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RIEMANN-STIELTJES TYPE INTEGRAL BASED ON GENERATED PSEUDO-OPERATIONS

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Abstract. We consider generated pseudo-operations of the following form: $x \oplus y = g^{(-1)}(g(x) + g(y))$, $x \odot y = g^{(-1)}(g(x)g(y))$, where g is a positive strictly monotone generating function and $g^{(-1)}$ is its pseudo-inverse. Using this type of pseudo-operations, the Riemann-Stieltjes type integral is introduced and investigated.

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1. Introduction

It is a well known fact that classical Riemann-Stieltjes integral has great application in several areas of analysis as well as in probability theory and physics (see [3, 4]). The main aim of this paper is to present pseudo-analysis' counterpart of this integral. By means of generalized generated pseudo-operations and corresponding measure-like set function, pseudo Riemann-Stieltjes integral has been constructed. This new Riemann type of integral belongs to the area of pseudo-analysis, i.e., to the theory that combines approaches from many different fields and is capable of supplying solutions that were not achieved by the classical tools. Some of important results concerning pseudo-analysis, both theory and application, can be found in [1, 2, 5, 6, 7, 10, 13, 16, 18].

Section 2 contains preliminary notions, such as semiring, generalized generated pseudo-operations and measure-like set function. Operations in question are generalizations of pseudo-operations that are in the core of g-calculus ([9, 11, 12, 13, 14]). Set function defined in this section is based on generating function g and, under some additional conditions, has properties of fuzzy measure ([13, 18]). In the third section is presented the construction of pseudo Riemann-Stieltjes integral. Some of its basic properties are given for case when generating function g is strictly monotone and continuous function (we should stress that in this case used operations are operations from g-semiring). Specifically, for strictly increasing bijection g strong connection between the proposed integral and Lebesque integral is obtained. Additionally, we give connections

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with g-integral ([12, 13]) and another Lebesgue type of integral known as pseudo-Lebegue-Stieltjes integral ([8]).

2. Preliminary notions

The basic preliminary notions needed in this paper are notions of generated pseudo-operations. More precisely, the operations in question are both the so-called g-operations, i.e. operations from g-semiring and the generalizations of g-operations. Before introducing this form of generated pseudo-operations we give a short overview of the semiring ([7, 10, 13]).

Let $[a, b]$ be a closed subinterval of $[-\infty, +\infty]$ (in some cases semiclosed subintervals will be considered) and let $\preceq$ be total order on $[a, b]$. A semiring is the structure $([a, b], \oplus, \odot)$ when the following holds:

- $\oplus$ is *pseudo-addition*, i.e., a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero element, denoted by $\mathbf{0}$;
- $\odot$ is *pseudo-multiplication*, i.e., a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing $(x \preceq y$ implies $x \odot z \preceq y \odot z,\ z \in [a, b]_+ = \{x : x \in [a, b],\ \mathbf{0} \preceq x\})$, associative and for which there exists a unit element denoted by $\mathbf{1}$;
- $\mathbf{0} \odot x = \mathbf{0}$;
- $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

There are three basic classes of semirings with continuous (up to some points) pseudo-operations. The first class contains semirings with idempotent pseudo-addition and non-idempotent pseudo-multiplication. Semirings with strict pseudo-operations defined by strictly monotone and continuous generator function $g : [a, b] \to [0, +\infty]$, i.e. g-semirings, form the second class, and semirings with both idempotent operations belong to the third class. More on this structure as well as on measures and integrals constructed on it can be found in [7, 10, 12, 13, 14, 15].

As already mentioned, of special interest for this paper are g-operations, i.e., operations given by the strictly monotone and continuous generator $g : [a, b] \to [0, +\infty]$ in the following manner

$$(1) \qquad x \oplus y = g^{-1}(g(x) + g(y)) \quad \text{and} \quad x \odot y = g^{-1}(g(x)g(y))$$

(see [12, 13]), as well as the generalization of g-operations. In this paper, in order to give more general definition of integral in question, constrains normally put on the generating function g are weakened in such sense that generating function g does not need to be continuous, just strictly monotone. Therefore, in order to define generalized g-operations, it is necessary to use the pseudo-inverse function of the generator g instead of the classical inverse function.

Remark 1. For non-decreasing function $f : [a,b] \to [a_1,b_1]$, where $[a,b]$ and $[a_1,b_1]$ are closed subintervals of extended real line $[-\infty,+\infty]$, the pseudo-inverse is $f^{(-1)}(y) = \sup\{x \in [a,b] \mid f(x) < y\}$. If f is non-increasing function, its pseudo-inverse is $f^{(-1)}(y) = \sup\{x \in [a,b] \mid f(x) > y\}$. More on this subject can be found in [6].

