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AN EXISTENCE RESULT FOR A KIRCHHOFF $p(x)$-LAPLACIAN EQUATION

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Abstract. In this article, using Mountain Pass Theorem, we investigate the existence of a nontrivial weak solution for nonlocal equations driven by $p(x)$-Laplacian, under Dirichlet boundary condition.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and $N \geq 3$. In this paper, we consider the problem

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \mu f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous map, $p \in C_+(\bar{\Omega})$ with $1 < p^- := \inf_\Omega p(x) \leq p(x) \leq p^+ := \sup_\Omega p(x) < N$, $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$is a Carathéodory function satisfying some certain conditions and μ is a parameter.

The problem (1.1) is related to the stationary problem

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2\, dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

for $0 < x < L$, $t \geq 0$, where $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E is the Young modulus, ρ is the mass density, h is the cross-section area, L is the length and p_0 the initial axial tension, proposed by Kirchhoff [19] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Such nonlinear Kirchhoff model can also be used for describing the dynamics of an axially moving string. In recent years, axially moving string-like continua such as wires, belts, chains, band-saws have been subjects of the study of researchers (see [24]).

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In recent years, elliptic problems involving p-Kirchhoff type operators have been studied in many papers, we refer to [2, 3, 5, 10, 15–18, 20, 23, 25], in which the authors have used different methods to get the existence of the solutions for (1.1) in the case when $p(x) = p$ is a constant.

If $p : \bar{\Omega} \to \mathbb{R}$ is a continuous function, the problem (1.1) has been firstly studied by variational methods in [6, 7]. The $p(x)$-Laplacian possesses more complicated nonlinearities than p-Laplacian, for example it is not homogeneous. The study of differential equations and variational problems involving $p(x)$-growth conditions is a consequence of their applications.

Infinitely many solution of the problem (1.1) in the special case when $M(t) = a+bt$, has been studied by Dai and Liu in [7], by using a direct variational approach. In [4], the author considered the problem (1.1) in the case when $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying the following conditions:

$(M_1{}')$ there exists $m_2 \geq m_1 > 0, \delta_2 \geq \delta_1 > 1$ such that

$$m_1 t^{\delta_1 - 1} \leq M(t) \leq m_2 t^{\delta_2 - 1}$$

for all $t \in \mathbb{R}^+$;

$(M_2{}')$ for all $t \in \mathbb{R}^+$, $\hat{M}(t) \geq M(t)t$ holds, where $\hat{M}(t) = \int_0^t M(s)ds$;

and the special case

$$f(x,u) = \lambda \left(a(x)|u|^{\alpha(x)-2}u + b(x)|u|^{\beta(x)-2}u \right),$$

where $p, \alpha, \beta \in C(\bar{\Omega})$ satisfy

$$1 < \alpha^- \leq \alpha^+ < \delta_1 p^- < \delta_2 p^+ < \beta^- \leq \beta^+ < \min\left\{ N, \frac{Np^-}{N-p^-} \right\}.$$

Using the Mountain Pass Theorem and Ekeland variational principle, he has proved that the problem (1.1) has at least two distinct, nontrivial weak solution.

In the present paper, we establish the existence of a nontrivial weak solution of the problem (1.1) on a certain range of λ. For this purpose, we will adapt some arguments developed in [21]. In fact, we will make use of the Palais-Smale condition introduced by Ambrosetti-Rabinowitz in [1] to prove the existence of a nontrivial weak solution for the problem (1.1) which corresponds to the local minimum of the energy functional.

We assume that:

(M_1) there exists a constant m_0 such that $0 < m_0 \leq M(t)$, $\forall t \in [0, \infty)$;

(M_2) there exists $t_0 \geq 0$ such that $\hat{M}(t) \geq tM(t)$, for every $t \in [t_0, \infty)$, where $\hat{M}(t) := \int_0^t M(s)\, ds$.

