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GLOBAL SMOOTHNESS PRESERVATION BY SOME NONLINEAR MAX-PRODUCT OPERATORS

Lucian Coroianu and Sorin G. Gal

Abstract. In this paper we study the problem of partial global smoothness preservation in the cases of max-product Bernstein approximation operators, max-product Hermite-Féjer interpolation operators based on the Chebyshev nodes of first kind and max-product Lagrange interpolation operators based on the Chebyshev nodes of second kind.

1. Introduction

In several recent papers, the approximation and shape preserving properties for the so-called max-product Bernstein operators (see [2, 3, 6]), max-product Hermite-Féjer interpolation operators (see [4]) and max-product Lagrange interpolation operators (see [5, 7]) were studied. One of the main characteristic is that these max-product operators present much better approximation properties than their linear counterpart (especially than the Hermite-Féjer and Lagrange polynomials).

In this paper we extend these studies for the above mentioned max-product operators, to the global smoothness preservation property.

The (partial) global smoothness preservation property can be described as follows. We say that the sequence of operators $L_n : C[a,b] \to C[a,b], n \in \mathbb{N}$, (partially) preserves the global smoothness of f, if for any $\alpha \in (0,1]$ and

$$f \in Lip\,\alpha = \{f : [a,b] \to \mathbb{R}; \exists M > 0, \text{ such that } |f(x) - f(y)| \leq M|x-y|^{\alpha}\},$$

there exists $0 < \beta \leq \alpha$ independent of f and n, such that $L_n(f) \in Lip\,\beta$, for all $n \in \mathbb{N}$.

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Equivalently, the property $L_n(f) \in Lip\,\beta$, for all $n \in \mathbb{N}$ means that there exists $C > 0$ independent of n but possibly depending on f, such that

$$\omega_1(L_n(f); h) \le Ch^\beta, \text{ for all } h \in [0,1], n \in \mathbb{N}.$$

Here $\omega_1(f;\delta) = \sup\{|f(x+h) - f(x)|; 0 \le h \le \delta, x, x+h \in [a,b]\}$ is the uniform modulus of continuity, and of course, it can be replaced by other kinds of moduli of continuity too.

When $\beta = \alpha$ we have a complete global smoothness preservation.

It is well-known that, in general, if $(L_n(f)(x))_{n\in\mathbb{N}}$ is a sequence of linear Bernstein-type operators, then the complete global smoothness preservation holds (see e.g. the book [1]), while if $(L_n(f)(x))_{n\in\mathbb{N}}$ is a sequence of linear interpolation operators (in the sense that each $L_n(f)(x)$ coincides with $f(x)$ on a system of given nodes), then excepting for example some particular Shepard operators, the interpolation conditions do not allow to have a complete global smoothness preservation property, i.e. in this case in general we have $\beta < \alpha$ (see [10] or [8, Chapter 1]).

In the present paper we study the global smoothness preservation property for the max-product Bernstein operator in Section 2, for the max-product Hermite-Féjer operator on the Chebyshev nodes of first kind in Section 3 and for the max-product Lagrange operator on the Chebyshev nodes of second kind in Section 4.

As a conclusion, we will derive that these max-product operators have the nice property that the images of the Lipschitz classes $Lip\,\alpha$, $0 < \alpha < 1$, is the same Lipschitz class $Lip\,\beta$, with $\beta = \frac{\alpha}{4+\alpha}$.

2. Max-product Bernstein operator

In this section we study the global smoothness preservation for the max-product Bernstein operator.

For a function $f : [0,1] \to \mathbb{R}_+$, the Bernstein approximation operator of max-product kind is given by the formula (see e.g. [9, p. 326])

$$B_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} p_{n,k}(x)},$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and $\bigvee_{k=0}^{n} p_{n,k}(x) = \max_{k=\{0,\ldots,n\}}\{p_{n,k}(x)\}$.

Remark. As it was proved in [3], $B_n^{(M)}(f)$ is a nonlinear (more exactly sublinear on the space of positive functions) operator, well-defined for all $x \in \mathbb{R}$, piecewise rational function on $\mathbb{R}$. Also, as it was proved in [2], $B_n^{(M)}(f)$ possesses some interesting approximation and shape preserving properties. For example, the order of uniform approximation is $\omega_1(f; 1/\sqrt{n})$ However, for some subclasses of functions including for example the class of concave functions and also a subclass of the convex functions, the essentially better order $\omega_1(f; 1/n)$ is obtained. In addition, $B_n^{(M)}(f)$ is continuous for any positive function f, preserves the monotonicity and the quasi-convexity.

For the main results of this paper we need the following five lemmas.

Lemma 2.1. [2, Lemma 3.4] *For $n \in N$, $n \geq 1$, we have*

$$\bigvee_{k=0}^{n} p_{n,k}(x) = p_{n,j}(x), \text{ for all } x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right],\ j = 0, 1, \ldots, n.$$

Remark. It easily follows that

$$p_{n,j}\left(\frac{j+1}{n+1}\right) = p_{n,j+1}\left(\frac{j+1}{n+1}\right) \text{ for all } j \in \{0, 1, \ldots, n\}.$$

Lemma 2.2. *Let $n \in N$, $n \geq 1$ and $j \in \{0, 1, \ldots, n\}$. The following assertions hold:*

(i) If $j \leq \frac{n}{2}$ then $p_{n,j}\left(\frac{j}{n+1}\right) \geq p_{n,j}\left(\frac{j+1}{n+1}\right)$;

(ii) If $j \geq \frac{n}{2}$ then $p_{n,j}\left(\frac{j}{n+1}\right) \leq p_{n,j}\left(\frac{j+1}{n+1}\right)$.

