Novi Sad J. Math.
Vol. 38, No. 3, 2008, ??-??

CHAOS EXPANSION OF GENERALIZED RANDOM PROCESSES ON FRACTIONAL WHITE NOISE SPACE[1]

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Abstract. We consider chaos expansion in generalized random variable spaces $\exp(\mathcal{S})_{\rho,H}$ and $\exp(\mathcal{S})_{-\rho,H}$ based on fractional white noise space, which correspond to the ones studied in [6]. Generalized stochastic processes with values in these spaces are proven to have a series expansion, and different Wick products are discussed.

AMS Mathematics Subject Classification (2000): 60H40, 60G20, 60G15.

Key words and phrases: fractional Brownian motion, Wick product, series expansions, generalized stochastic processes.

1. Introduction

In this paper we develop a version of chaos expansion theorems in spaces of generalized random variables of Kondratiev type, expansion of generalized random processes, and the Wick product. Our study will be based on fractional white noise spaces; for a detailed review of fractional white noise theory we refer to [1], [2] and [4]. Since there exists an isomorphism between the classical white noise space and the fractional white noise space, all theorems stated in this paper can be proven verbatim as in [6]; one just has to replace the orthogonal basis of the space of generalized random variables with the orthogonal basis of the corresponding fractional versions of these spaces. Also, we consider the general case $\rho \in [0,1]$, while in [6] we studied the case $\rho = 1$, $H = \frac{1}{2}$. We recall some basic definitions from [3], [4], [5] and [7].

1.1. Fractional Brownian motion on white noise spaces

Definition 1.1. Fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a Gaussian process $B^{(H)} = \{B_t^{(H)}(\cdot),\ t \in \mathbb{R}\}$ on $(\Omega, \mathcal{F}, \mu)$, with $B_0^{(H)} = 0$, zero expectation $E[B_t^{(H)}] = 0$ for all $t \in \mathbb{R}$, and covariance function

$$(1) \qquad E[B_s^{(H)} B_t^{(H)}] = \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\},\ \ s,t \in \mathbb{R}.$$

[1]The paper was supported by the project *Functional analysis methods, ODEs and PDEs with singularities*, No. 144016, financed by the Ministry of Science, Republic of Serbia.

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Fractional Brownian motion is a centered Gaussian process with non-independent stationary increments. Dependence structure of the fBm is modified by Hurst parameter $H \in (0,1)$. For $H = \frac{1}{2}$ the covariance function can be written as $E(B_t^{(\frac{1}{2})} B_s^{(\frac{1}{2})}) = \min\{s,t\}$, the process $B_t^{(\frac{1}{2})}$ becomes a standard Brownian motion and it has independent increments.
From (1) it follows that $E(B_t^{(H)} - B_s^{(H)})^2 = |t-s|^{2H}$ and according to Kolmogorov's theorem, fBm $B^{(H)}$ has a continuous modification. Fractional Brownian motion $B^{(H)}$ is an H self-similar process, i.e. $B_{\alpha t}^{(H)} = \alpha^H B_t^{(H)}, \alpha > 0$.
For $H \in (\frac{1}{2}, 1)$ the second partial derivative of the covariance function

$$\frac{\partial^2}{\partial t \partial s} E[B_s^{(H)} B_t^{(H)}] = H(2H-1)|t-s|^{2H-2}$$

is integrable and we can write

$$E[B_s^{(H)} B_t^{(H)}] = H(2H-1) \int_0^t \int_0^s |u-v|^{2H-2} du dv.$$

For any $n \in \mathbb{Z}$, $n \neq 0$ the autocovariance function is given by

$$r(n) := E[B_1^{(H)}(B_{n+1}^{(H)} - B_n^{(H)})] \sim \ H(2H-1)|n|^{2H-1}, \ \text{ when } |n| \to \infty$$

and for $H \in (\frac{1}{2}, 1)$ fBm has the long-range dependence property $\sum\limits_{n=1}^{\infty} r(n) = \infty$.

