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ON NUMERICAL SOLUTION OF SEMILINEAR SINGULAR PERTURBATION PROBLEMS BY USING THE HERMITE SCHEME ON A NEW BAKHVALOV-TYPE MESH [1]

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Abstract. A fourth-order finite-difference method for a semilinear singularly perturbed boundary value problem is studied. This method is based on Hermitian approximation of the second derivative on special new discretization mesh of Bakhvalov type. Numerical examples which demonstrate the effectiveness of the method are presented.

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This paper is concerned with the following singularly perturbed semilinear boundary value problem:

$$(1) \qquad -\varepsilon^2 u'' + c\,(x,u) = 0, \quad x \in I = [0,1]\,, \qquad u\,(0) = u\,(1) = 0,$$

where $\varepsilon \in (0, \varepsilon_0)\,,\ \varepsilon_0 << 1$, is a small perturbation parameter. For simplicity, we shall assume that $c \in C^\infty\,(I \times \mathbb{R})\,$, and

$$(2) \qquad 0 < \gamma^2 \le c_u\,(x,u)\,, \quad x \in I,\ u \in \mathbb{R}.$$

The condition (2) is the standard stability condition, which implies that both (1) and reduced problem $c\,(x,u) = 0$, have unique solutions u_ε and u_0, respectively, which are both in $C^\infty\,(I)$. If $u_0\,(0) \neq 0$ and $u_0\,(1) \neq 0$, the solution u_ε has a boundary layer of exponential type at $x = 0$ and $x = 1$. In general, the following estimate holds:

$$(3) \quad |u_\varepsilon^{(k)}(x)| \le \begin{cases} M\left(1 + \varepsilon^{-k} e^{-\gamma x/\varepsilon}\right), & x \in [0, 0.5]\,, \\ M\left(1 + \varepsilon^{-k} e^{-\gamma(1-x)/\varepsilon}\right), & x \in [0.5, 1]\,, \end{cases} \qquad k = 0, 1, \ldots,$$

see [11]. Here and throughout the paper, M, sometimes subscripted, denotes a generic positive constant, indepedent of ε and number of discretization subintervals n that will be used to solve (1) numerically.

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Problems of type (1) are probably the most frequently studied singular perturbation problems, both asymptotically and numerically, see [1], [2], [3], [4], [10], [11], [13], [15] and the references therein. This interest can be justified by several model problems arising in applications.

In this paper we shall consider the numerical method for (1) on a new discretization mesh of Baklhvalov type. The method considered was introduced by Herceg [4], and then studied and improved by Vulanović and Herceg [15], Vulanović [13], and Sun and Stynes [10].

The method discretizes the problem (1) on a special nonequidistant mesh that is dense in the boundary layers. It uses a nonequidistant generalization of the fourth order three-point finite-difference scheme known as the Hermite scheme (which we shall call the H-scheme), and a combination of the H-scheme with the standard central scheme (a combination of this kind will be denoted by HC). The HC-scheme is used also in [15] and [13].

Our mesh (which we shall call the H-mesh), and the meshes used in [4], [15] and [13] belong to those of Bakhvalov type, [1] and [11].

Herceg [4] assumed the following constraint on c in addition to (2) :

$$c_u(x,u) \le G(x), \quad x \in I, \ u \in \mathbb{R}, \qquad \min_{x\in I}\left\{5\gamma^2 - 2G(x)\right\} > 0.$$

This unpleasant condition here is eliminated. So, in this paper we consider only condition (2), as in the other papers that consider the same problems.

Our numerical results are obtained by solving boundary value problems which were considered in many papers. These results show that the theoretical order of convergence is also established numerically.

Let be noted that the Richardson extrapolation can also be used to improve the basic fourth order accuracy of the above scheme, on our new mesh, similarly as in [5] and [16].

1. Discretization mesh

In this paper we shall consider the discretization of problem (1) on the discretization mesh

$$I_h = \{x_i = \lambda(ih), \, i = 0, 1, \ldots, n\}, \qquad h = \frac{1}{n}, \qquad n \in \mathbb{N}. \tag{4}$$

The mesh generating function $\lambda(t)$ is given by

$$\lambda(t) = \begin{cases} \mu(\beta) + \mu'(\beta)(t-\beta), & t \in [0,\beta], \\ \mu(t) := \frac{a\varepsilon t}{q-t} + \delta, & t \in [\beta,\alpha], \\ \mu(\alpha) + \mu'(\alpha)(t-\alpha) & t \in [\alpha, 0.5], \\ 1 - \lambda(1-t), & t \in [0.5, 1]. \end{cases}$$

The constants a and q are independent of ε and satisfy:

$$q \in (0, 0.5), \qquad 0 < \varepsilon \leq \varepsilon_0 << 1, \qquad a > 0, \qquad a\varepsilon_0 < q. \tag{5}$$

The last condition guarantees the existence of the point $\alpha \in (0, q)$. The constant $\delta \geq 0$ is chosen so that $1 - 2\delta > 0$ and $\beta < \alpha$:

$$\delta = ap^2\varepsilon, \qquad p = \frac{1}{\kappa q - 1}, \qquad \kappa > \frac{1}{q - \sqrt{aq\varepsilon_0}}, \quad \kappa \in \mathbb{N}, \tag{6}$$

For simplicity, we assume that n is even and divisible by κ :

$$n = 2m, \quad m \in \mathbb{N}, \qquad n = 0 \,(\bmod\ \kappa). \tag{7}$$

From (7) follows that β belongs to I_h and the interval $(0, \beta]$ contains $\frac{n}{\kappa}$ points of discretization mesh.

The condition

$$h = \frac{1}{n} < \frac{q}{Q}, \qquad Q = 2\left(1 + \frac{\sqrt{3}}{3}\right) \approx 3.1547. \tag{8}$$

implies only that set the $I'_h \subset I_h$ can be constructed:

$$I'_h = \{t_i \in I_h : q - Qh < (i-1)h < \alpha \quad \text{or} \quad 1 - \alpha < (i+1)h < 1 - Qh\}.$$

Let us note that the set I'_h can be empty.

