ON THE $W_2$-CURVATURE TENSOR OF THE SEMI-SYMMETRIC NON-METRIC CONNECTION IN A KENMOTSU MANIFOLD

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Abstract. The objective of the present paper is to study the $W_2$-curvature tensor of the semi-symmetric non-metric connection in a Kenmotsu manifold. It is shown that if in $M^n$, $W_2^2 = 0$, then $M^n$ is isometric to the hyperbolic space $H^n(-1)$, where $W_2^2$ is the $W_2$-curvature tensor of the semi-symmetric non-metric connection. Also, locally $W_2$-symmetric Kenmotsu manifold and $W_2$-recurrent Kenmotsu manifold with respect to the semi-symmetric non-metric connection have been studied.

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1. Introduction

In 1924, A. Friedmann and J.A. Schouten \cite{6} introduced the notion of semi-symmetric linear connection on a differentiable manifold. In 1930, Bartolotti \cite{4} gave a geometrical meaning of such a connection. In 1932, H.A. Hayden \cite{7} defined and studied semi-symmetric metric connection. In 1970, K. Yano \cite{19}, started a systematic study of the semi-symmetric metric connection in a Riemannian manifold, and this was further studied by various authors.

A linear connection $\nabla^*$ on a Riemannian manifold $M^n$ is called semi-symmetric if its torsion tensor

$$T^*(X,Y) = \nabla^*_X Y - \nabla^*_Y X - [X,Y]$$

satisfies

$$T^*(X,Y) = \eta(Y)X - \eta(X)Y,$$

where $\eta$ is a non-zero 1-form associated with a vector field $\xi$ defined by $\eta(X) = g(X,\xi)$. A semi-symmetric connection $\nabla^*$ is called semi-symmetric metric connection \cite{19} if it further satisfies $\nabla^*_X g = 0$; otherwise it is non-metric.

In 1975, Prvanović \cite{14} introduced the concept of semi-symmetric non-metric connection with the name pseudo-metric, which was further studied by Andonie \cite{2, 3}. The study of semi-symmetric non-metric connection

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is much older than the nomenclature 'non-metric' was introduced. In 1992, Agashe and Chafle \[1\] introduced a semi-symmetric connection $\nabla^*$ satisfying $\nabla^*_X g \neq 0$, and called such a connection as \emph{semi-symmetric non-metric connection}. Semi-symmetric connections were further studied by several authors such as Sengupta, De and Binh \[15\], Pathak and De \[11\], Singh and Pandey \[16\], Singh, Pandey and Pandey \[17\], and many others.

Semi-symmetric connections play an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, if a man is moving on the surface of the earth always facing one definite point, say Jerusalem or Mekka or the North pole, then this displacement is semi-symmetric and metric \[6\].

On the other hand, in 1972, K. Kenmotsu \[9\] studied a class of contact Riemannian manifolds satisfying some special conditions. We call it Kenmotsu manifold. Kenmotsu proved that if in a Kenmotsu manifold the condition $R(X, Y).R = 0$ holds, then the manifold is of negative curvature $-1$, where $R$ is the curvature tensor of type $(1, 3)$ and $R(X, Y)$ denotes the derivations of the tensor algebra at each point of the tangent space. A Riemannian manifold satisfying the condition $R(X, Y).R = 0$ is called semi-symmetric \[18\]. In \[8\], Jun, De and Pathak have studied some relations about semi-symmetric, Ricci semi-symmetric or Weyl semi-symmetric conditions in Riemannian manifolds. In \[20\], Yildiz and De have studied $W_2$-semi-symmetric Kenmotsu manifolds. They have classified Kenmotsu manifolds which satisfy $P.W_2 = 0$, $I.W_2 = 0$, $C.W_2 = 0$ and $\tilde{C}.W_2 = 0$, where $P$, $I$, $C$ and $\tilde{C}$ are the projective curvature tensor, concircular curvature tensor, conformal curvature tensor and quasi-conformal curvature tensor respectively.

In 1970, Pokhariyal and Mishra \[13\] have introduced new tensor fields, called $W_2$ and $E$-tensor fields in a Riemannian manifold and studied their properties. Again, Pokhariyal \[12\] have studied some properties of these tensor fields in a Sasakian manifolds. Recently, Matsumoto, Ianus and Mihai \[11\] have studied $P$-Sasakian manifolds admitting $W_2$ and $E$-tensor fields. Also, De and Sarkar \[5\], Yildiz and De \[20\] have studied $W_2$-curvature tensor. The curvature tensor $\nabla^* W_2$ is defined by

\begin{equation}
\nabla^* W_2(X, Y, Z, U) = \nabla^* R(X, Y, Z, U) + \frac{1}{n - 1} \{ g(X, Z) Ric(Y, U) - g(Y, Z) Ric(X, U) \},
\end{equation}

where $Ric$ is the Ricci tensor of type $(0, 2)$ and

\begin{equation}
\nabla^* W_2(X, Y, Z, U) = g(W_2(X, Y)Z, U)
\end{equation}

and

\begin{equation}
\nabla^* R(X, Y, Z, U) = g(R(X, Y)Z, U),
\end{equation}

for the arbitrary vector fields $X$, $Y$, $Z$ and $U$.

