

SPACE-LIKE AND TIME-LIKE HYPERSPHERES IN REAL PSEUDO-RIEMANNIAN 4-SPACES WITH ALMOST CONTACT B-METRIC STRUCTURES

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Abstract. We considered the 4-dimensional pseudo-Riemannian spaces with inner products of signature (3,1) and (2,2). The objects of investigation are space-like and time-like hyperspheres in the respective cases. These hypersurfaces are equipped with almost contact B-metric structures. The constructed manifolds are characterized geometrically.

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Introduction

The geometry of 4-dimensional Riemannian spaces is well developed. When the metric is generalized to pseudo-Riemannian there are two significant cases: the Lorentz-Minkowski space $\mathbb{R}^{3,1}$ and the neutral pseudo-Euclidean 4-space $\mathbb{R}^{2,2}$. These spaces are object of special interest because of their importance in physics. The space $\mathbb{R}^{3,1}$ has applications in the general relativity and the space $\mathbb{R}^{2,2}$ is connected to the string theory.

Hyperspheres in an even-dimensional space are known as a fundamental example of almost contact metric manifolds (cf. [1]). We are interested in almost contact B-metric structures, introduced in [4]. Almost contact B-metric manifolds are the odd-dimensional counterpart of almost complex manifolds with Norden metric (cf. [3, 5]), where the almost complex structure acts as an anti-isometry with respect to the metric. The investigation of almost contact B-metric manifolds is developed by many works, for example [7, 8, 9, 10, 11, 12].

In the present work we consider space-like and time-like hyperspheres in $\mathbb{R}^{3,1}$ and $\mathbb{R}^{2,2}$, known also as 3-dimensional de Sitter and anti-de Sitter spacetimes, respectively (cf. [2]). Our goal is to construct almost contact B-metric manifolds on these hypersurfaces and to study their geometrical properties. These explicit examples contribute for the study of the considered manifolds in the lowest dimension.

The paper is organized as follows. In Sect. 1 we recall some preliminary facts about the studied manifolds. In Sect. 2 we are interested in space-like spheres in $\mathbb{R}^{3,1}$. Sect. 3 is devoted to time-like spheres in $\mathbb{R}^{2,2}$.

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1. Preliminaries

Let us denote an *almost contact B-metric manifold* by $(M, \varphi, \xi, \eta, g)$, i.e. M is a $(2n + 1)$ -dimensional differentiable manifold with an almost contact structure (φ, ξ, η) consisting of an endomorphism φ of the tangent bundle, a Reeb vector field ξ , its dual contact 1-form η as well as M is equipped with a pseudo-Riemannian metric g of signature $(n + 1, n)$, such that the following algebraic relations are satisfied [4]:

$$\begin{aligned} \varphi\xi = 0, \quad \varphi^2 = -\text{Id} + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1, \\ g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y), \end{aligned}$$

where Id is the identity. In the latter equality and further, x, y, z, w will stand for arbitrary elements of $\mathfrak{X}(M)$, the Lie algebra of tangent vector fields, or vectors in the tangent space T_pM of M at an arbitrary point p in M .

A classification of almost contact B-metric manifolds, consisting of eleven basic classes $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{11}$, is given in [4]. This classification is made with respect to the tensor F of type $(0,3)$ defined by

$$F(x, y, z) = g((\nabla_x \varphi) y, z),$$

where ∇ is the Levi-Civita connection of g . The following properties are valid in general:

$$\begin{aligned} (1.1) \quad F(x, y, z) &= F(x, z, y) \\ &= F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi), \\ F(x, \varphi y, \xi) &= (\nabla_x \eta)y = g(\nabla_x \xi, y). \end{aligned}$$

The intersection of the basic classes is the special class \mathcal{F}_0 , determined by the condition $F(x, y, z) = 0$, and it is known as the class of the *cosymplectic B-metric manifolds*.

Let $\{\xi; e_i\}$ ($i = 1, 2, \dots, 2n$) be a basis of T_pM and let (g^{ij}) be the inverse matrix of (g_{ij}) . Then with F are associated the 1-forms θ, θ^*, ω , called *Lee forms*, defined by:

$$\theta(z) = g^{ij}F(e_i, e_j, z), \quad \theta^*(z) = g^{ij}F(e_i, \varphi e_j, z), \quad \omega(z) = F(\xi, \xi, z).$$

Now let us consider the case of the lowest dimension of the considered manifolds, i.e. $\dim M = 3$.