Definition 1.1. We say that $u \in X = W_0^{1,p(x)}(\Omega)$ is a weak solution of the problem (1.1) if

$$M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v\, dx - \mu \int_\Omega f(x,u) v dx = 0,$$

for any $v \in X$.

Let us associate with the problem (1.1) the functional energy $\varphi : X = W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\varphi(u) = \Phi(u) - \mu\Psi(u),$$

where

$$\Phi(u) = \hat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right), \quad \Psi(u) = \int_\Omega F(x,u)\,dx, \tag{1.3}$$

and $F(x,u) = \int_0^u f(x,s)\,ds$. The functional φ associated with problem (1.1) is well defined and of C^1 class on X. Then

$$\begin{aligned}\langle \varphi'(u), u\rangle &= M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx - \mu\int_\Omega f(x,u)v\,dx \\ &= \langle \Phi'(u), v\rangle - \mu\langle \Psi'(u), v\rangle\end{aligned}$$

for all $u, v \in X$. Thus, weak solutions of the problem (1.1) are exactly the critical points of the functional φ.

2. Preliminaries

Recall that for a real Banach space X with topological dual X^*, we say that a C^1 -functional $\varphi : X \to \mathbb{R}$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ (briefly $(PS)_c$) when every sequence $\{u_n\}$ in X such that

$$\varphi(u_n) \to c \text{ and } \|\varphi'(u_n)\|_{X^*} \to 0,$$

as $n \to \infty$, possesses a convergent subsequence. We say that φ satisfies the Palais-Smale condition (in short (PS)) if $(PS)_c$ holds for every $c \in \mathbb{R}$.

Theorem 2.1. [1, 22] *Let $(X, \|\cdot\|)$ be a real Banach space and let $\varphi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function, such that $\varphi(0_X) = 0$ and satisfying the (PS) condition. Suppose that:*

(I_1) *there exist constants $\rho, \alpha > 0$ such that $\varphi(u) \geq \alpha$ if $\|u\| = \rho$,*

(I_2) *there exists $e \in X$ with $\|e\| > \rho$ such that $\varphi(e) \leq 0$.*

Then φ possesses a critical value $C \geq \alpha$, which can be characterized as

$$C := \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} \varphi(u),$$

where

$$\Gamma := \{\gamma \in C([0,1]; X) : \gamma(0) = 0 \wedge \gamma(1) = e\}.$$

For the reader's convenience, we recall some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spacea. We refer the reader to [8, 9, 11, 14] for details.

Let Ω be a bounded domain of $\mathbb{R}^N$. Denote

$$C_+(\Omega) = \{p : p \in C(\bar{\Omega}), p(x) > 1, \text{ for all } x \in \bar{\Omega}\};$$
$$p^+ = \max\{p(x); x \in \bar{\Omega}\}, \quad p^- = \min\left\{p(x); x \in \bar{\Omega}\right\};$$

$L^{p(x)}(\Omega) = \{u : u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)}dx < \infty\}$. Under the norm

$$\|u\|_{L^{p(x)}}(\Omega) = \|u\|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}$$

it becomes a Banach space [11]. We also define the space

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u(x)\|_{L^{p(x)}(\Omega)} + \|\nabla u(x)\|_{L^{p(x)}(\Omega)}.$$

We denote by ${W_0}^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Then $\|u\| = \|\nabla u\|_{L^{p(x)}(\Omega)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$.

Proposition 2.2. [11] *The space $({W_0}^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable Banach space. Moreover, if $q \in C_+(\bar{\Omega})$ and $1 \le q(x) < p^*(x)$ for all $x \in \bar{\Omega}$ then the embedding ${W_0}^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^*(x) = \infty$ if $p(x) \ge N$.*

Proposition 2.3. [8, 14] *(i) The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|u\|_{p(x)}\|v\|_{p'(x)}.$$

(ii) If $p_1, p_2 \in C_+(\Omega)$ and $p_1(x) \le p_2(x)$ for all $x \in \bar{\Omega}$, then

$$L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$$

and the embedding is continuous.