Proof. After elementary calculus, $p_{n,j}(\frac{j}{n+1}) \geq p_{n,j}(\frac{j+1}{n+1})$ is equivalent with

$$\left(\frac{j}{j+1}\right)^{j} \geq \left(\frac{n-j}{n-j+1}\right)^{n-j}.$$

Let us consider the functions $g : [0, n] \to \mathbb{R}$, $g(x) = \left(\frac{x}{x+1}\right)^{x}$ and $h : [0, n] \to \mathbb{R}$, $h(x) = \left(\frac{n-x}{n-x+1}\right)^{n-x}$. We have

$$g'(x) = \left(\frac{x}{x+1}\right)^{x} \left(\frac{1}{x+1} - (\ln(x+1) - \ln x)\right) \leq 0$$

for all $x \in (0, 1]$, where we used the well-known inequality $\frac{1}{x+1} \leq \ln(x+1) - \ln x$, $x \in (0, \infty)$. Therefore, g is nonincreasing on $[0, 1]$. Since $h(x) = g(n-x)$ for all $x \in (0, n]$, it easily follows that h is nondecreasing on $[0, 1]$. Because $h(\frac{n}{2}) = g(\frac{n}{2})$ and noting the monotonicity of g and h, we conclude that both assertions of the lemma hold. ■

Throughout the paper, C, C_0, C_1, C_2, c will denote absolute positive constants which can be of different values at each occurrence (and of different independencies mentioned correspondingly).

Lemma 2.3. *Let $n \in N$, $n \geq 1$ and $j \in \{0, 1, \ldots, n\}$. Then*

$$\min\left\{p_{n,j}\left(\frac{j}{n+1}\right), p_{n,j}\left(\frac{j+1}{n+1}\right)\right\} \geq \frac{C}{\sqrt{n}},$$

where $C > 0$ is an absolute constant independent of n and j.

Proof. We distinguish two cases: (i) n is even and (ii) n is odd.

Case (i). By Lemma 2.2 and by the Remark after Lemma 2.1, it follows that

$$\min\left\{p_{n,j}\left(\frac{j}{n+1}\right), p_{n,j}\left(\frac{j+1}{n+1}\right)\right\} \geq p_{n,n_0}\left(\frac{n_0}{n+1}\right) = p_{n,n_0}\left(\frac{n_0+1}{n+1}\right)$$

where $n_0 = \frac{n}{2}$. By direct calculation we get

$$p_{n,n_0}\left(\frac{n_0}{n+1}\right) = \frac{(2n_0)!}{(n_0!)^2} \cdot \left(\frac{n_0(n_0+1)}{(2n_0+1)^2}\right)^{n_0} = \frac{(2n_0)!}{(n_0!)^2 4^{n_0}} \cdot \left(\frac{n_0^2+n_0}{n_0^2+n_0+1/4}\right)^{n_0}$$

By the Wallis's formula (see [12, p. 142])

$$\lim_{n\to\infty} \frac{2\cdot 4\cdot \ldots (2n)}{1\cdot 3\cdot \ldots (2n-1)\sqrt{2n+1}} = \sqrt{\frac{\pi}{2}},$$

it is immediate that

$$\frac{(2^n n!)^2}{(2n)!} \cdot \frac{1}{\sqrt{2n}} \sim \sqrt{\frac{\pi}{2}},$$

and therefore there exists two absolute constants $C_1, C_2 > 0$ (independent of n), such that

$$\frac{C_1}{\sqrt{n}} \leq \frac{(2n)!}{(n!)^2 4^n} \leq \frac{C_2}{\sqrt{n}}, \text{ for all } n \in \mathbb{N}.$$

On the other hand, we have

$$\left(\frac{n_0^2+n_0}{n_0^2+n_0+1/4}\right)^{n_0} \geq \left(\frac{n_0^2+n_0}{n_0^2+n_0+1}\right)^{n_0} \geq \left(\frac{2n_0}{2n_0+1}\right)^{n_0} \geq \frac{1}{\sqrt{e}}.$$

Taking into account these last two inequalities, we get $p_{n,n_0}(\frac{n_0}{n+1}) \geq \frac{C}{\sqrt{n}}$, which proves the lemma in this case.

