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SANDWICH-TYPE RESULTS FOR A CLASS OF FUNCTIONS DEFINED BY A GENERALIZED DIFFERENTIAL OPERATOR

Priyabrat Gochhayat

Abstract. By making use of a generalized differential operator a new class of non-Bazilevič functions is introduced. Differential sandwich-type theorem for the above class is investigated. Relevant connections of the results, which are presented in this paper, with various other known results are also pointed out.

1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the *open* unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and $\mathcal{H}[a,n]$ $(n \in \mathbb{N} := \{1,2,3,\cdots\})$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots .$$

Let $\mathcal{A}(\subset \mathcal{H})$ be the class of all analytic functions given by the power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{U}).$$

Recalling the principle of subordination between analytic functions, we say that f is *subordinate* to g, written as $f \prec g$ in $\mathcal{U}$ or $f(z) \prec g(z) \quad (z \in \mathcal{U})$, if there exists a function ω, analytic in $\mathcal{U}$ satisfying the conditions of the Schwarz lemma (i.e. $\omega(0) = 0$ and $|\omega(z)| < 1$) such that $f(z) = g(\omega(z))$ $(z \in \mathcal{U})$. It follows that

$$f(z) \prec g(z)\ (z \in \mathcal{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}).$$

In particular, if g is univalent in $\mathcal{U}$, then the reverse implication also holds (cf. [13]).

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Furthermore, f is said to be subordinate to g in the disk $\mathcal{U}_r := \{z : z \in \mathbb{C}$ and $|z| < r\}$, if the function $f_r(z) = f(rz)$ is subordinate to $g_r(z) = g(rz)$ in $\mathcal{U}$.

Definition 1.1. Let $p, h \in \mathcal{H}$ and let $\varphi(r, s, t; z) : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$. If $p(z)$ and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if $p(z)$ satisfies the second order superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \quad (z \in \mathcal{U}) \tag{1.1}$$

then $p(z)$ is a solution of the differential superordination (1.1). (Note that, if f is subordinate to F, then F is superordinate to f).

An analytic function q is called a subordinant of the differential superordination, or more precisely a *subordinant* if $q \prec p$, for all p satisfying (1.1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$, for all subordinants q of (1.1) is said to be the *best subordinant*. Note that the best subordinant is unique upto a rotation of $\mathcal{U}$. Recently in [10], Miller and Mocanu obtained conditions on h, q and φ for which the following implication holds:

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z) \quad (z \in \mathcal{U}).$$

Using the results due to Miller and Mocanu [10], Bulboaca [6] considered certain classes of first order differential superordination as well as superordination-preserving integral operators [5]. More recently using the result of Bulboaca [6], Ali et al. [1] obtained some sufficient conditions for functions to satisfy

$$q_1(z) \prec zf'(z)/f(z) \prec q_2(z), \quad (z \in \mathcal{U})$$

where q_1, q_2 are univalent in $\mathcal{U}$ with $q_1(0) = 1 = q_2(0)$.

We now introduce the generalized differential operator $D^{k,\alpha}_{\lambda,\delta} : \mathcal{A} \to \mathcal{A}$ defined by

$$D^{k,\alpha}_{\lambda,\delta} f(z) = z + \sum_{n=2}^{\infty} [n^\alpha + (n-1)n^\alpha\lambda]^k \binom{n+\delta-1}{\delta} a_n z^n$$
$$(z \in \mathcal{U}; \lambda, \delta \geq 0; k, \alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{1.2}$$

Note that the differential operator $D^{k,\alpha}_{\lambda,\delta}$ unifies many operators of $\mathcal{A}$.

In particular:

$D^{k,1}_{0,0} (= D^{k,0}_{1,1}) \equiv$ Sălăgean operator (cf. [12]),

$D^{0,\alpha}_{\lambda,\delta} \equiv$ Ruscheweyh differential operator of order δ (cf. [11]),

$D^{k,0}_{\lambda,0} \equiv$ Al-Oboudi operator(cf. [2]),

$D^{k,0}_{1,\delta} (= D^{k,1}_{0,\delta}) \equiv$ differential operator studied by Al-shaqsi and Darus (cf. [4]).