1.2. Zemanian spaces

We recall some basic notions from theory of Zemanian spaces (see [7]). Define

$$\mathcal{A}_k = \{f = \sum_{n=1}^{\infty} a_n \psi_n : \ \|f\|_k^2 = \sum_{n=1}^{\infty} |a_n|^2 \widetilde{\lambda}_n^{2k} < \infty\}, \ k \in \mathbb{Z},$$

$\mathcal{A} = \bigcap\limits_{k \in \mathbb{N}_0} \mathcal{A}_k$ and $\mathcal{A}' = \bigcup\limits_{k \in \mathbb{N}_0} \mathcal{A}_{-k}$, where $\widetilde{\lambda}_n$ and ψ_n are eigenvalues and eigenvectors, respectively, of the self-adjoint linear differential operator $\mathcal{R}$ of the form

$$\mathcal{R} = \theta_0 D^{n_1} \theta_1 \ldots D^{n_\nu} \theta_\nu = \overline{\theta}_\nu (-D)^{n_\nu} \ldots \overline{\theta}_1 (-D)^{n_1} \overline{\theta}_0,$$

where $D = \frac{d}{dx} \theta_k$ are smooth complex functions and $n_k \in \mathbb{Z}$, $k = 1, 2, ..., \nu$. The *Zemanian space of test functions* $\mathcal{A}$ is equipped with the projective topology and its dual $\mathcal{A}'$, the *Zemanian space of generalized functions,* is equipped with the inductive topology which is equivalent to the strong dual topology. The action of a generalized function $f = \sum\limits_{n=1}^{\infty} a_n \psi_n \in \mathcal{A}'$ onto a test function $\varphi = \sum\limits_{n=1}^{\infty} b_n \psi_n \in \mathcal{A}$ is given by the dual paring $\langle f, \varphi \rangle = \sum\limits_{n=1}^{\infty} a_n b_n$.

1.3. Fractional white noise space

In the sequel we denote by h_n, $n \in \mathbb{N}_0$, the Hermite polynomials and by ξ_n, $n \in \mathbb{N}$, the family of the Hermite functions. For properties of Hermite polynomials and Hermite functions we refer to [3].
We will adapt the classical white noise calculus to the fractional white noise as it is done in [2] and [4]. Fix a Hurst parameter $H \in [\frac{1}{2}, 1)$. Define

$$\phi(s,t) = H(2H-1)|s-t|^{2H-2}, \text{ for } s,t \in \mathbb{R}.$$

The space of measurable functions defined by

$$L^2_\phi(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : \|f\|^2_H = \int_{\mathbb{R}} \int_{\mathbb{R}} f(s)f(t)\phi(s,t)dsdt < \infty\},$$

equipped with the inner product $(f,g)_H := \int_{\mathbb{R}} \int_{\mathbb{R}} f(s)g(t)\phi(s,t)dsdt$, $f,g \in L^2_\phi(\mathbb{R})$, becomes a separable Hilbert space.

Lemma 1.1. ([4]) *Let $\Gamma_\phi : L^2_\phi(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by $\Gamma_\phi f(u) = c_H \int_u^\infty \frac{f(t)}{(t-u)^{\frac{3}{2}-H}} dt$, where c_H is the constant $c_H = \sqrt{\frac{H(2H-1)\Gamma(\frac{3}{2}-H)}{\Gamma(H-\frac{1}{2})\Gamma(2-2H)}}$ and Γ denotes the Gamma function. Then Γ_ϕ is an isometry between $L^2_\phi(\mathbb{R})$ and $L^2(\mathbb{R})$.*

The functions

$$e_n(x) = \Gamma_\phi^{-1}\xi_n(x), \quad n = 1, 2..., \tag{2}$$

belong to the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R})$ and form an orthonormal basis in $L^2_\phi(\mathbb{R})$.