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many aspects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \le p^+ < \infty$ and continuous functions are dense if $p^+ < \infty$. An important role in manipulating the generalized Lebesgue-Sobolev space is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

PROPOSITION 2.4. [11] *If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold*

$$\|u\|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^+}$$

provided $\|u\|_{p(x)} > 1$, while

$$\|u\|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^-}$$

provided $\|u\|_{p(x)} < 1$ and

$$\|u_n - u\|_{p(x)} \to 0 \iff \rho_{p(x)}(u_n - u) \to 0.$$

LEMMA 2.5. [12] *Denote*

$$A(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx, \text{ for all } u \in X.$$

Then $A(u) \in C^1(X, R)$ and the derivative operator A' of A is

$$\langle A'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v\, dx \text{ for all } u, v \in X,$$

and we have

(1) *A is a convex functional;*
(2) *$A' : X \to X^*$ is a bounded homeomorphism and strictly monotone operator;*
(3) *A' is a mapping of type S_+, namely: $u_n \rightharpoonup u$ and $\limsup \langle A'(u_n), u_n - u\rangle \leq 0$, imply $u_n \to u$ (strongly) in X.*

3. Proof of the main result

Put

$$\lambda_{1,p(x)} = \inf \left\{ \frac{\int_\Omega |\nabla u(x)|^{p(x)}\, dx}{\int_\Omega |u(x)|^{p(x)}\, dx} : u \in {W_0}^{1,p(x)}(\Omega) \setminus \{0\} \right\}.$$

In [13], the authors were interested in the eigenvalues of the $p(x)$-Laplacian Dirichlet problem. They showed that Λ, the set of eigenvalues, is a nonempty infinite set such that $\sup \Lambda = +\infty$. Moreover, they proved that if there is a vector $l \in \mathbb{R}^N \setminus \{0\}$ such that for any $x \in \Omega$, $p(x + tl)$ is monotone for $t \in I_x = \{t \mid x + tl \in \Omega\}$, then $\lambda_* = \inf \Lambda > 0$.

LEMMA 3.1. [13] $\lambda_* > 0 \iff \lambda_{1,p(x)} > 0$.

THEOREM 3.2. *Let us assume that $M : [0, \infty) \to [0, \infty)$ is a continuous map such that conditions (M_1) and (M_2) hold. Further, require that $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function that verifies:*

(GR) the subcritical growth condition:

$$|f(x,t)| \le c(1+|t|^{q(x)-1}), \forall x \in \bar{\Omega}, \quad \forall t \in \mathbb{R},$$

where $c > 0$ *and* $p(x) < q(x) < p^*(x)$*;*

(AR) the Ambrosetti-Rabinowitz condition: there exists $t^* > 0$ *such that*

$$0 < \theta F(x,\xi) \le f(x,\xi)\xi, \quad \forall x \in \bar{\Omega}, \ \forall |\xi| \ge t^*,$$

where $\theta > p^+$.

We assume that

$$\limsup_{t\to 0} \frac{f(x,t)}{|t|^{p(x)-2}t} \le \lambda, \tag{3.1}$$

uniformly for $x \in \bar{\Omega}$*, where* $\lambda < m_0\lambda_{1,p(x)}(\Omega)$*. Then there exists* $\mu^* > 0$ *such that the problem* (1.1) *has at least one nontrivial weak solution in* X*, for* $\mu \in (0,\mu^*)$.

Proof. We will complete the proof of this theorem in four steps.