Case (ii). By Lemma 2.2 and by the Remark after Lemma 2.1, it follows that

$$\min\left\{p_{n,j}(\frac{j}{n+1}), p_{n,j}(\frac{j+1}{n+1})\right\} \geq p_{n,n_1}(\frac{n_1+1}{n+1})$$

where $n_1 = \frac{n-1}{2}$. We have

$$\begin{aligned} p_{n,n_1}\left(\frac{n_1+1}{n+1}\right) &= \frac{(2n_1+1)!}{n_1!(n_1+1)!} \cdot \left(\frac{n_1+1}{2n_1+2}\right)^{n_1} \cdot \left(\frac{n_1+1}{2n_1+2}\right)^{n_1+1} \\ &= \frac{(2n_1)!}{(n_1!)^2 4^n} \cdot \frac{2n_1+1}{2n_1+2} \geq \frac{C}{\sqrt{n}}. \end{aligned}$$

Collecting the estimates from the above two cases we get the desired conclusion. ■

Lemma 2.4. *One has*

$$\bigvee_{k=0}^{n} p_{n,k}(x) \geq \frac{C}{\sqrt{n}}$$

for all $n \in N$, $n \geq 1$ and $x \in [0,1]$, where $C > 0$ is a constant independent of n and x.

Proof. Let $x \in [0,1]$ and $n \in \mathbb{N}$ be arbitrary fixed. Let us choose $j \in \{0,1,\dots,n\}$ such that $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. Then we have

$$\begin{aligned} p_{n,j}(x) &= \binom{n}{j} x^j (1-x)^{n-j} \geq \binom{n}{j} \left(\frac{j}{n+1}\right)^j \left(1 - \frac{j+1}{n+1}\right)^{n-j} \\ &= \binom{n}{j} \left(\frac{j}{n+1}\right)^j \left(\frac{n-j+1}{n+1}\right)^{n-j} \left(\frac{n-j}{n-j+1}\right)^{n-j} \\ &= p_{n,j} \left(\frac{j}{n+1}\right) \left(\frac{n-j}{n-j+1}\right)^{n-j} \geq p_{n,j} \left(\frac{j}{n+1}\right) \frac{1}{e}. \end{aligned}$$

But applying Lemma 2.3, we get $p_{n,j}(x) \geq \frac{C}{\sqrt{n}}$, which proves the present lemma. ∎

Remark. In fact, the lower estimate in Lemma 2.4 is the best possible. Indeed, by the proof of Lemma 2.3, there exists absolute constants C_1, C_2, such that

$$\frac{C_1}{\sqrt{n}} \leq \frac{(2n)!}{(n!)^2 4^n} \leq \frac{C_2}{\sqrt{n}},$$

for all $n \in \mathbb{N}$. Then, by Lemma 2.1 and by the proof of Lemma 2.2, it follows that $p_{n,n_0}(\frac{n_0}{n_0+1}) = \bigvee_{k=0}^{n} p_{n,k}(\frac{n_0}{n_0+1}) \leq \frac{C_0}{\sqrt{n}}$, where $n_0 = [\frac{n}{2}]$ and C_0 does not depend on n. This implies the desired conclusion.

Also, we have the following

Lemma 2.5. *For all bounded $f : [0,1] \to R_+$, $n \in N$ and $h > 0$, we have*

$$\omega_1(B_n^{(M)}(f); h) \leq C n^2 \|f\| h,$$

where $\|f\| = \sup\{|f(x)|; x \in [-1,1]\}$ and $C > 0$ is a constant independent of f, n and h.

Proof. By Lemma 2.4, it follows that $\bigvee_{k=0}^{n} p_{n,k}(x) \geq \frac{C}{\sqrt{n}}$, for all $x \in [0,1]$, with $C > 0$ independent of n and x. Then, we have

$$\begin{aligned} \left|B_n^{(M)}(f)(x) - B_n^{(M)}(f)(y)\right| &= \left| \frac{\bigvee_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} p_{n,k}(x)} - \frac{\bigvee_{k=0}^{n} p_{n,k}(y) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} p_{n,k}(y)} \right| \\ &= \frac{1}{\bigvee_{k=0}^{n} p_{n,k}(x) \bigvee_{k=0}^{n} p_{n,k}(y)} \times \\ \times &\left| \bigvee_{k=0}^{n} p_{n,k}(y) \bigvee_{k=0}^{n} p_{n,k}(x) f(\frac{k}{n}) - \bigvee_{k=0}^{n} p_{n,k}(x) \bigvee_{k=0}^{n} p_{n,k}(y) f(\frac{k}{n}) \right| \\ &\leq Cn \left| \bigvee_{k=0}^{n} p_{n,k}(y) \bigvee_{k=0}^{n} p_{n,k}(x) f(\frac{k}{n}) - \bigvee_{k=0}^{n} p_{n,k}(x) \bigvee_{k=0}^{n} p_{n,k}(y) f(\frac{k}{n}) \right|. \end{aligned}$$

Without loss of generality, let us suppose that $B_n^{(M)}(f)(x) \geq B_n^{(M)}(f)(y)$. Let $k_1, k_2 \in \{0,1,\dots,n\}$ be such that

$$\bigvee_{k=0}^{n} p_{n,k}(y) = p_{n,k_1}(y), \quad \bigvee_{k=0}^{n} p_{n,k}(x) f(\frac{k}{n}) = p_{n,k_2}(x) f(\frac{k_2}{n}).$$

