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# A METHOD FOR OBTAINING THIRD-ORDER ITERATIVE FORMULAS

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**Abstract.** We present a method for constructing new third-order methods for solving nonlinear equations. These methods are modifications of Newton's method. Also, we obtain some known methods as special cases, for example, Halley's method, Chebyshev's method, super-Halley method. Several numerical examples are given to illustrate the performance of the presented methods.

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## 1. Introduction

In this paper we consider a family of iterative methods for finding a simple root $\alpha$ of nonlinear equation$f(x) = 0$. We assume that $f$ satisfies

$$f \in C^3[a,b], \quad f'(x) \neq 0, \;\; x \in [a,b], \quad f(a) > 0 > f(b). \tag{1}$$

Under these assumptions the function $f$ has a unique root $\alpha \in (a,b)$.

Newton's method is a well-known iterative method for computing approximation of $\alpha$ by using

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \ldots$$

for some appropriate starting value $x_0$. Newton's method quadratically converges in some neighborhood of $\alpha$ if $f'(\alpha) \neq 0$, [4].

The classical Chebyshev-Halley methods which improve Newton's method are given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \cdot \left(1 + \frac{t(x_k)}{2(1 - \beta t(x_k))}\right),$$

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where

$$t(x) = \frac{f(x) f''(x)}{f'(x)^2}. \tag{2}$$

This family has third-order of convergence and includes Chebyshev's method ($\beta = 0$), Halley's method ($\beta = \frac{1}{2}$) and super-Halley method ($\beta = 1$), see [3, 5, 7].

Newton's and Chebyshev-Halley methods belong to the class of one-point iteration methods without memory [7]

$$x_{k+1} = F(x_k). \tag{3}$$

Here we consider the developing of third-order modifications of Newton's method. Using an iteration function of the form

$$F(x) = x - \frac{f(x)}{f'(x)} G(x),$$

we obtain for a specific function $G$ and some of its approximations iterative methods of the form (3), which are cubically convergent. Some known methods are members of our family of methods. So, our algorithm 2 is Chebyshev's method, our algorithm 5 is Halley's method, and our algorithm 6 is super-Halley method. Also, our algorithm 7 is

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'\left(x - \frac{f(x_n)}{f'(x_n)}\right)}$$

from [8] and [2], and our algorithm 9 is

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'\left(x - \frac{f(x_n)}{f'(x_n)}\right)}\right)$$

from [2] and [6]. The algorithm 1 is a class of algorithms depending on two parameters.

## 2. Main result

The crux of the present derivation is to obtain a specific function $G$ and some of its approximations such that the special iteration function $F$

$$F(x) = x - \frac{f(x)}{f'(x)} G(x) \tag{4}$$

produces a sequence $\{x_n\}$ by (3) which is cubically convergent.

One can see that Newton's and Chebyshev-Halley iteration functions are special cases of (3) with

$$G(x) = 1$$

and

$$G(x) = 1 + \frac{t(x)}{2(1 - \beta t(x))}$$

respectively.

If we define

$$G(x) = \sqrt{\frac{f'(x)}{f'(\alpha)}}, \tag{5}$$

and $F$ by (4) we obtain an iterative method of third-order. For our definition of the function $G$ we need the knowledge of the zero $\alpha$. Since the value of $\alpha$ is unknown, we can use appropriate approximations for $G$. In [1] another weight function $h$ is considered. Namely,

$$h(x) = 1 + \frac{1}{2}\ln\left(\left|\frac{f'(x)}{f'(\alpha)}\right|\right).$$

We shall consider three different possibilities for constructing the function $G$. Firstly, we approximate $\alpha$ in (5) only. In this way we obtain algorithm 1. The second possibility is to approximate $G$ using Taylor or Padé expansion and after that to use some approximations for $\alpha$, $f'(\alpha)$ and $f''(\alpha)$. In this way we construct algorithms 2-8. The third possibility is to approximate the square root in (5) and after that to approximate $f'(\alpha)$. This way we obtain algorithms 9 and 10. Obviously, using similar approximations one can also obtain other new third-order iterative methods.