The value α is a unique point from $(0, q)$ which is the abscissa of the contact point of the tangent line from $(0.5, 0.5)$ to $\mu(t)$, and it can be found exactly from $\mu(\alpha) + \mu'(\alpha)\left(\frac{1}{2} - \alpha\right) = \frac{1}{2}$:

$$\alpha = \frac{q(1-2\delta) - \sqrt{aq\varepsilon\left((1-2q)(1-2\delta) + 2a\varepsilon\right)}}{1 - 2\delta + 2a\varepsilon}. \tag{9}$$

The point β is a unique point from $(0, q)$ which is the abscissa of the contact point of the tangent line from $(0, 0)$ to $\mu(t)$, and it can be found exactly from $\mu(\beta) + \mu'(\beta)(-\beta) = 0$:

$$\beta = \frac{q\sqrt{\delta}}{\sqrt{\delta} + \sqrt{a\varepsilon}}. \tag{10}$$

Now, the mesh generating function $\lambda(t)$ is given by

$$\lambda(t) = \begin{cases} a\varepsilon\left(\frac{\beta}{q-\beta} + \frac{q(t-\beta)}{(q-\beta)^2}\right) + \delta, & t \in [0, \beta], \\ \mu(t) := \frac{a\varepsilon t}{q-t} + \delta, & t \in [\beta, \alpha], \\ a\varepsilon\left(\frac{\alpha}{q-\alpha} + \frac{q(t-\alpha)}{(q-\alpha)^2}\right) + \delta & t \in [\alpha, 0.5], \\ 1 - \lambda(1-t), & t \in [0.5, 1]. \end{cases} \tag{11}$$

Let α and β be given by (9) and (10) respectively, and λ by (11). Then the following lemmas can be proved.

Lemma 1.1. *It holds that* $1-2\delta>0$ *and*

$$\sqrt{aq\left(1-2q\right)}<\frac{q-\alpha}{\sqrt{\varepsilon}}<\sqrt{aq}. \tag{12}$$

Lemma 1.2. *For the mesh generating function* λ *it holds*

$$0<\lambda'\left(t\right)\le\frac{1}{1-2q},\quad t\in\left[0,1\right],$$

and

$$\frac{2a\varepsilon q}{\left(q-\beta\right)^{3}}\le\lambda''\left(t\right)<\frac{2}{\left(1-2q\right)\sqrt{aq\varepsilon\left(1-2q\right)}},\quad t\in\left[\beta,\alpha\right].$$

Lemma 1.3. *It holds*

i) $\kappa q>1$, *ii)* $p>0$,
iii) $\beta=\frac{qp}{1+p}=\frac{1}{\kappa}$, *iv)* $\beta<q-\sqrt{aq\varepsilon}<\alpha$.

Lemma 1.4. *The mesh generating function* λ *satisfies* $M_0\sqrt{\varepsilon}\le\lambda\left(\alpha\right)\le M_1\sqrt{\varepsilon}$.

Lemma 1.5. *Let*

$$f\left(\varepsilon\right)=\varepsilon^{-k}e^{-\gamma/\sqrt{\varepsilon}},$$

where k *and* γ *are positive constants. Then*

$$\lim_{\varepsilon->0}f\left(\varepsilon\right)=0\qquad\text{and}\qquad f\left(\varepsilon\right)\le\left(\frac{2k}{\gamma e}\right)^{2k},\quad\varepsilon\in\left[0,0.5\right].$$

Lemma 1.6. *Let*

$$f\left(x\right)=\frac{1}{\left(q-x\right)^{k}}e^{-\frac{\gamma ax}{q-x}},$$

where a,q *and* γ *are positive constants. Then*

$$f\left(x\right)\le e^{a\gamma}\left(\frac{k}{aq\gamma e}\right)^{k},\quad x\in\left[0,q\right].$$

2. Difference scheme

In order to form a discretization of the problem (1) we approximate the differential equation of (1) by a difference formula of Hermite type in $x_i\in I_h\backslash I_h'$. The coefficients in this formula are not constant, i.e. they depend on x_{i-1}, x_i and x_{i+1} for all $i=1,2,\dots n-1$. These coefficients one can obtain in a similar way as on an equidistant mesh. Let

$$T^hw_i=a_1\left(i\right)w_{i-1}+a_0\left(i\right)w_i+a_2\left(i\right)w_{i+1}+b_1\left(i\right)w''_{i-1}+b_0\left(i\right)w''_i+b_2\left(i\right)w''_{i+1},$$

where $w_i = w(x_i)$,and $w_i'' = w''(x_i)$ for a function $w(x)$.

We obtain the coefficients $a_j(i)$ and $b_j(i)$, $j = 0, 1, 2$ from the system

$$T^h x_i^k = 0, \quad k = 0, 1, 2, 3, 4,$$

$$b_1(i) + b_0(i) + b_2(i) = 1.$$

Let $h_i = x_i - x_{i-1}, \quad i = 1, 2, \ldots n-1$, then we have

$$a_1(i) = \tfrac{-2}{h_i(h_i+h_{i+1})}, \quad a_0(i) = \tfrac{2}{h_i h_{i+1}}, \quad a_2(i) = \tfrac{-2}{h_{i+1}(h_i+h_{i+1})},$$

$$b_0(i) = a_0(i)\frac{h_i^2 + h_{i+1}^2 + 3h_{i+1}h_i}{12},$$

$$b_1(i) = -a_1(i)\frac{h_i^2 - h_{i+1}^2 + h_{i+1}h_i}{12}, \qquad b_2(i) = -a_2(i)\frac{h_{i+1}^2 - h_i^2 + h_{i+1}h_i}{12}.$$

Using this we approximate the differential equation of (1) at $x_i \in I_h \backslash I_h'$ by

$$\begin{aligned} F_i := \quad & \varepsilon^2 (a_1(i) w_{i-1} + a_0(i) w_i + a_2(i) w_{i+1}) + b_1(i) c(x_{i-1}, w_{i-1}) \\ & + b_0(i) c(x_i, w_i) + b_2(i) c(x_{i+1}, w_{i+1}) = 0. \end{aligned}$$

If I_h' is non-empty, we approximate (1) at $x_i \in I_h'$ by

$$F_i := \quad \varepsilon^2 (a_1(i) w_{i-1} + a_0(i) w_i + a_2(i) w_{i+1}) + c(x_i, w_i) = 0,$$

see [4].