Step 1. We claim that there exist constants $m_1 > 0$ and $m_2 \ge 0$ such that

$$\frac{m_0\|u\|^{p^-}}{p^+} \le \phi(u) \le \frac{m_1\|u\|^{p^+}}{p^-} + m_2. \tag{3.2}$$

Pick $t_1 > t_0$, where t_0 appears in the relation (M_2). Then we have $\frac{M(t)}{\hat{M}(t)} \le \frac{1}{t}$ for every $t \in]t_1,\infty)$. So

$$\int_{t_1}^{t} \frac{M(s)}{\hat{M}(s)}\,ds = \log\frac{\hat{M}(t)}{\hat{M}(t_1)} \le \int_{t_1}^{t}\frac{ds}{s} = \log\frac{t}{t_1},$$

for every $t \in]t_1,\infty)$. Thus $\hat{M}(t) \le \frac{\hat{M}(t_1)}{t_1}t$ for every $t \in]t_1,\infty)$. Hence we can say that

$$\hat{M}(t) \le m_1 t + m_2,$$

for every $t \in [0,+\infty)$. For instance, $m_1 := \frac{\hat{M}(t_1)}{t_1}$ and $m_2 = \max_{t\in[0,t_1]}\hat{M}(t)$.

Noting this for constants m_1 and m_2, Step 1 is completed.

Step 2. We claim that every Palais-Smale sequence for the functional φ is bounded in $W_0^{1,p(x)}(\Omega)$.

Let $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ be a Palais-Smale sequence, that is, $\varphi(u_n) \to c$ for $c \in \mathbb{R}$ and

$$\|\varphi'(u_n)\|_{W^{-1,p(x)}} \to 0.$$

Suppose the contrary. Then passing to a subsequence if necessary, we may assume that $\|u_n\| \to +\infty$. By conditions M_1 and M_2, it follows that there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\varphi(u_n) - \frac{\langle\varphi'(u_n)\rangle}{\theta} &= \hat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right) \\
&\quad - \mu\int_\Omega F(x,u_n)\,dx - \frac{1}{\theta}\left(\langle\Phi'(u_n),u_n\rangle - \mu\int_\Omega f(x,u_n)u_n\,dx\right) \\
&\ge m_0\left(\frac{1}{p^+}-\frac{1}{\theta}\right)\|u_n\|^{p^-} + \mu\int_\Omega\left[\frac{f(x,u_n(x))u_n(x)}{\theta} - F(x,u_n(x))\right]dx
\end{aligned}$$

for every $n \geq n_0$. Thus

$$m_0\left(\frac{\theta-p^+}{p^+\theta}\right)\|u_n\|^{p^-} \leq \varphi(u_n) - \frac{1}{\theta}\langle\varphi'(u_n), u_n\rangle$$
$$-\mu\int_{|u_n|>t^*}\left(\frac{f(x,u_n)u_n}{\theta} - F(x,u_n)\,dx\right) + C\operatorname{meas}(\Omega), \quad \forall n \leq n_0,$$

where "meas(Ω)" denotes the standard Lebesgue measure of Ω and

$$C := \sup\left\{\left|\frac{f(x,t)t}{\theta} - F(x,t)\right| : x \in \bar{\Omega}, |t| < t^*\right\}.$$

Now, we observe that, the (AR) condition yields

$$\int_{|u_n(x)|>t^*}\left[\frac{f(x,u_n(x))u_n(x)}{\theta} - F(x,u_n(x))\right]\,dx \geq 0.$$

So, we deduce that

$$m_0\left(\frac{\theta-p^+}{p^+\theta}\right)\|u_n\|^{p^-} \leq \varphi(u_n) - \frac{<\varphi'(u_n), u_n>}{\theta} + \mu C\operatorname{meas}(\Omega),$$

for every $n \geq n_0$. Then, for every $n \geq n_0$ one has

$$C'\|u_n v\|^{p^-} \leq \left\{\varphi(u_n) + \|\varphi'(u_n)\|_{W^{-1,p(x)}(\Omega)}\frac{\|u_n\|}{\theta} + \mu C\operatorname{meas}(\Omega)\right\},$$

where $C' := m_0\left(\frac{\theta-p^+}{p^+\theta}\right) > 0$. In conclusion, dividing by $\|u_n\|$ and letting $n \to \infty$, we obtain a contradiction. This completes the proof of the claim.