### 2.1. Algorithm 1. Approximations of $\alpha$

We can use some quadratic approximation for $\alpha$,

$$\alpha \approx \varphi_{\beta,\gamma}(x),$$

where $\varphi_\beta$ is a suitable function depending on a real parameter $\beta$. For example, we can choose

$$\varphi_{\beta,\gamma}(x) = x - \frac{f(x)}{f'(x - \beta f(x)) + \gamma f(x)}. \tag{6}$$

One can see that for $\gamma = 0$ and $\beta = 1$ we have (7), for $\gamma = 0$ and $\beta = 0$ (8) and for $\gamma = 0$ and $\beta = -1$ we obtain (9), which are given in [1], i.e.

$$\varphi_1(x) = x - \frac{f(x)}{f'(x - f(x))} \tag{7}$$

$$\varphi_0(x) = x - \frac{f(x)}{f'(x)} \tag{8}$$

$$\varphi_{-1}(x) = x - \frac{f(x)}{f'(x + f(x))} \tag{9}$$

Now we define for real parameter $\beta$

$$G_{\beta,\gamma}(x) = \sqrt{\frac{f'(x)}{f'(\varphi_{\beta,\gamma}(x))}}.$$

## 2.2. Approximation of $G$ by using Taylor expansion

Using Taylor expansion from

$$\sqrt{\frac{f'(x)}{f'(\alpha)}}$$

we obtain

$$G(x) \approx 1 + \frac{(x-\alpha)f''(\alpha)}{2f'(\alpha)}. \tag{10}$$

Using this approximation, we can obtain some new functions:

### 2.2.1. Algorithm 2. Chebyshev method

In (10) instead of $x-\alpha$ we use Newton's correction $\frac{f(x)}{f'(x)}$ and approximate $f'(\alpha)$ with $f'(x)$ and approximate $f''(\alpha)$ with $f''(x)$. This way we obtain

$$G_{CH}(x) = 1 + \frac{f(x)f''(x)}{2f'(x)^2} = 1 + \frac{t(x)}{2}.$$

Iterative method (3) with $G_{CH}(x)$ and $F$ defined by (4) becomes Chebysev's iterative method.

### 2.2.2. Algorithm 3.

In (10) instead of $x-\alpha$ we use Newton's correction $\frac{f(x)}{f'(x)}$ and approximate $f'(\alpha)$ with $f'(x)$ and $f''(\alpha)$ is approximated with

$$f''(\alpha) \approx \frac{f'(x) - f'\left(x - \frac{f(x)}{f'(x)}\right)}{\frac{f(x)}{f'(x)}}.$$

So, we obtain

$$G_{D1}(x) = 1 + \frac{f'(x) - f'\left(x - \frac{f(x)}{f'(x)}\right)}{2f'(x)}.$$

#### 2.2.3. Algorithm 4.

In (10) instead of $x-\alpha$ we use Newton's correction $\frac{f(x)}{f'(x)}$ and approximate $f'(\alpha)$ with

$$f'(x-\frac{f(x)}{f'(x)}),$$

and approximate $f''(\alpha)$ with

$$f''(\alpha)\approx\frac{f'(x)-f'\left(x-\frac{f(x)}{f'(x)}\right)}{\frac{f(x)}{f'(x)}}.$$

This way we obtain

$$G_{D2}(x)=1+\frac{f'(x)-f'\left(x-\frac{f(x)}{f'(x)}\right)}{2f'\left(x-\frac{f(x)}{f'(x)}\right)}=\frac{f'(x)+f'\left(x-\frac{f(x)}{f'(x)}\right)}{2f'\left(x-\frac{f(x)}{f'(x)}\right)}.$$

### 2.3. Approximation of $G$ by using Padé expansion

Using Padé expansion from

$$\sqrt{\frac{f'(x)}{f'(\alpha)}}$$

we obtain

$$G(x)\approx\frac{1}{1-\frac{(x-\alpha)f''(\alpha)}{2f'(\alpha)}} \tag{11}$$

Using this approximation, we can obtain some new algorithms:

#### 2.3.1. Algorithm 5. Halley's method

In (11) instead of $x-\alpha$ we use Newton's correction $\frac{f(x)}{f'(x)}$ and approximate $f'(\alpha)$ with $f'(x)$ and $f''(\alpha)$ with $f''(x)$. In such way we obtain

$$G_{HL}(x)=\frac{1}{1-\frac{\left(\frac{f(x)}{f'(x)}\right)f''(x)}{2f'(x)}}=\frac{2}{2-t(x)}.$$

Iterative method (3) with $G_{CH}(x)$ and $F$ defined by (4) becomes Halley's iterative method.