Motivated by the above studies, in the present paper, we consider the $W_2$-curvature tensor of a semi-symmetric non-metric connection and study some curvature conditions. The present paper is organized as follows: In Section 2, some preliminary results regarding Kenmotsu manifold are recalled. In Section 3, we obtain the curvature tensor, Ricci tensor and scalar curvature of the
semi-symmetric non-metric connection. Section 4 is devoted to the study of the \( W_2^* \)-curvature tensor of semi-symmetric non-metric connection in the Kenmotsu manifold. In this section is shown that, if \( W_2^* = 0 \) in \( M^n \) then \( M^n \) is isomorphic to hyperbolic space \( H^n(-1) \), where \( W_2^* \) is the \( W_2^* \)-curvature tensor of the semi-symmetric non-metric connection \( \nabla^* \). Also, \( R^*(\xi, X).W_2^* = 0, W_2^*(\xi, X).R^* = 0 \) and \( W_2^*(\xi, X).Ric^* = 0 \) have been studied and obtained in each case that \( M^n \) is an Einstein manifold, where \( R^* \) and \( Ric^* \) are the curvature tensor and Ricci tensor respectively of the semi-symmetric non-metric connection \( \nabla^* \). In Section 5, a locally \( W_2^*\phi \)-symmetric Kenmotsu manifold with respect to semi-symmetric non-metric connection have been studied. The last section is devoted to the study of the \( W_2^*\phi \)-recurrent Kenmotsu manifold with respect to the semi-symmetric non-metric connection.

2. Preliminaries

If on an odd-dimensional differentiable manifold \( M^n \) of differentiability class \( C^{r+1} \), there exists a vector valued real linear function \( \phi \), a 1-form \( \eta \), the associated vector field \( \xi \) and the Riemannian metric \( g \) satisfying

\[
\phi^2 X = -X + \eta(X)\xi, \quad (2.1)
\]

\[
\eta(\phi X) = 0, \quad (2.2)
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.3)
\]

for arbitrary vector fields \( X \) and \( Y \), then the structure \((\phi, \xi, \eta, g)\) is called an almost contact metric structure and the manifold \( M^n \) with this structure is called an almost contact metric manifold. In view of equations (2.1), (2.2) and (2.3), we have

\[
\eta(\xi) = 1, g(X, \xi) = \eta(X), \phi \xi = 0. \quad (2.4)
\]

An almost contact metric manifold is called Kenmotsu manifold ([3]) if

\[
(\nabla_X \phi) = -\eta(Y)\phi X - g(X, \phi Y)\xi, \quad (2.5)
\]

\[
\nabla_X \xi = X - \eta(X)\xi, \quad (2.6)
\]

where \( \nabla \) is the Levi-Civita connection. Also the following relations hold in the Kenmotsu manifolds

\[
(\nabla_X \eta)(Y) = g(X, Y) + \eta(X)\eta(Y), \quad (2.7)
\]

\[
R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (2.8)
\]

\[
R(\xi, X)Y = -R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi, \quad (2.9)
\]
(2.10) \[ \eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \]

(2.11) \[ Ric(X,\xi) = -(n-1)\eta(X), \]

(2.12) \[ Q\xi = -(n-1)\xi, \quad r = -n(n-1), \]

where Q is the Ricci operator, i.e.

\[ g(QX,Y) = Ric(X,Y), \]

and r is the scalar curvature of the connection \( \nabla \),

(2.13) \[ Ric(\phi X,\phi Y) = Ric(X,Y) + (n-1)\eta(X)\eta(Y), \]

for the arbitrary vector fields \( X,Y,Z \) on \( M^n \).

A Kenmotsu manifold is said to be an \( \eta \)-Einstein manifold if its Ricci tensor \( Ric \) is of the form

(2.14) \[ Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \]

for the vector fields \( X \) and \( Y \), where \( a \) and \( b \) are functions on \( M^n \).

Let \( M^n \) be an \( n \)-dimensional Kenmotsu manifold and \( \nabla \) be the Levi-Civita connection on \( M^n \). A linear connection \( \nabla^* \) on \( M^n \) is given by

(2.15) \[ \nabla^* Y = \nabla X Y + \eta(Y)X. \]

Using equation (2.15), the torsion tensor \( T^* \) of \( M^n \) with respect to the connection \( \nabla^* \) is given by

(2.16) \[ T^*(X,Y) = \nabla^* X Y - \nabla^* Y X - [X,Y] = \eta(Y)X - \eta(X)Y, \]

which shows that the linear connection defined in equation (2.15) is a semi-symmetric connection.

Moreover, using equation (2.15) we have, for all vector fields \( X, Y, Z \)

(2.17) \[ \nabla^* g(Y,Z) = \nabla^* X g(Y,Z) - g(\nabla^* Y X, Z) - g(Y, \nabla^* X Z) \]
\[ = -\eta(Y)g(X,Z) - \eta(Z)g(X,Y). \]

A linear connection \( \nabla^* \) defined in equation (2.15) satisfies equations (2.11) and (2.17), and therefore we call \( \nabla^* \) a semi-symmetric non-metric connection.