Definition 2. *Let g be a positive strictly monotone function defined on $[a,b] \subset [-\infty,+\infty]$ such that $0 \in Ran(g)$.* The generalized generated pseudo-addition $\oplus$ *and* the generalized generated pseudo-multiplication $\odot$ *are given by*

$$x \oplus y = g^{(-1)}(g(x) + g(y)), \tag{2}$$

$$x \odot y = g^{(-1)}(g(x)g(y)), \tag{3}$$

where $g^{(-1)}$ is pseudo-inverse function for function g.

Remark 3. It is obvious that if the generator g from the previous definition is monotone bijection $g : [a,b] \to [0,\infty]$, g-semiring is obtained and operations are given by (1).

Remark 4. Importance of operations generated by monotone generating functions can be easily illustrated through the theory of triangular norms ([6]).

Let us first consider strictly monotone generators that are not necessary continuous. It is well known fact that if $t : [0,1] \to [0,\infty]$ is a strictly decreasing function such that $t(1) = 0$ and $t(x) + t(y) \in Ran(t) \cup [t(0^+),\infty]$ for all $(x,y) \in [0,1]^2$, function $T : [0,1]^2 \to [0,1]$ of the form

$$T(x,y) = t^{(-1)}\left(t(x) + t(y)\right) \tag{4}$$

is a t-norm. Therefore, some important t-norms can be considered as generalized generated pseudo-additions on $[0,1]$. For example, drastic product, i.e. t-norm of the form

$$T_D(x,y) = \begin{cases} \min\{x,y\}, & \text{if } \max\{x,y\} = 1, \\ 0, & \text{otherwise,} \end{cases}$$

is given by the generator $t : [0,1] \to [0,\infty]$ where $t(x) = 2 - x$ for $x \in [0,1)$ and $t(1) = 0$.

Strictly monotone and continuous generators give us continuous Archimedean t-norms and t-conorms, e.g., T is a continuous Archimedean t-norm if and only if there exists a continuous, strictly decreasing function $t : [0,1] \to [0,\infty]$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that for all $(x,y) \in [0,1]^2$ holds (4). It should be stressed that we can distinguish two parts of this group. If $t(0) = \infty$, i.e., t is a bijection, obtained t-norms are strict t-norms of the form

$$T(x,y) = t^{-1}\left(t(x) + t(y)\right),$$

and are examples of pseudo-operations from g-semiring on $[0, 1]$. On the other hand, if $t(0) < \infty$, obtained t-norms are nilpotent t-norms that remain in the form (4). Example for this nilpotent case is well known Lukasiewicz t-norm $T_L(x, y) = \max\{0, x + y - 1\}$. It is generated by function $t : [0, 1] \to [0, \infty]$, $t(x) = 1-x$, and since its pseudo-inverse $t^{(-1)} : [0, \infty] \to [0, 1]$ is $t^{(-1)}(x) = 1-x$ for $x \in [0, 1]$ and $t^{(-1)}(x) = 0$ for $x \in [1, \infty]$, it can be easily shown that (4) holds. This nilpotent case can be of special interest for future studies since it hasn't been considered in the g-calculus so far.

Regarding strictly monotone and continuous generators, some more general observation can be given. If $g : [a, b] \to [0, \infty]$ is strictly monotone and continuous function such that $0 \in Ran(g)$, operation given by (2) is an associative operation on $[a, b]^2$.

Results used in this remark and more on triangular norms, both properties and applications, can be found in [6].

As in the case of g-semiring, monotonicity of generating function g is closely connected with the order $\preceq$ on $[a, b]$, i.e., $x \preceq y \Leftrightarrow g(x) \leq g(y)$. Additionally, $x \prec y$ if and only if $g(x) \leq g(y)$ and $x \neq y$.

Further on, by pseudo-addition and pseudo-multiplication, if not stated otherwise, operations (2) and (3) will be considered.

It is obvious that operations (2) and (3) are commutative, however, they need not be associative. Some basic properties of pseudo-operations in question are given by the following proposition.

Proposition 5. *Let $\oplus$ and $\odot$ be pseudo-operations from Definition 2 and $x, y, z \in [a, b]$.*

(a) If $g(x) + g(y), g(z)g(x), g(z)g(y) \in Ran(g)$, $\odot$ is distributive over $\oplus$, i.e.,

$$z \odot (x \oplus y) = (z \odot x) \oplus (z \odot y).$$

(b) Neutral element for $\oplus$ is $\mathbf{0} = g^{(-1)}(0)$.

(c) If $1 \in Ran(g)$, the neutral element for $\odot$ is $\mathbf{1} = g^{(-1)}(1)$.