By using the operator $D^{k,\alpha}_{\lambda,\delta}$, we now define a new subclass of analytic functions as follows:

Definition 1.2. The function $f \in \mathcal{A}$ is said to be in the class $\mathcal{GD}^{k,\alpha}_{\lambda,\delta}(\phi)$ $(\lambda, \delta \geq 0;\ k, \alpha \in \mathbb{N}_0)$ if and only if it satisfies the condition

$$(D^{k,\alpha}_{\lambda,\delta} f)'(z) \left(\frac{z}{(D^{k,\alpha}_{\lambda,\delta} f)(z)} \right)^{1+\mu} \prec \phi(z) \qquad (z \in \mathcal{U}; 0 \leq \mu \leq 1), \tag{1.3}$$

where $\phi(z)$ is an analytic function with positive real part on $\mathcal{U}$ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disc $\mathcal{U}$ onto a region which is symmetric with respect to the real axis.

Note that for $k = 0$, $\delta = 0$ and $\phi = (1+z)/(1-z)$ the class reduces to the class of functions of non-Bazilevič type which is recently introduced and studied by Obradović [3] as follows:

$$f \in \mathcal{GD}^{\alpha}_{\lambda} \left(\frac{1+z}{1-z} \right) \Leftrightarrow f'(z) \left(\frac{z}{f(z)} \right)^{1+\mu} \prec \frac{1+z}{1-z} \qquad (z \in \mathcal{U}; 0 \leq \mu \leq 1).$$

In the present article we investigate certain subordination and superordination results of the class $\mathcal{GD}^{k,\alpha}_{\lambda,\delta}(\phi)$, together with differential sandwich type theorem as an interesting consequence of the results.

2. Preliminaries

To establish our main results, we need the following:

Definition 2.1. ([10, Definition 2, p. 817]; see also [9, Definition 2.2b, p. 21]) Let Q be the set of all functions f that are analytic and injective on $\overline{\mathcal{U}} \setminus E(f)$, where

$$E(f) := \left\{ \zeta : \zeta \in \partial\mathcal{U} \text{ and } \lim_{z \to \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{U} \setminus E(f)$.

Lemma 2.2. [9, Theorem 3.4h, p. 132] *Let q be univalent in the open unit disk $\mathcal{U}$ and θ and ϕ be analytic in a domain D containing $q(\mathcal{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

1. *Q is starlike in $\mathcal{U}$, and*
2. *$\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in \mathcal{U}$.*

If p is analytic in $\mathcal{U}$, with $p(0) = q(0)$, $p(\mathcal{U}) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$$

then $p \prec q$ and q is the best dominant.

Lemma 2.3. [6] *Let q be univalent in the open unit disk $\mathcal{U}$ and ϑ and φ be analytic in a domain D containing $q(\mathcal{U})$. Suppose that*

1. *$\Re\left(\frac{z\vartheta'(q(z))}{\varphi(q(z))}\right) > 0$ for $z \in \mathcal{U}$, and*

2. $zq'(z)\varphi(q(z))$ *is starlike univalent in* $\mathcal{U}$.

If $p \in \mathcal{H}[q(0),1] \cap Q$, *with* $p(\mathcal{U}) \subseteq D$, *and* $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ *is univalent in* $\mathcal{U}$ *and*

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \quad (z \in \mathcal{U})$$

then $q \prec p$ *and* q *is the best subordinant.*

LEMMA 2.4. [8] *If* $-1 \le B < A \le 1$, $\beta > 0$ *and the complex number* γ *is constrained by*

$$\Re(\gamma) \ge -\frac{\beta(1-A)}{(1-B)},$$

then the following differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U})$$

has a univalent solution in $\mathcal{U}$ *given by*

$$q(z) = \begin{cases} \frac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}dt} - \frac{\gamma}{\beta}; & B \neq 0 \\ \frac{z^{\beta+\gamma}\exp(\beta Az)}{\beta\int_0^z t^{\beta+\gamma-1}\exp(\beta At)dt} - \frac{\gamma}{\beta}; & B = 0. \end{cases} \tag{2.1}$$

If the function $\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *is analytic in* $\mathcal{U}$ *and satisfies*

$$\phi(z) + \frac{z\phi'(z)}{\beta\phi(z) + \gamma} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}),$$

then

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}) \tag{2.2}$$

and $q(z)$ *is the best dominant of* (2.2).