3. Curvature tensor of semi-symmetric non-metric connection in a Kenmotsu manifold

The curvature tensor \( R^* \) of the semi-symmetric non-metric connection \( \nabla^* \) in \( M^n \) is defined by

(3.1) \[ R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z. \]
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Using equations (2.7) and (2.15) in the above equation, we get

$$R^*(X, Y)Z = R(X, Y)Z + \{g(X, Z)Y - g(Y, Z)X\}$$
$$+ 2\eta(Z)\{\eta(Y)X - \eta(X)Y\},$$

where $R$ is the Riemannian curvature tensor of $\nabla$. From the above equation, we have

$$'R^*(X, Y, Z, U) = 'R(X, Y, Z, U) + \{g(X, Z)g(Y, U) - g(Y, Z)g(X, U)\}$$
$$+ 2\eta(Z)\{\eta(Y)g(X, U) - \eta(X)g(Y, U)\},$$

where $'R^*(X, Y, Z, U) = g(R^*(X, Y)Z, U)$.

Putting $X = U = e_i$ in the above equation and taking summation over $i$, $1 \leq i \leq n$, we get

$$R^*(Y, Z) = Ric(Y, Z) - (n - 1)g(Y, Z) + 2(n - 1)\eta(Y)\eta(Z),$$

where $R^*$ and $\text{Ric}$ are the Ricci tensors of the connections $\nabla^*$ and $\nabla$ respectively.

This gives

$$Q^*Y = QY - (n - 1)Y + 2(n - 1)\eta(Y)\xi.$$

Contracting the above equation, we get

$$r^* = r - n^2 + 3n - 2,$$

where $r^*$ and $r$ are the scalar curvatures of the connections $\nabla^*$ and $\nabla$ respectively. Putting $X = \xi$ in equation (3.2) and using equations (2.4) and (2.9), we get

$$R^*(\xi, Y)Z = -R^*(Y, \xi)Z = 2\{\eta(Y)\eta(Z) - g(Y, Z)\}\xi.$$

4. $W_2$-Curvature Tensor of Semi-Symmetric Non-Metric Connection in a Kenmotsu Manifold

From equation (1.1), we have

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{n - 1}\{g(X, Z)QY - g(Y, Z)QX\}.$$

The $W_2$- curvature tensor of the semi-symmetric non-metric connection $\nabla^*$ in a Kenmotsu manifold $M^n$ is given by

$$W_2^*(X, Y)Z = R^*(X, Y)Z + \frac{1}{n - 1}\{g(X, Z)Q^*Y - g(Y, Z)Q^*X\}.$$

Using equations (3.2) and (3.5) in the above equation, we get

$$W_2^*(X, Y)Z = R^*(X, Y)Z + 2\eta(Z)\{\eta(Y)X - \eta(X)Y\}$$
$$+ 2\{g(X, Z)\eta(Y) - g(Y, Z)\eta(X)\}\xi + \frac{1}{n - 1}\{g(X, Z)QY - g(Y, Z)QX\},$$
which on using equation (4.1), gives

\begin{equation}
W^*_2(X, Y)Z = W_2(X, Y)Z + 2\{\eta(Y)X - \eta(X)Y\}\eta(Z) + 2\{g(X, Z)\eta(Y) - g(Y, Z)\eta(X)\}\xi.
\end{equation}

Putting \(Z = \xi\) in the above equation and using equations (2.4), (2.8) and (4.1), we get

\begin{equation}
W^*_2(X, Y)\xi = \{\eta(Y)X - \eta(X)Y\} + \frac{1}{n-1}\{\eta(X)QY - \eta(Y)QX\},
\end{equation}

which gives

\begin{equation}
\eta(W^*_2(X, Y)\xi) = 0.
\end{equation}

Again, putting \(X = \xi\) in equation (4.3) and using equations (2.4), (2.9), (3.5) and (4.1), we get

\begin{equation}
W^*_2(\xi, Y)Z = -W^*_2(Y, \xi)Z
= -\eta(Z)Y + \frac{1}{(n-1)}\eta(Z)QY + 4\eta(Y)\eta(Z)\xi - 2g(Y, Z)\xi.
\end{equation}

**Lemma 4.1.** An \(\eta\)-Einstein Kenmotsu manifold of the form

\[\text{Ric}(X, Y) = a\eta(X)Y + b\eta(X)\eta(Y)\]

is an Einstein manifold, where \(a\) or \(b\) are constants \([8]\).

**Theorem 4.2.** In a Kenmotsu manifold \(M^n\), if \(W_2\)-curvature tensor of the semi-symmetric non-metric connection vanishes, then it is isomorphic to the hyperbolic space \(H^n(-1)\).

**Proof.** Let \(W^*_2 = 0\). In view of equation (1.4), we have

\begin{equation}
R(X, Y)Z = 2\{\eta(X)Y - \eta(Y)X\}\eta(Z) - 2\{g(X, Z)\eta(Y) - g(Y, Z)\eta(X)\}\xi.
\end{equation}

Taking the inner product of the above equation with \(\xi\) and using equation (2.4), we get

\begin{equation}
g(R(X, Y)Z, \xi) = -[g(X, Z)g(Y, \xi) - g(Y, Z)g(X, \xi)],
\end{equation}

which gives

\begin{equation}
R(X, Y, Z, U) = -[g(X, Z)g(Y, U) - g(Y, Z)g(X, U)].
\end{equation}

This shows that \(M^n\) is isomorphic to the hyperbolic space \(H^n(-1)\).

