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ON FOCAL CURVES IN EUCLIDEAN n-SPACE $\mathbb{R}^n$

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Abstract. In this paper we consider the focal curves of the curves in the Euclidean n-space $\mathbb{R}^n$. First we give some basic results on Darboux vector of these curves. Later, we prove some results on the order of contact of these curves. Further, we give necessary and sufficient conditions for a focal curve to become 2-planar. We also show that if the ratios of the curvatures of a curve γ are constant then the ratios of the curvatures of the focal curve C_γ are also constant.

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1. Introduction

Let $\gamma = \gamma(s) : I \to \mathbb{R}^n$ be a regular curve in $\mathbb{R}^n$ (i.e. $\|\gamma'\|$ is nowhere zero), where I is interval in $\mathbb{R}$. γ is called a *Frenet curve of osculating order* r (or generic curve [12]) ($r \in \mathbb{N}_0$) if $\gamma'(s)$, $\gamma''(s)$,...,$\gamma^{(r)}(s)$ are linearly independent and $\gamma'(s)$, $\gamma''(s)$,...,$\gamma^{(r+1)}(s)$ are no longer linearly independent for all s in I [13]. In this case, $Im(\gamma)$ lies in an r-dimensional Euclidean subspace of $\mathbb{R}^n$. To each Frenet curve of rank r there can be associated orthonormal r-frame $\{t, n_1, ..., n_{r-1}\}$ along γ, the Frenet r-frame, and $r-1$ functions $\kappa_1, \kappa_2, ..., \kappa_{r-1}$:$I \longrightarrow \mathbb{R}$, the Frenet curvature, such that

$$\begin{bmatrix} t' \\ n_1' \\ n_2' \\ \cdots \\ n_{r-1}' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa_1 & 0 & \cdots & 0 \\ -\kappa_1 & 0 & \kappa_2 & \cdots & 0 \\ 0 & -\kappa_2 & 0 & \cdots & 0 \\ \cdots & & & & \kappa_{r-1} \\ 0 & 0 & \cdots & -\kappa_{r-1} & 0 \end{bmatrix} \begin{bmatrix} t \\ n_1 \\ n_2 \\ \cdots \\ n_{r-1} \end{bmatrix} \tag{1.1}$$

where v is the speed of the curve.

In fact, to obtain $t, n_1, ..., n_{r-1}$ it is sufficient to apply the Gram-Schmidt orthonormalization process to $\gamma'(s)$, $\gamma''(s)$,...,$\gamma^{(r)}(s)$. Moreover, the functions $\kappa_1, \kappa_2, ..., \kappa_{r-1}$ are easily obtained as by-product during this calculation. More

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precisely, $t, n_1, ..., n_{r-1}$ and $\kappa_1, \kappa_2, ..., \kappa_{r-1}$ are determined by the following formulas [11]:

$$
\begin{aligned}
v_1(s) &: = \gamma\,'(s) \quad ; t := \frac{v_1(s)}{\|v_1(s)\|}, \\
v_k(s) &: = \gamma^{(k)}(s) - \sum_{i=1}^{k-1} < \gamma^{(k)}(s), v_i(s) > \frac{v_i(s)}{\|v_i(s)\|^2}, \\
\kappa_{k-1}(s) &: = \frac{\|v_k(s)\|}{\|v_{k-1}(s)\| \, \|v_1(s)\|}, \\
n_{k-1} &: = \frac{v_k(s)}{\|v_k(s)\|},
\end{aligned} \tag{1.2}
$$

where $k \in \{2, 3, ..., r\}$. It is natural and convenient to define Frenet curvatures $\kappa_r = \kappa_{r+1} = ... = \kappa_{n-1} = 0$. It is clear that $t, n_1, ..., n_{r-1}$ and $\kappa_1, \kappa_2, ..., \kappa_{r-1}$ can be defined for any regular curve (not necessarily a Frenet curve) in the neighborhood of a point s_0 for which $\gamma\,'(s_0)$, $\gamma''(s_0)$,...,$\gamma^{(r)}(s_0)$ are linearly independent.