#### 2.3.2. Algorithm 6. Super-Halley method

In (11) instead of $x-\alpha$ we use Halley's correction

$$\frac{f(x)}{f'(x)}\frac{2}{2-t(x)}$$

and approximate $f'(\alpha)$ with $f'(x)$ and $f''(\alpha)$ with $f''(x)$. This way we obtain super-Halley method.

$$G_{SH}(x) = \frac{1}{1 - \frac{\frac{f(x)}{f'(x)} \frac{1}{1-\frac{t(x)}{2}} f''(x)}{2f'(x)}} = \frac{1}{1 - \frac{t(x)}{2} \frac{1}{1-\frac{t(x)}{2}}} = \frac{1}{1 - \frac{t(x)}{2-t(x)}} = \frac{2 - t(x)}{2(1 - t(x))}.$$

### 2.3.3. Algorithm 7.

In (11) instead of $x - \alpha$ we use Newton's correction $\frac{f(x)}{f'(x)}$ and approximate $f'(\alpha)$ with $f'(x)$ and $f''(\alpha)$ with

$$f''(\alpha) \approx \frac{f'(x) - f'\left(x - \frac{f(x)}{f'(x)}\right)}{\frac{f(x)}{f'(x)}}.$$

So, we obtain

$$G_{D3}(x) = \frac{2f'(x)}{f'(x) + f'\left(x - \frac{f(x)}{f'(x)}\right)}.$$

Iterative method (3) with $G_{D3}(x)$ and $F$ defined by (4) is considered in [8] and [2].

$$F(x) = x - \frac{f(x)}{f'(x)} G_{D3}(x).$$

### 2.3.4. Algorithm 8.

In (11) instead of $x - \alpha$ we use Newton's correction$\frac{f(x)}{f'(x)}$, we approximate $f'(\alpha)$ with

$$f'(x - \frac{f(x)}{f'(x)})$$

and $f''(\alpha)$ with

$$f''(\alpha) \approx \frac{f'(x) - f'\left(x - \frac{f(x)}{f'(x)}\right)}{\frac{f(x)}{f'(x)}}.$$

Now, we have

$$G_{D4}(x) = \frac{-2f'\left(x - \frac{f(x)}{f'(x)}\right)}{f'(x) - 3f'\left(x - \frac{f(x)}{f'(x)}\right)}.$$

## 2.4. Approximation of $G$ by using square root approximation

For approximating square root of a real number there are many different formulas. We shall use only two to demonstrate a way for obtaining some new iterative methods of form (3) with $F$ given by (4) where $G$ is replaced with $G_{HR}$ or $G_{LB}$.

### 2.4.1. Algorithm 9.

Using Heron's approximation of square root

$$\sqrt{\frac{f'(x)}{f'(\alpha)}} \approx \frac{1}{2}\left(1+\frac{f'(x)}{f'(\alpha)}\right)$$

and

$$f'(\alpha) \approx f'\left(x-\frac{f(x)}{f'(x)}\right),$$

we obtain

$$G_{HR}(x) = \frac{1}{2} + \frac{f'(x)}{2f'\left(x-\frac{f(x)}{f'(x)}\right)}.$$

Iterative method (3) with $G_{HR}(x)$ and $F$ defined by (4) is considered in [2] and [6].

### 2.4.2. Algorithm 10.

Using Lambert's approximation of square root, i.e.

$$\sqrt{\frac{f'(x)}{f'(\alpha)}} \approx \frac{1+3\frac{f'(x)}{f'(\alpha)}}{3+\frac{f'(x)}{f'(\alpha)}} = \frac{3f'(x)+f'(\alpha)}{f'(x)+3f'(\alpha)}$$

and

$$f'(\alpha) \approx f'\left(x-\frac{f(x)}{f'(x)}\right),$$

we obtain

$$G_{LB}(x) = \frac{3f'(x)+f'\left(x-\frac{f(x)}{f'(x)}\right)}{f'(x)+3f'\left(x-\frac{f(x)}{f'(x)}\right)}.$$

Let us consider the iterative procedure (3) where $F$ is given by (4). Our conditions imply that $f$ has exactly one root in $(a,b)$.

**Theorem 1.** *Let us assume that the function $f$ is sufficiently smooth in a neighborhood of its simple root $\alpha$ and $f'(\alpha) \neq 0$. Then the iterative method $x_{k+1} = F(x_k)$, where*

$$F(x) = x - \frac{f(x)}{f'(x)}G(x)$$

*and function $G$ is some of our functions $G_{\beta,\gamma}$, $G_{CH}$, $G_{HL}$, $G_{SH}$, $G_{HR}$, $G_{LB}$, $G_{D1}$, $G_{D2}$, $G_{D3}$, $G_{D4}$, converges cubically to the unique solution $\alpha$ of $f(x)=0$ in a neighborhood of $\alpha$.*

*Proof.* It is well known that the iterative method (3) is cubically convergent if

$$F(\alpha) = \alpha, \quad F'(\alpha) = F''(\alpha) = 0, \quad F'''(\alpha) \neq 0.$$

Differentiating (4) we get

$$F'(x) = 1 - u'(x) G(x) - u(x) G'(x)$$

and

$$F''(x) = -u''(x) G(x) - 2u'(x) G'(x) - u(x) G''(x)$$

where

$$u(x) = \frac{f(x)}{f'(x)}.$$

It is easy to see that for all our functions $G$ it holds $G(\alpha) = 1$. After simple calculations one can obtain that

$$G'(\alpha) = \frac{f''(\alpha)}{2f'(\alpha)}.$$

We have $u'(x) = 1 - t(x)$, where $t$ is defined by (2). It follows that
$u(\alpha) = 0$ and $u'(\alpha) = 1$.