STEP 3. We claim that the functional φ satisfies the compactness (PS) condition.

Take $\{u_n\} \subset X$ to be a Palais Smale sequence. Thus, by Step 2 the sequence $\{u_n\}$ is necessarily bounded in X. Since X is reflexive, we may extract a subsequence, that for simplicity we call again $\{u_n\}$, such that $u_n \rightharpoonup u$ in X. We will prove that u_n strongly converges to $u \in X$. Exploiting the derivative $\varphi(u_n)(u_n - u)$, we obtain

$$\langle\varphi'(u_n), u_n - u\rangle = \langle\Phi'(u_n), u_n - u\rangle + \mu\int_\Omega f(x,u_n(x))(u_n-u)\,dx,$$
$$\langle\Phi'(u_n), u_n - u\rangle = \langle\varphi'(u_n), u_n - u\rangle + \mu\int_\Omega f(x,u_n(x))(u_n-u)\,dx.$$

Since $\|\varphi'(u_n)\|_{W^{-1,p'(x)}(\Omega)} \to 0$ and the sequence $\{u_n - u\}$ is bounded in X, taking into account that

$$|\langle\varphi'(u_n), u_n - u\rangle| \leq \|\varphi'(u_n)\|_{X^*}\|u_n - u\|,$$

one has

$$\langle\varphi'(u_n), u_n - u\rangle \to 0.$$

Further, by (GR) and taking into account that $u_n \to u$ in $L^{q(x)}$ we obtain

$$\int_\Omega |f(x, u_n(x))|(u_n - u)\, dx \to 0.$$

We can conclude (by (M_1)) that

$$\langle A(u_n), u_n - u\rangle \to 0,$$

as $n \to \infty$. Since the operator A has the (S_+) property, in conclusion, $u_n \to u$ strongly in X. Hence, as claimed, the functional φ fulfills condition (PS).

Step 4. We claim that the functional φ has the geometry of the Mountain Pass Theorem. More precisely:

1) there exist $\mu^* > 0$ and $\rho, r > 0$ such that for any $\mu \in (0, \mu^*)$, we have

$$\varphi(u) > r > 0 \ \ \forall u \in X \text{ with } \|u\| = \rho;$$

2) for some $u_0 \in {W_0}^{1,p(x)}(\Omega) = X$ one has $\varphi(zu_0) \to -\infty$, as $z \to +\infty$.

We choose $\varepsilon > 0$ small enough, verifying

$$m_0 > \frac{\lambda + \varepsilon}{\lambda_1, p(x)(\Omega)}.$$

By condition (3.1) there exists $\delta_\varepsilon > 0$ such that $\frac{f(x,t)}{|t|^{p(x)-2}} \leq \lambda + \varepsilon$, for every $x \in \bar{\Omega}$ and $|t| < \delta_\varepsilon$. Hence,

$$F(x, \xi) \leq \frac{\lambda + \varepsilon}{p^+}|\xi|^{p^+},$$

for every $|\xi| \leq \delta_\varepsilon$. As a consequence of the above inequality, using (GR) condition, the Sobolev embedding $X \hookrightarrow L^{q(x)}(\Omega)$ and (3.2), we can write

$$\begin{aligned}
\varphi(u) &= \Phi(u) - \mu \int_\Omega F(x, u)\, dx \\
&\geq \hat{M}\left(\int_\Omega \frac{1}{p(x)}|\bigtriangledown u|^{p(x)} dx\right) - \mu \int_{|u| \leq \delta_\varepsilon} \frac{\lambda + \varepsilon}{p^+}|u|^{p^+} dx - D\mu \int_{|u| > \delta_\varepsilon} |u|^{q^+} dx \\
&\geq \frac{m_0}{p^+}\|u\|^{p^-} - \mu \frac{\lambda + \varepsilon}{p^+ \lambda_{1,p(x)}}\|u\|^{p^+} - D\mu\|u\|^{q^+},
\end{aligned}$$

for a suitable positive constant D. Hence, for any $u \in X$ with $\|u\| = 1$, we get

$$\varphi(u) \geq \frac{1}{p^+}\left(m_0 - \mu \frac{\lambda + \varepsilon}{\lambda_{1,p(x)}}\right) - D\mu. \tag{3.3}$$