We introduce an almost contact B-metric structure (φ, ξ, η, g) on M by

$$(1.2) \quad \begin{aligned} \varphi e_1 = 0, \quad \varphi e_2 = e_3, \quad \varphi e_3 = -e_2, \quad \xi = e_1, \\ \eta(e_1) = 1, \quad \eta(e_2) = \eta(e_3) = 0, \end{aligned}$$

$$(1.3) \quad g(e_1, e_1) = g(e_2, e_2) = -g(e_3, e_3) = 1, \quad g(e_i, e_j) = 0, \quad i \neq j \in \{1, 2, 3\}.$$

The components of $F, \theta, \theta^*, \omega$ with respect to the φ -basis $\{e_1, e_2, e_3\}$ are denoted by $F_{ijk} = F(e_i, e_j, e_k), \theta_k = \theta(e_k), \theta_k^* = \theta^*(e_k), \omega_k = \omega(e_k)$.

According to [6], we have:

$$\begin{aligned} \theta_1 &= F_{221} - F_{331}, & \theta_2 &= F_{222} - F_{332}, & \theta_3 &= F_{223} - F_{322}, \\ \theta_1^* &= F_{231} + F_{321}, & \theta_2^* &= F_{223} + F_{322}, & \theta_3^* &= F_{222} + F_{332}, \\ \omega_1 &= 0, & \omega_2 &= F_{112}, & \omega_3 &= F_{113}. \end{aligned}$$

If F^s ($s = 1, 2, \dots, 11$) are the components of F in the corresponding basic classes \mathcal{F}_s then: [6]

$$\begin{aligned} (1.4) \quad & F^1(x, y, z) = (x^2\theta_2 - x^3\theta_3) (y^2z^2 + y^3z^3), \\ & \theta_2 = F_{222} = F_{233}, \quad \theta_3 = -F_{322} = -F_{333}; \\ & F^2(x, y, z) = F^3(x, y, z) = 0; \\ & F^4(x, y, z) = \frac{1}{2}\theta_1 \{x^2 (y^1z^2 + y^2z^1) - x^3 (y^1z^3 + y^3z^1)\}, \\ & \frac{1}{2}\theta_1 = F_{212} = F_{221} = -F_{313} = -F_{331}; \\ & F^5(x, y, z) = \frac{1}{2}\theta_1^* \{x^2 (y^1z^3 + y^3z^1) + x^3 (y^1z^2 + y^2z^1)\}, \\ & \frac{1}{2}\theta_1^* = F_{213} = F_{231} = F_{312} = F_{321}; \\ & F^6(x, y, z) = F^7(x, y, z) = 0; \\ & F^8(x, y, z) = \lambda \{x^2 (y^1z^2 + y^2z^1) + x^3 (y^1z^3 + y^3z^1)\}, \\ & \lambda = F_{212} = F_{221} = F_{313} = F_{331}; \\ & F^9(x, y, z) = \mu \{x^2 (y^1z^3 + y^3z^1) - x^3 (y^1z^2 + y^2z^1)\}, \\ & \mu = F_{213} = F_{231} = -F_{312} = -F_{321}; \\ & F^{10}(x, y, z) = \nu x^1 (y^2z^2 + y^3z^3), \quad \nu = F_{122} = F_{133}; \\ & F^{11}(x, y, z) = x^1 \{ (y^2z^1 + y^1z^2) \omega_2 + (y^3z^1 + y^1z^3) \omega_3 \}, \\ & \omega_2 = F_{121} = F_{112}, \quad \omega_3 = F_{131} = F_{113}, \end{aligned}$$

where $x = x^i e_i$, $y = y^j e_j$, $z = z^k e_k$. Obviously, the class of 3-dimensional almost contact B-metric manifolds is

$$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}.$$

Three natural connections on an arbitrary $(M, \varphi, \xi, \eta, g)$ are considered in [7], i.e. linear connections which preserve φ , ξ , η , g . They are called a φ B-connection, a φ -canonical connection and a φ KT-connection. The φ B-connection is defined by

$$(1.5) \quad D_x y = \nabla_x y + \frac{1}{2} \{ (\nabla_x \varphi) \varphi y + (\nabla_x \eta) y \cdot \xi \} - \eta(y) \nabla_x \xi.$$

The φ -canonical connection is determined by an identity for its torsion with respect to the structure tensors and the φ KT-connection is characterized as the natural connection with totally antisymmetric torsion.