3. Main results

We have the following subordination and superordination results:

THEOREM 3.1. *Let the function* q *be analytic in* $\mathcal{U}$ *such that* $q(z) \neq 0$ $(z \in \mathcal{U})$. *Suppose that* $\frac{zq'(z)}{q(z)}$ *is univalent starlike in* $\mathcal{U}$. *Let*

$$\Re\left\{1 + \frac{q(z)}{\beta} + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0 \quad (\beta \in \mathbb{C};\ \beta \neq 0) \tag{3.1}$$

and

$$\begin{aligned}\Omega(\alpha,k,\lambda,\delta;f)(z) := {} & \left(D_{\lambda,\delta}^{\alpha,k} f\right)'(z)\left(\frac{z}{(D_{\lambda,\delta}^{\alpha,k} f)(z)}\right)^{1+\mu} \\ & + \beta\left[\frac{z(D_{\lambda,\delta}^{\alpha,k} f)''(z)}{(D_{\lambda,\delta}^{\alpha,k} f)'(z)} + (1+\mu)\left(1 - \frac{z(D_{\lambda,\delta}^{\alpha,k} f)'(z)}{(D_{\lambda,\delta}^{\alpha,k} f)(z)}\right)\right].\end{aligned} \tag{3.2}$$

If q satisfies

$$\Omega(\alpha,k,\lambda,\delta;f)(z) \prec q(z)+\beta\frac{zq'(z)}{q(z)} \quad (\beta\in\mathbb{C};\ \beta\neq 0),$$

then

$$\left(D_{\lambda,\delta}^{\alpha,k}f\right)'(z)\left(\frac{z}{(D_{\lambda,\delta}^{\alpha,k}f)(z)}\right)^{1+\mu} \prec q(z) \quad (0\le\mu\le 1) \tag{3.3}$$

and q is the best dominant.

Proof. Let the function p be defined by

$$p(z) := \left(D_{\lambda,\delta}^{\alpha,k}f\right)'(z)\left(\frac{z}{(D_{\lambda,\delta}^{\alpha,k}f)(z)}\right)^{1+\mu} \quad (z\in\mathcal{U};\ z\neq 0; f\in\mathcal{A}).$$

Logarithmic differentiation yields

$$\frac{zp'(z)}{p(z)} = \frac{z(D_{\lambda,\delta}^{\alpha,k}f)''(z)}{(D_{\lambda,\delta}^{\alpha,k}f)'(z)} + (1+\mu)\left(1-\frac{z(D_{\lambda,\delta}^{\alpha,k}f)'(z)}{(D_{\lambda,\delta}^{\alpha,k}f)(z)}\right).$$

Let $\theta(w) := w$ and $\phi(w) := \beta/w$; by letting $Q(z) = zq'(z)\phi(q(z)) = \beta\frac{zq'(z)}{q(z)}$ and $h(z) = \theta(q(z)) + Q(z) = q(z)+\beta\frac{zq'(z)}{q(z)}$, we observe that $Q(z)$ is univalent starlike in $\mathcal{U}$ and $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$. Thus assertion (3.3) of Theorem 3.1 follows by an applications of Lemma 2.2. This completes the proof. ■

By taking $q(z) = \frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$ and $q(z) = \left(\frac{1+z}{1-z}\right)^{\tau}$, $0<\tau\le 1$, in Theorem 3.1, we get the following:

Corollary 3.2. *Assume that* (3.1) *holds. If $f\in\mathcal{A}$ and*

$$\Omega(\alpha,k,\lambda,\delta;f)(z) \prec \frac{1+Az}{1+Bz}+\beta\frac{(A-B)z}{(1+Az)(1+Bz)} \quad (\beta\in\mathbb{C};\ \beta\neq 0),$$

where $\Omega(\alpha,k,\lambda,\delta;f)$ is defined in (3.2)*, then*

$$\left(D_{\lambda,\delta}^{\alpha,k}f\right)'(z)\left(\frac{z}{(D_{\lambda,\delta}^{\alpha,k}f)(z)}\right)^{1+\mu} \prec \frac{1+Az}{1+Bz} \quad (0\le\mu\le 1)$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Corollary 3.3. *Assume that* (3.1) *holds. If $f\in\mathcal{A}$ and*

$$\Omega(\alpha,k,\lambda,\delta;f)(z) \prec \left(\frac{1+z}{1-z}\right)^{\tau}+\frac{2\beta\tau z}{(1-z^2)} \quad (\beta\in\mathbb{C};\ \beta\neq 0),$$

where $\Omega(\alpha,k,\lambda,\delta;f)$ *is defined in* (3.2)*, then*

$$\left(D_{\lambda,\delta}^{\alpha,k}f\right)'(z)\left(\frac{z}{(D_{\lambda,\delta}^{\alpha,k}f)(z)}\right)^{1+\mu} \prec \left(\frac{1+z}{1-z}\right)^{\tau} \quad (0\le\mu\le1)$$

and $\left(\frac{1+z}{1-z}\right)^{\tau}$ *is the best dominant.*

Theorem 3.4. *Let the function* q *be analytic in* $\mathcal{U}$ *such that* $q(z)\neq0$ $(z\in\mathcal{U})$*. Suppose that* $\frac{zq'(z)}{q(z)}$ *is univalent starlike in* $\mathcal{U}$*. Furthermore assume that*

$$\Re\left\{\frac{q(z)}{\beta}\right\}>0 \quad (\beta\in\mathbb{C};\ \beta\neq0). \tag{3.4}$$

If $f\in\mathcal{A}$*, with*

$$0\neq\left(D_{\lambda,\delta}^{\alpha,k}f\right)'(z)\left(\frac{z}{(D_{\lambda,\delta}^{\alpha,k}f)(z)}\right)^{1+\mu}\in\mathcal{H}[q(0),1]\cap Q,$$

and $\Omega(\alpha,k,\lambda,\delta;f)$ *is univalent in* $\mathcal{U}$*, then*

$$q(z)+\beta\frac{zq'(z)}{q(z)}\prec\Omega(\alpha,k,\lambda,\delta;f)(z) \quad (\beta\in\mathbb{C};\ \beta\neq0),$$

implies

$$q(z)\prec\left(D_{\lambda,\delta}^{\alpha,k}f\right)'(z)\left(\frac{z}{(D_{\lambda,\delta}^{\alpha,k}f)(z)}\right)^{1+\mu} \quad (0\le\mu\le1) \tag{3.5}$$

where q *is the best subordinant and* $\Omega(\alpha,k,\lambda,\delta;f)$ *is given by* (3.2).

Proof. By setting $\vartheta(w):=w$ and $\varphi:=\beta/w$, we observe that ϑ and φ are analytic in $\mathbb{C}$ and $\mathbb{C}\setminus\{0\}$ respectively. $\varphi(w)\neq0$ $(w\in\mathbb{C}\setminus\{0\})$ and q is univalent convex yields

$$\Re\left(\frac{\vartheta'(q(z))}{\varphi(q(z))}\right)=\Re\left(\frac{q(z)}{\beta}\right)>0 \quad (\beta\in\mathbb{C};\ \beta\neq0).$$

Application of Lemma 2.3 gives the assertion (3.5) of Theorem 3,4. This completes the proof. ∎

Now by combining Theorems 3.1 and 3.4, we get the following differential Sandwich-type theorem:

Theorem 3.5. *Let the function* q_1 *and* q_2 *be univalent in* $\mathcal{U}$ *such that* $q_1(z)\neq0$ *and* $q_2(z)\neq0$ $(z\in\mathcal{U})$ *with* $\frac{zq_1'(z)}{q_1(z)}$ *and* $\frac{zq_2'(z)}{q_2(z)}$ *is univalent starlike. Furthermore assume that* q_1 *and* q_2 *satisfies* (3.1) *and* (3.4)*, respectively. If* $f\in\mathcal{A}$*, with*

$$0\neq\left(D_{\lambda,\delta}^{\alpha,k}f\right)'(z)\left(\frac{z}{(D_{\lambda,\delta}^{\alpha,k}f)(z)}\right)^{1+\mu}\in\mathcal{H}[q(0),1]\cap Q,\ \text{ and }\ \Omega(\alpha,k,\lambda,\delta;f)$$

is univalent in $\mathcal{U}$*, then*

$$q_1(z)+\beta\frac{zq_1'(z)}{q_1(z)}\prec\Omega(\alpha,k,\lambda,\delta;f)(z)\prec q_2(z)+\beta\frac{zq_2'(z)}{q_2(z)}\quad(\beta\in\mathbb{C};\ \beta\neq 0),$$

implies

$$q_1(z)\prec\left(D_{\lambda,\delta}^{\alpha,k}f\right)'(z)\left(\frac{z}{(D_{\lambda,\delta}^{\alpha,k}f)(z)}\right)^{1+\mu}\prec q_2(z)\quad(0\leq\mu\leq 1)\tag{3.6}$$

where q_1 *and* q_2 *are respectively the best subordinant and the dominant.*

Taking $k=0$, $\delta=0$; $\mu\to 0$ and 1 we have the following:

Corollary 3.6. *For* $k=0$, $\delta=0$; $\mu\to 0$, *let the functions* q_1 *and* q_2 *be univalent in* $\mathcal{U}$ *such that* $q_1(z)\neq 0$ *and* $q_2(z)\neq 0$ $(z\in\mathcal{U})$ *with* $\frac{zq_1'(z)}{q_1(z)}$ *and* $\frac{zq_2'(z)}{q_2(z)}$ *is univalent starlike. Furthermore assume that* q_1 *and* q_2 *satisfy* (3.1) *and* (3.4)*, respectively. If* $f\in\mathcal{A}$*, with*

$$\frac{zf'(z)}{f(z)}\in\mathcal{H}[q(0),1]\cap Q,$$

and

$$\frac{zf'(z)}{f(z)}+\beta\left[\frac{zf''(z)}{f'(z)}+\left(1-\frac{zf'(z)}{f(z)}\right)\right]$$

is univalent in $\mathcal{U}$*, then*

$$q_1(z)+\beta\frac{zq_1'(z)}{q_1(z)}\prec\frac{zf'(z)}{f(z)}\prec q_2(z)+\beta\frac{zq_2'(z)}{q_2(z)}\quad(\beta\in\mathbb{C};\ \beta\neq 0),$$

implies

$$q_1(z)\prec\frac{zf'(z)}{f(z)}\prec q_2(z)$$

where q_1 *and* q_2 *are respectively the best subordinant and the dominant.*

Corollary 3.7. [7] *For* $k=0$, $\delta=0$; $\mu\to 1$, *let the functions* q_1 *and* q_2 *be univalent in* $\mathcal{U}$ *such that* $q_1(z)\neq 0$ *and* $q_2(z)\neq 0$ $(z\in\mathcal{U})$ *with* $\frac{zq_1'(z)}{q_1(z)}$ *and* $\frac{zq_2'(z)}{q_2(z)}$ *is univalent starlike. Furthermore assume that* q_1 *and* q_2 *satisfies* (3.1) *and* (3.4)*, respectively. If* $f\in\mathcal{A}$*, with*

$$\frac{z^2f'(z)}{(f(z))^2}\in\mathcal{H}[q(0),1]\cap Q,$$

and

$$\frac{z^2f'(z)}{(f(z))^2}+\beta\left[\frac{zf''(z)}{f'(z)}+2\left(1-\frac{zf'(z)}{f(z)}\right)\right]$$

is univalent in $\mathcal{U}$*, then*

$$q_1(z)+\beta\frac{zq_1'(z)}{q_1(z)}\prec\frac{z^2f'(z)}{(f(z))^2}\prec q_2(z)+\beta\frac{zq_2'(z)}{q_2(z)}\quad(\beta\in\mathbb{C};\ \beta\neq 0),$$

implies

$$q_1(z) \prec \frac{z^2 f'(z)}{(f(z))^2} \prec q_2(z)$$

where q_1 and q_2 are respectively the best subordinant and the dominant.

