

BLOCH–PERIODIC GENERALIZED FUNCTIONS

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Abstract. Bloch-periodicity is a generalization of the notion of periodic and antiperiodic functions with much practical relevance for engineering science and especially condensed matter physics. In recent work we have considered this property in the setting of classical functional analysis, and also introduced the new notion of asymptotically Bloch-periodic functions. In this paper we formulate these properties and results in the framework of Colombeau-type generalized functions.

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1. Introduction

This paper introduces the notion of Bloch–periodic and asymptotically Bloch–periodic generalized functions and, as an application, we will display some solutions to semilinear differential equations with irregular coefficients in his framework.

Bloch periodicity,

$$f(x + p) = e^{i k \cdot p} f(x) ; \quad x, p, k \in \mathbb{R}^n ,$$

is a generalization of the usual periodicity and the recently extensively studied antiperiodicity (see, *e.g.*, [2, 3, 7, 8, 14, 25] and references therein), which is obviously of major relevance in particular in condensed matter and solid state physics, where quantum mechanical wavefunctions have this symmetry.

In [15] we have generalized this notion to that of asymptotic Bloch periodicity, which will be defined later in the paper. Solutions to various equations describing propagation of heat and of waves in solid matter are expected to have this property.

The novelty of this paper is to extend these definitions to the framework of algebras of generalized functions as first introduced by Colombeau [9, 10]. These algebras are quotient spaces of “moderate” sequences modulo the subset of “negligible” sequences; more recently this has been identified to be the associated Hausdorff space of the subspace of sequences for which multiplication is continuous with respect to a naturally arising “asymptotic” topology.

As an application we will consider solutions to differential equations with non-smooth coefficients, which have the considered periodicity.

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The applications in biology, physics, engineering, and other sciences make the study of Bloch-periodic solutions a very attractive topic in the qualitative theory of differential equations.

In [21], the authors studied the structure of several classes of functions ranging from periodic to bounded and continuous ones. Then they used the results obtained to investigate the existence of these types of solutions to the integro-differential equation $u' = Au + a * Au + f(\cdot, Cu(\cdot))$. Some properties of asymptotically periodic and S -asymptotically ω -periodic functions were also investigated.

In [24] authors extended these results to antiperiodic functions. They studied asymptotically antiperiodic functions in a Banach space and used the results to consider existence of antiperiodic mild solutions to a class of integrodifferential equations.

The paper [15] is a continuation of [21] and [24] considering the case of Bloch-periodic functions. It introduces the notion of and asymptotically Bloch-periodic functions in a Banach space, and investigates the existence of Bloch-periodic mild solutions to a class of integrodifferential equations.

This present paper will make the link between this line of research and earlier work of the author in the area of Colombeau generalized functions, which have gained substantial interest over the last years. Indeed, these algebras provide the adequate framework for posing and solving differential problems in cases where irregularities *and* non-linearity are both present. The classical theories of calculus and Schwartz distributions can handle one or the other, but not both at the same time; it is a well known proven fact that distributions cannot be multiplied unconditionnally. On the other hand, this case is most interesting for applications because many interesting natural phenomena are non-linear: in particular, while a linear system may well *propagate* singularities, it is unlikely (if not unable) to *produce* singularities. Conversely, while most natural laws are linear in first approximation, *i.e.*, for small forces, accelerations, velocities and deviation from equilibrium, this is not true in more extreme situations, and it may often be the case that irregularities as shocks etc. are associated with such conditions which make the linear approximation invalid.

2. Colombeau type generalized functions

The concept of Bloch periodicity is quite well known in physics and engineering, but seems not yet extensively studied in mathematics. After investigations in the framework of classical analysis [15] where we discussed Bloch-periodic solutions to integro-differential equations of the form $u' = Au + a * Au + f$, we introduce it here for the first time in the setting of Colombeau-type generalized functions.

We refer to [16] for details on the general construction of M -extensions, which include Colombeau's algebra of New Generalized Functions [9, 10] as a special case. Here, we just recall the basic ideas.

An asymptotic scale $M \subset R^\Lambda$ defines a natural topology $\tau_M(E)$ on the se-

quence space E^Λ , for any topological R -module $(E, \tau(E))$. Namely, the neighborhoods of zero of this topology are simply the “product” of the sequences defining the growth scale, with the zero neighborhoods of the base space E .

Colombeau’s simplified algebra is obtained by choosing $\Lambda = (0, 1]$, $M = \{(\varepsilon^m)_{\varepsilon \in \Lambda}; m \in \mathbb{Z}\}$, and $E = \mathcal{C}^\infty(\mathbb{R}^n)$ with topology defined by seminorms $p_{K,\ell}$ of sup norm of ℓ -th derivatives on compact sets K .