Since the considered manifold is 3-dimensional and the class $\mathcal{F}_3 \oplus \mathcal{F}_7$ is empty, then the φ KT-connection does not exist and the φ -canonical connection coincides with the φ B-connection.

The square norm of $\nabla \varphi$ is defined in [7] as follows

$$(1.6) \quad \|\nabla \varphi\|^2 = g^{ij} g^{ks} g((\nabla_{e_i} \varphi) e_k, (\nabla_{e_j} \varphi) e_s).$$

An almost contact B-metric manifold having a zero square norm of $\nabla\varphi$ is called an *isotropic-cosymplectic B-metric manifold* ([7]). Obviously, the equality $\|\nabla\varphi\|^2 = 0$ is valid if $(M, \varphi, \xi, \eta, g)$ is a \mathcal{F}_0 -manifold, but the inverse implication is not always true.

The Nijenhuis tensor N of the almost contact structure is defined as usual by $N = [\varphi, \varphi] + d\eta \otimes \xi$, where $[\varphi, \varphi](x, y) = [\varphi x, \varphi y] + \varphi^2[x, y] - \varphi[\varphi x, y] - \varphi[x, \varphi y]$ for $[x, y] = \nabla_x y - \nabla_y x$ and $d\eta$ is the exterior derivative of η . According to [11], the associated Nijenhuis tensor \widehat{N} has the following form $\widehat{N} = \{\varphi, \varphi\} + (\mathcal{L}_\xi g) \otimes \xi$, where $\{\varphi, \varphi\}(x, y) = \{\varphi x, \varphi y\} + \varphi^2\{x, y\} - \varphi\{\varphi x, y\} - \varphi\{x, \varphi y\}$ for $\{x, y\} = \nabla_x y + \nabla_y x$ and $\mathcal{L}_\xi g$ is the Lie derivative of g with respect to ξ .

The corresponding tensors of type (0,3) on $(M, \varphi, \xi, \eta, g)$ are determined by $N(x, y, z) = g(N(x, y), z)$ and $\widehat{N}(x, y, z) = g(\widehat{N}(x, y), z)$. From [11], it is known that the tensors $N(x, y, z)$ and $\widehat{N}(x, y, z)$ are expressed by F as follows

$$\begin{aligned}
 (1.7) \quad N(x, y, z) &= F(\varphi x, y, z) - F(x, y, \varphi z) + \eta(z)F(x, \varphi y, \xi) \\
 &\quad - F(\varphi y, x, z) + F(y, x, \varphi z) - \eta(z)F(y, \varphi x, \xi), \\
 \widehat{N}(x, y, z) &= F(\varphi x, y, z) - F(x, y, \varphi z) + \eta(z)F(x, \varphi y, \xi) \\
 &\quad + F(\varphi y, x, z) - F(y, x, \varphi z) + \eta(z)F(y, \varphi x, \xi).
 \end{aligned}$$

Let $R = [\nabla, \nabla] - \nabla_{[\cdot, \cdot]}$ be the curvature (1,3)-tensor of ∇ and the corresponding curvature (0,4)-tensor be denoted by the same letter: $R(x, y, z, w) = g(R(x, y)z, w)$. The following properties are valid in general:

$$\begin{aligned}
 (1.8) \quad R(x, y, z, w) &= -R(y, x, z, w) = -R(x, y, w, z), \\
 R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) &= 0.
 \end{aligned}$$

The Ricci tensor ρ and the scalar curvature τ for R and g as well as their associated quantities are defined as follows

$$\begin{aligned}
 (1.9) \quad \rho(y, z) &= g^{ij}R(e_i, y, z, e_j), \quad \rho^*(y, z) = g^{ij}R(e_i, y, z, \varphi e_j), \\
 \tau &= g^{ij}\rho(e_i, e_j), \quad \tau^* = g^{ij}\rho^*(e_i, e_j), \quad \tau^{**} = g^{ij}\rho^*(e_i, \varphi e_j).
 \end{aligned}$$

Each non-degenerate 2-plane α in T_pM with respect to g and R has the following sectional curvature

$$(1.10) \quad k(\alpha; p) = \frac{R(x, y, y, x)}{g(x, x)g(y, y)},$$

where $\{x, y\}$ is an orthogonal basis of α .