**Theorem 4.3.** A Kenmotsu manifold \(M^n\) with the semi-symmetric non-metric connection \(\nabla^*\) satisfying \(R^*(\xi, X).W^*_2 = 0\), is an \(\eta\)-Einstein manifold.
\textbf{Proof.} Let \((R^*(\xi, Z), W_2^*)(X, Y)U = 0\). Then, we have

\begin{equation}
R^*(\xi, Z)W_2^*(X, Y)U - W_2^*(R^*(\xi, Z)X, Y)U = 0,
\end{equation}

which on using equation (3.10), gives

\begin{equation}
\eta(Z)\eta(W_2^*(X, Y)U)\xi - g(Z, W_2^*(X, Y)U)\xi - \eta(X)\eta(Z)W_2^*(\xi, Y)U + g(X, Z)W_2^*(\xi, Y)U - \eta(Z)\eta(U)W_2^*(X, Y)\xi = 0.
\end{equation}

Now, using equations (3.13), (4.3) and (4.7) in the above equation, we get

\begin{align}
\eta(Z)\eta(W_2^*(X, Y)U)\xi &- g(Z, W_2^*(X, Y)U)\xi - 2g(Z, Y)\eta(X)\eta(U)\xi \\
&+ 2g(Z, X)\eta(Y)\eta(U)\xi + 2\eta(X)\eta(Z)\eta(U)Y - \frac{2}{n-1}\eta(X)\eta(Z)\eta(U)QY \\
&+ 2g(Y, U)\eta(X)\eta(Z)\xi - g(Z, X)\eta(U)Y + \frac{1}{n-1}g(Z, X)\eta(U)QY \\
&- 2g(Z, X)g(Y, U)\xi - 2\eta(Y)\eta(Z)\eta(U)X + \frac{2}{n-1}\eta(Y)\eta(Z)\eta(U)QX \\
&- 2g(X, U)\eta(Y)\eta(Z)\xi + g(Z, Y)\eta(U)X - \frac{1}{n-1}g(Z, Y)\eta(U)QX \\
&+ 2g(Z, Y)g(X, U)\xi + g(Z, U)\eta(Y)X - g(Z, U)\eta(X)Y \\
&+ \frac{1}{n-1}g(Z, U)[\eta(X)QY - \eta(Y)QX] = 0.
\end{align}

Taking the inner product of the above equation with \(\xi\), we get

\begin{equation}
\eta(Z)\eta(W_2^*(X, Y)U) - g(Z, W_2^*(X, Y)U) - 2g(Z, X)\eta(Y)\eta(Z) \\
+ 2g(Y, U)\eta(X)\eta(Z) - 2g(Z, X)g(Y, U) + 2g(Z, Y)g(X, U) = 0.
\end{equation}

Using equation (4.12) in the above equation, we get

\begin{equation}
'R(X, Y, U, Z) = 2[g(Y, U)\eta(X) - g(X, U)\eta(Y)]\eta(Z) \\
- \frac{1}{n-1}[Ric(Y, Z)g(X, U) - Ric(X, Z)g(Y, U)] \\
- 2[g(Z, X)g(Y, U) - g(Y, Z)g(X, U)].
\end{equation}

Putting \(X = Z = e_i\) in the above equation and taking summation over \(i\), \(1 \leq i \leq n\), we get

\begin{equation}
Ric(Z, U) = ag(Y, U) + b\eta(Y)\eta(U),
\end{equation}

where \(a = \frac{(4-3n)(n-1)}{n}\) and \(b = \frac{-2(n-1)}{n}\). This shows that \(M^n\) is an \(\eta\)-Einstein manifold.

This completes the proof. \(\square\)

Now, in view of Lemma 4.4, we can state as follows
Corollary 4.4. A Kenmotsu manifold $M^n$ with the semi-symmetric non-metric connection $\nabla^*$ satisfying $R^*(\xi, X)W^*_2 = 0$, is an Einstein manifold.

Theorem 4.5. A Kenmotsu manifold $M^n$ with the semi-symmetric non-metric connection $\nabla^*$ satisfying $W^*_2(\xi, Z)R^* = 0$, is an $\eta$-Einstein manifold.

Proof. Consider $(W^*_2(\xi, Z)R^*)(X, Y)\xi = 0$. Then, we have

\begin{equation}
W^*_2(\xi, Z)R^*(X, Y)\xi - R^*(W^*_2(\xi, Z)X, Y)\xi
- R^*(X, W^*_2(\xi, Z)Y)\xi - R^*(X, Y)W^*_2(\xi, Z)\xi = 0,
\end{equation}

which on using equation (4.17), gives

\begin{equation}
- \eta(R^*(X, Y, U)Z) + \frac{1}{n-1}\eta(R^*(X, Y, U)QZ) + 4\eta(Z)\eta(R^*(X, Y, U)\xi)
- 2g(Z, R^*(X, Y, U))\xi + \eta(X)R^*(Z, Y, U) - \frac{1}{n-1}\eta(X)R^*(QZ, Y, U)
- 4\eta(X)\eta(Z)R^*(\xi, Y, U) + 2g(Z, X)R^*(\xi, Y, U) + \eta(Y)R^*(X, Z, U)
- \frac{1}{n-1}\eta(Y)R^*(X, QZ, U) - 4\eta(Y)\eta(Z)R^*(X, \xi, U) + 2g(Y, Z)R^*(X, \xi, U)
+ \eta(U)R^*(X, Y, Z) - \frac{1}{n-1}\eta(U)R^*(X, Y, QZ) - 4\eta(Z)\eta(U)R^*(X, Y)\xi
+ 2g(Z, U)R^*(X, Y)\xi = 0.
\end{equation}