(d) $g^{(-1)}(0) \odot x = x \odot g^{(-1)}(0) = g^{(-1)}(0)$ for all $x \in [a, b]$.

(e) $\oplus$ is a non-decreasing function, i.e., for $x \preceq y$ we have $x \oplus z \preceq y \oplus z$.

(f) $\odot$ is a non-decreasing function, i.e., for $x \preceq y$ we have $x \odot z \preceq y \odot z$.

(g) In the general case, associativity does not hold for $\oplus$.

(h) In the general case, the cancellation law does not hold for $\oplus$.

Proof. Proofs for properties (a)-(f) follow directly from Definition 2 and properties of the pseudo-inverse function (see [6]). Claims (g) and (h) are illustrated by the following example. □

Example 6. Let $g : [0, +\infty] \to [0, +\infty]$ given by

$$g(x) = \begin{cases} \ln(x+1), & x \in [0, 2], \\ e^x, & x \in (2, +\infty] \end{cases}$$

be a generating function for pseudo-addition $\oplus$. Its pseudo-inverse is

$$g^{(-1)}(x) = \begin{cases} e^x - 1, & x \in [0, \ln 3], \\ \ln x, & x \in (e^2, +\infty], \\ 2, & x \in (\ln 3, e^2]. \end{cases}$$

This pseudo-inverse function is continuous and strictly increasing on $Ran(g)$.

Now, for this choice of generating function and corresponding pseudo-operation it can be easily shown that the following holds:

$$\left(\frac{3}{2} \oplus \frac{1}{2}\right) \oplus 3 = \ln\left(\ln 3 + e^3\right) \neq \ln\left(\ln \frac{15}{4} + e^3\right) = \frac{3}{2} \oplus \left(\frac{1}{2} \oplus 3\right),$$

therefore, associativity, in the general case, does not hold.

Also, it can be easily shown that $\frac{3}{2} \oplus \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{3}$, i.e., the cancellation law does not hold either.

Remark 7. Due to the strict monotonicity of generating function, left neutral element for $\oplus$ is a, if the generating function is increasing, or b for g decreasing. This implies that $[a, b]_+ = \{x \mid x \in [a, b],\ g^{(-1)}(0) \preceq x\}$ is the whole interval $[a, b]$ and positive non-decreasingness for $\odot$ from the "classical" case is equivalent to the non-decreasing property of $\odot$ from the generalized g-semiring.

Since $\oplus$ is not necessarily an associative operation, further on the following notation will be used:

$$\bigoplus_{i=1}^{n} \alpha_i = (\ldots((\alpha_1 \oplus \alpha_2) \oplus \alpha_3) \oplus \ldots) \oplus \alpha_n,$$

where $\alpha_i \in [a, b]$ $i \in \{1, 2, \ldots n\}$.

Also, by means of generating function g it is possible to introduce a metric. Let $d : [a, b]^2 \to [0, +\infty]$ be a function given by

$$d(x, y) = |g(x) - g(y)|, \tag{5}$$

where $x, y \in [a, b]$ and g is a generating function for $\oplus$. It is easy to check that d fulfills all conditions for being a metric.

2.1. Measure-like set function given by generator g

Another notion essential for the construction of pseudo Riemann-Stieltjes integral is the notion of measure-type set function. Therefore, we shall consider a set function that is given by means of generating function g and defined on family of subintervals of the real line.

Let $\mathcal{C}$ be a family of semiclosed subintervals $(c, d]$ of $\mathbb{R}$ where $c \leq d$, then $\mathcal{C}$ is semiring of sets, i.e., $\emptyset \in \mathcal{C}$, if $A, B \in \mathcal{C}$ then $A \cap B \in \mathcal{C}$ and for all $A, B \in \mathcal{C}$ there exists $C_1, ..., C_n$ from $\mathcal{C}$ such that $C_i \cap C_j = \emptyset$ for $i \neq j$ and $A \backslash B = \cup_{k=1}^{n} C_k$.

Definition 8. *Let g be a generating function from Definition 2 and let ϕ be a bounded function defined on the real line. A mapping $m : \mathcal{C} \to [a, b]$ is called g_ϕ-*set-function *if*

$$m((c,d]) = g^{(-1)}\left(\phi(d) - \phi(c)\right). \tag{6}$$

Example 9. Let us consider a generating function from Example 6 and the function ϕ given by

$$\phi(x) = \begin{cases} e^x, & x \in (-\infty, 0], \\ 3, & x \in (0, \infty). \end{cases}$$

The g_ϕ-set-function for an interval $(c, d]$ where $c, d \leq 0$, since $e^d - e^c \in [0, 1)$, is

$$m((c,d]) = g^{(-1)}\left(e^d - e^c\right) = e^{e^d - e^c} - 1.$$

If $c, d > 0$ then $\phi(c) = \phi(d) = 3$ and $m((c, d]) = g^{(-1)}(0) = 0$. For an interval $(c, d]$ where $c \leq 0$ and $d > 0$, holds $m((c, d]) = g^{(-1)}(3 - e^c) = 2$, since $3 - e^c \in [2, 3)$.