A 2-plane α is said to be a φ -holomorphic section (respectively, a ξ -section) if $\alpha = \varphi\alpha$ (respectively, $\xi \in \alpha$).

2. Space-like hyperspheres in $\mathbb{R}^{3,1}$

In this section we consider a hypersurface of the Lorentz-Minkowski space $\mathbb{R}^{3,1}$. Let $\langle \cdot, \cdot \rangle$ be the Lorentzian inner product, i.e.

$$\langle x, y \rangle = x^1y^1 + x^2y^2 + x^3y^3 - x^4y^4,$$

where $x(x^1, x^2, x^3, x^4)$, $y(y^1, y^2, y^3, y^4)$ are arbitrary vectors in $\mathbb{R}^{3,1}$. Let us consider a space-like hypersphere S_1^3 at the origin with a real radius r identifying the point p in $\mathbb{R}^{3,1}$ with its position vector z , i.e.

$$\langle z, z \rangle = r^2.$$

It is parameterized by

$$z(r \cos u^1 \cos u^2, r \cos u^1 \sin u^2, r \sin u^1 \cosh u^3, r \sin u^1 \sinh u^3),$$

where u^1, u^2, u^3 are real parameters such as $u^1 \neq \frac{k\pi}{2} (k \in \mathbb{Z})$, $u^2 \in [0; 2\pi]$. Then for the local basic vectors $\partial_i = \frac{\partial z}{\partial u^i}$ we have the following

$$\begin{aligned} \langle \partial_1, \partial_1 \rangle &= r^2, & \langle \partial_2, \partial_2 \rangle &= r^2 \cos^2 u^1, & \langle \partial_3, \partial_3 \rangle &= -r^2 \sin^2 u^1, \\ \langle \partial_i, \partial_j \rangle &= 0, & i &\neq j. \end{aligned}$$

By substituting $e_i = \frac{1}{\sqrt{|\langle \partial_i, \partial_i \rangle|}} \partial_i$ we obtain a basis $\{e_i\}$, $i \in \{1, 2, 3\}$ as follows

$$(2.1) \quad e_1 = \frac{1}{r} \partial_1, \quad e_2 = \frac{\varepsilon_1}{r \cos u^1} \partial_2, \quad e_3 = \frac{\varepsilon_2}{r \sin u^1} \partial_3,$$

where $\varepsilon_1 = \text{sgn}(\cos u^1)$, $\varepsilon_2 = \text{sgn}(\sin u^1)$. We equip it with an almost contact structure determined as in (1.2). The metric on the hypersurface, denoted by g , is the restriction of $\langle \cdot, \cdot \rangle$ on the sphere. Then $\{e_1, e_2, e_3\}$ is an orthonormal φ -basis on the tangent space $T_p S_1^3$ at $p \in S_1^3$, i.e. for $g_{ij} = g(e_i, e_j)$, $i, j \in \{1, 2, 3\}$, we have (1.3). Thus, we get that $(S_1^3, \varphi, \xi, \eta, g)$ is a 3-dimensional almost contact B-metric manifold.

By virtue of (2.1) we obtain the commutators of the basic vectors e_i

$$(2.2) \quad [e_1, e_2] = \frac{1}{r} \tan u^1 e_2, \quad [e_1, e_3] = -\frac{1}{r} \cot u^1 e_3, \quad [e_2, e_3] = 0.$$

Using the well-known Koszul identity for ∇ of g we get

$$(2.3) \quad \begin{aligned} \nabla_{e_2} e_1 &= -\frac{1}{r} \tan u^1 e_2, & \nabla_{e_2} e_2 &= \frac{1}{r} \tan u^1 e_1, \\ \nabla_{e_3} e_1 &= \frac{1}{r} \cot u^1 e_3, & \nabla_{e_3} e_3 &= \frac{1}{r} \cot u^1 e_1 \end{aligned}$$

and the other components are zero.