Now, we can see that $F(\alpha) = \alpha$ and $F'(\alpha) = 0$. Since

$$u''(\alpha) = -t'(\alpha) = -\frac{f''(\alpha)}{f'(\alpha)}$$

and

$$F''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)} G(\alpha) - 2G'(\alpha) = \frac{f''(\alpha)}{f'(\alpha)} - 2\frac{f''(\alpha)}{2f'(\alpha)} = 0,$$

we conclude that

$$F(\alpha) = \alpha, \quad F'(\alpha) = F''(\alpha) = 0,$$

which is sufficient to complete the proof. □

## 3. Numerical examples

We present some numerical test results for our cubically convergent methods and the Newton's method. Methods with iteration functions $F$ were compared, where

$$F(x) = x - \frac{f(x)}{f'(x)} G(x),$$

and $G$ is one of our functions 1, $G_{\beta,\gamma}$, $G_{CH}$, $G_{HL}$, $G_{SH}$, $G_{HR}$, $G_{LB}$, $G_{D1}$, $G_{D2}$, $G_{D3}$ , $G_{D4}$. So, we have the following 13 iterative functions:

$$F_1(x) = x - \frac{f(x)}{f'(x)},$$

$$F_2(x) = x - \frac{f(x)}{f'(x)} G_{\beta,\gamma}(x), \ \ \beta = 1, \ \gamma = 0,$$

$$F_3(x) = x - \frac{f(x)}{f'(x)} G_{\beta,\gamma}(x), \ \ \beta = 0, \ \gamma = 0,$$

$$F_4(x) = x - \frac{f(x)}{f'(x)} G_{\beta,\gamma}(x), \ \ \beta = -1, \ \gamma = 0,$$

$$F_5(x) = x - \frac{f(x)}{f'(x)} G_{CH}(x),$$

$$F_6(x) = x - \frac{f(x)}{f'(x)} G_{D1}(x)$$

$$F_7(x) = x - \frac{f(x)}{f'(x)} G_{D2}(x)$$

$$F_8(x) = x - \frac{f(x)}{f'(x)} G_{HL}(x),$$

$$F_9(x) = x - \frac{f(x)}{f'(x)} G_{SH}(x),$$

$$F_{10}(x) = x - \frac{f(x)}{f'(x)} G_{D3}(x),$$

$$F_{11}(x) = x - \frac{f(x)}{f'(x)} G_{D4}(x),$$

$$F_{12}(x) = x - \frac{f(x)}{f'(x)} G_{HR}(x),$$

$$F_{13}(x) = x - \frac{f(x)}{f'(x)} G_{LB}(x).$$

The order of convergence COC can be approximated using the formula

$$COC \approx \frac{\ln |(x_{n+1} - \alpha) / (x_n - \alpha)|}{\ln |(x_n - \alpha) / (x_{n-1} - \alpha)|}.$$

All computations were performed in *Mathematica* 6.0. When SetPrecision is used to increase the precision of a number, we can choose number *prec* of digits in floating point arithmetics. In our tables we give the value of *prec.* We use the following stopping criteria in our calculations: $|x_k - \alpha| < \varepsilon$ and$|f(x_k)| < \varepsilon$, where $\alpha$ is exact solution of considered equation. With *it* we denote number of iteration steps. For numerical illustrations in this section we used the fixed stopping criteria $\varepsilon = 10^{-15}$ and$prec = 1000$.

We present some numerical test results for our iterative methods in Table 1. We used the following functions:

$$f_1(x) = \sin x - \frac{1}{2}, \quad \alpha_1 * \approx 0.5235987755982988731,$$

$$f_2(x) = x^3 - 10, \quad \alpha_2* \approx 2.1544346900318837218,$$

$$f_3(x) = e^x - x^2, \quad \alpha_3* \approx 0.9100075724887090607,$$

$$f_4(x) = x^3 + 4x^2 - 10, \quad \alpha_4* \approx 1.3652300134140968458,$$

$$f_5(x) = (x-1)^3 - 1, \quad \alpha_5 = 2,$$

$$f_6(x) = \sin x - \frac{x}{2}, \quad \alpha_6* \approx 1.8954942670339809471.$$

We also display the approximation $\alpha*$ of exact root $\alpha$ for each equation. $\alpha*$ is calculated with precision *prec*, but only 20 digits are displayed.