Then

$$
\begin{aligned}
&\left|B_n^{(M)}(f)(x) - B_n^{(M)}(f)(y)\right| \\
&\quad \le Cn\left(\bigvee_{k=0}^{n} p_{n,k}(y)\bigvee_{k=0}^{n} p_{n,k}(x)f(\frac{k}{n}) - \bigvee_{k=0}^{n} p_{n,k}(x)\bigvee_{k=0}^{n} p_{n,k}(y)f(\frac{k}{n})\right) \\
&\quad = Cn\left(p_{n,k_1}(y)p_{n,k_2}(x)f(\frac{k_2}{n}) - \bigvee_{k=0}^{n} p_{n,k}(x)\bigvee_{k=0}^{n} p_{n,k}(y)f(\frac{k}{n})\right) \\
&\quad \le Cn\left(p_{n,k_1}(y)p_{n,k_2}(x)f(\frac{k_2}{n}) - p_{n,k_1}(x)p_{n,k_2}(y)f(\frac{k_2}{n})\right) \\
&\quad = Cnf(\frac{k_2}{n})[p_{n,k_1}(y)p_{n,k_2}(x) - p_{n,k_1}(x)p_{n,k_2}(y)] \\
&\quad = Cnf(\frac{k_2}{n})[(p_{n,k_1}(y)p_{n,k_2}(x) - p_{n,k_1}(x)p_{n,k_2}(x)) \\
&\qquad + (p_{n,k_1}(x)p_{n,k_2}(x) - p_{n,k_1}(x)p_{n,k_2}(y))] \\
&\quad = Cnf(\frac{k_2}{n})[p_{n,k_2}(x)(p_{n,k_1}(y) - p_{n,k_1}(x)) + p_{n,k_1}(x)(p_{n,k_2}(x) - p_{n,k_2}(y))].
\end{aligned}
$$

Taking into account that $p_{n,k_1}(x) \le 1$ and $p_{n,k_2}(x) \le 1$, we get

$$
\begin{aligned}
&\left|B_n^{(M)}(f)(x) - B_n^{(M)}(f)(y)\right| \\
&\quad \le Cn\,\|f\|\,(|p_{n,k_1}(y) - p_{n,k_1}(x)| + |p_{n,k_2}(x) - p_{n,k_2}(y)|) \\
&\quad \le Cn\,\|f\|\left(\left\|p'_{n,k_1}\right\||x-y| + \left\|p'_{n,k_2}\right\||x-y|\right).
\end{aligned}
$$

If $k = 0$ or $k = n$, then $p_{n,k}(x) = x^n$ and we get $\|p'_{n,k}\| = n$. If $k \in [1, 2, \ldots, n-1\}$, then it is known that $p'_{n,k}(x) = n(p_{n-1,k-1}(x) - p_{n-1,k}(x))$. Consequently, we obtain $\|p'_{n,k}\| \le 2n$ for all $k \in \{0, 1, \ldots, n\}$. Clearly, this implies

$$\left|B_n^{(M)}(f)(x) - B_n^{(M)}(f)(y)\right| \le Cn^2\,\|f\|\,|x-y|.$$

Passing to supremum with $|x-y| \le h$, the lemma is proved. ■

We are now in position to prove the main result of this section.

THEOREM 2.6. *Let $f : [0,1] \to R_+$. If $f \in Lip_M\alpha$ with $0 < \alpha \le 1$, then for all $n \in N$ and $0 \le h \le 1$ we have*

$$\omega_1(B_n^{(M)}(f); h) \le ch^{\alpha/(4+\alpha)},$$

where $c > 0$ is independent of n and h (but depends on f).

Proof. By Lemma 2.5 we get

$$\omega_1(B_n^{(M)}(f); h) \le Cn^2 h, \text{ for all } h \in [0,1],$$

where $C > 0$ is independent of n and h.

On the other hand, for $|x-y| \le h$, by [2, Theorem 4.1], we get

$$
\begin{aligned}
&|B_n^{(M)}(f)(x) - B_n^{(M)}(f)(y)| \\
&\quad \le |B_n^{(M)}(f)(x) - f(x)| + |f(x) - f(y)| + |f(y) - B_n^{(M)}(f)(y)| \\
&\qquad\qquad \le 2\|B_n^{(M)}(f) - f\| + Ch^\alpha \le c\left[\frac{1}{n^{\alpha/2}} + h^\alpha\right].
\end{aligned}
$$

Passing to supremum with $|x-y| \le h$, it follows

$$\omega_1(B_n^{(M)}(f);h) \le C\left[\frac{1}{n^{\alpha/2}} + h^{\alpha}\right].$$

Therefore, for all $n \in \mathbb{N}$ and $0 \le h \le 1$ we get

$$\omega_1(B_n^{(M)}(f);h) \le c\min\left\{n^2 h, \frac{1}{n^{\alpha/2}} + h^{\alpha}\right\},$$

where $c > 0$ is independent of n and h. The optimal choice here is obtained when $n^2 h = \frac{1}{n^{\alpha/2}}$, that is if $h = \frac{1}{n^{2+\alpha/2}}$. Indeed, if $h < \frac{1}{n^{2+\alpha/2}}$ then the minimum is the first term, and when $h > \frac{1}{n^{2+\alpha/2}}$ then is the second term. This therefore implies $n = \frac{1}{h^{1/(2+\alpha/2)}}$ and replacing above we obtain

$$\omega_1(B_n^{(M)}(f);h) \le ch^{\alpha/(4+\alpha)}, \text{ for all } n \in \mathbb{N}, h \in [0,1],$$

which proves the theorem. ■

REMARKS. 1) Theorem 2.6 shows that the images of the class $Lip\,\alpha$, $\alpha \in (0,1]$, through all the max-product Bernstein operators $B_n^{(M)}, n \in \mathbb{N}$, belong to the same class $Lip\,\beta$, with $\beta = \frac{\alpha}{4+\alpha}$.