Let us compute the components of the natural connection denoted by D in (1.5). Then, using (1.2), (1.3), (1.5), (2.3), we establish that

$$(2.4) \quad D_{e_i} e_j = 0, \quad i, j \in \{1, 2, 3\}.$$

According to (1.2), (1.3) and (2.3), we obtain the value of the square norm of $\nabla\varphi$ as follows

$$(2.5) \quad \|\nabla\varphi\|^2 = -\frac{2}{r^2} (\tan^2 u^1 + \cot^2 u^1).$$

Taking into account (1.2), (1.3) and (2.3), we compute the components F_{ijk} of F with respect to the basis $\{e_1, e_2, e_3\}$. They are

$$(2.6) \quad F_{213} = F_{231} = -\frac{1}{r} \tan u^1, \quad F_{312} = F_{321} = \frac{1}{r} \cot u^1$$

and the other components of F are zero.

Using (1.7) and (2.6), we find the basic components $N_{ijk} = N(e_i, e_j, e_k)$ and $\widehat{N}_{ijk} = \widehat{N}(e_i, e_j, e_k)$ of the Nijenhuis tensor and its associated tensor, respectively,

$$\begin{aligned} N_{122} &= -N_{212} = N_{133} = -N_{313} = -\frac{1}{r}(\cot u^1 + \tan u^1), \\ \widehat{N}_{122} &= \widehat{N}_{212} = \widehat{N}_{133} = \widehat{N}_{313} = \frac{1}{r}(\cot u^1 + \tan u^1), \\ \widehat{N}_{221} &= -\widehat{N}_{331} = -\frac{4}{r} \tan u^1, \end{aligned}$$

as well as their square norms, according to (1.6), as follows

$$(2.7) \quad \begin{aligned} \|N\|^2 &= \frac{4}{r^2}(\cot^2 u^1 + \tan^2 u^1 + 2), \\ \|\widehat{N}\|^2 &= \frac{4}{r^2}(\cot^2 u^1 + 9 \tan^2 u^1 + 2). \end{aligned}$$

Bearing in mind (1.4) and (2.6), we establish the equality

$$(2.8) \quad F(x, y, z) = (F^5 + F^9)(x, y, z),$$

where F^5 and F^9 are the components of F in the basic classes \mathcal{F}_5 and \mathcal{F}_9 , respectively. The nonzero components of F^5 and F^9 with respect to $\{e_1, e_2, e_3\}$ are the following

$$(2.9) \quad \begin{aligned} F_{213}^5 &= F_{231}^5 = F_{312}^5 = F_{321}^5 = \frac{1}{2}\theta_1^* = \frac{1}{2r}(\cot u^1 - \tan u^1), \\ F_{213}^9 &= F_{231}^9 = -F_{312}^9 = -F_{321}^9 = \mu = -\frac{1}{2r}(\cot u^1 + \tan u^1). \end{aligned}$$

Let us remark that the above components of F^5 and F^9 are nonzero for all values of u^1 in its domain. By virtue of (2.8), (2.9) and (1.1), we get that

$$(2.10) \quad d\eta = 0, \quad \nabla_\xi \xi = 0.$$

Using the equalities (1.3), (2.2) and (2.3), we compute the components $R_{ijk\ell} = R(e_i, e_j, e_k, e_\ell)$ of the curvature tensor R with respect to $\{e_1, e_2, e_3\}$. The nonzero components are given by the following ones and the symmetries of R in (1.8)

$$(2.11) \quad R_{1221} = -R_{1331} = -R_{2332} = \frac{1}{r^2}.$$

By virtue of (1.3), (1.9) and (2.11), the basic components $\rho_{jk} = \rho(e_j, e_k)$ and $\rho_{jk}^* = \rho^*(e_j, e_k)$ of the Ricci tensor ρ and its associated tensor ρ^* , respectively, as well as the values of the scalar curvature τ and its associated curvatures τ^* , τ^{**} are the following

$$\begin{aligned} \rho_{11} &= \rho_{22} = -\rho_{33} = \frac{2}{r^2}, & \rho_{23}^* &= \rho_{32}^* = \frac{1}{r^2}, \\ \tau &= \frac{6}{r^2}, & \tau^* &= 0, & \tau^{**} &= \frac{2}{r^2}. \end{aligned}$$

Moreover, using (1.3), (1.10) and (2.11), we obtain the basic sectional curvatures $k_{ij} = k(e_i, e_j)$ determined by the basis $\{e_i, e_j\}$ of the corresponding 2-plane as follows

$$(2.12) \quad k_{12} = k_{13} = k_{23} = \frac{1}{r^2}.$$

Let us remark that (1.3), (2.11) and (2.12) imply the following form of the curvature tensor

$$(2.13) \quad R(x, y, z, w) = \frac{1}{r^2} \{g(y, z)g(x, w) - g(x, z)g(y, w)\}.$$