Consider $\Omega := \mathcal{S}'(\mathbb{R})$ and the Borel $\sigma-$algebra $\mathcal{B}$ generated by the weak topology on Ω. By the Bochner-Minlos theorem, there exists a unique probability measure μ_H on $(\mathcal{S}'(\mathbb{R}), \mathcal{B})$ such that the relation

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle \omega, f\rangle} d\mu_H(\omega) = e^{-\frac{1}{2}\|f\|^2_H} \tag{3}$$

holds for each $f \in \mathcal{S}(\mathbb{R})$. In equality (3) $\langle \omega, f\rangle := \omega(f)$ represents the usual dual paring between a tempered distribution $\omega \in \mathcal{S}'(\mathbb{R})$ and a test function $f \in \mathcal{S}(\mathbb{R})$. The triplet $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu_H)$ is called *the fractional white noise probability space* and μ_H *the fractional white noise measure.*

From now on we assume that the basic probability space is the space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu_H)$. Denote $(L)^2_H = L^2(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu_H)$, the space of square integrable functions on $\mathcal{S}'(\mathbb{R})$ with respect to the measure μ_H. It is a Hilbert space equipped with the inner product

$$(f,g)_{(L)^2_H} = E_{\mu_H}(fg) = \int_{\mathcal{S}'(\mathbb{R})} f(\omega)g(\omega)d\mu_H(\omega).$$

Note that (3) implies the isometry $E_{\mu_H}(\langle\omega,f\rangle^2)=\|f\|^2_H, \quad f\in\mathcal{S}(\mathbb{R})$.
Now we will define fractional Brownian motion as an element of $(L)^2_H$ by

$$\widetilde{B_t^{(H)}}(\omega):=\langle\omega,\chi_{[0,t]}(\cdot)\rangle_{(L)^2_H},\ \ \omega\in\mathcal{S}'(\mathbb{R}),$$

where $\chi_{[0,t]}$ represents the characteristic function of $[0,t]$, $t\in\mathbb{R}$. From Kolmogorov's continuity theorem, the process $\widetilde{B^{(H)}}$ has a $t-$continuous version, which we will denote by $B_t^{(H)}$. It follows directly that $B_t^{(H)}$ is a Gaussian process with expectation $E_{\mu_H}(B_t^{(H)})=0$ and covariance function

$$E_{\mu_H}[B_t^{(H)}B_s^{(H)}]=\int_0^t\int_0^s\phi(u,v)dudv=\frac{1}{2}(|t|^{2H}+|s|^{2H}-|s-t|^{2H}),$$

i.e. $B_t^{(H)}$ is fractional Brownian motion.
For definition of the integral with respect to fractional Brownian motion we refer to [1] and [3]. If $f\in L^2_\phi(\mathbb{R})$ then

$$\langle\omega,f\rangle=\int_{\mathbb{R}}f(t)\,dB_t^{(H)}(\omega).$$

1.4. Wiener-Itô chaos expansion

Let $\mathcal{J}=(\mathbb{N}_0^{\mathbb{N}})_c$ denote the set of all multi-indices α of non-negative integers, which have finitely many nonzero components. If $\alpha=(\alpha_1,\cdots\alpha_m,0,0,\cdots)\in\mathcal{J}$ then

$$\mathcal{H}_\alpha(\omega):=h_{\alpha_1}(\langle\omega,e_1\rangle)\cdots h_{\alpha_m}(\langle\omega,e_m\rangle) \tag{4}$$

represents the αth *Fourier-Hermite polynomial.* In particular, for the kth unit vector $\varepsilon^{(k)}=(0,\cdots,0,1,0,\cdots)$, the sequence of zeros with the number 1 as the kth component, $k=1,2,...$ we get

$$\mathcal{H}_{\varepsilon^{(k)}}(\omega)=h_1(\langle\omega,e_k\rangle)=\langle\omega,e_k\rangle=\int_{\mathbb{R}}e_k\,dB_t^{(H)}(\omega).$$