Then the “moderate” sequences can be defined as the subspace E_M on which multiplication $R^\Lambda \times E^\Lambda \rightarrow E^\Lambda$ is continuous. In the Colombeau case these are nets of functions $(f_\varepsilon)_\varepsilon$ whose seminorms $p_{K,\ell}$ are bounded by some negative power of the parameter $\varepsilon \in \Lambda = (0, 1]$.

Finally, the M -extension $\mathcal{G}_M(E)$ is simply the Hausdorff space $\tilde{E}_M = E_M / \{o\}$; *i.e.*, Colombeau’s set \mathcal{N} of “negligible” sequences is nothing else than the intersection of the above mentioned neighborhoods of zero, which amounts to require that all seminorms tend to zero faster than *any* power of ε .

Together with the canonical extension of morphisms $\phi : E \rightarrow F$ to $\mathcal{G}_M(\phi) : \mathcal{G}_M(E) \rightarrow \mathcal{G}_M(F)$, based on a component-wise definition which passes to the quotient, this yields a functor $\mathcal{G}_M : \mathbf{ModTop}_R \rightarrow \mathbf{ModTop}_{\mathcal{G}_M(R)}$.

Furthermore, thanks to the functoriality of the construction, if E is a presheaf (resp., a fine sheaf), then $\mathcal{G}_M(E) : \Omega \rightarrow \mathcal{G}_M(E(\Omega))$ is also a presheaf (resp., a fine sheaf).

In the sequel we will use as base space $E = \mathcal{E}(\mathbb{R}^n; X)$ where \mathcal{E} is the sheaf of smooth functions defined on \mathbb{R}^n , with values in the Banach space X . Since the scale is often fixed once for ever, we will sometimes also drop the subscript M and simply write $\mathcal{G}(E)$ for the space (or sheaf) of generalized functions.

3. Bloch-periodic functions

Definition 1 (Bloch-periodic functions). Assume given a complex linear space X , vectors $p, k \in \mathbb{R}^n$, and a subset $\mathbb{J} = \mathbb{J}_p \subset \mathbb{R}^n$ such that $\mathbb{J} + p \subset \mathbb{J}$. A function $f : \mathbb{J} \rightarrow X$ is said to be (p, k) -periodic, or Bloch-periodic with period p and Bloch wave vector or Floquet exponent k , iff

$$f(x + p) = e^{i k \cdot p} f(x), \quad \text{for all } x \in \mathbb{J}.$$

We denote by $P_{p,k}(\mathbb{J}; X)$, the space of all (p, k) -periodic functions $\mathbb{J} \rightarrow X$, and we drop \mathbb{J} in the case $\mathbb{J} = \mathbb{R}^n$.

Example 2. If f is (p, k) -periodic with $k \cdot p = 2\pi$, then f is simply p -periodic; if $k \cdot p = \pi$, then f is p -antiperiodic.

One could define Bloch-periodic functions without imposing a domain such that $\mathbb{J} + p \subset \mathbb{J}$, by requiring the given property only whenever $\{x, x + p\} \subset D_f$. However, in the sequel, to obtain several results especially for asymptotically Bloch-periodic functions, we must be able to “go to infinity”; yet in applications

we don't always have functions defined on the whole of \mathbb{R}^n . Important nontrivial examples include $\mathbb{J} = \mathbb{R}_+^n$ (for $p \in \mathbb{R}_+^n$), or a strip in \mathbb{R}^2 , parallel to p , $\mathbb{J} = [a, b] + \mathbb{R}p$.

Example 3. Examples of Bloch periodic functions are given for any sequence $\{a_n\} = O(1/n^2)$ by $f(x) = \sum_{n \in J} e^{i n k \cdot x} a_n$, $x, k \in \mathbb{R}^n$. This is (p, k) -Bloch-periodic when $J \subset 1 + m\mathbb{Z}$, $m = \frac{2\pi}{k \cdot p}$, e.g., the set of odd numbers for $k \cdot p = \pi$, or $J = \{1, 4, 7, 10, \dots\}$ for $k \cdot p = 2\pi/3$.

3.1. Bloch-periodic Colombeau-type generalized functions

Definition 4. The space of (p, k) -periodic generalized functions $\mathbb{J} \rightarrow X$, in $\mathcal{G}_M(\mathcal{E})$ and for given $p, k \in \mathbb{R}^n$, is

$$\mathcal{G}P_{p,k}\mathcal{E}(\mathbb{J}; X) := \{f \in \mathcal{G}_M(\mathcal{E}(\mathbb{J}; X)) \mid \mathbf{t}_p f = e^{i k \cdot p} f\},$$

the eigenspace, for eigenvalue $e^{i k \cdot p}$, of the translation operator $\mathbf{t}_p : f \mapsto \mathbf{t}_p f$ with $\mathbf{t}_p f(x) = f(x + p)$, acting on $\mathcal{G}_M(\mathcal{E}(\mathbb{J}; X))$. Again, we will omit the subscript M and \mathbb{J} when they are tacitly understood, or not relevant, or $\mathbb{J} = \mathbb{R}^n$.