Now, using equation (3.2) in the above equation, we get

\begin{equation}
2R(X, Y, U, Z)\xi = -\eta(R(X, Y, U)Z) + \frac{1}{n-1}\eta(R(X, Y, U))QZ
+ 4\eta(Z)\eta(R(X, Y, U))\xi - 2g(Y, U)g(Z, X)\xi - 4\eta(Y)\eta(U)g(Z, X)\xi
+ \eta(X)R(Z, Y, U) + g(Z, U)\eta(X)Y - 4\eta(X)\eta(Z)\eta(U)Y
- \frac{1}{n-1}\eta(X)R(QZ, Y, U) - \frac{1}{n-1}\eta(X)\eta(Z)g(Y, U)\xi
- 4g(X, Z)g(Y, U)\xi + 2g(X, Z)\eta(Y)Y + 4g(X, Z)\eta(Y)\eta(U)\xi
+ \eta(Y)R(X, Z, U) - g(Z, U)\eta(Y)X - \frac{1}{n-1}\eta(Y)R(X, QZ, U)
+ \frac{1}{n-1}\eta(Y)Ric(Z, U)X - 4\eta(Y)\eta(Z)g(X, U)\xi + 2g(Y, Z)g(X, U)\xi
+ \eta(U)R(X, Y, Z) + 4\eta(Y)\eta(Z)\eta(U)X - \frac{1}{n-1}\eta(U)R(X, Y, QZ)
- \frac{1}{n-1}Ric(X, Z)\eta(U)Y + \frac{1}{n-1}Ric(Y, Z)\eta(U)X.
\end{equation}
Taking the inner product of the above equation with \( \xi \), we get

\[
2' R(X, Y, U, Z) = 2\eta(Z)\eta(R(X, Y, U)) - 6g(X, Z)g(Y, U)
+ 2\eta(X)\eta(R(Z, Y, U)) - \frac{1}{n-1}Ric(Z, U)\eta(X)\eta(Y) + 4\eta(X)\eta(Z)g(Y, U)
+ 2g(X, Z)\eta(Y)\eta(U) + 2g(Y, Z)g(X, U) + 2\eta(U)\eta(R(X, Y, Z))
- \frac{1}{n-1}Ric(Z, U)\eta(Y)\eta(U) + \frac{1}{n-1}Ric(Y, Z)\eta(X)\eta(U).
\]

Putting \( X = Z = e_i \) in the above equation and taking summation over \( i \), \( 1 \leq i \leq n \), we get

\[
Ric(Y, U) = ag(Y, U) + b\eta(Y)\eta(U),
\]

where \( a = -(3n - 1) \) and \( b = \frac{3n-1}{2} \). This shows that \( M^n \) is an \( \eta \)-Einstein manifold.

This completes the proof. \( \square \)

Now by Lemma 4.1, we can state as follows

**Corollary 4.6.** A Kenmotsu manifold \( M^n \) with the semi-symmetric non-metric connection \( \nabla^* \) satisfying

\[
W^*_2(\xi, Z).R^* = 0,
\]

is an Einstein manifold.

**Theorem 4.7.** A Kenmotsu manifold \( M^n \) with the semi-symmetric non-metric connection \( \nabla^* \) satisfying

\[
(W^*_2(\xi, Z).Ric^*)(X, Y) = 0,
\]

is an \( \eta \)-Einstein manifold.

**Proof.** Consider \((W^*_2(\xi, Z).Ric^*)(X, Y) = 0\). Then, we have

\[
Ric^*(W^*_2(\xi, Z)X, Y) + Ric^*(X, W^*_2(\xi, Z)Y) = 0,
\]

which on using equations (3.3) and (1.4), gives

\[
-3Ric(Y, Z)\eta(X) - 3Ric(X, Z)\eta(Y)
+ (n - 1)g(Y, Z)\eta(X) + (n - 1)g(X, Z)\eta(Y) = 0.
\]

Now, putting \( X = \xi \) in the above equation and using equations (2.4) and (2.11), we get

\[
Ric(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z),
\]

where \( a = \frac{(n-1)}{3} \) and \( b = \frac{4(n-1)}{3} \).

This shows that \( M^n \) is an \( \eta \)-Einstein manifold.

This completes the proof. \( \square \)

Now by Lemma 4.1, we can state as follows

**Corollary 4.8.** A Kenmotsu manifold \( M^n \) with the semi-symmetric non-metric connection \( \nabla^* \) satisfying \( W^*2(\xi, Z).Ric^*(X, Y) = 0 \), is an Einstein manifold.
5. Locally $W_2$-$\phi$-symmetric Kenmotsu manifold with semi-symmetric non-metric connection

**Definition 5.1.** An n-dimensional Kenmotsu manifold $M^n$ is said to be locally $W_2$-$\phi$-symmetric, if

\begin{equation}
\phi^2((\nabla_U W_2)(X, Y)Z) = 0,
\end{equation}

for all vector fields $X$, $Y$, $Z$ and $U$ orthogonal to $\xi$.

