A NOTE ON MODULAR CURVES AND
FUNDAMENTAL UNITS OF NEGATIVE NORM

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Abstract. We re-prove the fact that the fundamental unit of the ring of integers of a real quadratic number field is of negative norm whenever the discriminant is a prime number congruent to 1 mod 4.

Let $d \in \mathbb{Z}$ be a square-free, positive integer and let $K = \mathbb{Q}(\sqrt{d})$ be the associated real quadratic number field. The question whether the fundamental unit of the corresponding ring of integers $\mathcal{O}$ has positive or negative norm has important consequences, both from a (number-)theoretical point of view (see e.g. [3]) and for applications (see e.g. [4]). Given a fixed d, there is in general no known criterion to decide this question despite of explicitly solving the negative Pell equation. However, it has already been known in the 19th century that the fundamental unit has negative norm if the discriminant D of $\mathcal{O}$ is a prime number congruent to 1 mod 4. In this note, we give a new proof of this fact using modular curves on Hilbert modular surfaces.

Let D be a square-free positive integer $\equiv 0$ or $1 \mod 4$ and let $\mathcal{O}_D$ denote the ring of integers of discriminant D and let $^\sigma$ denote the Galois automorphism. The corresponding Hilbert modular surface is defined as

$$X_D := \mathbb{H} \times \mathbb{H}/\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$$

where $\mathbb{H}$ denotes the complex upper half plane, $\mathcal{O}_D^\vee$ is the inverse different, and $\mathrm{SL}_2(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ acts on $\mathbb{H} \times \mathbb{H}$ via the Möbius transformations given by the two embeddings $\mathcal{O}_D \hookrightarrow \mathbb{R}$. We call a matrix in $\mathrm{GL}_2(K)$ primitive if it is of the form

$$\begin{pmatrix} \mu & bD \\ -a & -\mu^\sigma \end{pmatrix}$$

where $a, b \in \mathbb{Z}$, $\mu \in \mathcal{O}_D$ do not have a common divisor in $\mathbb{Z} \subset \mathcal{O}_D$. Given such a primitive matrix, we define

$$\mathbb{H}_U := \left\{ (\tau_1, \tau_2) \mid \tau_2 = -\frac{-\mu^\sigma \tau_1 + bD}{-a\tau_1 + \mu} \right\}$$

and let F_U denote its projection to X_D. Finally, let

$$F_N(\nu) = \{F_U \mid U \text{ is primitive, } \det(U) = N, \ \nu(U) = \pm\nu\}$$

where $\nu(U)$ is the image of μ in $\mathcal{O}_D/(\mathcal{O}_D^\vee)^{-1}$ and let

$$F_N = \bigcup \{F_U \mid U \text{ is primitive, } \det(U) = N\}$$

be the modular curve. From now on D will be a prime number $\equiv 1 \mod 4$. In [2], McMullen reproved a result originally going back to Franke [1] in a way which is easier to apply for us.

PROPOSITION 1. (Franke, [1, Corollary 3.7]) *If D is a prime then $F_N(\nu') = \emptyset$, whenever $F_N(\nu) \neq \emptyset$ and $\nu' \neq \pm\nu$.*

Now let us assume that $\mathcal{O}_D$ has a fundamental unit ϵ of positive norm. Then ϵ acts on $\mathbb{H} \times \mathbb{H}$ by its two embeddings into $\mathbb{R}$.

LEMMA 2. *We have*

$$\epsilon F_N(\nu) = F_N(\epsilon\nu).$$

Proof. If $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\epsilon \mathbb{H}_U = \epsilon \left\{ \left(\tau_1, \frac{d\tau_1 + b}{c\tau_1 + a}\right) \right\} = \left\{ \left(\epsilon\tau_1, \epsilon^\sigma \frac{d\tau_1 + b}{c\tau_1 + a}\right) \right\} = \left\{ \left(\tau_1, \frac{\epsilon^\sigma d\tau_1 + b}{c\tau_1 + a\epsilon}\right) \right\}. \quad \blacksquare$$

Moreover, neither ϵ nor $-\epsilon$ may preserve the class of ν in $\mathcal{O}_D/\mathcal{D}$ since $\epsilon \notin \mathcal{D}$ as one can e.g. see by considering the norm of $\sqrt{D}$. This is a contradiction. So we have proven:

THEOREM 3. *If the discriminant $D \equiv 1 \mod 4$ is a prime, then $\mathcal{O}_D$ has a fundamental unit of negative norm.*

REFERENCES

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