As a convergence criterion it was required that distance of two consecutive approximations $\delta$ for the zero be less than $10^{-15}$. Also displayed are the number of iterations to approximate root ($it$), the computational order of convergence (COC), the value $f(x_{it})$ and$|x_{it} - \alpha|$.

Table 1: Numerical results

| | IT | COC | $\Delta x_*$ | $f(x_*)$ | $\delta$ |
|---|---|---|---|---|---|
| $f_1,\ x_0 = 0.05$ | | | | | |
| $F_1$ | 5 | 2 | $3.6 \cdot 10^{-35}$ | $-3.1 \cdot 10^{-35}$ | $1.1 \cdot 10^{-17}$ |
| $F_2$ | 4 | 3 | $1.2 \cdot 10^{-58}$ | $-1.0 \cdot 10^{-58}$ | $8.7 \cdot 10^{-20}$ |
| $F_3$ | 4 | 3 | $1.3 \cdot 10^{-76}$ | $-1.1 \cdot 10^{-76}$ | $1.5 \cdot 10^{-25}$ |
| $F_4$ | 4 | 3 | $8.9 \cdot 10^{-65}$ | $7.7 \cdot 10^{-65}$ | $9.5 \cdot 10^{-22}$ |
| $F_5$ | 4 | 3 | $3.1 \cdot 10^{-24}$ | $-2.7 \cdot 10^{-54}$ | $2.1 \cdot 10^{-18}$ |
| $F_6$ | 4 | 3 | $2.4 \cdot 10^{-78}$ | $2.1 \cdot 10^{-78}$ | $3.1 \cdot 10^{-26}$ |
| $F_7$ | 4 | 3 | $4.3 \cdot 10^{-71}$ | $-3.7 \cdot 10^{-71}$ | $8.0 \cdot 10^{-24}$ |
| $F_8$ | 4 | 3 | $8.0 \cdot 10^{-56}$ | $-7.0 \cdot 10^{-56}$ | $6.9 \cdot 10^{-19}$ |
| $F_9$ | 4 | 3 | $5.0 \cdot 10^{-58}$ | $-4.3 \cdot 10^{-58}$ | $1.4 \cdot 10^{-19}$ |
| $F_{10}$ | 4 | 4 | $2.0 \cdot 10^{-158}$ | $1.7 \cdot 10^{-158}$ | $5.9 \cdot 10^{-40}$ |
| $F_{11}$ | 4 | 3 | $3.3 \cdot 10^{-64}$ | $-2.8 \cdot 10^{-64}$ | $1.3 \cdot 10^{-21}$ |
| $F_{12}$ | 4 | 3 | $1.2 \cdot 10^{-76}$ | $-1.0 \cdot 10^{-76}$ | $1.4 \cdot 10^{-25}$ |
| $f_1,\ x_0 = 1.0$ | | | | | |
| $F_1$ | 6 | 2 | $2.8 \cdot 10^{-45}$ | $-2.4 \cdot 10^{-45}$ | $9.8 \cdot 10^{-23}$ |
| $F_2$ | 4 | 3 | $1.5 \cdot 10^{-51}$ | $1.3 \cdot 10^{-51}$ | $2.0 \cdot 10^{-17}$ |
| $F_3$ | 4 | 3 | $6.2 \cdot 10^{-82}$ | $5.4.10^{-82}$ | $2.5 \cdot 10^{-27}$ |
| $F_4$ | 4 | 3 | $5.1 \cdot 10^{-60}$ | $-4.5 \cdot 10^{-60}$ | $3.7 \cdot 10^{-20}$ |
| $F_5$ | 5 | 3 | $6.9 \cdot 10^{-81}$ | $5.9 \cdot 10^{-81}$ | $2.7 \cdot 10^{-27}$ |
| $F_6$ | 5 | 3 | $5.1 \cdot 10^{-131}$ | $4.4 \cdot 10^{-131}$ | $8.5 \cdot 10^{-44}$ |
| $F_7$ | 4 | 3 | $2.7 \cdot 10^{-59}$ | $2.4 \cdot 10^{-59}$ | $7.0 \cdot 10^{-20}$ |
| $F_8$ | 5 | 3 | $1.7 \cdot 10^{-127}$ | $1.4 \cdot 10^{-127}$ | $8.7 \cdot 10^{-43}$ |
| $F_9$ | 4 | 3 | $3.3 \cdot 10^{-90}$ | $2.9 \cdot 10^{-90}$ | $2.7 \cdot 10^{-30}$ |
| $F_{10}$ | 4 | 4 | $7.0 \cdot 10^{-138}$ | $6.1 \cdot 10^{-138}$ | $8.0 \cdot 10^{-35}$ |
| $F_{11}$ | 4 | 3 | $2.7 \cdot 10^{-47}$ | $2.3 \cdot 10^{-47}$ | $5.4 \cdot 10^{-16}$ |
| $F_{12}$ | 4 | 3 | $2.8 \cdot 10^{-59}$ | $2.4 \cdot 10^{-59}$ | $7.0 \cdot 10^{-20}$ |
| $F_{13}$ | 4 | 3 | $6.4 \cdot 10^{-77}$ | $5.5 \cdot 10^{-77}$ | $1.2 \cdot 10^{-25}$ |