**Definition 5.2.** An n-dimensional Kenmotsu manifold $M^n$ is said to be locally $W_2$-$\phi$-symmetric with respect to the semi-symmetric non-metric connection if

\begin{equation}
\phi^2((\nabla_U W_2^*)(X, Y)Z) = 0,
\end{equation}

for all vector fields $X$, $Y$, $Z$ and $U$ orthogonal to $\xi$, where $W_2^*$ is the $W_2$-curvature tensor of the semi-symmetric non-metric connection $\nabla^*$.

**Theorem 5.3.** A Kenmotsu manifold $M^n$ is locally $W_2$-$\phi$-symmetric with respect to the semi-symmetric non-metric connection $\nabla^*$ if and only if it is so with respect to the Levi-Civita connection $\nabla$.

**Proof.** From equation (2.13), we have

\begin{equation}
(\nabla_U W_2^*)(X, Y)Z = (\nabla_U W_2^*)(X, Y)Z + \eta(W_2^*(X, Y)Z)U.
\end{equation}

Now, differentiating equation (5.3) covariantly with respect to $U$, we get

\begin{align}
(\nabla_U W_2^*)(X, Y)Z &= (\nabla_U W_2)(X, Y)Z - 2(\nabla_U \eta)(X)\eta(Z)Y \\
&\quad - 2(\nabla_U \eta)(Z)\eta(X)Y + 2(\nabla_U \eta)(Y)\eta(Z)X + 2(\nabla_U \eta)(Z)\eta(Y)X \\
&\quad + 2(\nabla_U \eta)(Y)g(X, Z)\xi - 2(\nabla_U \eta)(X)g(Y, Z)\xi.
\end{align}

Now, using equation (5.4) in equation (5.3), we get

\begin{align}
(\nabla_U W_2^*)(X, Y)Z &= (\nabla_U W_2)(X, Y)Z - 2(\nabla_U \eta)(X)\eta(Z)Y \\
&\quad - 2(\nabla_U \eta)(Z)\eta(X)Y + 2(\nabla_U \eta)(Y)\eta(Z)X + 2(\nabla_U \eta)(Z)\eta(Y)X \\
&\quad + 2(\nabla_U \eta)(Y)g(X, Z)\xi - 2(\nabla_U \eta)(X)g(Y, Z)\xi \\
&\quad + 2g(X, Z)\eta(Y)U - 2g(Y, Z)\eta(X)U.
\end{align}

Using equation (2.11) in the above equation, we get

\begin{align}
(\nabla_U W_2^*)(X, Y)Z &= (\nabla_U W_2)(X, Y)Z - 2g(U, X)\eta(Z)Y \\
&\quad + 2g(U, Y)\eta(Z)X + 2g(U, Z)\eta(Y)X - 2g(U, Z)\eta(X)Y \\
&\quad + 2g(X, Z)\eta(Y)U - 2g(Y, Z)\eta(X)U + 2g(X, Z)g(Y, U)\xi \\
&\quad - 2g(Y, Z)g(U, X)\xi + 2g(Y, Z)\eta(U)\eta(X)\xi - 2g(X, Z)\eta(Y)\eta(U)\xi \\
&\quad + 4\eta(X)\eta(Z)\eta(U)Y - 4\eta(Y)\eta(Z)\eta(U)X.
\end{align}
Applying $\phi^2$ on both sides of the above equation and using equations (2.1) and (2.2), we get 

$$\phi^2((\nabla^*_U W^*_2)(X,Y)Z) = \phi^2((\nabla_U W_2)(X,Y)Z) + 2g(U,X)\eta(Z)Y$$

$$- 2g(U,X)\eta(Y)\eta(Z)\xi - 2g(U,Y)\eta(Z)X + 2g(U,Y)\eta(X)\eta(Z)\xi$$

$$- 2g(U,Z)\eta(Y)\eta(X)Y - 2g(X,Z)\eta(Y)U$$

$$+ 2g(X,Z)\eta(Y)\eta(U)\xi + 2g(Y,Z)\eta(X)U - 2g(Y,Z)\eta(X)\eta(U)\xi$$

$$- 4\eta(X)\eta(Z)\eta(U)Y + 4\eta(Y)\eta(Z)\eta(U)X.$$

Now, if X, Y, Z, U are orthogonal to $\xi$, then the above equation reduces to 

$$\phi^2((\nabla^*_U W^*_2)(X,Y)Z) = \phi^2((\nabla_U W_2)(X,Y)Z).$$

This completes the proof. \[\Box\]

6. $W_2$-$\phi$-recurrent Kenmotsu manifold with semi-symmetric non-metric connection

**Definition 6.1.** An n-dimensional Kenmotsu manifold $M^n$ is said to be $W_2$-$\phi$-recurrent, if 

$$\phi^2((\nabla_W W_2)(X,Y)Z) = A(W)W_2(X,Y)Z,$$

for the arbitrary vector fields X, Y, Z and W, where A is non-zero 1-form.