In the above, we use the translation operator on $\mathcal{G}_M(\mathcal{E}(\mathbb{J}; X))$, which is actually the canonical extension of the (linear and continuous) translation operator acting on the base space $\mathcal{E}(\mathbb{J}; X)$. As mentioned in section 2, due to the functoriality of the construction, this extension is again a well defined, linear and continuous operator on the space of generalized functions.

We prefer the above way of writing $f(x + p) = e^{i k \cdot p} f(x)$, since the use of the linear and continuous translation operator and notion of eigenvalue, both certainly well-defined, avoids the doubts that might arise with more elementary notation, concerning ‘‘point values’’ on one hand, and the exponential function on the other hand.

This way of writing things also makes quite evident that we have the following generalization of results from the classical case:

Proposition 5. *Some purely algebraic properties of Bloch-periodic generalized functions include:*

- (i) $\mathcal{G}P_{p,k}\mathcal{E}(X) \subset \mathcal{G}P_{mp,k}\mathcal{E}(X)$, for any $m \in \mathbb{N}$ (or $m \in \mathbb{Z}$ if $-\mathbb{J} \subset \mathbb{J}$).
- (ii) $\mathcal{G}P_{p,k}\mathcal{E}(X)$ is a \mathbb{C} -vector space and $\tilde{\mathbb{C}}$ -module, where $\tilde{\mathbb{C}} = \mathcal{G}_M(\mathbb{C})$ is the ring of generalized complex numbers.
- (iii) $\mathcal{G}P_{p,k}\mathcal{E}(X)$ is stable under linear operators $A \in \mathcal{L}(X)$ (acting ‘‘pointwise’’ on \mathcal{E} , $Af = x \mapsto Af(x)$, and canonically extending to $\mathcal{G}P_{p,k}\mathcal{E}(X)$).

If X and thus $\mathcal{E}(X)$ are algebras:

- (iv) If $f \in \mathcal{G}P_{p,k}\mathcal{E}(X)$ and $g \in \mathcal{G}P_{p,m}\mathcal{E}(X)$ for some $m \in \mathbb{R}^n$, then the product $fg \in \mathcal{G}P_{p,k+m}\mathcal{E}(X)$.

(v) If $f^{-1} = 1/f$ is well defined, then f^{-1} is $(p, -k)$ -periodic.

Furthermore, if we restrict ourselves, for simplicity, to $\mathbb{J} = \mathbb{R}^n$, we have the following basic properties of $\mathcal{G}P_{p,k}\mathcal{E}(X)$:

Proposition 6. • $\mathcal{G}P_{p,k}\mathcal{E}(X)$ is stable under translations $\mathbf{t}_a : f \mapsto f_a := x \mapsto f(x + a)$, $a \in \mathbb{R}^n$ arbitrary.

- $\mathcal{G}P_{p,k}\mathcal{E}(X)$ is stable under $I_\Delta : f \mapsto I_\Delta f := x \mapsto \int_{x+\Delta} f(y) \, d^n y$, where Δ is a compact set. In the one-dimensional case, we replace Δ by $a \in \mathbb{R}$ and have $I_a f := x \mapsto \int_x^{x+a} f(y) \, dy$.
- $\mathcal{G}P_{p,k}\mathcal{E}(X)$ is stable under convolution by integrable functions g , $f \mapsto f * g := x \mapsto \int f(x - y) g(y) \, d^n y$.

Proof. All of the given operations are continuous linear operators, well defined on the space of generalized functions, according to the standard textbooks [10, 13]. So there is no issue concerning the growth conditions, and the preservation of the Bloch periodicity can be shown using the standard change of variable as in the classical setting, if one wishes, on the level of representatives. \square

3.2. Existence and properties of (p, k) -periodic generalized functions

Example 7. Examples of Bloch periodic *generalized* functions are given for any sequence of generalized numbers $(a_n)_{n \in \mathbb{N}}$ with only a finite number of nonzero terms, by

$$f(x) = \sum_{n \in J} e^{i n k \cdot x} a_n, \quad x, k \in \mathbb{R}^n.$$

As before, this is (p, k) -Bloch-periodic when $J \subset 1 + m\mathbb{Z}$, $m = \frac{2\pi}{k \cdot p}$, e.g., the set of odd numbers for $k \cdot p = \pi$, or $J = \{1, 4, 7, 10, \dots\}$ for $k \cdot p = 2\pi/3$. Other suitable, less restrictive conditions on the coefficients a_n , via the convergence of the series $\sum_{n \in \mathbb{N}} a_n$ in the topological ring of generalized numbers $\tilde{\mathbb{C}}$, could be possible.