This paper is organized as follows: Section 2 gives some basic concepts of the Darboux vector of curves in $\mathbb{R}^n$. Section 3 explains some geometric properties about the order of a contact of curves in $\mathbb{R}^n$. Section 4 tells about the focal curves in $\mathbb{R}^n$. Further this section provides some basic concepts of these curves. Some results are also presented in this section.

2. Darboux vector of curves in $\mathbb{R}^n$

Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a unit speed curve. γ is called generic if the derivatives of γ of order $1, \ldots, (n-1)$, are linearly independent [12]. When the unit speed vector $\gamma'(s) = t$ of a curve γ in the Euclidean space $\mathbb{R}^n$, $n > 2$, is translated to an arbitrary fixed point O, the end point of translated vector t, describes a curve T on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, called the tangent indicatrix of γ [15]. A flattening of a curve γ in $\mathbb{R}^n$ is a point where the derivatives of γ of order $1, \ldots, (n-1)$, are linearly independent and those of order $(1, \ldots, n)$ are linearly dependent [15]. A point of a curve γ in $\mathbb{R}^n$, $n > 2$, is called twisting if the tangent indicatrix of γ has a flattening at the corresponding point [9].

Proposition 2.1. *[14] The closed curve $\gamma : S^1 \to \mathbb{R}^{2k}$, given by*

$$\gamma(s) = (\cos(s), \sin(s), \cos(2s), \sin(2s), \ldots, \cos(ks), \sin(ks)),$$

has no twisting.

Proposition 2.2. *The number of twistings of a closed curve in $\mathbb{R}^{2k+1}$ is at least equal to the number of its flattenings [14].*

Definition 2.3. For a generic curve with osculating order of $2k$, the curvatures are positive, and only the last curvature can vanish at some isolated points (at the flattenings). Let γ be a smoothly immersed curve in $\mathbb{R}^{2k+1}$, $k \geq 1$ with curvatures $\kappa_1, \kappa_2, ..., \kappa_{2k-1}, \kappa_{2k}$ where $\kappa_{2k} \neq 0$. Denote by

$$\begin{array}{rcll} a_0 = & \kappa_2\kappa_4...\kappa_{2k} & & \\ a_1 = & \frac{\kappa_1}{\kappa_2}a_0 & & \\ ... & ... & & \\ a_j = & \frac{\kappa_{2j-1}}{\kappa_{2j}}a_{j-1} & & \\ a_k = & \frac{\kappa_{2k-1}}{\kappa_{2k}}a_{k-1} & = \kappa_1\kappa_3...\kappa_{2k-1}. \end{array} \tag{2.1}$$

The Darboux vector in $\mathbb{R}^{2k+1}$ is defined by

$$\widetilde{d}(s) = a_0 t + a_1 n_2 + ... + a_k n_{2k}, \tag{2.2}$$

where $\{t = \gamma\,'(s), n_1, n_2, ..., n_{2k}\}$ is the Frenet Frame of γ [15].

Lemma 2.4. *[14]. The derivative of $\widetilde{d}(s)$ is*

$$\widetilde{d}\,'(s) = a_0' t + a_1' n_2 + ... + a_k' n_{2k}. \tag{2.3}$$

Definition 2.5. (Darboux vertex): The point $\gamma(s_0)$ is called a Darboux vertex of γ if the first derivative of the Darboux vector $\widetilde{d}(s)$ vanishes at that point [14].

Theorem 2.6. *[14] Let γ be a smoothly immersed curve in $\mathbb{R}^{2k+1}$ $(k \geq 1)$, with $\kappa_1, \kappa_2, ..., \kappa_{2k}$ for its curvatures. The curve γ has a Darboux vertex at the point $\gamma(s_0)$ if and only if*

$$(\frac{\kappa_2}{\kappa_1})' = 0, (\frac{\kappa_4}{\kappa_3})' = 0, ..., (\frac{\kappa_{2k}}{\kappa_{2k-1}})' = 0, \tag{2.4}$$

at the point $\gamma(s_0)$.