**Definition 6.2.** An n-dimensional Kenmotsu manifold $M^n$ is said to be $W_2$-$\phi$-recurrent with respect to the semi-symmetric non-metric connection if 

$$\phi^2((\nabla^*_W W^*_2)(X,Y)Z) = A(W)W^*_2(X,Y)Z,$$

for arbitrary vector fields X, Y, Z and W.

**Theorem 6.3.** A $W_2$-$\phi$-recurrent Kenmotsu manifold with respect to a semi-symmetric non-metric connection is an $\eta$-Einstein manifold.

**Proof.** From equations (2.1) and (6.2), we have 

$$-((\nabla^*_W W^*_2)(X,Y)Z) + \eta((\nabla_W W^*_2)(X,Y)Z)\xi = A(W)W^*_2(X,Y)Z,$$

which reduces to 

$$-g((\nabla^*_W W^*_2)(X,Y)Z,U) + \eta((\nabla_W W^*_2)(X,Y)Z)\eta(U) = A(W)g(W^*_2(X,Y)Z,U).$$
Using equations (6.3) and (6.6) in the above equation, we get

\[(6.5)\]
\[-g((\nabla_W R)(X, Y)Z, U) - \frac{1}{n-1}[[\nabla_W Ric](Y, U)g(X, Z) - (\nabla_W Ric)(X, U)g(Y, Z)]
+ 2g(W, X)g(Y, U)\eta(Z) - 2g(W, Y)g(X, U)\eta(Z) - 2g(W, Z)g(X, U)\eta(Y)
+ 2g(W, Z)g(Y, U)\eta(X) - 2g(X, Z)g(W, U)\eta(Y) + 2g(Y, Z)g(W, U)\eta(X)
- 4g(Y, U)\eta(X)\eta(Z)\eta(W) + 4g(X, U)\eta(Y)\eta(Z)\eta(W) + 2g(X, Z)\eta(Y)\eta(W)\eta(U)
- 2g(Y, Z)\eta(X)\eta(W)\eta(U) + \eta((\nabla_W R)(X, Y)Z)\eta(U) - (\nabla_W \eta)(Y)\eta(U)g(X, Z)
+ (\nabla_W \eta)(X)\eta(U)g(Y, Z) - 2g(W, X)\eta(Y)\eta(Z)\eta(U) + 2g(W, Y)\eta(X)\eta(Z)\eta(U)
= A(W)g(W_2(X, Y)Z, U) - 2A(W)\{\eta(X)g(Y, U) - \eta(Y)g(X, U)\}\eta(Z)
+ 2A(W)\{\eta(Y)g(X, Z) - \eta(X)g(Y, Z)\}\eta(U).\]

Let \(\{e_i\}, i = 1, 2, \ldots, n\) be an orthonormal basis of the tangent space at any point of the manifold. Then, putting \(X = e_i\) in equation (6.5) and taking summation over \(i, 1 \leq i \leq n\), we get

\[(6.6)\]
\[-\frac{n}{n-1}(\nabla_W Ric)(Y, Z) + \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i) + \frac{1}{n-1}g(Y, Z)(\nabla_W r)
+ (4n - 4)\eta(Y)\eta(Z)\eta(W) - 2ng(W, Z)\eta(Y) - (\nabla_W \eta)(Y)\eta(Z)
+ (4 - 2n)g(W, Y)\eta(Z) = \frac{n}{n-1}A(W)Ric(Y, Z) - \frac{(r + 2n - 2)}{n-1}A(W)g(Y, Z)
+ 2nA(W)\eta(Y)\eta(Z).\]

Putting \(Z = \xi\) in the above equation, we get

\[(6.7)\]
\[-\frac{n}{n-1}(\nabla_W Ric)(Y, \xi) + \eta((\nabla_W R)(e_i, Y)\xi)\eta(e_i) + \frac{1}{n-1}\eta(Y)(\nabla_W r)
+ (2n - 4)\eta(Y)\eta(W) - (\nabla_W \eta)(Y) + (4 - 2n)g(W, Y)
= \frac{n^2 - 3n - r + 2}{n-1}A(W)\eta(Y).\]

The second term on L.H.S. of equation (6.7) takes the form

\[(6.8)\]
\[E = \eta((\nabla_W R)(e_i, Y)\xi)\eta(e_i) = g((\nabla_W R)(e_i, Y)\xi, \xi)g(e_i, \xi),\]

which is denoted by \(\lambda\). In this case \(\lambda\) vanishes. Namely, we have

\[(6.9)\]
\[g((\nabla_W R)(e_i, Y)\xi, \xi) = g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(\nabla_W e_i, Y)\xi, \xi)
- g(R(e_i, \nabla_W Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi).\]

at \(p \in \mathcal{M}^n\). In the local coordinates \(\nabla_W e_i = W^j \Gamma^h_{ji} e_h\), where \(\Gamma^h_{ji}\) are the Christoffel symbols. Since \(\{e_i\}\) is an orthonormal basis, the metric tensor \(g_{ij} = \delta_{ij}\), where \(\delta_{ij}\) is the Kronecker delta and hence the Christoffel symbols are zero. Therefore, \(\nabla_W e_i = 0\). Also, we have

\[(6.10)\]
\[g(R(e_i, \nabla_W Y)\xi, \xi) = 0,\]
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since $R$ is skew-symmetric. Using equation (6.11) and $\nabla W e_i = 0$ in equation (6.7), we get

\[ g((\nabla W R)(e_i, Y)\xi, \xi) = g(\nabla W R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla W, \xi, \xi). \]