Remark 10. A similar problem of the measure-type set function applied on the family of intevals has been investigated in [17]. Operations used in [17] are also generalized generated pseudo-operations, however, of the following form

$$x \oplus y = g^{-1}(\varepsilon g(x) + g(y)) \quad \text{and} \quad x \odot y = g^{-1}(g^\gamma(x) g(y)), \tag{7}$$

where ε and γ are arbitrary but fixed positive real numbers and g is a positive strictly monotone continuous generating function. These operations need not be commutative nor associative. The $\oplus$-measures associated with operations (7) are defined on partitions of some interval $[c, d]$ and depend on the number of subintervals and the position of the subinterval, i.e., $\mu_{P_n} : P_n \to [0, \infty)$ and

$$\mu_{P_n}((x_i, x_{i+1}]) = g^{-1}\left(\frac{x_{i+1} - x_i}{\varepsilon^{n-i-1}}\right), \tag{8}$$

where $P_n = \{(x_i, x_{i+1}]\}_{i=0}^{n-1}$ and $c = x_0 < x_1 < \ldots < x_n = d$. Specifically, for a monotone bijection g, $\varepsilon = 1$ and $\phi(x) = x$, set functions (6) and (8) coincide.

Some basic properties of the set function given by (6) are presented in the following proposition.

Proposition 11. *Let $m : \mathcal{C} \to [a, b]$ be a g_ϕ-*set-function *given by (6). Then, the following holds:*

a) $m(\emptyset) = g^{(-1)}(0)$,

b) if functions g and ϕ are of the same monotonicity, m is a monotone set function, i.e., fuzzy measure.

Proof. Since an empty set can always be represented as interval $(c, c]$, claim a) is obvious. Claim b) follows directly from Definition 8, monotonicity of the functions g and ϕ and fact that pseudo-inverse function remains of the same monotonicity as the source function (see [6]). □

Example 12. Let m be the g_ϕ-set-function from Example 9. Functions g and ϕ are of the same monotonicity. Let $(c_1, d_1]$ be a subinterval of $\mathbb{R}$ such that $c_1 \leq 0$ and $d_1 > 0$, and let $(c_2, d_2] \subseteq (c_1, d_1]$. Since $m((c_1, d_1]) = 2$ (see Example 9), we have the following:

(i) if $c_2 \leq 0$ and $d_2 > 0$, then $m((c_2, d_2]) = 2$ and $m((c_2, d_2]) = m((c_1, d_1])$;

(ii) if $c_2 \leq 0$ and $d_2 \leq 0$, then $m((c_2, d_2]) = e^{e^{d_2} - e^{c_2}} - 1 \in [0, e - 1)$ and $m((c_2, d_2]) < m((c_1, d_1])$;

(iii) if $c_2 > 0$ and $d_2 > 0$, then $m((c_2, d_2]) = 0$ and $m((c_2, d_2]) < m((c_1, d_1])$.

Also, if $c_1 \leq c_2 < d_2 \leq d_2 \leq 0$ it can be easily shown that $m((c_2, d_2]) \leq m((c_1, d_1])$, and if $0 < c_1 \leq c_2 < d_2 \leq d_2$ we have $m((c_2, d_2]) = m((c_1, d_1])$. Therefore, the g_ϕ-set-function in question is a monotone set function.

Also, the following types of pseudo-(super)additivety can be proved.

Proposition 13. *If $\mathcal{P} = \{(x_i, x_{i+1}]\}_{i=0}^{n-1}$ is an n-partition of some interval $(c, d] \in \mathcal{C}$ such that $c = x_0 \leq x_1 \leq \ldots \leq x_n = d$, then the following holds:*

a) if $g : [a, b] \to [0, \infty]$ is a monotone bijection and ϕ monotone on $[c, d]$, m is pseudo-additive on $\mathcal{P}$, i.e.,

$$m((c, d]) = m(\cup_{i=0}^{n-1}(x_i, x_{i+1}]) = \bigoplus_{i=0}^{n-1} m((x_i, x_{i+1}]),$$

b) if g is either strictly increasing left-continuous or strictly decreasing right-continuous function and ϕ monotone on $[c, d]$, m is pseudo-superadditive on $\mathcal{P}$, i.e.,

$$m((c, d]) = m(\cup_{i=0}^{n-1}(x_i, x_{i+1}]) \succeq \bigoplus_{i=0}^{n-1} m((x_i, x_{i+1}]),$$