3. The order of contact of Curves

Definition 3.1. Let M be a d-dimensional submanifold of $\mathbb{R}^n$, considered as a complete intersection:

$$M = \{x \in \mathbb{R}^n : g_1(x) = \ldots = g_{n-d}(x) = 0\}.$$

We say that k is the order of contact of a (regularly parametrized) smooth curve $\gamma :\to \gamma(s) \in \mathbb{R}^n$ with the submanifold M, or that γ and M have k-point contact, at a point of intersection $\gamma(s_0)$, if each function $g_1 \circ \gamma, \ldots, g_{n-d} \circ \gamma$ has a zero of multiplicity at least k at $s = s_0$, and at least one of them has a zero of multiplicity k at $s = s_0$ [14].

Definition 3.2. The osculating hyperplane of γ at s is the subspace generated by $\{t(s), n_1(s), n_2(s), \ldots, n_{n-1}(s)\}$ that passes through $\gamma(s)$. The unit vector $n_n(s)$ is called binormal vector of γ at s. The normal hyperplane of γ at s is defined to be the one generated by $\{n_1(s), n_2(s), \ldots, n_{n-1}(s), n_n(s)\}$ passing through $\gamma(s)$ [10].

Remark 3.3. Let γ be a curve in $\mathbb{R}^n$. The order of contact of a curve with its osculating hyperplane at a flattening is at least $n+1$, whereas at an ordinary point it is n [15].

Theorem 3.4. *[6] Let $\gamma = \gamma(s) : I \to \mathbb{R}^{n+1}$ be a unit speed curve given the Frenet frame field $\{t(s), n_1(s), \dots, n_n(s)\}$. If $m_i, 1 \leq i \leq n+1$ are the coordinates of centers of the osculating spheres then the following hold;*

i) $\det(m_2', m_3', \dots, m_{n+1}') = 0 \iff \gamma$ *is a curvature line (or generalized helix),*

ii) $\det(m_2', m_3', \dots, m_{n+1}') = 0 \iff \sum\limits_{i=1}^{n} m_i^2 =$ *constant.*

Definition 3.5. Let M be a smooth m-dimensional submanifold in $(m+d)$-dimensional Euclidean space $\mathbb{R}^{m+d}$. For $p \in M$ and a non-zero vector X in T_pM, we define the $(d+1)$-dimensional affine subspace $E(p,X)$ of $\mathbb{R}^{m+d}$ by $E(p,X) = p + span(X, T_p^{\perp}M)$. In a neighborhood of p, the intersection $M \cap E(p,X)$ is a regular curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$. We suppose the parameter $s \in (-\varepsilon, \varepsilon)$ is a multiple of the arc-length such that $\gamma(0) = p$ and $\gamma\,'(s) = X$. Each choice of $X \in T_p(M)$ yields a different curve which is called the normal section of M at p in the direction of X ([1], [4]).

For each normal section γ if $\gamma\,'(s_0)$, $\gamma\,''(s_0)$,...,$\gamma^{(d)}(s_0)$ are linearly independent and $\gamma\,'(s_0)$, $\gamma\,''(s_0)$,...,$\gamma^{(d+1)}(s_0)$ are not linearly independent, then M is said to have d-planar normal sections $(d < m)$.

Definition 3.6. Let $\gamma = \gamma(s) : I \to \mathbb{R}^n$ be a regular unit speed curve in $\mathbb{R}^n$. If γ is of osculating order $(n-1)$ at the point $p = \gamma(s_0)$ then p is called a flattening point of γ [15].

We prove the following result.