2) It is an open question if the exponent $\alpha/(4+\alpha)$ in the statement of Theorem 2.6 is the best possible.

3) Comparing with the complete global smoothness property of the linear Bernstein polynomials (see e.g. [1, p. 231, relation (7.1)]), the result in Theorem 2.6 is weaker. But this is not an unexpected result, taking into account that each max-product Bernstein operator $B_n^{(M)}(f)$, has a finite number of points where is not differentiable.

3. Max-product Hermite-Féjer operator

In this section we find global smoothness preservation for the max-product Hermite-Féjer interpolation operator based on the Chebyshev nodes of first kind.

Let $f : [-1,1] \to \mathbb{R}$ and $x_{n,k} = \cos(\frac{2k+1}{2(n+1)}\pi) \in (-1,1)$, $k \in \{0,\dots,n\}$, $-1 < x_{n,n} < x_{n,n-1} < \dots < x_{n,0} < 1$, be the roots of the first kind Chebyshev polynomial $T_{n+1}(x) = \cos[(n+1)arccos(x)]$. Denoting

$$h_{n,k}(x) = (1 - x x_{n,k}) \cdot \left(\frac{T_{n+1}(x)}{(n+1)(x-x_{n,k})}\right)^2,$$

it is well known that the max-product Hermite-Fejér interpolation operator is given by the formula (see [5])

$$H_{2n+1}^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} h_{n,k}(x) f(x_{n,k})}{\bigvee_{k=0}^{n} h_{n,k}(x)},$$

where $\bigvee_{k=0}^{n} h_{n,k}(x) = \max_{k=\{0,\dots,n\}}\{h_{n,k}(x)\}$.

REMARK. As it was proved in [5], $H_{2n+1}^{(M)}(f)(x)$ is a nonlinear (more exactly sublinear on the space of positive functions) operator, well-defined for all $x \in \mathbb{R}$ and a continuous, piecewise rational function on $\mathbb{R}$. Also, $H_{2n+1}^{(M)}(f)(x_{n,j}) = f(x_{n,j})$ for all $n \in \mathbb{N}$ and $j = 0, 1, \dots, n$, that is interpolatory on the points $x_{n,j}, n \in \mathbb{N}, j \in \{0, \dots, n\}$.

Firstly, we need the following auxiliary result.

THEOREM 3.1. *For all bounded $f : [-1, 1] \to \mathbb{R}_+$, $n \in N$ and $h > 0$, we have*

$$\omega_1(H_{2n+1}^{(M)}(f); h) \le Cn^4 \|f\| h,$$

where $\|f\| = \sup\{|f(x)|; x \in [-1, 1]\}$ and $C > 0$ is independent of n and h.

Proof. Since $\sum_{k=0}^n h_{n,k}(x) = 1$ for all $x \in [-1, 1]$, it follows that $\bigvee_{k=0}^n h_{n,k}(x) \ge 1/(n+1) \ge 1/(2n)$, for all $x \in [-1, 1]$. Then, we have

$$\begin{aligned}
&|H_{2n+1}^{(M)}(f)(x) - H_{2n+1}^{(M)}(f)(y)| \\
&\quad = \left| \frac{\bigvee_{k=0}^n h_{n,k}(x) f(x_{n,k})}{\bigvee_{k=0}^n h_{n,k}(x)} - \frac{\bigvee_{k=0}^n h_{n,k}(y) f(x_{n,k})}{\bigvee_{k=0}^n h_{n,k}(y)} \right| \\
&\quad = \frac{1}{\bigvee_{k=0}^n h_{n,k}(x) \bigvee_{k=0}^n h_{n,k}(y)} \times \\
&\quad \times \left| \bigvee_{k=0}^n h_{n,k}(y) \bigvee_{k=0}^n h_{n,k}(x) f(x_{n,k}) - \bigvee_{k=0}^n h_{n,k}(x) \bigvee_{k=0}^n h_{n,k}(y) f(x_{n,k}) \right| \\
&\quad \le 4n^2 \left| \bigvee_{k=0}^n h_{n,k}(y) \bigvee_{k=0}^n h_{n,k}(x) f(x_{n,k}) - \bigvee_{k=0}^n h_{n,k}(x) \bigvee_{k=0}^n h_{n,k}(y) f(x_{n,k}) \right|.
\end{aligned}$$