| | | | | | |
|---|---|---|---|---|---|
| $f_2,\ x_0 = 2.2$ | | | | | |
| $F_1$ | 8 | 2 | $5.0 \cdot 10^{-216}$ | $4.1 \cdot 10^{-216}$ | $2.9 \cdot 10^{-108}$ |
| $F_2$ | 6 | 3 | $7.9 \cdot 10^{-520}$ | $-6.5 \cdot 10^{-520}$ | $1.1 \cdot 10^{-173}$ |
| $F_3$ | 6 | 3 | $2.2 \cdot 10^{-757}$ | $-1.8 \cdot 10^{-757}$ | $1.2 \cdot 10^{-252}$ |
| $F_4$ | 6 | 3 | $1.9 \cdot 10^{-506}$ | $-1.6 \cdot 10^{-506}$ | $3.6 \cdot 10^{-169}$ |
| $F_5$ | 6 | 3 | $3.3 \cdot 10^{-503}$ | $-2.7 \cdot 10^{-503}$ | $3.5 \cdot 10^{-168}$ |
| $F_6$ | 6 | 3 | $2.0 \cdot 10^{-537}$ | $-1.7 \cdot 10^{-537}$ | $1.5 \cdot 10^{-179}$ |
| $F_7$ | 5 | 3 | $2.0 \cdot 10^{-370}$ | $1.6 \cdot 10^{-370}$ | $1.8 \cdot 10^{-123}$ |
| $F_8$ | 6 | 3 | $4.4 \cdot 10^{-571}$ | $-3.6 \cdot 10^{-571}$ | $1.0 \cdot 10^{-190}$ |
| $F_9$ | 6 | 3 | $5.7 \cdot 10^{-742}$ | $-4.6 \cdot 10^{-742}$ | $2.1 \cdot 10^{-247}$ |
| $F_{10}$ | 6 | 3 | $8.9 \cdot 10^{-639}$ | $-7.3 \cdot 10^{-639}$ | $3.1 \cdot 10^{-213}$ |
| $F_{11}$ | 6 | 3 | $2.2 \cdot 10^{-592}$ | $-1.8 \cdot 10^{-592}$ | $8.5 \cdot 10^{-198}$ |
| $F_{12}$ | 5 | 3 | $2.0 \cdot 10^{-370}$ | $1.6 \cdot 10^{-370}$ | $1.8 \cdot 10^{-123}$ |
| $F_{13}$ | 6 | 3 | $9.6 \cdot 10^{-751}$ | $-7.9 \cdot 10^{-751}$ | $1.9 \cdot 10^{-250}$ |
| $f_3,\ x_0 = 1.27$ | | | | | |
| $F_1$ | 6 | 2 | $2.3 \cdot 10^{-51}$ | $-6.8 \cdot 10^{-51}$ | $6.2 \cdot 10^{-26}$ |
| $F_2$ | 5 | 3 | $1.0 \cdot 10^{-90}$ | $3.0 \cdot 10^{-90}$ | $7.7 \cdot 10^{-31}$ |
| $F_3$ | 4 | 3 | $6.5 \cdot 10^{-89}$ | $-1.9 \cdot 10^{-88}$ | $8.5 \cdot 10^{-30}$ |
| $F_4$ | 5 | 3 | $1.9 \cdot 10^{-131}$ | $5.7 \cdot 10^{-131}$ | $2.1 \cdot 10^{-44}$ |
| $F_5$ | 4 | 3 | $7.4 \cdot 10^{-51}$ | $-2.2 \cdot 10^{-50}$ | $2.1 \cdot 10^{-17}$ |
| $F_6$ | 4 | 3 | $2.0 \cdot 10^{-58}$ | $-6.1 \cdot 10^{-58}$ | $6.9 \cdot 10^{-20}$ |