It is known that $E_{\mu_H}(\mathcal{H}_\alpha\mathcal{H}_\beta)=\alpha!\delta_{\alpha\beta}$, where $\alpha!=\alpha_1!\alpha_2!\cdots$, for $\alpha=(\alpha_1,\alpha_2,\cdots)\in\mathcal{J}$, and $\delta_{\alpha\beta}$ denotes the Kronecker symbol. The family of Fourier-Hermite polynomials $\{\frac{1}{\sqrt{\alpha!}}\mathcal{H}_\alpha\}_{\alpha\in\mathcal{J}}$ forms an orthonormal basis of $(L)^2_H$.

Theorem 1.1 ([4]: Wiener-Itô chaos expansion theorem)**.** *For each element $F\in(L)^2_H$ there exists a unique family of real constants $\{c_\alpha\}_{\alpha\in\mathcal{J}}$ such that F has a representation of the form*

$$F(\omega)=\sum_{\alpha\in\mathcal{J}}c_\alpha\mathcal{H}_\alpha(\omega) \tag{5}$$

and $\|F\|^2_{(L)^2_H}=\sum_{\alpha\in\mathcal{J}}c_\alpha^2\alpha!$. Moreover, $c_\alpha=\frac{1}{\alpha!}E_{\mu_H}(F\mathcal{H}_\alpha(\omega))$.

Example 1. Fractional Brownian motion is a random variable $B_t^{(H)}(\omega) \in (L)_H^2$, given by the chaos expansion

$$B_t^{(H)}(\omega) = \sum_{k=1}^{\infty} \int_0^t \left(\int_{\mathbb{R}} e_k(v)\phi(u,v)\, dv \right) du\, \mathcal{H}_{\varepsilon^{(k)}}(\omega). \tag{6}$$

2. Fractional spaces $\exp(\mathcal{S})_{\rho,\,H}$ and $\exp(\mathcal{S})_{-\rho,\,H}$

Following [6], we define fractional spaces of generalized random variables of Hurst index $H \in [\frac{1}{2}, 1)$. Let $\rho \in [0,1]$.

Definition 2.1.

- The space of fractional stochastic test functions $\exp(\mathcal{S})_{\rho,\,H}$ consists of elements $f(\omega) = \sum_{\alpha\in\mathcal{J}} c_\alpha \mathcal{H}_\alpha(\omega) \in (L)_H^2,\ c_\alpha \in \mathbb{R},\ \alpha \in \mathcal{J}$, such that

  $$\|f\|^2_{\rho,p,\exp,\,H} = \sum_{\alpha\in\mathcal{J}} c_\alpha^2 (\alpha!)^{1+\rho} e^{p(2\mathbb{N})^\alpha} < \infty,\ \mathit{for\ all}\ p \in \mathbb{N}_0, \tag{7}$$

  where $(2\mathbb{N})^\alpha = \prod_{j=1}^{\infty} (2j)^{\alpha_j},\ \mathit{for}\ \alpha = (\alpha_1, \alpha_2, \cdots) \in \mathcal{J}$.

- The space of fractional stochastic generalized functions $\exp(\mathcal{S})_{-\rho,\,H}$ consist of formal expansions $F(\omega) = \sum_{\alpha\in\mathcal{J}} b_\alpha \mathcal{H}_\alpha \in (L)_H^2,\ b_\alpha \in \mathbb{R}\ ,\ \alpha \in \mathcal{J}$ such that

  $$\|F\|^2_{-\rho,-p,\exp,\,H} = \sum_{\alpha\in\mathcal{J}} b_\alpha^2 (\alpha!)^{1-\rho} e^{-p(2\mathbb{N})^\alpha} < \infty,\ \mathit{for\ some}\ p \in \mathbb{N}_0. \tag{8}$$