Proof.

a) For bijection g its pseudo-inverse coincides with regular inverse function ([6]), therefore for some n-partition $\mathcal{P}$ of the interval $(c, d] \in \mathcal{C}$, such that $c = x_0 \leq x_1 \leq \ldots \leq x_n = d$ and ϕ non-decreasing the following holds

$$\begin{aligned} \bigoplus_{i=0}^{n-1} m((x_i, x_{i+1}]) &= g^{-1}\left(\textstyle\sum_{i=0}^{n-1} g\left(m((x_i, x_{i+1}])\right)\right) \\ &= g^{-1}\left(\textstyle\sum_{i=0}^{n-1} \left(\phi(x_{i+1}) - \phi(x_i)\right)\right) = m\left((c, d]\right). \end{aligned}$$

For ϕ non-increasing, the equality in question trivially holds.

b) For g strictly increasing left-continuous or strictly decreasing right-continuous generating function holds $g \circ g^{(-1)}(x) \leq x$, for all $x \in [0, +\infty]$. Now, for some n-partition $\mathcal{P}$ ϕ non-decreasing, this implies

$$\begin{aligned} \bigoplus_{i=0}^{n-1} m((x_i, x_{i+1}]) &\preceq g^{(-1)} \left(\textstyle\sum_{i=0}^{n-1} g\left(m((x_i, x_{i+1}])\right) \right) \\ &\preceq g^{(-1)} \left(\textstyle\sum_{i=0}^{n-1} \phi(x_{i+1}) - \phi(x_i) \right) = m\left((c, d]\right). \end{aligned}$$

For ϕ non-increasing, the inequality in question trivially holds. □

Example 14. *Let us consider a generating function from Example 6 and the function ϕ given by*

$$\phi(x) = \begin{cases} e^x, & x \in (-\infty, 0], \\ x^3 + 1, & x \in (0, 2], \\ 9, & x \in (2, \infty). \end{cases}$$

Now, since $g \circ g^{(-1)}(x) < x$ for all $x \in (\ln 3, e^2]$, strict inequality in claim b) from the previous proposition can be obtained in some cases:

$$m((0, 1.5]) \oplus m((1.5, 3]) = g^{(-1)}(\ln 9) = 2 \prec \ln 8 = g^{(-1)}(8) = m((0, 3]).$$

However, it is also possible to obtain the equality:

$$m((0, 2]) \oplus m((2, 3]) = g^{(-1)}(8) = m((0, 3]).$$

3. Pseudo Riemann-Stieltjes integral

The main aim of this section is to introduce a pseudo-analysis' counterpart of the well known Riemann-Stieltjes integral (see [4]). For this construction, operations, metric and measure presented in the previous section will be used.

Let g be a generating function from Definition 2 defined on the interval $[a, b]$ and $\oplus$ and $\odot$ pseudo-operations given by (2) and (3), respectively. Let ϕ be a bounded function defined on the real line and let m be a g_ϕ-set function given by (6). If $\mathcal{P} = \{(\omega_i, (x_{i-1}, x_i])\}_{i=1}^n$ is a tagged partition of $[c, d]$, i.e., $c = x_0 \leq x_1 \leq \ldots \leq x_n = d$ and $\omega_i \in (x_{i-1}, x_i]$, *the Riemann-Stieltjes pseudo-sum of f with respect to ϕ for the tagged partition $\mathcal{P}$* is

$$\bigoplus_{\mathcal{P}} f = \bigoplus_{i=1}^{n} f(\omega_i) \odot m((x_{i-1}, x_i]),$$

where $f : [c, d] \to [a, b]$.

Definition 15. *Function $f : [c,d] \to [a,b]$ is pseudo Riemann-Stieltjes integrable with respect to ϕ on $[c,d]$ whenever there is a real number PI satisfying the following condition: for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d\left(\bigoplus_{\mathcal{P}} f, PI\right) < \varepsilon,$$

for all tagged partitions $\mathcal{P}$ of $[c,d]$ that fulfills $\max\{x_i - x_{i-1} \,|\, 1 \le i \le n\} < \delta$.

It is easy to check that the number PI from previous definition, if exists, is uniquely determined. This number PI is pseudo Riemann-Stieltjes integral of f on $[c,d]$, and it will be denoted by $(pRS)\int_{[c,d]}^{(\oplus,\odot)} f\, d\phi$.

Remark 16. Specially, for $g(x) = x$ the previous definition will give the classical Riemann-Stieltjes integral $(RS)\int_c^d f\, d\phi$. More on this integral can be found in [4].