Proposition 3.7. *Let M be a smooth m-dimensional submanifold in Euclidean space $\mathbb{R}^{m+d}$. If M has pointwise $d-$planar normal sections at the point $\gamma(0) = p$, then each point p is a flattening point of γ.*

Proof. Let γ be a normal section of the submanifold $M \subset \mathbb{R}^{m+d}$. Then the normal section is also a curve in $(d+1)-$dimensional affine subspace $E(p,X)$ of $\mathbb{R}^{m+d}$. If M has pointwise $d-$planar normal sections at point $\gamma(0) = p$ then γ has osculating order d. So by previous definition p is a flattening point of order d. □

Definition 3.8. Let M be a Riemannian manifold and ∇ a Riemannian connection on M. For the curve $\alpha :]-\varepsilon, \varepsilon[\subset \mathbb{R} \to M$ on M if $\nabla_{\alpha'(s)}\alpha'(s) = 0$ then α is said to be a geodesic on M. For each tangent vector field X of $\chi(M)$, if $\alpha(0) = p$ and $\alpha'(0) = X_p$ then α is a geodesic of M with respect to (p, X_p) [3].

Definition 3.9. The geodesic γ_u and the normal section β_u at (p,u) are said to be in contact of order k if $\gamma_u^{(i)}$ and $\beta_u^{(i)}$ denote the ith derivatives of γ_u and β_u with respect to their arclength functions. A submanifold M in a Euclidean space is said to be in contact of order k if, for each $p \in M$ and $u \in U_pM$, the geodesic γ_u and the normal section β_u at (p, u) are in contact of order k.

If the submanifold M is in contact of order k for every natural number k, the contact number $C_{\#}(M)$ of M is defined to be ∞. That is;

$$\begin{aligned} \gamma_u'(0) &= \beta_u'(0) \\ &\vdots \\ \gamma_u^{(k)}(0) &= \beta_u^{(k)}(0). \end{aligned}$$

Otherwise, the contact number $C_{\#}(M)$ is defined to be the largest natural number k such that M is in contact of order k and but not of order $k+1$ [2].

Example 3.10. Let $\psi_j : M \to \mathbb{R}^{m_j}$, $(j = 1, \ldots, r)$ be an isometric immersion with geodesic normal sections. For any real numbers $c_1, c_2, \ldots, c_r$ with $c_1^2 + c_2^2 + \ldots + c_r^2 = 1$, the diagonal immersion,

$$(c_1\psi_1, \ldots, c_r\psi_r) : M \to \mathbb{R}^{m_1+\cdots+m_r} : p \to (c_1\psi_1(p), \ldots, c_r\psi_r(p))$$

satisfies $C_{\#}(M) = \infty$ [2].

The following theorems are proved.

Theorem 3.11. *[2] All submanifolds M in $\mathbb{R}^n$ with geodesic normal sections satisfy $C_{\#}(M) = \infty$.*

Theorem 3.12. *[2] For each $M \subset \mathbb{R}^{n+k}$, the contact number of M is at least* 2.

We have the following result.

Proposition 3.13. *Let M be a smooth m-dimensional submanifold in $\mathbb{R}^{m+d}$ and γ be the normal section of M. If the contact number $C_{\#}(M)$ of M at point p is ∞ then γ has a Darboux vertex at that point.*

Proof. If $C_{\#}(M) = \infty$ then by Theorem 3.11, M has geodesic normal sections. Further, in [5] M is helical submanifold and γ is a helical normal section. So the Frenet curvatures of γ are constant. Thus by Theorem 2.6, $\gamma\ (0) = p$ is a Darboux vertex of γ. □

4. The Focal Curve of A Curve

The focal set or caustic of a submanifold of positive codimension in Euclidean space $\mathbb{R}^{n+1}$ (for instance, of a curve in $\mathbb{R}^3$) is defined as the envelope of the family of normal lines to the submanifold.

The hyperplane normal to γ at a point is the union of all lines normal to γ at that point. The envelope of all hyperplanes normal to γ is thus a component of the focal set that we call the main component (the other component is the curve γ itself, but we will not consider it) [13].

Definition 4.1. Given a generic curve $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$, let $F : \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}$ be the $(n+1)$-parameter family of real functions given by

$$F(q,\theta) = \frac{1}{2} \left\| q - \gamma(\theta) \right\|^2 .$$

The caustic of the family F is given by the set

$$\left\{ q \in \mathbb{R}^{n+1} : \exists \theta \in \mathbb{R} : F_q'(\theta) = 0 \text{ and } F_q''(\theta) = 0 \right\}.$$

[13].