We form a discrete analogue of problem (1) in the form $F(w) = 0$, where $F = (F_0, F_1, \ldots, F_n)$, and

$$\begin{aligned} F_0 := \quad & w_0 = 0, \\ F_i := \quad & \varepsilon^2 (a_1(i) w_{i-1} + a_0(i) w_i + a_2(i) w_{i+1}) + b_1(i) c(x_{i-1}, w_{i-1}) \\ & + b_0(i) c(x_i, w_i) + b_2(i) c(x_{i+1}, w_{i+1}) = 0, \quad i = 1, 2, \ldots n-1, \\ F_n := \quad & w_n = 0. \end{aligned} \tag{13}$$

We note that it holds $b_1(i) = b_2(i) = 0$, $b_0(i) = 1$ for $x_i \in I_h'$, and then we have a central difference scheme.

The obtained discrete analogue combines Hermite difference scheme (at $x_i \in I_h \backslash I_h'$) and central difference scheme (at $x_i \in I_h'$). This combination we shall call HC-scheme.

The solution $w^* = [w_0^*, w_1^*, \ldots, w_n^*]^\top$ to $F(w) = 0$, is an approximation to the exact solution u_ε of (1).

Let

$$u_{\varepsilon,h} = \left[u_\varepsilon(x_0), u_\varepsilon(x_1), \ldots, u_\varepsilon(x_n)\right]^\top,$$

be the restriction of u_ε on the discretization mesh. Our aim is to prove

$$\|u_{\varepsilon,h} - w^*\|_\infty \le Mh^4. \tag{14}$$

It is easy to see that for $i = 1, 2, \ldots, n-1$

$$\varepsilon^2 T^h u_\varepsilon(x_i) = \varepsilon^2 T^h u_\varepsilon(x_i) - F_i(w^*) = F_i(u_\varepsilon),$$

since $F_i(w^*) = 0$. From $F(u_\varepsilon) - F(w^*) = F(u_\varepsilon)$ it follows

$$F'(v)(u_\varepsilon - w^*) = F(u_\varepsilon) \tag{15}$$

for some $v = [v_0, v_1, \ldots, v_n]^\top$. From (15) we obtain (14) if $F'(v)^{-1}$ exists and if the following two estimates hold: $\left\|F'(v)^{-1}\right\|_\infty \le M$, and $\|F(u_\varepsilon)\|_\infty \le Mh^4$.

Let us prove the existence of w^* and (14). As the first step, we prove the following Lemma and Theorem.

Lemma 2.1. *On the H-mesh it holds that* $|u_\varepsilon(x_i) - u_\varepsilon(x_{i-1})| \le Mh$, $i = 1, 2, \ldots, n$.

Proof. We consider only the set $I_h \bigcap [0, 0.5]$ since for $I_h \bigcap [0.5, 1]$ the proof is analogous. Let us consider the following three sets:

$$\begin{aligned} \tau_\beta &= \{x \in I_h : 0 < x \le \lambda(\beta)\}, \\ \tau_c &= \{x \in I_h : \lambda(\beta) < x \le \lambda(\alpha)\}, \\ \tau_\alpha &= \{x \in I_h : \lambda(\alpha) < x \le 0.5\}. \end{aligned}$$

Since

$$u(x_i) - u(x_{i-1}) = \int_{x_{i-1}}^{x_i} u'(t)\,dt, \quad i = 1, 2, \ldots, \frac{n}{2}, \tag{16}$$

and, from (3),

$$|u'(x)| \le M\left(1 + \frac{1}{\varepsilon}e^{-\gamma x/\varepsilon}\right), \quad x \in [0, 0.5],$$

it follows

$$\left|\int_{x_{i-1}}^{x_i} u'(t)\,dt\right| \le M\left(1 + \frac{1}{\gamma}e^{-\gamma x/\varepsilon}\right)\Bigg|_{x_{i-1}}^{x_i} \le M(x_i - x_{i-1}). \tag{17}$$

For $x_i \in \tau_\beta$ it holds

$$x_j = a\varepsilon\left(\frac{\beta}{q-\beta} + \frac{q(jh-\beta)}{(q-\beta)^2}\right) + \delta, \quad j = i-1, i,$$

and

$$x_i - x_{i-1} = \frac{a\varepsilon q}{n\left(q-\beta\right)^2} = \frac{a\varepsilon\left(1+p\right)^2}{nq} \le Mh. \tag{18}$$

For $x_i \in \tau_c$ it holds

$$x_i - x_{i-1} = \mu\left(ih\right) - \mu\left((i-1)h\right) = \mu'\left(\theta\right)h, \quad \theta \in \left((i-1)h, ih\right).$$

Since

$$0 < \mu'\left(t\right) \le \frac{1}{1-2q}, \quad t \in [0,1],$$

it follows

$$\left|x_i - x_{i-1}\right| \le Mh. \tag{19}$$

For $x_i \in \tau_\alpha$ it holds

$$x_j = a\varepsilon\left(\frac{\alpha}{q-\alpha} + \frac{q\left(jh-\alpha\right)}{\left(q-\alpha\right)^2}\right) + \delta, \quad j = i-1, i,$$

and

$$x_i - x_{i-1} = \frac{a\varepsilon q}{n\left(q-\alpha\right)^2} \le \frac{a\varepsilon q}{nM\varepsilon} \le Mh, \tag{20}$$

since $M\sqrt{\varepsilon} \le q-\alpha$. Now, the proof of this Lemma follows from (17) and $(18), (19)$ and (20). □

Theorem 2.2. *Let the conditions* (5)-(8) *be satisfied. Then it holds*

$$\left\|F\left(u_\varepsilon\right)\right\|_\infty \le Mh^4.$$

Proof. We consider truncation error in the form from [4]. For a function $y \in C^6\left(I\right)$ it holds

$$\begin{aligned} Ry_i &= \frac{y^{(5)}\left(x_i\right)}{120}\left(-a_1\left(i\right)h_i^5 + a_2\left(i\right)h_{i+1}^5\right) + \frac{y^{(5)}\left(x_i\right)}{6}\left(-b_1\left(i\right)h_i^3 + b_2\left(i\right)h_{i+1}^3\right) \\ &\quad + \frac{y^{(6)}\left(\theta_i\right)}{720}\left(a_1\left(i\right)h_i^6 + a_2\left(i\right)h_{i+1}^6\right) + \frac{y^{(6)}\left(\sigma_i\right)}{24}\left(b_1\left(i\right)h_i^4 + b_2\left(i\right)h_{i+1}^4\right), \end{aligned}$$

where $\theta_i, \sigma_i \in \left(x_{i-1}, x_{i+1}\right)$. Simple calculation shows that

$$Ry_i = P_i y^{(5)}\left(x_i\right) + Q_i y^{(6)}\left(\theta_i\right) + S_i y^{(6)}\left(\sigma_i\right),$$

where

$$\begin{aligned} P_i &= \frac{1}{180}\left(h_{i+1} - h_i\right)\left(2h_i^2 + 2h_{i+1}^2 + 5h_i h_{i+1}\right), \\ Q_i &= -\frac{h_{i+1}^5 + h_i^5}{360\left(h_{i+1} + h_i\right)}, \qquad S_i = \frac{1}{144}\left(h_i^4 + h_{i+1}^4 - h_i^2 h_{i+1}^2\right). \end{aligned}$$