If we equip $\exp(\mathcal{S})_{\rho,\,H}$ with the projective topology and $\exp(\mathcal{S})_{-\rho,\,H}$ with the inductive topology then $\exp(\mathcal{S})_{-\rho,\,H}$ can be identified with the dual of $\exp(\mathcal{S})_{\rho,\,H}$ and the action of $F \in \exp(\mathcal{S})_{-\rho,\,H}$ onto a test function $f \in \exp(\mathcal{S})_{\rho,\,H}$ is given by

$$\ll F\,, f\ \gg := \sum_{\alpha\in\mathcal{J}} \alpha! b_\alpha c_\alpha. \tag{9}$$

For $H = \frac{1}{2}$ these spaces reduce from fractional ones to the corresponding classical spaces $\exp(\mathcal{S})_\rho$ and $\exp(\mathcal{S})_{-\rho}$ defined in [6]. The family of functions

$$\{(\alpha!)^{-\frac{1+\rho}{2}} e^{-\frac{p}{2}(2\mathbb{N})^\alpha} \mathcal{H}_\alpha\}_{\alpha\in\mathcal{J}}$$

forms an orthonormal basis of the Hilbert space $\exp(\mathcal{S})_{\rho,H}$.
The spaces $\exp(\mathcal{S})_{\rho,\,H}$ and $\exp(\mathcal{S})_{-\rho,\,H}$ are generalization of the fractional Kondratiev spaces (see [1], [3]), which are denoted by $(\mathcal{S})_{\rho,H}$ and $(\mathcal{S})_{-\rho,H}$ respectively. Following relationship of fractional Kondratiev spaces can be proved in the same way like in [6]:

$$\exp(\mathcal{S})_{\rho,\,H} \subseteq (\mathcal{S})_{\rho,H} \subseteq (L)_H^2 \subseteq (\mathcal{S})_{-\rho,H} \subseteq \exp(\mathcal{S})_{-\rho,\,H}.$$

Definition 2.2. Let $Z:\mathbb{R}\to\exp(\mathcal{S})_{-\rho,H}$ be a function such that $\ll Z_t,\,\varphi\gg\,\in L^1(\mathbb{R},dt)$ for all $\varphi\in\exp(\mathcal{S})_{\rho,H}$. Then $\int_{\mathbb{R}} Z_t\,dt$ is the unique element of $\exp(\mathcal{S})_{-\rho,H}$ such that

$$\ll \int_{\mathbb{R}} Z_t dt,\,\varphi \gg = \int_{\mathbb{R}} \ll Z_t,\,\varphi\gg dt, \quad \textit{for all } \varphi\in\exp(\mathcal{S})_{\rho,H}.$$

Example 2. Fractional white noise is given by the chaos expansion

$$W_t^{(H)}(\omega)=\sum_{k=1}^{\infty}\left(\int_{\mathbb{R}} e_i(v)\phi(t,v)dv\right)\mathcal{H}_{\epsilon^{(k)}}(\omega) \tag{10}$$

and it is an element of $(\mathcal{S})_{-\rho,H}$, for $\rho=0$, for all $t>0$. It is integrable in the sense of Definition 2.2 and the relation $\frac{d}{dt}B_t^{(H)}=W_t^{(H)}$ holds (see [1] *)*.

2.1. Wick product

Definition 2.3. The Wick product of two fractional stochastic generalized functions $X=\sum_{\alpha\in\mathcal{J}} a_\alpha\mathcal{H}_\alpha(\omega)$, $a_\alpha\in\mathbb{R}$, $\alpha\in\mathcal{J}$, and $Y=\sum_{\beta\in\mathcal{J}} b_\beta\mathcal{H}_\beta(\omega)$, $b_\beta\in\mathbb{R}$, $\beta\in\mathcal{J}$, from $\exp(\mathcal{S})_{-\rho,\,H}$ is defined by

$$X\lozenge Y(\omega)=\sum_{\alpha,\beta\in\mathcal{J}} a_\alpha b_\beta\mathcal{H}_{\alpha+\beta}(\omega)=\sum_{\gamma\in\mathcal{J}}(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta)\mathcal{H}_\gamma(\omega).$$

The following theorem is a generalization of the Wick product in Kondratiev spaces provided for $\rho=1$ in [4] and for $\rho=1, H=\frac{1}{2}$ in [6].