Proposition 4.2. *[13] The caustic of the family $F(q,\theta) = \frac{1}{2} \left\| q - \gamma(\theta) \right\|^2$ coincides with the focal set of the curve $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$.*

Definition 4.3. The center of the osculating hypersphere of γ at a point lies in the hyperplane normal to the γ at that point. So we can write

$$C_\gamma = \gamma + c_1 n_1 + c_2 n_2 + \cdots + c_n n_n,$$

which is called focal curve of γ, where $c_1, c_2, \ldots, c_n$ are smooth functions of the parameter of the curve γ. We call the function c_i the i^{th} focal curvature of γ. Moreover, the function c_1 never vanishes and $c_1 = \frac{1}{\kappa_1}$ [13].

Proposition 4.4. *[15] The focal curvatures of γ, parametrized by arc length s, satisfy the following "scalar Frenet equations" for $c_n \neq 0$:*

$$\begin{pmatrix} 1 \\ c_1' \\ c_2' \\ c_3' \\ \vdots \\ c_{n-2}' \\ c_{n-1}' \\ c_n' - \frac{(R_n^2)'}{2c_n} \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 & \cdots & 0 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & \cdots & 0 & 0 & 0 \\ 0 & -\kappa_2 & 0 & \ddots & & & \\ 0 & 0 & -\kappa_3 & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & & & & \\ & & & & 0 & \kappa_{n-1} & 0 \\ & & & & -\kappa_{n-1} & 0 & \\ 0 & 0 & & \cdots & 0 & -\kappa_n & 0 \end{pmatrix} \begin{pmatrix} 0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-2} \\ c_{n-1} \\ c_n \end{pmatrix}$$

Remark 4.5. If the curve is spherical then the last component of the left hand side vector is just c_n [15].

Proposition 4.6. *[8] The spherical curve in $\mathbb{R}^4$ is parametrized by*

$$\alpha(s) = m - \frac{R}{\kappa_1} n_1(s) + R \frac{\kappa_1'}{\kappa_1^2 \kappa_2} n_2(s) + \frac{R}{\kappa_3} \left(\frac{\kappa_1'}{\kappa_1^2 \kappa_2} \right)' \frac{\kappa_2}{\kappa_1} n_3(s)$$

where m is the center and R is the radius of the sphere.

We prove the following result.

Theorem 4.7. *Let γ be a normal section of M and C_γ is the generalized evolute of γ. Then the velocity vector of C_γ is proportional with the last Frenet vector of γ.*

Theorem 4.8. *Let γ be a normal section of M and C_γ is the generalized evolute of γ. If $C'_\gamma = 0$ then R_n is constant and the curve γ is spherical.*

Proof. Let C_γ be the generalized evolute of the normal section curve γ. If $C'_\gamma(s) = 0$ then from the previous theorem $(c_{n-1}\kappa_n + c'_n) = 0$. So, from the Remark 4.5, the normal section γ is spherical. □

Remark 4.9. Let C_γ be the generalized evolute of γ. We say that s_0 is a vertex of γ if $\left\|C'_\gamma(s_0)\right\| = 0$. A vertex of a curve in $\mathbb{R}^{n+1}$ is a point at which the curve has at least $(n+3)-$point contact with its hypersurface.

We prove the following result.

Theorem 4.10. *C_γ is 2-planar if and only if*
i) $\kappa_{n-1} = 0$, *or*
ii) $\kappa_n = 0$, *or*
iii) $C'_\gamma = 0$ *that is γ is a spherical curve and point p is vertex of γ.*

Proof. Let us denote $A = c_{n-1}\kappa_n + c'_n$. Differentiating C_γ we get

$$\begin{aligned} C'_\gamma &= A n_n \\ C''_\gamma &= A' n_n - A\kappa_n n_{n-1} \\ C'''_\gamma &= \underbrace{(A'' - A\kappa_n^2)}_{B} n_n + \underbrace{(-2A'\kappa_n - A\kappa'_n)}_{C} n_{n-1} + \underbrace{A\kappa_n\kappa_{n-1}}_{D} n_{n-2}. \end{aligned}$$