Since

$$\varepsilon^2 T^h u_\varepsilon(x_i) = \varepsilon^2 R u_\varepsilon(x_i)$$

$$R u_\varepsilon(x_i) = a_1(i)\, u_\varepsilon(x_{i-1}) + a_0(i)\, u_\varepsilon(x_i) + a_2(i)\, u_\varepsilon(x_{i+1})$$

$$+ b_1(i)\, c(x_{i-1}, u_\varepsilon(x_{i-1})) + b_0(i)\, c(x_i, u_\varepsilon(x_i)) + b_2(i)\, c(x_{i+1}, u_\varepsilon(x_{i+1})),$$

it follows

$$F(u_\varepsilon) = \varepsilon^2 R u_\varepsilon(x_i) = \varepsilon^2 \left(P_i u_\varepsilon^{(5)}(x_i) + Q_i u_\varepsilon^{(6)}(\theta_i) + S_i u_\varepsilon^{(6)}(\sigma_i) \right).$$

We shall prove Theorem by considering the following three parts of the truncation error $\varepsilon^2 R u_\varepsilon(x_i)$:

$$\varepsilon^2 P_i u_\varepsilon^{(5)}(x_i), \qquad \varepsilon^2 Q_i u_\varepsilon^{(6)}(\theta_i), \qquad \varepsilon^2 S_i u_\varepsilon^{(6)}(\sigma_i).$$

We consider only the set $I_h \bigcap [0, 0.5]$ since for $I_h \bigcap [0.5, 1]$ the proof is analogous. In the following we shall consider the sets:

$$\begin{aligned} \tau_1 &= \{ j \in I_n : 0 \le (j-1)\, h < \beta \}, \\ \tau_2 &= \{ j \in I_n : \beta \le (j-1)\, h < \alpha \}, \\ \tau_3 &= \{ j \in I_n : \alpha \le (j-1)\, h \le 0.5 \}. \end{aligned}$$

Case I. Suppose that $i \in \tau_1$, i.e.

$$0 \le (i-1)\, h < \beta. \tag{21}$$

There are three possibilities:

I.1) $ih < \beta$,

I.2) $ih = \beta < (i+1)\, h = \beta + h \le \alpha$,

I.3) $ih = \beta < \alpha < (i+1)\, h = \beta + h$.

Before we disscusing these three cases, we consider case I when the central difference scheme is applied.

Obviously, if $ih \le \beta$ then

$$q - Qh \le (i-1)\, h \le \beta - h \Longleftrightarrow h \ge \frac{q-\beta}{Q-1} = \frac{q}{(Q-1)(1+p)}.$$

From this we conclude that

$$q - Qh \le (i-1)\, h < \alpha, \tag{22}$$

since $\beta < \alpha$. It means that at x_i the central difference scheme is used. From (22) it follows

$$h > \frac{q-\alpha}{Q} > M\sqrt{\varepsilon}, \qquad \text{and} \qquad \varepsilon^2 \le M h^4.$$

In this case we have

$$Ru_\varepsilon(x_i) = a_1(i)\,u_\varepsilon(x_{i-1}) + a_0(i)\,u_\varepsilon(x_i) + a_2(i)\,u_\varepsilon(x_{i+1}) + c(x_i, u_\varepsilon(x_i)),$$

$$\varepsilon^2 u''_\varepsilon(x_i) = c(x_i, u_\varepsilon(x_i)),$$

and for some $\theta_i \in (x_{i-1}, x_{i+1})$

$$u''_\varepsilon(\theta_i) = a_1(i)\,u_\varepsilon(x_{i-1}) + a_0(i)\,u_\varepsilon(x_i) + a_2(i)\,u_\varepsilon(x_{i+1}) + c(x_i, u_\varepsilon(x_i)).$$

Using this we conclude

$$\begin{aligned}\left|\varepsilon^2 Ru_\varepsilon(x_i)\right| &\leq \varepsilon^2 M \max\{|u''_\varepsilon(x)| : x \in I\} \\ &\leq M\varepsilon^2\left(1 + \varepsilon^{-2}e^{-\gamma x_{i-1}/\varepsilon}\right) \leq M\left(\varepsilon^2 + e^{-Mn}\right) \leq Mh^4,\end{aligned}$$

which completes the proof in this case.

In the following we assume that

$$h < \frac{q-\beta}{Q-1} = \frac{q}{(Q-1)(1+p)}.$$

Because of (8) it must be $h < q/Q$, so we shall consider

$$h < \min\left\{\frac{q}{Q}, \frac{q}{(Q-1)(1+p)}\right\}$$

and prove

(23) $$h_i = x_i - x_{i-1} \leq M\varepsilon h, \quad i = 1, 2, \ldots, n$$

and

(24) $$h_{i+1} - h_i \leq M\varepsilon h^2, \quad i = 1, 2, \ldots, n.$$

Using these estimates we have in case **I.1** $P_i = 0$ and

$$\begin{aligned}\left|Q_i u_\varepsilon^{(6)}(\theta_i) + S_i u_\varepsilon^{(6)}(\theta_i)\right| &\leq M\varepsilon^4 h^4 \max_{x_{i-1}\leq x\leq x_{i+1}} \left|u_\varepsilon^{(6)}(x)\right| \\ &\leq M\varepsilon^4 h^4\left(1 + \frac{1}{\varepsilon^6}e^{-\gamma\frac{x_{i-1}}{\varepsilon}}\right) \\ &\leq M\varepsilon^4 h^4\left(1 + \frac{1}{\varepsilon^6}\right),\end{aligned}$$

i.e.

$$\left|\varepsilon^2 Ru_\varepsilon(x_i)\right| \leq M\varepsilon^6 h^4\left(1 + \frac{1}{\varepsilon^6}\right) \leq Mh^4.$$

In cases **I.2** and **I.3** it holds

$$\varepsilon^2 \left| P_i u_\varepsilon^{(5)}(x_i) \right| \le \varepsilon^2 M \varepsilon^3 h^4 \max_{x_{i-1} \le x \le x_{i+1}} \left| u_\varepsilon^{(5)}(x) \right| \le M \varepsilon^5 h^4 \left(1 + \frac{1}{\varepsilon^5} \right) \le M h^4.$$

So, Theorem is proved in this case too.