Put $\mu^* = \frac{m_0 \lambda_{1,p(x)}}{\lambda + \varepsilon - Dp^+ \lambda_{1,p(x)}}$. Using (3.3), for any $\mu \in (0, \mu^*)$ we have $\varphi(u) > 0$ for all $u \in X$.

Next, pick $u_0 \in X$ such that $\text{meas}(\{x \in \Omega : u_0(x) \geq t^*\}) > 0$. Being $F(x,\xi)$ a θ-superhomogeneous function if $|\xi| > t^*$, for $z > 1$, we obtain

$$\begin{aligned}
\varphi(zu_0) &= \hat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla zu_0|^{p(x)}\,dx\right) - \mu\int_\Omega F(x,zu_0)\,dx \\
&\leq \frac{m_1}{p^-}\|zu_0\|^{p^+} + m_2 - \mu\int_\Omega F(x,zu_0)\,dx \\
&\leq \frac{m_1}{p^-}\|u_0\|^{p^+}z^{p^+} - z^\theta\mu\int_{|u_0|\geq t^*} F(x,u_0)dx + m_2 + \mu C\,\text{meas}(\Omega),
\end{aligned}$$

where $M := \{|F(x,\xi)| : x \in \bar{\Omega}, |\xi| \leq t^*\}$. Thus, the (AR) condition implies that $\varphi(zu_0) \to -\infty$ as $z \to +\infty$. This concludes the claim and completes the proof of the main theorem.

Note that in the last inequality we use the fact that

$$F(x,z\xi) \geq F(x,\xi)z^\theta,$$

for every $x \in \bar{\Omega}$, $|\xi| \geq t^*$ and $z \geq 1$. Indeed, for $z = 1$, clearly the equality hold. Otherwise, fix $|\xi| \geq t^*$ and define $g(x,z) := F(x,z\xi)$, for every $x \in \bar{\Omega}$ and $z \in]1,\infty)$. By (AR) condition it follows that

$$\frac{g'(x,z)}{g(x,z)} \geq \frac{\theta}{z} \text{ and } \theta F(x,z) \leq F'(x,z)z,$$

for every $x \in \bar{\Omega}$ and $z > 1$. Integrating in $]1,z]$ it follows that

$$\int_1^z \frac{g'(x,s)}{g(x,s)}\,ds = \log\frac{g(x,z)}{g(x,1)} \geq \theta\int_1^z \frac{ds}{s} = \log z^\theta.$$

In conclusion, since for every $x \in \bar{\Omega}, |\xi| \geq t^*$ and $z > 1$ one has

$$F(x,z\xi) =: g(x,z) \geq g(x,1)z^\theta = F(x,\xi)z^\theta. \quad \blacksquare$$

Remark. The (AR) condition that has appeared in this paper plays an important role in studying the existence of nontrivial solutions of many quasilinear elliptic boundary value problems. It is quite natural and important not only to insure that the Euler functional associated to problem (1.1) has a mountain pass geometry, but also to guarantee that Palais-Smale sequence is bounded. But this condition is very restrictive eliminating many nonlinearities. There are always many functions that do not satisfy this condition. For example, for the sake of simplicity, we consider $f(x,t) = 2t\ln(1+|t|)$, in the special case $p(x) \equiv 2$.

The procedure used in this paper can be applied for some other well known solvability conditions (see [26]).

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