If C_γ is 2-planar then C'_γ, C''_γ and C'''_γ are linearly dependent. So we get

$$\begin{vmatrix} A & 0 & 0 \\ A' & -A\kappa_n & 0 \\ B & C & D \end{vmatrix} = D(-A^2\kappa_n) = A^3\kappa_n^2\kappa_{n-1} = 0,$$

which completes the proof. □

Proposition 4.11. *[4] Let M be pointwise planar normal sections and each normal section at p has one of its vertices then M has parallel second fundamental form.*

Proposition 4.12. *[15] The curvatures of a generic curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^{n+1}$, parametrized by arc length, may be obtained in terms of the focal curvatures of γ by the formula:*

$$\kappa_i = \frac{c_1c'_1 + c_2c'_2 + \cdots + c_{i-1}c'_{i-1}}{c_{i-1}c_i}, \ \textit{for } i \geq 2.$$

Remark 4.13. For a generic curve, the functions c_i or c_{i-1} can vanish at isolated points. At these points the function $c_1c'_1 + c_2c'_2 + \cdots + c_{i-1}c'_{i-1}$ also vanishes, and the corresponding value of the function κ_i may be obtained by l'Hospital rule. Denote by R_l the radius of the osculating l-sphere. Obviously $R_l^2 = c_1^2 + c_2^2 + \ldots + c_l^2$. In particular, $R_l^2 = \|C_\gamma - \gamma\|^2$ [15].

Theorem 4.14. *[15] The radius R_l of the osculating l-sphere of a generic curve, parametrized by arc length, in the Euclidean space $\mathbb{R}^{n+1}$, $n > 1$, is critical if and only if*

a) $c_2 = 0$, for $l = 1$;

b) either $c_l = 0$ or $c_{l+1} = 0$, for $1 < l < n$;

c) either $c_n = 0$ or $c_n' + c_{n-1}\kappa_n = 0$, for $l = n$.

Corollary 4.15. *[15] If the l^{th} focal curvature c_l vanishes at a point, then R_l and R_{l-1} are critical at that point.*

Definition 4.16. We define a generalized helix as a curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ such that its tangent vector forms a constant angle with a given direction v at $\mathbb{R}^n$ [9].

Proposition 4.17. *[9] A curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ is a generalized helix if and only if the function $\det(\gamma''(s), \gamma'''(s), \ldots, \gamma^{(n+1)}(s))$ is identically zero, where $\gamma^{(i)}$ represents the ith derivative of γ with respect to its arc length.*

Definition 4.18. The point s_0 is a flattening of γ if the contact of γ with the osculating hyperplane at s_0 is of order at least n [9].

Proposition 4.19. *[9] A point s_0 is a flattening of γ if and only if*

$$\det(\gamma'(s_0), \gamma''(s_0), \ldots, \gamma^{(n)}(s_0)) = 0,$$

where $\gamma^{(i)}$ represents the ith derivative of γ with respect to its arc length.

Definition 4.20. A conformal flattening or vertex of γ is a point at which γ has contact of order at least $n + 1$ with its osculating hypersphere [9].

Definition 4.21. A twisting of $\gamma : \mathbb{R} \to \mathbb{R}^n$ is a flattening of its tangent indicatrix $\gamma_T : \mathbb{R} \to \mathbb{S}^{n-1}$. It follows that if γ is parametrized by its arc length s, then s_0 is a twisting of γ if and only if $\det(\gamma''(s_0), \gamma'''(s_0), \ldots, \gamma^{(n+1)}(s_0)) = 0$ [9].

We shall see now that the twistings of γ can also be characterized as points at which it has higher order of contact with some generalized helix [9].