Now, we shall prove estimates (23) and (24) in all tree subcases of case **I**.

I.1) If $ih < \beta$, then $(i+1)h \le \beta$ and

$$h_{i+1} = h_i = x_i - x_{i-1} = \frac{a\varepsilon q}{n(q-\beta)^2} = \frac{a\varepsilon(1+p)^2}{nq} = M\varepsilon h$$

and $h_{i+1} - h_i = 0$.

I.2a) If $ih = \beta < (i+1)h = \beta + h \le \alpha$ and

$$h < \frac{q-\beta}{Q-1} = \frac{q}{(Q-1)(1+p)} \le \frac{q}{Q},$$

hold, then

$$h_i = x_i - x_{i-1} = \frac{a\varepsilon q}{n(q-\beta)^2} = \frac{a\varepsilon(1+p)^2}{nq} \le M\varepsilon h$$

and

$$\begin{aligned} h_{i+1} &= \lambda(\beta+h) - \lambda(\beta) = \frac{a\varepsilon h(1+p)^2}{q\left(1 - h\frac{1+p}{q}\right)} < \frac{a\varepsilon h(1+p)^2}{q\left(1 - \frac{1}{Q-1}\right)} \\ h_{i+1} &< \frac{a\varepsilon h(1+p)^2}{q\left(1 - \frac{1}{Q-1}\right)} \le \frac{a\varepsilon h(1+p)^2(Q-1)}{q(Q-2)} \le M\varepsilon h. \end{aligned}$$

For $h_{i+1} - h_i$ we obtain

$$\begin{aligned} h_{i+1} - h_i &= \frac{a\varepsilon h^2(1+p)^3}{q^2\left(1 - h\frac{1+p}{q}\right)} < \frac{a\varepsilon h^2(1+p)^3}{q^2\left(1 - \frac{1}{Q-1}\right)} \\ &< \frac{a\varepsilon h^2(1+p)^3(Q-1)}{q^2(Q-2)} \le M\varepsilon h^2. \end{aligned}$$

I.2b) If $ih = \beta < (i+1)h = \beta + h \le \alpha$ and

$$h < \frac{q}{Q} \le \frac{q-\beta}{Q-1} = \frac{q}{(Q-1)(1+p)},$$

then it holds

$$p \le \frac{1}{Q-1}, \qquad 1 + p \le \frac{Q}{Q-1}, \qquad Q - 1 - p \ge \frac{(Q-2)Q}{Q-1},$$

$$h_{i+1} = \lambda(\beta+h) - \lambda(\beta) = \frac{a\varepsilon h (1+p)^2}{q\left(1 - h\frac{1+p}{q}\right)}$$

$$h_{i+1} < \frac{a\varepsilon h (1+p)^2}{q\left(1 - \frac{1+p}{Q}\right)} \le \frac{a\varepsilon h Q^2}{q(Q-2)(Q-1)} \le M\varepsilon h.$$

For $h_{i+1} - h_i$ we obtain

$$h_{i+1} - h_i = \frac{a\varepsilon h^2 (1+p)^3}{q^2\left(1 - h\frac{1+p}{q}\right)} < \frac{a\varepsilon h^2 (1+p)^3}{q^2\left(1 - \frac{1+p}{Q}\right)}$$

$$h_{i+1} - h_i < \frac{a\varepsilon h^2 (1+p)^3 (Q-1)}{q^2 (Q-2)} \le \frac{a\varepsilon h^2 Q^3}{q^2 (Q-1)^2 (Q-2)} \le M\varepsilon h^2.$$

I.3a) If $ih = \beta < \alpha < (i+1)h = \beta + h$ and

$$h < \frac{q-\beta}{Q-1} = \frac{q}{(Q-1)(1+p)} \le \frac{q}{Q},$$

then

$$h_i = x_i - x_{i-1} = \frac{a\varepsilon q}{n(q-\beta)^2} = \frac{a\varepsilon (1+p)^2}{nq} \le M\varepsilon h$$

and

$$h_{i+1} = \lambda(\beta+h) - \lambda(\beta) = \frac{a\varepsilon h q\left(q - \beta - \frac{(\alpha-\beta)^2}{h}\right)}{(q-\beta)(q-\alpha)^2} < \frac{a\varepsilon h q}{(q-\alpha)^2},$$

since $\alpha - \beta < h$, $\alpha - \beta < q - \beta$ and

$$\frac{q - \beta - \frac{(\alpha-\beta)^2}{h}}{q-\beta} = 1 - \frac{(\alpha-\beta)^2}{(q-\beta)h} < 1.$$

Because of $\alpha < \beta + h$ it follows

$$\alpha < q\left(\frac{p}{1+p} + \frac{1}{(Q-1)(1+p)}\right)$$

and

$$h_{i+1} < \frac{a\varepsilon h q}{(q-\alpha)^2} < \frac{a\varepsilon h (1+p)^2 (Q-1)^2}{q(Q-2)^2} \le M\varepsilon h.$$

For $h_{i+1} - h_i$ we find

$$h_{i+1} - h_i = \frac{a\varepsilon q(\alpha-\beta)\left((2q-\alpha-\beta)h - (\alpha-\beta)(q-\beta)\right)}{(q-\alpha)^2 (q-\beta)^2}$$

$$h_{i+1} - h_i < \frac{aq\varepsilon h^2 \left(2q - \alpha - \beta - (q - \beta)\right)}{(q-\alpha)^2 (q-\beta)^2} = \frac{aq\varepsilon h^2}{(q-\alpha)(q-\beta)^2}$$

$$h_{i+1} - h_i < \frac{a\varepsilon h^2 (1+p)^3 (Q-1)}{q^2 (Q-2)} \leq M\varepsilon h^2.$$