Theorem 2.1.

i) If $X,Y\in\exp(\mathcal{S})_{-\rho,\,H}$ *then* $X\lozenge Y\in\exp(\mathcal{S})_{-\rho,\,H}$.

ii) If $X,Y\in\exp(\mathcal{S})_{\rho,\,H}$ *then* $X\lozenge Y\in\exp(\mathcal{S})_{\rho,\,H}$.

Proof: The proof can be carried out as in [6] by replacing the basis H_α, $\alpha\in\mathcal{J}$, by the orthogonal basis $\mathcal{H}_\alpha$, $\alpha\in\mathcal{J}$, of the fractional white noise space $(L)^2_H$, as defined in (2) and (4). □

The Wick product is commutative, associative and distributive with respect to addition.

Definition 2.4. Let $Y:\mathbb{R}\to\exp(\mathcal{S})_{-\rho,H}$ such that $Y_t\lozenge W_t^{(H)}$ is $dt-$integrable in $\exp(\mathcal{S})_{-\rho,H}$. Then Y_t is $dB^{(H)}$-integrable and its fractional stochastic integral of Itô type $\int_{\mathbb{R}} Y_t(\omega)dB_t^{(H)}(\omega)$ is defined by

$$\int_{\mathbb{R}} Y_t(\omega)\,dB_t^{(H)}(\omega):=\int_{\mathbb{R}} Y_t\lozenge W_t^{(H)}(\omega)\,dt. \tag{11}$$

3. Generalized stochastic processes (GRP)

Definition 3.1. Generalized stochastic processes of type (O) are measurable mappings $Y : \mathbb{R} \to \exp(\mathcal{S})_{-\rho,H}$.

Example 3. Fractional Brownian motion $B_t^{(H)}$ and fractional white noise $W_t^{(H)}$, defined by the chaos expansions (6) and (10), are examples of GRPs of type (O).

Theorem 3.1. *Let* $Y_t : \mathbb{R} \to \exp(\mathcal{S})_{-\rho,H}$.

(i) A mapping Y_t *is a GRP of type (O) if and only if it has an expansion*

$$Y_t(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha(t) \mathcal{H}_\alpha(\omega), \ t \in \mathbb{R}, \tag{12}$$

where $c_\alpha : \mathbb{R} \to \mathbb{R}$, $\alpha \in \mathcal{J}$, *are measurable functions and there exists* $p \in \mathbb{N}_0$ *such that for each* $t \in \mathbb{R}$

$$\|Y_t\|^2_{-\rho,-p,exp,H} = \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} c_\alpha(t)^2 e^{-p(2\mathbb{N})^\alpha} < \infty.$$

(ii) If $c_\alpha \in L^1(\mathbb{R})$, $\alpha \in \mathcal{J}$, *and if there exists* $p \in \mathbb{N}_0$ *such that*

$$\int_{\mathbb{R}} \|Y_t\|^2_{-\rho,-p,exp,H} \, dt = \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} |c_\alpha(t)|^2_{L^1(\mathbb{R})} e^{-p(2\mathbb{N})^\alpha} < \infty, \tag{13}$$

then $Y_t \in L^1(\mathbb{R}, \exp(\mathcal{S})_{-\rho,H})$ *i.e.* Y_t *is an integrable process in sense of Definition 2.2 and*

$$\int_{\mathbb{R}} Y_t \, dt = \sum_{\alpha \in \mathcal{J}} \left(\int_{\mathbb{R}} c_\alpha(t) dt \right) \mathcal{H}_\alpha(\omega). \tag{14}$$