Proposition 4.22. *[9] Given a curve $\gamma(s)$ parametrized by arc length in $\mathbb{R}^n$, there exists for each point $\gamma(s_0) = p$ of this curve some generalized helix $\gamma_p(s)$ whose contact with γ at p is of order at least n. Moreover, if s_0 is a flattening point of γ_T then we have that γ_p has order of contact at least $n+1$ with γ at p.*

Theorem 4.23. *[13] Let $\gamma : s \to \gamma(s) \in \mathbb{R}^{n+1}$ be a good curve without its flattenings. Write $\kappa_1, \kappa_2, ..., \kappa_n$ for its Euclidean curvatures and $\{t, n_1, n_2, ..., n_n\}$ for its Frenet Frame. For each non-vertex $\gamma(s)$ of γ, write $\varepsilon(s)$ for the sign of $(c_n' + c_{n-1}\kappa_n)(s)$ and $\delta_k(s)$ for the sign of $(-1)^k \varepsilon(s)\kappa_n(s)$, $k = 1, \ldots, n$. For any non-vertex of γ the following holds:*

a) The Frenet frame $\{T, N_1, N_2, ..., N_n\}$ of C_γ at $C_\gamma(s)$ is well-defined and its vectors are given by $T = \varepsilon n_n$, $N_k = \delta_k n_{n-k}$, for $k = 1, \ldots, n-1$, and $N_n = \pm t$. The sign in $\pm t$ is chosen in order to obtain a positive basis.

b) The Euclidean curvatures $K_1, K_2, ..., K_n$ *of the parametrized focal curve of* γ. $C_\gamma : s \to C_\gamma(s)$, *are related to those of* γ *by:*

$$\frac{K_1}{|\kappa_n|} = \frac{K_2}{\kappa_{n-1}} = \cdots = \frac{|K_n|}{\kappa_1} = \frac{1}{|c_n' + c_{n-1}\kappa_n|},$$

the sign of K_n *is equal to* δ_n *times the sign chosen in* $\pm t$.

That is the Frenet matrix of C_γ *at* $C_\gamma(s)$ *is*

$$\frac{1}{|c_n' + c_{n-1}\kappa_n|}\begin{pmatrix} 0 & |\kappa_n| & 0 & \cdots & 0 & 0 & 0 \\ -|\kappa_n| & 0 & \kappa_{n-1} & \cdots & 0 & 0 & 0 \\ 0 & -\kappa_{n-1} & 0 & \ddots & & & \\ 0 & 0 & -\kappa_{n-2} & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & & & & \\ & & & & 0 & \kappa_2 & 0 \\ & & & & -\kappa_2 & 0 & \mp\delta_n\kappa_1 \\ 0 & 0 & & \cdots & 0 & \pm\delta_n\kappa_1 & 0 \end{pmatrix}.$$

We have the following result.

Theorem 4.24. *Let* $\gamma : s \to \gamma(s) \in \mathbb{R}^{n+1}$ *be a good curve without its flattenings and* C_γ *be its focal curve. If the ratios of the curvatures of* γ *are constant then the ratios of the curvatures of* C_γ *are also constant.*

Proof. Let $\kappa_1, \kappa_2, ..., \kappa_n$ and $K_1, K_2, ..., K_n$ be Euclidean curvatures of γ and its focal curve C_γ respectively. Then from Theorem 4.23, we get

$$\begin{aligned} \frac{K_1}{K_2} &= \frac{|\kappa_n|}{\kappa_{n-1}} = \frac{1}{|c_n' + c_{n-1}\kappa_n|} \\ \frac{K_3}{K_4} &= \frac{\kappa_{n-1}}{\kappa_{n-2}} \\ &\vdots \\ \frac{K_{n-1}}{|K_n|} &= \frac{\kappa_2}{\kappa_1}. \end{aligned}$$

Further, if $\frac{\kappa_{2k}}{\kappa_{2k-1}} = const.$, then $\frac{K_{2k-1}}{K_{2k}} = const.$ for $1 \leq k \leq \frac{n}{2}$, our theorem is thus proved. □

For more details on curves of $\mathbb{R}^n$ with constant curvature ratios (ccr-curves) see also [7].

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