I.3b) If $ih = \beta < \alpha < (i+1)h = \beta + h$ and

$$h < \frac{q}{Q} \leq \frac{q-\beta}{Q-1} = \frac{q}{(Q-1)(1+p)},$$

then

$$h_i = x_i - x_{i-1} = \frac{a\varepsilon q}{n(q-\beta)^2} = \frac{a\varepsilon (1+p)^2}{nq} \leq M\varepsilon h$$

and

$$h_{i+1} = \lambda(\beta + h) - \lambda(\beta) = \frac{a\varepsilon hq \left(q - \beta - \frac{(\alpha-\beta)^2}{h}\right)}{(q-\beta)(q-\alpha)^2} < \frac{a\varepsilon hq}{(q-\alpha)^2},$$

as in case **I.3a**). Further

$$\alpha < \beta + h < q\left(\frac{p}{1+p} + \frac{q}{Q}\right)$$

and

$$h_{i+1} < \frac{a\varepsilon hq}{(q-\alpha)^2} < \frac{a\varepsilon h (1+p)^2 Q^2}{q(Q-1-p)^2} \leq \frac{a\varepsilon h Q^2}{q(Q-2)^2}.$$

For $h_{i+1} - h_i$ in this case we obtain as in **I.3a**)

$$h_{i+1} - h_i \leq \frac{aq\varepsilon h^2}{(q-\alpha)(q-\beta)^2} = \frac{a\varepsilon h^2 (1+p)^2}{q(q-\alpha)}$$

$$h_{i+1} - h_i < \frac{a\varepsilon h^2 (1+p)^3 Q}{q^2 (Q-1-p)} \leq \frac{a\varepsilon h^2 Q^3}{q^2 (Q-1)^2 (Q-2)} \leq M\varepsilon h^2.$$

Case II. Let us assume $i \in \tau_2$, i.e. $\beta \leq (i-1)h < \alpha$. There are two subcases:

II.1) $q - Qh \leq (i-1)h < \alpha$.

The proof of Theorem in this case is the same as in case I assuming the same condition.

II.2) $(i-1)h < \min\{\alpha, q - Qh\}$.

If $(i-1)h < \alpha$ and $(i-1)h < q - Qh$, then holds

$$(i+1)h < q \qquad \text{i} \qquad q - (i+1)h \geq (q - (i-1)h)\frac{Q-2}{Q}.$$

Because of that and

$$h_{i+1} = \lambda\left(\left(i+1\right)h\right) - \lambda\left(ih\right) \le \lambda'\left(\left(i+1\right)h\right) \le Mh$$

we obtain

$$h_{i+1} \le \frac{Mh\varepsilon}{\left(q-\left(i-1\right)h\right)^2}.$$

Now, it holds

$$h_{i+1} - h_i \le Mh^2\lambda''\left(\left(i+1\right)h\right) \le \frac{Mh^2\varepsilon}{\left(q-\left(i-1\right)h\right)^3} \le \frac{Mh^2}{\sqrt{\varepsilon}}.$$

Since $q-\left(i-1\right)h > q-\alpha > M\sqrt{\varepsilon}$, it follows

$$\left|P_i u_\varepsilon^{(5)}\left(x_i\right)\right| \le M\varepsilon^3 h^4 \max_{x_{i-1}\le x\le x_{i+1}} \left|u_\varepsilon^{(5)}\left(x\right)\right|$$

and

$$\begin{aligned}\left|P_i u_\varepsilon^{(5)}\left(x_i\right)\right| &\le M\varepsilon^3 h^4\left(1+\varepsilon^{-5}e^{-\gamma ih/\varepsilon}\right)\frac{1}{\left(q-\left(i-1\right)h\right)^7}\\ &\le Mh^4\left(\frac{1}{\sqrt{\varepsilon}}+\varepsilon^{-2}\frac{e^{-\gamma a(i-1)h/(q-(i-1)h)}}{\left(q-\left(i-1\right)h\right)^7}\right)\end{aligned}$$

and, by Lemma (1.6),

$$\varepsilon^2\left|P_i u_\varepsilon^{(5)}\left(x_i\right)\right| \le Mh^4.$$

For $\varepsilon^2\left|Q_i u_\varepsilon^{(6)}\left(\theta_i\right)+S_i u_\varepsilon^{(6)}\left(\theta_i\right)\right|$ we obtain

$$\varepsilon^2\left|Q_i u_\varepsilon^{(6)}\left(\theta_i\right)+S_i u_\varepsilon^{(6)}\left(\theta_i\right)\right| \le Mh^4\left(1+\frac{e^{-\gamma a(i-1)h/(q-(i-1)h)}}{\left(q-\left(i-1\right)h\right)^8}\right) \le Mh^4,$$

which completes the proof in case **II**.

Case III. Let be assumed $i\in\tau_3$, i.e. $\left(i-1\right)h \ge \alpha$. Then is $h_i = h_{i+1}$ and

$$\varepsilon^{-4}e^{-\gamma(i-1)h/\varepsilon} \le \varepsilon^{-4}e^{-\lambda(\alpha)/\varepsilon} \le \varepsilon^{-4}e^{-M/\sqrt{\varepsilon}} \le M,$$

since $\lambda\left(\alpha\right) \ge M\sqrt{\varepsilon}$. For h_{i+1} we obtain again

$$h_{i+1} = \lambda\left(\left(i+1\right)h\right) - \lambda\left(ih\right) \le \lambda'\left(\left(i+1\right)h\right) \le Mh.$$

Since in this case $P_i = 0$ and

$$\left|Q_i u_\varepsilon^{(6)}\left(\theta_i\right)+S_i u_\varepsilon^{(6)}\left(\theta_i\right)\right| \le Mh^4\left(1+\varepsilon^{-6}e^{-M/\sqrt{\varepsilon}}\right)$$

we obtain by Lemma (1.5) ,

$$\varepsilon^2 \left|Ru_\varepsilon(ih)\right| = \varepsilon^2 \left|Q_i u_\varepsilon^{(6)}(\theta_i) + S_i u_\varepsilon^{(6)}(\theta_i)\right| \le Mh^4 \left(1 + \varepsilon^{-4} e^{-M/\sqrt{\varepsilon}}\right) \le Mh^4.$$

The proof of existence of solution of $F(w) = 0$ is based on the proof of the following relation: $\left\|F'(v)^{-1}\right\|_\infty \le M$, where $F'(v)$ is Frechet-derivative of F.