Theorem 17. *Let $g : [a,b] \to [0,\infty]$ be a strictly increasing left-continuous or strictly decreasing right-continuous generating function and $f : [c,d] \to [a,b]$ a pseudo Riemann-Stieltjes integrable function on $[c,d]$ with respect to a bounded function ϕ. Then*

$$g\left((pRS)\int_{[c,d]}^{(\oplus,\odot)} f\, d\phi\right) \le (RS)\int_c^d g \circ f\, d\phi, \tag{9}$$

if the right side of the inequality exists.

Proof. Let f be a pseudo Riemann-Stieltjes integrable function on $[c,d]$ and $g \circ f$ Riemann-Stieltjes integrable function on $[c,d]$, i.e., for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each partition $\mathcal{P} = \{(\omega_i, (x_i, x_{i+1}])\}_{i=0}^{n-1}$ of $[c,d]$, where $\max\{x_i - x_{i-1} \,|\, 1 \le i \le n\} < \delta$, we have

$$d\left(\bigoplus_{\mathcal{P}} f, (pRS)\int_{[c,d]}^{(\oplus,\odot)} f\, d\phi\right) < \varepsilon$$

and $|\sum_{i=1}^{n} g \circ f(\omega_i)\,(\phi(x_i) - \phi(x_{i-1})) - (RS)\int_c^d g \circ f\, d\phi| < \varepsilon$. Since g is either a strictly increasing left-continuous or strictly decreasing right-continuous function, holds $g\left(\bigoplus_{\mathcal{P}} f\right) \le \sum_{i=1}^{n} g \circ f(\omega_i)\,(\phi(x_i) - \phi(x_{i-1}))$. Now,

$$g\left((pRS)\int_{[c,d]}^{(\oplus,\odot)} f\, d\phi\right) - (RS)\int_c^d g \circ f\, d\phi$$

$$< g\left(\bigoplus_{\mathcal{P}} f\right) + \varepsilon - \sum_{i=1}^{n} g \circ f(\omega_i)\,(\phi(x_i) - \phi(x_{i-1})) + \varepsilon \le 2\varepsilon.$$

This holds for all $\varepsilon > 0$ and, after allowing $\varepsilon \to 0$, inequality (9) is obtained. □

For $g : [a, b] \to [0, \infty]$ being a monotone bijection, i.e, g is a generating function for g-semiring and therefore generated pseudo-operations are given by (1), stronger connection between pseudo Riemann-Stieltjes integral and Riemann-Stieltjes integral can be proved.

Corollary 18. *If f is pseudo Riemann-Stieltjes integrable on $[c, d]$ then $g \circ f$ is a Riemann-Stieltjes integrable function on $[c, d]$ and*

$$(pRS) \int_{[c,d]}^{(\oplus,\odot)} f \, d\phi = g^{-1} \left((RS) \int_a^b g \circ f \, d\phi \right), \tag{10}$$

where g is a monotone bijection.

Also, if f is Riemann-Stieltjes integrable on $[c, d]$ it can be proved that $g^{-1} \circ f$ is a pseudo Riemann-Stieltjes integrable function on $[c, d]$.

Remark 19. Let $[u, v] \subseteq [c, d]$, let $1 \in Ran(g)$, and let $\chi_{[u,v]}$ be the pseudo-characteristic function of $[u, v]$, i.e. $\chi_{[u,v]}(x) = \mathbf{0}$ for all $x \notin [u, v]$ and $\chi_{[u,v]}(x) = \mathbf{1}$ for all $x \in [u, v]$. In light of the Definition 15, pseudo Riemann-Stieltjes integral of $\chi_{[u,v]}$ depends only on pseudo-sums of g_ϕ-set-function m of subintervals of $[u, v]$. Even more, if g is either strictly increasing left-continuous or strictly decreasing right-continuous function and ϕ monotone on $[c, d]$ it can be shown that $(pRS) \int_{[c,d]}^{(\oplus,\odot)} \chi_{[u,v]} \, d\phi \preceq m((u, v])$. Specifically, $(pRS) \int_{[c,d]}^{(\oplus,\odot)} \chi_{[u,v]} \, d\phi = m((u, v])$ holds for g being a monotone bijection from g-semiring. Therefore, this g-semiring case is of special interest for further investigation.

Some pseudo-linear properties of the pseudo Riemann-Stieltjes based on g-semiring can be proved.