Without loss of generality, let us suppose that $H_{2n+1}^{(M)}(f)(x) \ge H_{2n+1}^{(M)}(f)(y)$. Let $k_1, k_2 \in \{0, 1, \dots, n\}$ be such that

$$\bigvee_{k=0}^n h_{n,k}(y) = h_{n,k_1}(y),$$

$$\bigvee_{k=0}^n h_{n,k}(x) f(x_{n,k}) = h_{n,k_2}(x) f(x_{n,k_2}).$$

Then

$$\begin{aligned}
&\left| H_{2n+1}^{(M)}(f)(x) - H_n^M(f)(y) \right| \\
&\quad \le 4n^2 \left(\bigvee_{k=0}^n h_{n,k}(y) \bigvee_{k=0}^n h_{n,k}(x) f(x_{n,k}) - \bigvee_{k=0}^n h_{n,k}(x) \bigvee_{k=0}^n h_{n,k}(y) f(x_{n,k}) \right) \\
&\quad = 4n^2 \left(h_{n,k_1}(y) h_{n,k_2}(x) f(x_{n,k_2}) - \bigvee_{k=0}^n h_{n,k}(x) \bigvee_{k=0}^n h_{n,k}(y) f(x_{n,k}) \right)
\end{aligned}$$

$$\leq 4n^2 \left(h_{n,k_1}(y)h_{n,k_2}(x)f(x_{n,k_2}) - h_{n,k_1}(x)h_{n,k_2}(y)f(x_{n,k_2})\right)$$
$$= 4n^2 f(x_{n,k_2})[h_{n,k_1}(y)h_{n,k_2}(x) - h_{n,k_1}(x)h_{n,k_2}(y)]$$
$$= 4n^2 f(x_{n,k_2})[(h_{n,k_1}(y)h_{n,k_2}(x) - h_{n,k_1}(x)h_{n,k_2}(x))$$
$$+ (h_{n,k_1}(x)h_{n,k_2}(x) - h_{n,k_1}(x)h_{n,k_2}(y))]$$
$$= 4n^2 f(x_{n,k_2})[h_{n,k_2}(x)(h_{n,k_1}(y) - h_{n,k_1}(x)) + h_{n,k_1}(x)(h_{n,k_2}(x) - h_{n,k_2}(y))]$$

Taking into account that $h_{n,k_1}(x) \leq 1$ and $h_{n,k_2}(x) \leq 1$, we get

$$|H^{(M)}_{2n+1}(f)(x) - H^{(M)}_{2n+1}(f)(y)|$$
$$\leq 4n^2 \|f\| \left(|h_{n,k_1}(y) - h_{n,k_1}(x)| + |h_{n,k_2}(x) - h_{n,k_2}(y)|\right)$$
$$\leq 4n^2 \|f\| \left(\left\|h'_{n,k_1}\right\| |x-y| + \left\|h'_{n,k_2}\right\| |x-y|\right).$$

But by [10] (see also [8], first inequality on page 6) we have $\left\|h'_{n,j}\right\| \leq Cn^2$, for all $n \in N$ and $j \in \{0,1,\ldots,n\}$, where $C > 0$ is an absolute constant independent of n and j, which implies that

$$\left|H^{(M)}_{2n+1}(f)(x) - H^{(M)}_{2n+1}(f)(y)\right| \leq Cn^4 \|f\| |x-y|.$$

Passing to supremum with $|x-y| \leq h$, the theorem is proved. ∎

The main result of this section is the following.

THEOREM 3.2. *Let $f : [-1,1] \to \mathbb{R}_+$. If $f \in Lip_M \alpha$ with $0 < \alpha \leq 1$, then for all $n \in \mathbb{N}$ and $0 < h < 1$ we have*

$$\omega_1(H^{(M)}_{2n+1}(f); h) \leq ch^{\alpha/(4+\alpha)},$$

where $c > 0$ is independent of n and h (but depends on f).

Proof. By Theorem 3.1 we get

$$\omega_1(H^{(M)}_{2n+1}(f); h) \leq Cn^4 h, \text{ for all } h \in (0,1),$$

where $C > 0$ is independent of n and h.

On the other hand, for $|x-y| \leq h$, by [4, Theorem 3.1], we get

$$|H^{(M)}_{2n+1}(f)(x) - H^{(M)}_{2n+1}(f)(x)| \leq |H^{(M)}_{2n+1}(f)(x) - f(x)| + |f(x) - f(y)|$$
$$+ |f(y) - H^{(M)}_{2n+1}(f)(y)| \leq 2\|H^{(M)}_{2n+1}(f) - f\| + Ch^\alpha \leq c\left[\frac{1}{n^\alpha} + h^\alpha\right],$$

where $c > 0$ is independent of n and h. Passing to supremum with $|x-y| \leq h$ it follows

$$\omega_1(H^{(M)}_{2n+1}(f); h) \leq C\left[\frac{1}{n^\alpha} + h^\alpha\right].$$

Therefore, for all $n \in \mathbb{N}$ and $0 < h < 1$ we get

$$\omega_1(H^{(M)}_{2n+1}(f); h) \leq c \min\left\{n^4 h, \frac{1}{n^\alpha} + h^\alpha\right\}.$$

The optimal choice here is obtained when $n^4 h = \frac{1}{n^\alpha}$, that is if $h = \frac{1}{n^{4+\alpha}}$. Indeed, if $h < \frac{1}{n^{4+\alpha}}$ then the minimum is the first term, and when $h > \frac{1}{n^{4+\alpha}}$ then is the second term. This therefore implies $n = \frac{1}{h^{1/(4+\alpha)}}$ and replacing above we obtain

$$\omega_1(H^{(M)}_{2n+1}(f); h) \leq ch^{\alpha/(4+\alpha)}, \text{ for all } n \in \mathbb{N}, h \in (0,1),$$

which proves the theorem. ∎

REMARKS. 1) Theorem 3.2 shows that the images of the class $Lip\,\alpha$, $\alpha \in (0,1]$, through all the max-product Hermite-Féjer operators $H_{2n+1}^{(M)}, n \in \mathbb{N}$, belong to the same class $Lip\beta$, with $\beta = \frac{\alpha}{4+\alpha}$.