Obviously, the mapping F defined by (13) is continuously differentiable in $\mathbb{R}^{n+1}$. The Frechet-derivative $F'(v)$ of F for an arbitrary $v = [v_0, v_1, \ldots, v_n]^\top \in \mathbb{R}^{n+1}$ is the tridiagonal matrix

$$F'(v) = \begin{bmatrix} 1 & & & & & \\ A_1 & B_1 & C_1 & & & \\ & A_2 & B_2 & C_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_{n-1} & B_{n-1} & C_{n-1} \\ & & & & & 1 \end{bmatrix},$$

where

$$\begin{aligned} A_i &= \varepsilon^2 a_1(i) + b_1(i)\, c_u(x_{i-1}, v_{i-1}), \\ B_i &= \varepsilon^2 a_0(i) + b_0(i)\, c_u(x_i, v_i), \\ C_i &= \varepsilon^2 a_2(i) + b_2(i)\, c_u(x_{i+1}, v_{i+1}). \end{aligned}$$

Let

$$B_0 = B_n = 1, \qquad A_0 = A_n = C_0 = C_n = 0,$$

and

$$\sigma = \min\{B_i - |A_i| - |C_i| : i = 0, 1, \ldots, n\}.$$

Our aim is to find $\sigma_* > 0$, independent of n and ε, such that

$$\sigma \ge \sigma_* > 0, \quad v \in S\left(u_\varepsilon, Mh^4\right) = \left\{y \in \mathbb{R}^{n+1} : \|y - u_\varepsilon\| < Mh^4\right\}. \tag{25}$$

Because of that $F'(v)^{-1}$ exists and it holds

$$\left\|F'(v)^{-1}\right\|_\infty \le \frac{1}{\sigma_*}. \tag{26}$$

Now by the Hadamard Theorem [7], it follows that the equation $F(w) = 0$ has a unique solution w^*.

Theorem 2.3. *Suppose that the condition* (2) *is satisfied. There exists a constant $M_0 > 0$ independent of n and ε and positive integer n_0 which depends on M_0 but is independent of ε, such that the discrete analogue $F(w) = 0$ has a unique solution w^* for which holds*

$$\|u_{\varepsilon,h} - w^*\|_\infty \le M_0 h^4 = \frac{M_0}{n^4}, \quad n \ge n_0. \tag{27}$$

Proof. On the H-mesh holds, 2.2,

$$\|F(u_{\varepsilon,h})\|_\infty \leq M_1 h^4,$$

where the constant M_1 is independent of n and ε. Let

$$M_0 = \frac{2M_1}{\min\{1, \gamma^2/6\}},$$

then

$$\frac{1}{\min\{1, \gamma^2/6\}} \|F(u_{\varepsilon,h})\|_\infty < M_0 h^4. \tag{28}$$

We shall prove the existence of a positive integer n_0 which depends on M_0 but is independent of ε, such that

$$\left\|F'(v)^{-1}\right\|_\infty \leq \frac{1}{\min\{1, \gamma^2/6\}}, \quad v \in S\left(u_\varepsilon, Mh^4\right). \tag{29}$$

Since $a_1(i) + a_0(i) + a_2(i) = 0$, $i = 1, 2, \ldots, n-1$, and

$$s_i = B_i - |A_i| - |C_i|, \quad i = 1, 2, \ldots, n-1$$

we obtain

$$\begin{aligned} s_i &\geq b_0(i)\, c_u(x_i, v_i) - |b_1(i)|\, c_u(x_{i-1}, v_{i-1}) - |b_2(i)|\, c_u(x_{i+1}, v_{i+1}) \\ &\geq \frac{h_i^2 + h_{i+1}^2 + 3h_{i+1}h_i}{6h_{i+1}h_i} c_u(x_i, v_i) - \frac{h_i^2 + h_{i+1}^2 + h_{i+1}h_i}{6h_i(h_{i+1} + h_i)} c_u(x_{i-1}, v_{i-1}) \\ &\quad - \frac{h_i^2 + h_{i+1}^2 + h_{i+1}h_i}{6h_{i+1}(h_{i+1} + h_i)} c_u(x_{i+1}, v_{i+1}) \end{aligned}$$

and

$$\begin{aligned} s_i \geq\ & \frac{1}{3} c_u(x_i, v_i) \\ & - \frac{h_i^2 + h_{i+1}^2 + h_{i+1}h_i}{6h_i(h_{i+1} + h_i)} \left(h_i c_{xu}(\bar{t}_i, \bar{v}_i) - (v_i - v_{i-1})\, c_{uu}(\bar{t}_i, \bar{v}_i)\right) \\ & - \frac{h_i^2 + h_{i+1}^2 + h_{i+1}h_i}{6h_{i+1}(h_{i+1} + h_i)} \left(h_{i+1} c_{xu}\left(\tilde{t}_i, \tilde{v}_i\right) - (v_{i+1} - v_i)\, c_{uu}\left(\tilde{t}_i, \tilde{v}_i\right)\right), \end{aligned} \tag{30}$$

for some $(\bar{x}_i, \bar{v}_i)$ and $(\tilde{x}_i, \tilde{v}_i)$.

From 3 it follows the existence of M_2 such that

$$\max|u_\varepsilon(x) : x \in I| \leq M_2.$$

Now

$$(31)\qquad |c_{xu}(x,v)| + |c_{uu}(x,v)| \le M, \quad (x,v) \in I \times [-M_2 - 1, M_2 + 1].$$

Let n_1 be chosen so that $\frac{M_0}{n_1^4} \le 1$. From now on we assume that $n \ge n_1$ and $v = [v_0, v_1, \ldots, v_n]^\top \in S(u_\varepsilon, Mh^4)$. Then

$$(32)\qquad |v_i| \le M_2 + 1, \quad i = 0, 1, \ldots, n$$

and $|\bar{v}_i| \le M_2 + 1$ and $|\tilde{v}_i| \le M_2 + 1, \quad i = 0, 1, \ldots, n$. For $i = 1, 2, \ldots, n$ by Lemma 2.1 it follows

$$(33)\qquad \begin{aligned} |v_i - v_{i-1}| &\le |v_i - u_\varepsilon(x_i)| + |v_{i-1} - u_\varepsilon(x_{i-1})| \\ &\quad + |u_\varepsilon(x_i) - u_\varepsilon(x_{i-1})| \\ &\le 2M_0 h^4 + Mh. \end{aligned}$$

Using 30-33 we obtain for $n \ge n_1$

$$B_i - |A_i| - |C_i| \ge \frac{1}{3} c_u(x,v) - M_3 h, \quad i = 1, 2, \ldots, n-1,$$

where M_3 depends of M_0 but is independent of ε. If a positive integer n_2 is chosen so that

$$M_3 h < \frac{\gamma^2}{6}, \quad n \ge n_2,$$

and if $n_0 = \max\{n_1, n_2\}$, then for $n \ge n_0$ it holds

$$B_i - |A_i| - |C_i| \ge \frac{\gamma^2}{3} - \frac{\gamma^2}{6} = \frac{\gamma^2}{6}, \quad i = 1, 2, \ldots, n-1.$$

Now we have

$$B_i - |A_i| - |C_i| \ge \min\left\{1, \frac{\gamma^2}{6}\right\}, \quad i = 0, 1, 2, \ldots, n,$$

and (29) is true with $\sigma_* = \min\left\{1, \frac{\gamma^2}{6}\right\}$. □

Using (28) and (29) we conclude that F satisfies the conditions of Hadamard Theorem, [7] Because of that $F(w) = 0$ has a unique solution $w^* \in S(u_\varepsilon, Mh^4)$, i.e. it holds (27).