(iii) Moreover, if $c_\alpha \in L^2_\phi(\mathbb{R})$, $\alpha \in \mathcal{J}$, *and if* Y_t *is* $dB^{(H)}-$*integrable, then the expansion of the* $dB^{(H)}$*-integral of the process* Y_t *is given by*

$$\int_{\mathbb{R}} Y_t(\omega) \, dB_t^{(H)}(\omega) = \sum_{\alpha \in \mathcal{J}} \sum_{k \in \mathbb{N}} (c_\alpha, e_k)_H \, \mathcal{H}_{\alpha+\varepsilon^{(k)}}(\omega), \tag{15}$$

where the sum on the right hand side converges in $\exp(\mathcal{S})_{-\rho,H}$. *In particular, if* $\int_{\mathbb{R}} Y_t(\omega) \, dB_t^{(H)}(\omega) \in (L)^2_H$ *then* $E(\int_{\mathbb{R}} Y_t(\omega) \, dB_t^{(H)}(\omega)) = 0$.

Proof: (i) Since GRPs of type (O) with values in $\exp(\mathcal{S})_{-\rho,H}$ are defined pointwisely with respect to the parameter $t \in \mathbb{R}$, their chaos expansions follow directly from (5).
(ii) Following Definitions 2.1, (9) and condition (13) one can prove that Y_t is integrable and that $\int_{\mathbb{R}} Y_t dt$ is an unique element in $\exp(\mathcal{S})_{-\rho,H}$, which can be represented in the form (14).

(iii) The expansion (15) follows from

$$\begin{aligned}
\int_{\mathbb{R}} Y_t(\omega)\, dB_t^{(H)}(\omega) &= \int_{\mathbb{R}} Y_t \Diamond W_t^{(H)}(\omega)\, dt \\
&= \sum_{\alpha\in\mathcal{J}} \sum_{k\in\mathbb{N}} \Big(\int\!\!\int_{\mathbb{R}^2} c_\alpha(u) e_k(v) \phi(u,v) dv du \Big) \mathcal{H}_{\alpha+\epsilon^{(k)}}(\omega) \\
&= \sum_{\alpha\in\mathcal{J}} \sum_{k\in\mathbb{N}} (c_\alpha, e_k)_H\, \mathcal{H}_{\alpha+\varepsilon^{(k)}}(\omega). \qquad \square
\end{aligned}$$

Definition 3.2. Generalized stochastic processes of type (I) are linear continuous mappings from the Zemanian space $\mathcal{A}$ into the space $\exp(\mathcal{S})_{-\rho,H}$ i.e. elements of $\exp\mathcal{A}^* = \mathcal{L}(\mathcal{A}, \exp(\mathcal{S})_{-\rho,H})$.

The dual pairing in $\exp\mathcal{A}^*$ will be denoted by $[\,\cdot,\cdot\,]$. Since $\exp(\mathcal{S})_{-\rho,H}$ is nuclear, we may consider $\exp\mathcal{A}^*$ as the tensor product space $\mathcal{A}' \otimes \exp(\mathcal{S})_{-\rho,H}$. The dual action in $\exp\mathcal{A}^*$ is defined as

$$\left[\sum_{\alpha\in\mathcal{J}} f_\alpha \otimes \mathcal{H}_\alpha, \varphi\right] = \sum_{\alpha\in\mathcal{J}} \langle f_\alpha, \varphi\,\rangle\, \mathcal{H}_\alpha, \;\; \varphi \in \mathcal{A}.$$