THEOREM 3.8. *For $-1 \leq B < A \leq 1$ and $0 \leq \mu \leq 1$, if*

$$\begin{aligned} &\left(D_{\lambda,\delta}^{\alpha,k} f\right)'(z) \left(\frac{z}{(D_{\lambda,\delta}^{\alpha,k} f)(z)}\right)^{1+\mu} \\ &+ \beta \left[\frac{z(D_{\lambda,\delta}^{\alpha,k} f)''(z)}{(D_{\lambda,\delta}^{\alpha,k} f)'(z)} + (1+\mu)\left(1 - \frac{z(D_{\lambda,\delta}^{\alpha,k} f)'(z)}{(D_{\lambda,\delta}^{\alpha,k} f)(z)}\right)\right] \prec \frac{1+Az}{1+Bz} \quad (\beta > 0) \end{aligned} \tag{3.7}$$

then

$$\left(D_{\lambda,\delta}^{\alpha,k} f\right)'(z) \left(\frac{z}{(D_{\lambda,\delta}^{\alpha,k} f)(z)}\right)^{1+\mu} \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}), \tag{3.8}$$

where

$$q(z) = \begin{cases} \frac{\beta z^{1/\beta}(1+Bz)^{(A-B)/\beta B}}{\int_0^z t^{(1-\beta)/\beta}(1+Bt)^{(A-B)/\beta B} dt}; & (B \neq 0) \\ \frac{\beta z^{1/\beta} \exp(Az/\beta)}{\int_0^z t^{(1-\beta)/\beta} \exp(At/\beta) dt}; & (B = 0), \end{cases} \tag{3.9}$$

and q is the best dominant of (3.8).

Proof. Write

$$\phi(z) = \left(D_{\lambda,\delta}^{\alpha,k} f\right)'(z) \left(\frac{z}{(D_{\lambda,\delta}^{\alpha,k} f)(z)}\right)^{1+\mu} \quad (z \in \mathcal{U};\ z \neq 0;\ f \in \mathcal{A}).$$

Therefore, we observe that $\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathcal{U}$. Taking logarithmic differentiation and using (3.7) gives

$$\phi(z) + \beta \frac{z\phi'(z)}{\phi(z)} \prec \frac{1+Az}{1+Bz}$$

Application of Lemma 2.4 yields

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}),$$

where $q(z)$ is defined by (3.3) is the best dominant. This completes the proof. ∎

Taking $k = 0$, $\delta = 0$; $\mu \to 0$ and 1, we get the following corollaries, respectively:

COROLLARY 3.9. *For $-1 \leq B < A \leq 1$ and $\mu \to 0$, if $\beta \in \mathbb{C}$*

$$\frac{zf'(z)}{f(z)} + \beta \left[\frac{zf''(z)}{f'(z)} + \left(1 - \frac{zf'(z)}{f(z)}\right)\right] \prec \frac{1+Az}{1+Bz} \quad (\beta > 0)$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}),$$

where q is the best dominant.

COROLLARY 3.10. *For $-1 \leq B < A \leq 1$ and $\mu \to 1$, if $\beta \in \mathbb{C}$*

$$\frac{z^2 f'(z)}{(f(z))^2} + \beta \left[\frac{zf''(z)}{f'(z)} + 2\left(1 - \frac{zf'(z)}{f(z)}\right)\right] \prec \frac{1+Az}{1+Bz} \quad (\beta > 0)$$

then

$$\frac{z^2 f'(z)}{(f(z))^2} \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}),$$

where q is the best dominant.

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Department of Mathematics, Sambalpur University, Jyoti Vihar, Burla, Sambalpur, 768019, Odisha, India

E-mail: pb_gochhayat@yahoo.com