| $F_7$ | 4 | 3 | $1.0 \cdot 10^{-92}$ | $-3.0 \cdot 10^{-92}$ | $5.3 \cdot 10^{-31}$ |
| $F_8$ | 4 | 3 | $1.9 \cdot 10^{-56}$ | $-5.7 \cdot 10^{-56}$ | $3.4 \cdot 10^{-19}$ |
| $F_9$ | 4 | 3 | $9.5 \cdot 10^{-68}$ | $-2.8 \cdot 10^{-67}$ | $8.8 \cdot 10^{-23}$ |
| $F_{10}$ | 4 | 3 | $4.3 \cdot 10^{-71}$ | $-1.3 \cdot 10^{-70}$ | $5.4 \cdot 10^{-24}$ |
| $F_{11}$ | 4 | 3 | $3.7 \cdot 10^{-60}$ | $-1.1 \cdot 10^{-59}$ | $2.1 \cdot 10^{-20}$ |
| $F_{12}$ | 4 | 3 | $1.0 \cdot 10^{-92}$ | $-3.0 \cdot 10^{-92}$ | $5.3 \cdot 10^{-31}$ |
| $F_{13}$ | 4 | 3 | $1.4 \cdot 10^{-87}$ | $-4.2 \cdot 10^{-87}$ | $2.4 \cdot 10^{-29}$ |
| $f_4,\ x_0 = 1.8$ [1] | | | | | |
| $F_1$ | 5 | 2 | $1.6 \cdot 10^{-42}$ | $2.7 \cdot 10^{-41}$ | $1.8 \cdot 10^{-21}$ |
| $F_2$ | 4 | 3 | $8.9 \cdot 10^{-57}$ | $-1.5 \cdot 10^{-55}$ | $1.0 \cdot 10^{-19}$ |
| $F_3$ | 4 | 3 | $1.8 \cdot 10^{-115}$ | $-2.9 \cdot 10^{-114}$ | $1.1 \cdot 10^{-38}$ |
| $F_4$ | 5 | 3 | $3.4 \cdot 10^{-53}$ | $5.7 \cdot 10^{-52}$ | $1.6 \cdot 10^{-18}$ |
| $F_5$ | 4 | 3 | $1.5 \cdot 10^{-96}$ | $-2.4 \cdot 10^{-95}$ | $1.5 \cdot 10^{-32}$ |
| $F_6$ | 4 | 3 | $5.4 \cdot 10^{-93}$ | $-8.9 \cdot 10^{-92}$ | $2.2 \cdot 10^{-31}$ |
| $F_7$ | 3 | 3 | $2.7 \cdot 10^{-49}$ | $-4.4 \cdot 10^{-48}$ | $2.1 \cdot 10^{-16}$ |
| $F_8$ | 4 | 3 | $3.7 \cdot 10^{-112}$ | $-6.2 \cdot 10^{-111}$ | $1.3 \cdot 10^{-37}$ |
| $F_9$ | 4 | 3 | $5.4 \cdot 10^{-130}$ | $-9.0 \cdot 10^{-129}$ | $2.1 \cdot 10^{-43}$ |
| $F_{10}$ | 4 | 3 | $7.3 \cdot 10^{-105}$ | $-1.2 \cdot 10^{-103}$ | $3.0 \cdot 10^{-35}$ |
| $F_{11}$ | 4 | 3 | $2.3 \cdot 10^{-109}$ | $-3.8 \cdot 10^{-108}$ | $1.0 \cdot 10^{-36}$ |
| $F_{12}$ | 3 | 3 | $2.7 \cdot 10^{-49}$ | $-4.4 \cdot 10^{-48}$ | $2.1 \cdot 10^{-16}$ |
| $F_{13}$ | 4 | 3 | $9.8 \cdot 10^{-116}$ | $-1.6 \cdot 10^{-114}$ | $8.7 \cdot 10^{-39}$ |