In view of $g(R(e_i, Y)\xi, \xi) = -g(R(\xi, \xi)Y, e_i) = 0$ and $\nabla W g = 0$, we have

\[ g(\nabla W R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla W \xi) = 0, \]

which implies

\[ g((\nabla W R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, \nabla W \xi) - g(R(e_i, Y)\nabla W, \xi, \xi). \]

Since $R$ is skew-symmetric, we have

\[ g((\nabla W R)(e_i, Y)\xi, \xi) = 0. \]

Using equation (6.13) in equation (6.7), we get

\[ (\nabla W \text{Ric})(Y, \xi) = \eta(Y)(\nabla W r) + \frac{(2n - 4)(n - 1)}{n} \eta(Y)\eta(W) \]
\[ - \frac{n - 1}{n} (\nabla W \eta)(Y) - \frac{(2n - 4)(n - 1)}{n} g(W, Y) + \frac{(n^2 - 3n - r + 2)}{n} A(W)\eta(Y). \]

Now, we have

\[ (\nabla W \text{Ric})(Y, \xi) = \nabla W \text{Ric}(Y, \xi) - \text{Ric}(\nabla W Y, \xi) - \text{Ric}(Y, \nabla W \xi), \]

which on using equations (2.10) and (2.11) takes the form

\[ (\nabla W \text{Ric})(Y, \xi) = -(n - 1)g(Y, W) - \text{Ric}(Y, W). \]

Form equations (6.13) and (6.7), we have

\[ \text{Ric}(Y, W) = \left( \frac{n^2 - 5n + 4}{n} \right) g(Y, W) - \left( \frac{2n^2 - 6n + 4}{n} \right) \eta(Y)\eta(W) \]
\[ - \left( \frac{n^2 - 3n - r + 2}{n} \right) \eta(W) + \frac{n - 1}{n} (\nabla W \eta)(Y). \]

Replacing $Y$ and $W$ by $\phi Y$ and $\phi W$ respectively in the above equation and using equations (2.2), (2.3) and (2.13), we get

\[ \text{Ric}(Y, W) = \frac{n^2 - 5n + 4}{n} g(Y, W) - \frac{2n^2 - 6n + 4}{n} \eta(Y)\eta(W), \]

which shows that $M^n$ is an $\eta$-Einstein manifold. \hfill \Box

**Theorem 6.4.** In a $W_2$-$\phi$-recurrent Kenmotsu manifold $M^n$ admitting semi-symmetric non-metric connection, the characteristic vector field $\xi$ and the vector field $\rho$ associated with 1-form $A$ are co-directional and the 1-form $A$ is given by

\[ A(W) = \eta(\rho)\eta(W). \]
Proof. By virtue of equations (6.1) and (6.2), we have
\[(\nabla^*_W W_2^*)(X, Y)Z = \eta(\nabla^*_W W_2^*)(X, Y)Z\xi - A(W)W_2^*(X, Y)Z.\]

Using equations (6.3) and (6.10) in the above equation, we get
\[
(\nabla^*_W W_2^*)(X, Y)Z = \eta(\nabla^*_W W_2^*)(X, Y)Z\xi - A(W)W_2^*(X, Y)Z.
\]

Taking the inner product of the above equation with \(\xi\) and using equation (6.2), we get
\[
A(W)\eta(R(X, Y)Z) = A(W)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)].
\]

Writing two more equations by the cyclic permutations of \(W, X\) and \(Y\) from equation (6.21) and adding them to equation (6.21), we get
\[
A(W)\eta(R(X, Y)Z) + A(X)\eta(R(Y, W)Z) + A(Y)\eta(R(W, X)Z)
\]
\[
= A(W)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + A(X)[g(W, Z)\eta(Y)
\]
\[
- g(Y, Z)\eta(W)] + A(Y)[g(X, Z)\eta(W) - g(W, Z)\eta(X)].
\]

Using equation (2.11) in the above equation, we get
\[
A(W)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)] + A(X)[g(Y, Z)\eta(W) - g(W, Z)\eta(Y)]
\]
\[
+ A(Y)[g(W, Z)\eta(X) - g(X, Z)\eta(W)] = 0.
\]

Putting \(Y = Z = e_i\) in the above equation and taking summation over \(i, 1 \leq i \leq n\), we get
\[
A(W)\eta(X) = A(X)\eta(W),
\]
for all vector fields \(X\) and \(W\). Replacing \(X\) by \(\xi\) in the above equation, we get
\[
A(W) = \eta(\rho)\eta(W),
\]
for all vector fields \(W\), where \(A(\xi) = g(\xi, \rho) = \eta(\rho)\), \(\rho\) being the vector field associated to the 1-form \(A\), i.e.
\[
g(X, \rho) = A(X).
\]

This completes the proof. \(\square\)
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References


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