vee (\varphi(b), \varphi(a)) \in \varrho$
$\Longleftrightarrow (a,b) \in \varphi^{-1}(\varrho) \vee (b,a) \in \varphi^{-1}(\varrho)$. ■

Remark. Let S be a semigroup with apartness. A relation σ on S is a quasi-antiorder on S if and only if there exists an ordered semigroup T under the linear quasi-antiorder ϱ and a strongly extensional homomorphism f of S into T such that $\sigma = \varphi^{-1}(\varrho)$.

Suppose that $(S, =, \neq, \cdot)$ and $(T, =, \neq, \cdot, \varrho)$ are semigroups, where ϱ is a quasi-antiorder on T, and that $\varphi : S \longrightarrow T$ is a strongly extensional homomorphism. In the following proposition we will describe condition for the decomposition of the homomorphism φ.

Theorem 3.4. *Let* $(S, =, \neq, \cdot, \sigma)$ *and* $(T, =, \neq, \cdot, \varrho)$ *be semigroups, where* ϱ *is a linear quasi-antiorder on* T, *and* $\varphi : S \longrightarrow T$ *is a strongly extensional homomorphism. If* σ *is a quasi-antiorder in* S *such that* $\sigma \supseteq \varphi^{-1}(\varrho)$, *and if the apartness on semigroup* T *is tight, then the mapping* $f : S/(\sigma \cup \sigma^{-1}) \longrightarrow T$ *is a strongly extensional homomorphism of semigroups such that* $f \circ \pi(\sigma) = \varphi$. *Conversely, if* σ *is a quasi-antiorder on* S *for which there exists a strongly extensional homomorphism* $f : S/(\sigma \cup \sigma^{-1}) \longrightarrow T$ *such that* $f \circ \pi(\sigma) = \varphi$, *then* $\sigma \supseteq \varphi^{-1}(\varrho)$.

Proof. We will verify first that the mapping $f : S/(\sigma \cup \sigma^{-1}) \longrightarrow T$ defined by $f(aq) = \varphi(a)$, where $q = \sigma \cup \sigma^{-1}$, is a strongly extensional homomorphism of semigroups such that $f \circ \pi(\sigma) = \varphi$.

Let θ be a quasi-antiorder on the semigroup S/q. Then:

(1) If aq and bq are elements of S/q such that $aq = bq$, that is such that $(a,b) \bowtie q$, then $(a,b) \bowtie \sigma$ and $(b,a) \bowtie \sigma$. So, $(a,b) \bowtie \varphi^{-1}(\varrho)$ and $(b,a) \bowtie \varphi^{-1}(\varrho)$. Suppose that $\varphi(a) \neq \varphi(b)$. Then $(\varphi(a), \varphi(b)) \in \varrho$ or $(\varphi(b), \varphi(a)) \in \varrho$, i.e., $(a,b) \in \varphi^{-1}(\varrho)$ or

$(b,a) \in \varphi^{-1}(\varrho)$, which is impossible. So, we have $\neg(\varphi(a) \neq \varphi(b))$ and necessarily $\varphi(a) = \varphi(b)$ because the apartness in T is tight. Hence, we have $f(aq) = f(bq)$.

(2) The relation $f : S/(\sigma \cup \sigma^{-1}) \longrightarrow T$, defined by $f(aq) = \varphi(a)$, is strongly extensional. In fact, we have:

$$\begin{aligned}
f(aq) \neq f(bq) &\Longleftrightarrow \varphi(a) \neq \varphi(b) \\
&\Longrightarrow (\varphi(a), \varphi(b)) \in \varrho \vee (\varphi(b), \varphi(a)) \in \varrho \\
&\Longleftrightarrow (a,b) \in \varphi^{-1}(\varrho) \vee (b,a) \in \varphi^{-1}(\varrho) \\
&\Longrightarrow (a,b) \in \sigma \vee (b,a) \in \sigma \\
&\Longleftrightarrow (aq, bq) \in \theta = \pi(\sigma)^{-1} \vee (bq, aq) \in \theta = \pi(\sigma)^{-1} \\
&\Longrightarrow aq \neq bq.
\end{aligned}$$

(3) The strongly extensional function f is compatible with the semigroup operation. Indeed, let a and b be elements of S. We have

$$f(aq \cdot bq) = f((ab)q) = \varphi(ab) = \varphi(a) \cdot \varphi(b) = f(aq) \cdot f(bq).$$

(4) Let a be an arbitrary element of S. From the equality $f(aq) = \varphi(a)$ we conclude $(f \circ \pi(q))(a) = \varphi(a)$. So, $f \circ \pi(q) = \varphi$.