| | | | | | |
|---|---|---|---|---|---|
| $f_5,\ x_0 = 1.8$ [1] | | | | | |
| $F_1$ | 6 | 2 | $9.6 \cdot 10^{-42}$ | $2.9 \cdot 10^{-41}$ | $3.1 \cdot 10^{-21}$ |
| $F_2$ | 5 | 3 | $4.4 \cdot 10^{-98}$ | $1.3 \cdot 10^{-97}$ | $2.0 \cdot 10^{-33}$ |
| $F_3$ | 4 | 3 | $5.8 \cdot 10^{-61}$ | $-1.7 \cdot 10^{-60}$ | $9.5 \cdot 10^{-21}$ |
| $F_4$ | 6 | 3 | $4.0 \cdot 10^{-105}$ | $-1.2 \cdot 10^{-104}$ | $8.4 \cdot 10^{-36}$ |
| $F_5$ | 5 | 3 | $1.7 \cdot 10^{-118}$ | $-5.0 \cdot 10^{-118}$ | $4.6 \cdot 10^{-40}$ |
| $F_6$ | 5 | 3 | $2.1 \cdot 10^{-99}$ | $-6.4 \cdot 10^{-99}$ | $9.9 \cdot 10^{-34}$ |
| $F_7$ | 4 | 3 | $4.6 \cdot 10^{-107}$ | $1.4 \cdot 10^{-106}$ | $6.5 \cdot 10^{-36}$ |
| $F_8$ | 4 | 3 | $5.8 \cdot 10^{-61}$ | $-1.7 \cdot 10^{-60}$ | $9.5 \cdot 10^{-21}$ |
| $F_9$ | 4 | 3 | $1.3 \cdot 10^{-69}$ | $-3.9 \cdot 10^{-69}$ | $1.6 \cdot 10^{-23}$ |
| $F_{10}$ | 4 | 3 | $1.3 \cdot 10^{-49}$ | $-4.0 \cdot 10^{-49}$ | $4.9 \cdot 10^{-17}$ |
| $F_{11}$ | 4 | 3 | $3.5 \cdot 10^{-56}$ | $-1.1 \cdot 10^{-55}$ | $3.5 \cdot 10^{-19}$ |
| $F_{12}$ | 4 | 3 | $4.6 \cdot 10^{-107}$ | $1.4 \cdot 10^{-106}$ | $6.5 \cdot 10^{-36}$ |
| $F_{13}$ | 4 | 3 | $9.5 \cdot 10^{-63}$ | $-2.8 \cdot 10^{-62}$ | $2.4 \cdot 10^{-21}$ |
| $f_6,\ x_0 = 2.3$ [1] | | | | | |
| $F_1$ | 6 | 2 | $3.0 \cdot 10^{-48}$ | $-2.5 \cdot 10^{-48}$ | $2.3 \cdot 10^{-24}$ |
| $F_2$ | 4 | 3 | $1.1 \cdot 10^{-51}$ | $-8.9 \cdot 10^{-52}$ | $1.2 \cdot 10^{-17}$ |
| $F_3$ | 4 | 3 | $4.1 \cdot 10^{-77}$ | $-3.4 \cdot 10^{-77}$ | $6.7 \cdot 10^{-26}$ |
| $F_4$ | 5 | 3 | $1.7 \cdot 10^{-136}$ | $1.4 \cdot 10^{-136}$ | $7.4 \cdot 10^{-46}$ |
| $F_5$ | 4 | 3 | $6.9 \cdot 10^{-49}$ | $-5.7 \cdot 10^{-49}$ | $9.8 \cdot 10^{-17}$ |
| $F_6$ | 4 | 3 | $3.1 \cdot 10^{-53}$ | $-2.5 \cdot 10^{-53}$ | $3.6 \cdot 10^{-18}$ |
| $F_7$ | 4 | 3 | $3.6 \cdot 10^{-115}$ | $-2.9 \cdot 10^{-115}$ | $2.2 \cdot 10^{-38}$ |
| $F_8$ | 4 | 3 | $1.6 \cdot 10^{-55}$ | $-1.3 \cdot 10^{-55}$ | $7.4 \cdot 10^{-19}$ |
| $F_9$ | 4 | 3 | $6.5 \cdot 10^{-72}$ | $-5.3 \cdot 10^{-72}$ | $4.6 \cdot 10^{-24}$ |
| $F_{10}$ | 4 | 3 | $4.3 \cdot 10^{-64}$ | $-3.5 \cdot 10^{-64}$ | $1.1 \cdot 10^{-21}$ |
| $F_{11}$ | 4 | 3 | $3.9 \cdot 10^{-58}$ | $-3.2 \cdot 10^{-58}$ | $1.0 \cdot 10^{-19}$ |
| $F_{12}$ | 4 | 3 | $3.6 \cdot 10^{-115}$ | $-2.9 \cdot 10^{-115}$ | $2.2 \cdot 10^{-38}$ |
| $F_{13}$ | 4 | 3 | $3.1 \cdot 10^{-76}$ | $-2.6 \cdot 10^{-76}$ | $1.3 \cdot 10^{-25}$ |

## Conclusions

In this paper we presented the family of third-order iterative methods. Some well known methods belong to this family, for example, Halley's method, Chebyshev's method and super-Halley method from [3, 5, 7]. The first method in our tables is the Newton's method. The test results in Table 1 show that the computed order of convergence of the presented iterative methods is three, which supports the theoretical result obtained in this paper.

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