3. Numerical results

Let us present numerical results for the following test problem:

$$-\varepsilon^2 u'' + u + \cos^2(\pi x) + 2(\varepsilon\pi)^2 \cos(2\pi x) = 0, \qquad u(0) = u(1) = 0,$$

which was considered also in papers: [4], [16] and [12]. The exact solution of this problem is

$$u_\varepsilon(x) = \frac{e^{-\frac{x}{\varepsilon}} + e^{-\frac{1-x}{\varepsilon}}}{1 + e^{-\frac{1}{\varepsilon}}} - \cos^2(\pi x).$$

The errors $E_n = \|u_{\varepsilon,h} - w^*\|_\infty$, where w^* is the numerical solution on a mesh with n subintervals, are given in Tables 1-4. Also, we define in the usual way the order of convergence Ord for the two successive values of n with respective errors E_n and E_{2n} :

$$Ord = \frac{\ln E_n - \ln E_{2n}}{\ln 2}.$$

We expect that $Ord = 4$.

The H-mesh was used with $a = 2$ and $q = 0.48$. We used $\delta = d\varepsilon$, where $d = \frac{a}{(\kappa q - 1)^2}$ and κ is positive integer greater than $\frac{1}{q - \sqrt{aq\varepsilon}}$.

$n\|\varepsilon$	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}	2^{-11}	2^{-12}
64	2.63143(-6)	7.3109(-6)	4.86673(-6)	4.86663(-6)	3.60429(-5)	1.20245(-5)	4.86663(-6)	4.86663(-6)	4.86663(-6)
128	1.65607(-7) 3.99001	4.608(-7) 3.98784	3.07766(-7) 3.98305	3.07761(-7) 3.98304	3.07761(-7) 6.87176	3.29464(-6) 1.86778	1.57585(-6) 1.6268	4.574(-7) 3.4114	3.07761(-7) 3.98304
256	1.03697(-8) 3.99733	2.87822(-8) 4.00089	1.9247(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 7.41937	4.01002(-7) 1.97445	1.57775(-7) 1.53559	4.97625(-8) 2.62868
512	6.48871(-10) 3.99733	1.79873(-9) 4.00089	1.20314(-9) 3.99913	1.20312(-9) 3.99913	1.33201(-9) 3.99913	1.20312(-9) 7.41937	1.20312(-9) 1.97445	1.14457(-8) 1.53559	1.51622(-8) 2.62868
1024	4.04372(-11) 4.00418	1.12431(-10) 3.99987	7.51835(-11) 4.00025	7.51853(-11) 4.00018	8.55763(-11) 3.96025	7.69261(-11) 3.96716	7.51813(-11) 4.00026	7.51852(-11) 7.25015	7.51823(-11) 7.65587

$n\|\varepsilon$	2^{-14}	2^{-15}	2^{-16}	2^{-17}	2^{-18}	2^{-19}	2^{-20}	2^{-21}	2^{-22}
64	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)
128	3.07761(-7) 3.98304	3.07761(-7) 3.98304	3.07761(-7)) 3.98304	3.07761(-7) 3.98304	3.07761(-7) 3.98304	3.07761(-7) 3.98304	3.07761(-7) 3.98304	3.07761(-7) 3.98304	3.07761(-7) 3.98304
256	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913
512	1.20312(-9) 3.99976	1.20311(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976
1024	2.95545(-10) 2.02533	7.71797(-11) 3.96241	7.51814(-11) 4.00026	7.51857(-11) 4.00017	7.51821(-11) 4.00025	7.51849(-11) 4.00019	7.51815(-11) 4.00026	7.51857(-11) 4.00017	7.51822(-11) 4.00025

$n\|\varepsilon$	2^{-24}	2^{-26}	2^{-28}	2^{-30}	2^{-32}	2^{-34}	2^{-36}	2^{-38}	2^{-40}
64	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)
128	3.07761(-7) 3.98304	3.07761(-7) 3.98304	3.07761(-7)) 3.98304	3.07761(-7) 3.98304	3.07761(-7) 3.98304	3.07761(-7) 3.98304	3.07761(-7) 3.98304	3.07761(-7) 3.98304	3.07761(-7) 3.98304
256	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913
512	1.20312(-9) 3.99976	1.20311(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976
1024	7.51824(-11) 4.00024	7.51823(-11) 4.00024	7.51822(-11) 4.00025	7.5182(-11) 4.00025	7.5182(-11) 4.00025	7.5182(-11) 4.00025	7.5182(-11) 4.00025	7.5182(-11) 4.00025	7.5182(-11) 4.00025

$n\|\varepsilon$	2^{-42}	2^{-44}	2^{-46}	2^{-48}	2^{-50}	2^{-52}	2^{-54}	2^{-56}	2^{-58}
64	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.86663(-6)	4.9409(-3)
128	3.07761(-7) 3.98304	3.07761(-7) 3.98304	3.07761(-7)) 3.98304	3.07761(-7) 3.98304	3.07761(-7) 3.98304	3.07761(-7) 3.98304	3.07761(-7) 3.98304	3.07761(-7) 3.98304	3.07761(-7) 13.9707
256	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913	1.92467(-8) 3.99913
512	1.20312(-9) 3.99976	1.20311(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976	1.20312(-9) 3.99976
1024	7.51824(-11) 4.00024	7.51823(-11) 4.00024	7.51822(-11) 4.00025	7.5182(-11) 4.00025	7.5182(-11) 4.00025	7.5182(-11) 4.00025	7.5182(-11) 4.00025	7.5182(-11) 4.00025	7.5182(-11) 4.00025

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