2) It is an open question if the exponent $\alpha/(4+\alpha)$ in the statement of Theorem 3.2 is the best possible.

4. Max-product Lagrange operator

In this section we find global smoothness preservation properties for the max-product Lagrange interpolation operator based on the Chebyshev nodes of second kind, plus the endpoints.

Let $f : [-1,1] \to \mathbb{R}$ and $x_{n,k} = \cos(\frac{n-k}{n-1}\pi) \in [-1,1]$, $k \in \{1,\ldots,n\}$ be the Chebyshev knots of second kind in $[-1,1]$, plus the endpoints. More exactly, it is known that $x_{n,k}$ are the roots of $\omega_n(x) = sin[(n-1)t]sint$, $x = \cos t$ (which represents in fact the Chebyshev polynomial of second kind of degree $n-2$, multiplied by $1-x^2$) and that in this case for the fundamental Lagrange polynomials we can write (see [11, p. 377])

$$l_{n,k}(x) = \frac{(-1)^{k-1}\omega_n(x)}{(1+\delta_{k,1}+\delta_{k,n})(n-1)(x-x_{n,k})}, \qquad n \geq 2, \quad k = 1,\ldots,n,$$

where $\omega_n(x) = \Pi_{k=1}^n (x - x_{n,k})$ and $\delta_{i,j}$ denotes the Kronecker's symbol, that is $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.

Then, the max-product Lagrange interpolation operator is given by the formula (see [4])

$$L_n^{(M)}(f)(x) = \frac{\bigvee_{k=1}^{n} l_{n,k}(x) f(x_{n,k})}{\bigvee_{k=1}^{n} l_{n,k}(x)}, \qquad x \in [-1,1],$$

where $\bigvee_{k=1}^n l_{n,k}(x) = \max_{k=\{1,\ldots,n\}}\{l_{n,k}(x)\}$.

REMARK. As it was proved in [5], $L_n^{(M)}(f)(x)$ is a nonlinear (more exactly sublinear on the space of positive functions) operator, well-defined for all $x \in \mathbb{R}$ and a continuous, piecewise rational function on $\mathbb{R}$. Also, $L_n^{(M)}(f)(x_{n,j}) = f(x_{n,j})$ for all $n \in \mathbb{N}$ and $j = 1,\ldots,n$, that is interpolatory on the points $x_{n,j}, n \in \mathbb{N}, j \in \{0,\ldots,n\}$.

Firstly, we need the following result.

THEOREM 4.1. *For all bounded $f : [-1,1] \to \mathbb{R}_+$, $n \in N$ and $h > 0$, we have*

$$\omega_1(L_n^{(M)}(f);h) \leq Cn^4\|f\|h,$$

where C is an absolute constant independent of f, h and n.

Proof. Since $\sum_{k=1}^n l_{n,k}(x) = 1$ for all $x \in [-1,1]$, it follows that $\bigvee_{k=1}^n l_{n,k}(x) \geq 1/n$ for all $x \in [-1,1]$. Then, we have

$$\left|L_n^{(M)}(f)(x) - L_n^{(M)}(f)(y)\right|$$

$$= \left| \frac{\bigvee_{k=1}^{n} l_{n,k}(x) f(x_{n,k})}{\bigvee_{k=1}^{n} l_{n,k}(x)} - \frac{\bigvee_{k=1}^{n} l_{n,k}(y) f(x_{n,k})}{\bigvee_{k=1}^{n} l_{n,k}(y)} \right|$$

$$= \frac{1}{\bigvee_{k=1}^{n} l_{n,k}(x) \bigvee_{k=1}^{n} l_{n,k}(y)} \times$$

$$\times \left| \bigvee_{k=1}^{n} l_{n,k}(y) \bigvee_{k=1}^{n} l_{n,k}(x) f(x_{n,k}) - \bigvee_{k=1}^{n} l_{n,k}(x) \bigvee_{k=1}^{n} l_{n,k}(y) f(x_{n,k}) \right|$$

$$\leq n^2 \left| \bigvee_{k=1}^{n} l_{n,k}(y) \bigvee_{k=1}^{n} l_{n,k}(x) f(x_{n,k}) - \bigvee_{k=1}^{n} l_{n,k}(x) \bigvee_{k=1}^{n} l_{n,k}(y) f(x_{n,k}) \right|.$$