Bearing in mind the above results, we establish the truthfulness of the following

Theorem 2.1. *Let $(S_1^3, \varphi, \xi, \eta, g)$ be the space-like sphere in the Lorentz-Minkowski space $\mathbb{R}^{3,1}$ equipped with an almost contact B-metric structure. Then*

1. *the manifold is in the class $\mathcal{F}_5 \oplus \mathcal{F}_9$ but it belongs neither to \mathcal{F}_5 nor \mathcal{F}_9 and it is not an isotropic-cosymplectic B-metric manifold;*
2. *the φ B-connection which coincides with the φ -canonical connection vanishes in the basis $\{e_1, e_2, e_3\}$;*
3. *the square norm of $\nabla\varphi$ is negative;*
4. *the square norms of the Nijenhuis tensor and its associated are positive;*
5. *the contact form η is closed and the integral curves of ξ are geodesic;*
6. *the manifold is a space-form with positive constant sectional curvature.*

Proof. The proposition (1) follows from (2.5), (2.8) and (2.9). The truthfulness of the propositions (2), (3), (4), (5), (6) follows from (2.4), (2.5), (2.7), (2.10), (2.13), respectively. □

3. Time-like hyperspheres in $\mathbb{R}^{2,2}$

In [4], it is considered a unit time-like hypersphere S in $(\mathbb{R}^{2n+2}, J, G)$, where \mathbb{R}^{2n+2} is a complex Riemannian manifold with a canonical complex structure J and a Norden metric G . There is introduced an almost contact B-metric structure on S in appropriate way by means of J and G . The constructed hypersphere with the considered structure belongs to the class $\mathcal{F}_4 \oplus \mathcal{F}_5$.

In this section we use a different approach for equipping a time-like hypersphere in \mathbb{R}^{2n+2} for $n = 1$ with an almost contact B-metric structure.

Let us consider the neutral pseudo-Euclidean 4-space $\mathbb{R}^{2,2}$. Let $\langle \cdot, \cdot \rangle$ be the inner product defined by

$$\langle x, y \rangle = x^1y^1 + x^2y^2 - x^3y^3 - x^4y^4$$

for arbitrary vectors $x(x^1, x^2, x^3, x^4)$, $y(y^1, y^2, y^3, y^4)$ in $\mathbb{R}^{2,2}$. Let us consider a time-like hypersphere H_1^3 at the origin with a real radius r identifying the point p in $\mathbb{R}^{2,2}$ with its position vector z , i.e.

$$\langle z, z \rangle = -r^2.$$

It is parameterized by

$$z(r \sinh u^1 \cos u^2, r \sinh u^1 \sin u^2, r \cosh u^1 \cos u^3, r \cosh u^1 \sin u^3),$$

where $u^1, u^2, u^3 \in \mathbb{R}$ such as $u^1 \neq 0$. Then, for the local basic vectors ∂_i , we have the following

$$\begin{aligned} \langle \partial_1, \partial_1 \rangle &= r^2, & \langle \partial_2, \partial_2 \rangle &= r^2 \sinh^2 u^1, & \langle \partial_3, \partial_3 \rangle &= -r^2 \cosh^2 u^1, \\ \langle \partial_i, \partial_j \rangle &= 0, & i &\neq j. \end{aligned}$$

Similarly as in the previous section, we substitute $e_i = \frac{1}{\sqrt{|\langle \partial_i, \partial_i \rangle|}} \partial_i$ and we obtain an orthonormal basis $\{e_i\}$, $i \in \{1, 2, 3\}$, as follows

$$e_1 = \frac{1}{r} \partial_1, \quad e_2 = \frac{\varepsilon}{r \sinh u^1} \partial_2, \quad e_3 = \frac{1}{r \cosh u^1} \partial_3,$$

where $\varepsilon = \text{sgn}(u^1)$. As for S_1^3 , we introduce an almost contact B-metric structure on H_1^3 determined by (1.2) and (1.3). Hence, we get that $(H_1^3, \varphi, \xi, \eta, g)$ is a 3-dimensional almost contact B-metric manifold.