Proposition 20. *Let $\oplus$ and $\odot$ be pseudo-operations from Definition 2 given by a monotone bijection $g : [a, b] \to [0, \infty]$. If $f, h : [c, d] \to (a, b)$ are pseudo Riemann-Stieltjes integrable functions on $[c, d]$ with respect to ϕ, then:*

a) for some $\alpha \in (a, b)$, $\alpha \odot f$ is pseudo Riemann-Stieltjes integrable on $[c, d]$ with respect to ϕ, f is pseudo Riemann-Stieltjes integrable on $[c, d]$ with respect to $g(\alpha)\phi$ and

$$(pRS) \int_{[c,d]}^{(\oplus,\odot)} \alpha \odot f \, d\phi = \alpha \odot (pRS) \int_{[c,d]}^{(\oplus,\odot)} f \, d\phi = (pRS) \int_{[c,d]}^{(\oplus,\odot)} f \, d\, (g(\alpha)\phi) \, ;$$

b) $f \oplus h$ is pseudo Riemann-Stieltjes integrable on $[c, d]$ with respect to ϕ, and

$$(pRS) \int_{[c,d]}^{(\oplus,\odot)} f \oplus h \, d\phi = (pRS) \int_{[c,d]}^{(\oplus,\odot)} f \, d\phi \oplus (pRS) \int_{[c,d]}^{(\oplus,\odot)} h \, d\phi.$$

Proof. a) Since the generating function is bijection, this proof is based on Corollary 18 and algebraic properties of the classical Riemann-Stieltjes integral:

$$(pRS)\int_{[c,d]}^{(\oplus,\odot)} \alpha \odot f\, d\phi \;=\; g^{-1}\left((RS)\int_c^d g(\alpha)(g\circ f)\, d\phi\right)$$

$$= g^{-1}\left(g(\alpha)\left((RS)\int_c^d g\circ f\, d\phi\right)\right) = \alpha \odot (pRS)\int_{[c,d]}^{(\oplus,\odot)} f\, d\phi$$

and

$$(pRS)\int_{[c,d]}^{(\oplus,\odot)} \alpha \odot f\, d\phi \;=\; g^{-1}\left((RS)\int_c^d g(\alpha)(g\circ f)\, d\phi\right)$$

$$= g^{-1}\left((RS)\int_c^d g\circ f\, d\left(g(\alpha)\phi\right)\right) = (pRS)\int_{[c,d]}^{(\oplus,\odot)} f\, d\left(g(\alpha)\phi\right).$$

Proof for b) is analogous. □

Also, if $f : [c,d] \to (a,b)$ is pseudo Riemann-Stieltjes integrable on $[c,d]$ with respect to ϕ and φ than f is pseudo Riemann-Stieltjes integrable on $[c,d]$ with respect to $\phi + \varphi$ and

$$(pRS)\int_{[c,d]}^{(\oplus,\odot)} f\, d(\phi+\varphi) = (pRS)\int_{[c,d]}^{(\oplus,\odot)} f\, d\phi \oplus (pRS)\int_{[c,d]}^{(\oplus,\odot)} f\, d\varphi,$$

where $\oplus$ and $\odot$ are given by a monotone bijection. Additionally, f is pseudo Riemann-Stieltjes integrable on each subinterval of $[c,d]$ and, if $u \in (c,d)$, the following holds

$$(pRS)\int_{[c,d]}^{(\oplus,\odot)} f\, d\phi = (pRS)\int_{[c,u]}^{(\oplus,\odot)} f\, d\phi \oplus (pRS)\int_{[u,d]}^{(\oplus,\odot)} f\, d\phi.$$

There is a useful relationship between the Lebesgue integral and the pseudo Riemann-Stieltjes integral given by the following theorem. Analog theorem for the classical case has been given in [4]. Proofs are similar.

Theorem 21. *Let generator g be a strictly increasing bijection and $f : [c,d] \to [a,b]$ a measurable function. Let μ be the Lebesgue measure and ϕ_f distribution function of f given by $\phi_f(x) = \mu\left(\{t \in [c,d] \mid f(t) > x\}\right)$. Then*

$$(L)\int_c^d g\circ f\, d\mu = -g\left((pRS)\int_{[a,b]}^{(\oplus,\odot)} x\, d\phi_f\right), \tag{11}$$

if the integrals exist.