Without loss of generality let us suppose that $L_n^{(M)}(f)(x) \geq L_n^{(M)}(f)(y)$. Let $k_1, k_2 \in \{1, 2, \ldots, n\}$ be such that

$$\bigvee_{k=1}^{n} l_{n,k}(y) = l_{n,k_1}(y),$$

$$\bigvee_{k=1}^{n} l_{n,k}(x) f(x_{n,k}) = l_{n,k_2}(x) f(x_{n,k_2}).$$

Then

$$\begin{aligned}
&\left| L_n^{(M)}(f)(x) - L_n^{(M)}(f)(y) \right| \\
&\quad \leq n^2 \left(\bigvee_{k=1}^{n} l_{n,k}(y) \bigvee_{k=1}^{n} l_{n,k}(x) f(x_{n,k}) - \bigvee_{k=1}^{n} l_{n,k}(x) \bigvee_{k=1}^{n} l_{n,k}(y) f(x_{n,k}) \right) \\
&\quad = n^2 \left(l_{n,k_1}(y) l_{n,k_2}(x) f(x_{n,k_2}) - \bigvee_{k=1}^{n} l_{n,k}(x) \bigvee_{k=1}^{n} l_{n,k}(y) f(x_{n,k}) \right) \\
&\quad \leq n^2 \left(l_{n,k_1}(y) l_{n,k_2}(x) f(x_{n,k_2}) - l_{n,k_1}(x) l_{n,k_2}(y) f(x_{n,k_2}) \right) \\
&\quad = n^2 f(x_{n,k_2}) [l_{n,k_1}(y) l_{n,k_2}(x) - l_{n,k_1}(x) l_{n,k_2}(y)] \\
&\quad = n^2 f(x_{n,k_2}) [(l_{n,k_1}(y) l_{n,k_2}(x) - l_{n,k_1}(x) l_{n,k_2}(x)) \\
&\qquad + (l_{n,k_1}(x) l_{n,k_2}(x) - l_{n,k_1}(x) l_{n,k_2}(y))] \\
&\quad = n^2 f(x_{n,k_2}) [l_{n,k_2}(x)(l_{n,k_1}(y) - l_{n,k_1}(x)) + l_{n,k_1}(x)(l_{n,k_2}(x) - l_{n,k_2}(y))].
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
&|L_n^{(M)}(f)(x) - L_n^{(M)}(f)(y)| \\
&\qquad \leq C_0 n^2 \|f\| \left(|l_{n,k_1}(y) - l_{n,k_1}(x)| + |l_{n,k_2}(x) - l_{n,k_2}(y)| \right) \\
&\qquad \leq C_0 n^2 \|f\| \left(\left\| l'_{n,k_1} \right\| |x - y| + \left\| l'_{n,k_2} \right\| |x - y| \right).
\end{aligned}$$

By [8, the proof of Theorem 1.2.3, p. 13], we have $\left| l'_{n,k}(x) \right| \leq C_0 n^2$, for all $x \in [-1, 1]$, $n \in \mathbb{N}$ and $k \in \{1, 2, \ldots, n\}$, where C_0 is an absolute constant independent of f and n.

Replacing this above and passing to supremum with $|x-y| \leq h$, the theorem is proved. ■

The main result of this section is the following.

THEOREM 4.2. *Let $f : [-1,1] \to \mathbb{R}_+$. If $f \in Lip_M \alpha$ with $0 < \alpha \leq 1$, then for all $n \in \mathbb{N}$ and $0 \leq h \leq 1$ we have*

$$\omega_1(L_n^{(M)}(f); h) \leq ch^{\alpha/(4+\alpha)},$$

where $c > 0$ is independent of n and h (but depends on f).

Proof. By Theorem 4.1 we get

$$\omega_1(L_n^{(M)}(f); h) \leq Cn^4 h, \text{ for all } h \in [0,1],$$

where $C > 0$ is independent of n and h.

On the other hand, for $|x-y| \leq h$, by [5, Theorem 3.3], we get

$$\begin{aligned}|L_n^{(M)}(f)(x) - L_n^{(M)}(f)(x)| &\leq |L_n(f)(x) - f(x)| + |f(x) - f(y)| + |f(y) - L_n(f)(y)| \\ &\leq 2\|L_n(f) - f\| + Ch^\alpha \leq c\left[\frac{1}{n^\alpha} + h^\alpha\right],\end{aligned}$$

where $c > 0$ is independent of n and h. Reasoning in continuation exactly as in the proof of Theorem 3.2 we get the desired conclusion. ■

REMARKS. 1) Theorem 4.2 shows that the images of the class $Lip\,\alpha$, $\alpha \in (0,1]$, through all the max-product Lagrange operators $L_n^{(M)}$, $n \in \mathbb{N}$, belong to the same class $Lip\,\beta$, with $\beta = \frac{\alpha}{4+\alpha}$.

2) It is an open question if the exponent $\alpha/(4+\alpha)$ in the statement of Theorem 4.2 is the best possible.

3) Let us note that although they have better approximation properties (of Jackson type $\omega_1(f; 1/n)$, pointed out in [4] and [5]) than their linear counterpart polynomials, the above max-product Hermite-Féjer and max-product Lagrange operators satisfy weaker global smoothness preservation properties that their linear counterpart polynomials (compare above Theorem 3.2 with Corollary 1.2.1, pp. 7-8 in [8] and above Theorem 4.2 with Corollary 1.2.2, p. 15 in [8]). These are consequences of the fact that each max-product Hermite-Féjer operator, $H_{2n+1}^{(M)}(f)$, and each max-product Lagrange interpolation operator $L_n^{(M)}(f)$, obviously has a finite number of points where it is not differentiable.

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University of Oradea, Department of Mathematics and Computer Science, Str. Universitatii 1, 410087 Oradea, Romania

E-mail: `lcoroianu@uoradea.ro, galso@uoradea.ro`