Note that $\exp\mathcal{A}^*$ is the inductive limit of the spaces $\exp\mathcal{A}_k^* = \mathcal{L}(\mathcal{A}_k, \exp(\mathcal{S})_{-\rho,H})$. Clearly, $F \in \exp\mathcal{A}_k^*$ if and only if there exists $k_0 \in \mathbb{N}$ such that $F \in \mathcal{L}(\mathcal{A}_k, \exp(\mathcal{S})_{-\rho,-k_0,H})$, where

$$\exp(\mathcal{S})_{-\rho,-k_0,H} = \{ f = \sum_{\alpha\in\mathcal{J}} c_\alpha \mathcal{H}_\alpha \,:\, \|f\|_{\rho,-k_0,exp,H} < \infty \}.$$

The space $\exp\mathcal{A}_k^*$ is Banach with dual norm

$$\|F\|_{-k}^* = sup\{ \|\, [F,f]\, \|_{-\rho,-k_0,exp,H} \;:\, f \in \mathcal{A}_k,\; \|f\|_k \le 1 \}.$$

Theorem 3.2. *Let $\phi \in \exp\mathcal{A}^*$. Then $\phi \in \exp\mathcal{A}_k^*$, $k \in \mathbb{N}_0$, if and only if ϕ has a formal expansion*

$$\phi = \sum_{\alpha\in\mathcal{J}} f_\alpha \otimes \mathcal{H}_\alpha, \;\; f_\alpha \in \mathcal{A}_{-k}, \alpha \in \mathcal{J},$$

and if exists $k_0 \in \mathbb{N}_0$ such that for all bounded sets $B \subseteq \mathcal{A}_k$

$$\sup_{\varphi\in B} \sum_{\alpha\in\mathcal{J}} |\langle f_\alpha, \varphi\rangle|^2 \|\mathcal{H}_\alpha\|_{-\rho,-k_0,exp,H} < \infty. \tag{16}$$

Condition (16) is equivalent to condition (17)

$$\sum_{\alpha\in\mathcal{J}} \|f_\alpha\|_{-k}^2\, e^{-k_0(2\mathbb{N})^\alpha} < \infty. \tag{17}$$

Proof: The proof can be provided analogue to the proof of Theorem 4.1 in [6], by replacing the basis H_α, $\alpha \in \mathcal{J}$, with the basis $\mathcal{H}_\alpha$, $\alpha \in \mathcal{J}$, of the fractional white noise space. $\square$

We can identify a locally integrable GRP (O) $U \in L^1_{loc}(\mathbb{R}, \exp(\mathcal{S})_{-\rho,H})$ with a corresponding GRP (I) $\widetilde{U} \in \mathcal{L}(\mathcal{A}, \exp(\mathcal{S})_{-\rho,H})$ as described in [6].
Assume the Zemanian spaces are nuclear, i.e. there exists p such that $\sum_{n=1}^{\infty} \widetilde{\lambda}_n^{-p} < \infty$. With $\diamond$ we denote the Wick-type product defined in Zemanian spaces $\mathcal{A}$ (see [6]).

Definition 3.3. Let $\phi \in \exp \mathcal{A}_k^*$, $\psi \in \exp \mathcal{A}_l^*$ be two GRPs (I) given by expansions $\phi = \sum_{\alpha \in \mathcal{J}} f_\alpha \otimes \mathcal{H}_\alpha$, $f_\alpha \in \mathcal{A}_{-k}$ and $\psi = \sum_{\beta \in \mathcal{J}} g_\beta \otimes \mathcal{H}_\beta$, $g_\beta \in \mathcal{A}_{-l}$, $\alpha, \beta \in \mathcal{J}$. The Wick product of ϕ and ψ is the unique element in $\exp \mathcal{A}^*_{k+l+p}$ defined by

$$\phi \blacklozenge \psi = \sum_{\gamma \in \mathcal{J}} \Big(\sum_{\alpha+\beta=\gamma} f_\alpha \diamond g_\beta \Big) \otimes \mathcal{H}_\gamma.$$

The Wick product given in the previous definition will be used in further investigations concerning stochastic differential equations.

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