Let σ be a quasi-antiorder in semigroup S, $f : S/(\sigma \cup \sigma^{-1}) \longrightarrow T$ be a strongly extensional homomorphism of semigroups such that $f \circ \pi(q) = \varphi$. Then $\sigma \supseteq \varphi^{-1}(\varrho)$. Indeed,

$$\begin{aligned}
(a,b) \in \varphi^{-1}(\varrho) &\Longleftrightarrow (\varphi(a), \varphi(b)) \in \varrho \\
&\Longleftrightarrow ((f \circ \pi(q))(a), (f \circ \pi(q))(b)) \in \varrho \\
&\Longleftrightarrow (\pi(q)(a), \pi(q)(b)) \in f^{-1}(\varrho) \quad (\text{by } f^{-1}(\varrho) \subseteq Coker(f)) \\
&\Longrightarrow (\pi(q)(a), \pi(q)(b)) \in \theta \\
&\Longleftrightarrow (a,b) \in \sigma. \ \blacksquare
\end{aligned}$$

For the next proposition we need a lemma in which we will describe the anticongruences α and β on a semigroup S such that $\beta \subseteq \alpha$.

LEMMA 3.1. [18, Lemma 2] *Let α and β be anticongruences on a semigroup S with apartness such that $\beta \subseteq \alpha$. Then the relation β/α on S/α, defined by $\beta/\alpha = \{(x\alpha, y\alpha) \in S/\alpha \times S/\alpha : (x,y) \in \beta\}$, is an anticongruence on S/α and $(S/\alpha)/(\beta/\alpha) \cong S/\beta$ holds.*

So, at the end of this article, we are in position to give a description of a quasi-antiorder and a semigroup S with apartness such that:

THEOREM 3.5. *Let $(S, =, \neq, \cdot)$ be a semigroup, and let ϱ and σ be quasi-antiorders on S such that $\sigma \subseteq \varrho$. Then the relation σ/ϱ, defined by*

$$\sigma/\varrho = \{(x(\varrho \cup \varrho^{-1}), y(\varrho \cup \varrho^{-1})) \in S/(\varrho \cup \varrho^{-1}) \times S/(\varrho \cup \varrho^{-1}) : (x,y) \in \sigma\},$$

is a quasi-antiorder on $S/(\varrho \cup \varrho^{-1})$ and

$$(S/(\varrho \cup \varrho^{-1}))/((\sigma \cup \sigma^{-1})/(\varrho \cup \varrho^{-1})) \cong S/(\sigma \cup \sigma^{-1}).$$

Proof. Put $q = \varrho \cup \varrho^{-1}$, and let a and b be elements of S. Then

(1) $$\begin{aligned}
(aq, bq) \in \sigma/\varrho &\Longleftrightarrow (a,b) \in \sigma \\
&\Longrightarrow (a,b) \in \varrho \ (\text{because } \sigma \subseteq \varrho)
\end{aligned}$$

$$
\begin{aligned}
&\Longleftrightarrow (aq, bq) \in \theta \text{ (by definition of } \theta) \\
&\Longrightarrow aq \neq bq; \\
(aq, cq) \in \sigma/\varrho &\Longleftrightarrow (a, c) \in \sigma \\
&\Longrightarrow (\forall b \in S)((a, b) \in \sigma \ \vee \ (b, c) \in \sigma) \\
&\Longleftrightarrow (\forall bq \in S/q)((aq, bq) \in \sigma/\varrho \ \vee \ (bq, cq) \in \sigma/\varrho); \\
(xqaqyq, xqbqyq) \in \sigma/\varrho &\Longleftrightarrow (xayq, xbyq) \in \sigma/\varrho \\
&\Longleftrightarrow (xay, xby) \in \sigma \\
&\Longrightarrow (a, b) \in \sigma \ .
\end{aligned}
$$

(2) From $\sigma \subseteq \varrho$ it follows $\varrho \cup \varrho^{-1} \supseteq \sigma \cup \sigma^{-1}$, and

$$(S/(\varrho \cup \varrho^{-1})/((\sigma \cup \sigma^{-1})/(\varrho \cup \varrho^{-1})) \cong S/(\sigma \cup \sigma^{-1})$$

holds, by Lemma 3.1. ■

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