By similar way as for S_1^3 we obtain successively the following results:

$$[e_1, e_2] = -\frac{1}{r} \coth u^1 e_2, \quad [e_1, e_3] = -\frac{1}{r} \tanh u^1 e_3, \quad [e_2, e_3] = 0,$$

$$\begin{aligned} \nabla_{e_2} e_1 &= \frac{1}{r} \coth u^1 e_2, & \nabla_{e_2} e_2 &= -\frac{1}{r} \coth u^1 e_1, \\ \nabla_{e_3} e_1 &= \frac{1}{r} \tanh u^1 e_3, & \nabla_{e_3} e_3 &= \frac{1}{r} \tanh u^1 e_1, \end{aligned}$$

$$(3.1) \quad D_{e_i} e_j = 0, \quad i, j \in \{1, 2, 3\},$$

$$(3.2) \quad \|\nabla\varphi\|^2 = -\frac{2}{r^2} (\tanh^2 u^1 + \coth^2 u^1),$$

$$F_{213} = F_{231} = \frac{1}{r} \coth u^1, \quad F_{312} = F_{321} = \frac{1}{r} \tanh u^1,$$

$$\begin{aligned} N_{122} &= -N_{212} = N_{133} = -N_{313} = \frac{2}{r \sinh 2u^1}, \\ \widehat{N}_{122} &= \widehat{N}_{212} = \widehat{N}_{133} = \widehat{N}_{313} = -\frac{2}{r \sinh 2u^1}, \\ \widehat{N}_{221} &= -\widehat{N}_{331} = \frac{2}{r} (\coth u^1 + \tanh u^1), \end{aligned}$$

$$(3.3) \quad \begin{aligned} \|N\|^2 &= \frac{4}{r^2} (\coth^2 u^1 + \tanh^2 u^1 + 2), \\ \|\widehat{N}\|^2 &= \frac{4}{r^2} (3 \coth^2 u^1 + 3 \tanh^2 u^1 + 2), \end{aligned}$$

$$(3.4) \quad F(x, y, z) = (F^5 + F^9)(x, y, z),$$

$$(3.5) \quad \begin{aligned} F_{213}^5 &= F_{231}^5 = F_{312}^5 = F_{321}^5 = \frac{1}{2} \theta_1^* = \frac{1}{2r} (\coth u^1 + \tanh u^1), \\ F_{213}^9 &= F_{231}^9 = -F_{312}^9 = -F_{321}^9 = \mu = \frac{1}{2r} (\coth u^1 - \tanh u^1), \end{aligned}$$

$$(3.6) \quad d\eta = 0, \quad \nabla_\xi \xi = 0,$$

$$(3.7) \quad R_{1221} = -R_{1331} = -R_{2332} = k_{12} = k_{13} = k_{23} = -\frac{1}{r^2},$$

$$\rho_{11} = \rho_{22} = -\rho_{33} = -\frac{2}{r^2}, \quad \rho_{23}^* = \rho_{32}^* = -\frac{1}{r^2},$$

$$\tau = -\frac{6}{r^2}, \quad \tau^* = 0, \quad \tau^{**} = -\frac{2}{r^2}.$$

Similarly to the case of S_1^3 , the obtained results could be interpreted in the following

Theorem 3.1. *Let $(H_1^3, \varphi, \xi, \eta, g)$ be the time-like sphere in the space $\mathbb{R}^{2,2}$ equipped with an almost contact B-metric structure. Then*

1. *the manifold is in the class $\mathcal{F}_5 \oplus \mathcal{F}_9$ but it belongs neither to \mathcal{F}_5 nor \mathcal{F}_9 and it is not an isotropic-cosymplectic B-metric manifold;*
2. *the φ B-connection which coincides with the φ -canonical connection vanishes in the basis $\{e_1, e_2, e_3\}$;*
3. *the square norm of $\nabla\varphi$ is negative;*
4. *the square norms of the Nijenhuis tensor and its associated are positive;*
5. *the contact form η is closed and the integral curves of ξ are geodesic;*
6. *the manifold is a space-form with negative constant sectional curvature.*

Proof. The proposition (1) follows from (3.2), (3.4) and (3.5). The truthfulness of the propositions (2), (3), (4), (5), (6) follows from (3.1), (3.2), (3.3), (3.6), (3.7), respectively. \square

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