Proof. Pseudo Riemann-Stieltjes integrability of x implies that for some $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d\left(\bigoplus_{\mathcal{P}} x, (pRS)\int_{[a,b]}^{(\oplus,\odot)} x\, d\phi_f\right) < \varepsilon \tag{12}$$

whenever $\mathcal{P} = \{(\omega_i, (x_{i-1}, x_i])\}_{i=1}^n$ is a tagged partition of the interval $[a, b]$ and $\max\{x_i - x_{i-1} \,|\, 1 \leq i \leq n\} < \delta$. Now, (12) gives us

$$\sum_{i=1}^n g(\omega_i)\,(\phi_f(x_{i-1}) - \phi_f(x_i)) > -g\left((pRS)\int_{[a,b]}^{(\oplus,\odot)} x\, d\phi_f\right) - \varepsilon \tag{13}$$

and

$$\sum_{i=1}^n g(\omega_i)\,(\phi_f(x_{i-1}) - \phi_f(x_i)) < -g\left((pRS)\int_{[a,b]}^{(\oplus,\odot)} x\, d\phi_f\right) + \varepsilon. \tag{14}$$

Let $A_i = \{x \in [c, d] \,|\, g(x_{i-1}) < g \circ f(x) \leq g(x_i)\}$, $i = 1, \ldots, n$, be subsets of the domain of function f. Since our generating function g is continuous and strictly increasing, the following can be easily shown $\mu(A_i) = \phi_f(x_{i-1}) - \phi_f(x_i)$ where μ is the Lebesgue measure (see [4]). Properties of the classical Lebesgue integral and inequalities (13) and (14) will give us the following

$$\begin{aligned}(L)\int_c^d g\circ f\, d\mu &\geq \sum_{i=1}^n g\left(f(x_{i-1})\right)\mu(A_i)\\ &= \sum_{i=1}^n g\left(f(x_{i-1})\right)(\phi_f(x_{i-1}) - \phi_f(x_i))\\ &> -g\left((pRS)\int_{[a,b]}^{(\oplus,\odot)} x\, d\phi_f\right) - \varepsilon\end{aligned} \tag{15}$$

and

$$\begin{aligned}(L)\int_c^d g\circ f\, d\mu &\leq \sum_{i=1}^n g\left(f(x_i)\right)\mu(A_i)\\ &= \sum_{i=1}^n g\left(f(x_i)\right)(\phi_f(x_{i-1}) - \phi_f(x_i))\\ &< -g\left((pRS)\int_{[a,b]}^{(\oplus,\odot)} x\, d\phi_f\right) + \varepsilon.\end{aligned} \tag{16}$$

Now, (15) and (16) imply $\left|(L)\int_c^d g\circ f\, d\mu + g\left((pRS)\int_{[a,b]}^{(\oplus,\odot)} x\, d\phi_f\right)\right| < \varepsilon$, for all $\varepsilon > 0$, which gives us (11). □

Remark 22. Some additional constrains put on the generator g from the previous theorem will ensure the following connection with the g-integral ([12, 13]):

- if g^{-1} is an even function, then $\int_{[c,d]} f \odot d\nu = (pRS)\int_{[a,b]}^{(\oplus,\odot)} x\, d\phi_f$;
- if g^{-1} is an odd function, then $\int_{[c,d]} f \odot d\nu = -(pRS)\int_{[a,b]}^{(\oplus,\odot)} x\, d\phi_f$;

where a g-integral of function f is denoted by $\int_{[c,d]} f \odot d\nu$.

The problem of pseudo-integration, similar to the one presented in this paper, that has been focused on pseudo-probability space (Ω, S, P), continuous generating function and Lebesgue type of integral has been investigated in [8]. For an increasing and continuous generator $g : \mathbb{R} \to \mathbb{R}$ and pseudo operations of the form (1), the pseudo Lebesgue-Stieltjes integral has been introduced in [8]. Also, for random variable ξ, its pseudo-distribution function $F_g = P(\{\omega \mid \xi(\omega) < x\})$ and Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, the following connection between pseudo Lebesgue-Stieltjes and pseudo-Lebesgue integrals has been given as:

$$\int^{\oplus} f \circ \xi\, dP = \int^{\oplus} f\, dF_g, \tag{17}$$

where the integral on the left is a pseudo-Lebesgue integral based on the pseudo-probability P ($g \circ P$ is the classical probability) and one on the right is the pseudo Lebesgue-Stieltjes integral (see [8]). Some additional conditions will ensure the following connection between the pseudo Lebesgue-Stieltjes integral from [8] and the pseudo Riemann-Stieltjes integral.

Corollary 23. *Let (Ω, S, P) be a pseudo-probability space, $\xi : \Omega \to [a, b]$ a random variable, $F_g = P(\{\omega \mid \xi(\omega) < x\})$ pseudo-distribution function of the random variable ξ and $f : [a, b] \to [a, b]$ a measurable function, then*

$$g\left((pLS)\int_{[a,b]}^{\oplus} f\, dF_g\right) = -g\left((pRS)\int_{[a,b]} x\, d\phi_{f\circ\xi}\right),$$

where the integral on the left is a pseudo Lebesgue-Stieltjes integral ([8]).

Corollary 23 is a direct consequence of the result (17) from [8] and Theorem 21.

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