Lemma 8. For all $(y, v) \in \mathbb{R}^n \times \mathcal{G}_M(X)$, there is $u \in \mathcal{G}P_{p,k}\mathcal{E}(X)$ such that $u(y) = v$.

Proof. A suitable function $u \in \mathcal{G}P_{p,k}\mathcal{E}(X)$ is explicitly given by $u(x) = e^{i k \cdot (x-y)} v$. \square

3.3. Applications: Heat and wave equations

Example 9. Consider the heat equation $u_t(x, t) = u_{xx}(x, t)$, $(x, t) \in \mathbb{R} \times \mathbb{R}_+$. Solutions with $u(\cdot, 0) = f \in \mathcal{G}\mathcal{E}(\mathbb{R})$ are given by

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4t}} f(s) \, ds.$$

If $f \in \mathcal{GP}_{p,k}\mathcal{E}(\mathbb{R})$ is Bloch-periodic, then $u(x, t)$ is also Bloch-periodic with the same period and Bloch wave vector, in the first variable x , for any fixed $t \in \mathbb{R}_+$.

Example 10. Now consider the wave equation in \mathbb{R}^2 , $u_{tt}(x, t) = u_{xx}(x, t)$. Solutions with $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, are given by

$$u(x, t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} g(s) ds.$$

If $f, g \in \mathcal{GP}_{p,k}\mathcal{E}(X)$ are (p, k) -periodic, then for fixed $t \in \mathbb{R}_+$, $u(\cdot, t)$ is also Bloch-periodic with the same period and wave vector (p, k) .

3.4. Asymptotically Bloch-periodic generalized functions

We denote again by $\tilde{X} = \mathcal{G}_M(X)$ the generalized X -valued constants, *i.e.*, the M -extension of X .

Definition 11. We define the space of generalized functions $\mathbb{J} \rightarrow X$ vanishing at infinity as

$$\mathcal{GE}_0(\mathbb{J}; X) = \{f \in \mathcal{G}(\mathcal{E}(\mathbb{J}; X)) \mid \forall U \in \mathcal{V}(0_{\tilde{X}}) \exists K \Subset \mathbb{J} : f(\mathbb{J} \setminus K) \subset U\}.$$

Here, $\mathcal{V}(0_{\tilde{X}})$ is the set of all neighborhoods of zero in \tilde{X} , and by $f(\mathbb{J} \setminus K)$ we mean the set of all point values $f(x)$, with $x \in \mathbb{J} \setminus K$. With this, we can state the following

Definition 12. The set of asymptotically (p, k) -periodic generalized functions on $\mathbb{J} \subset \mathbb{R}^n$ is $\mathcal{GAP}_{p,k}(\mathcal{E}(\mathbb{J}; X)) := \mathcal{GP}_{p,k}\mathcal{E}(\mathbb{J}; X) + \mathcal{GE}_0(\mathbb{J}; X)$

$$= \{f \in \mathcal{G}(\mathcal{E}(\mathbb{J}; X)) \mid \exists (g, h) \in \mathcal{GP}_{p,k}\mathcal{E}(\mathbb{J}; X) \times \mathcal{GE}_0(\mathbb{J}; X) : f = g + h\}.$$

The generalized functions g and h are called, respectively, the *principal* and the *corrective* part of f .

Analog to similar results for asymptotically periodic (classical) functions, we have the

Proposition 13 (Uniqueness of decomposition). *The decomposition of $f \in \mathcal{GAP}_{p,k}\mathcal{E}(\mathbb{J}, X)$, as sum of $g \in \mathcal{GP}_{p,k}\mathcal{E}(\mathbb{J}, X)$ and $h \in \mathcal{GE}_0(\mathbb{J}; X)$, is unique.*

For this proposition to hold, it is obviously crucial that \mathbb{J} satisfies the characteristic property $\mathbb{J} + p \subset \mathbb{J}$ mentioned when this set was introduced in the definition of Bloch periodicity.

Proof. Assume that $g_1 + h_1 = g_2 + h_2$ are two such decompositions of $f \in \mathcal{GAP}_{p,k}(\mathcal{E}(\mathbb{J}, X))$. Then $g_1 - g_2 = h_2 - h_1 \in \mathcal{GE}_0(\mathbb{J}; X)$ vanishes at infinity, *i.e.*, the difference $g_1 - g_2$ must have values in any arbitrarily small neighborhood of zero of generalized numbers, outside some compact \mathbb{K} . But using the Bloch periodicity of $g_1 - g_2 \in \mathcal{GP}_{p,k}(\mathcal{E}(\mathbb{J}, X))$, the difference $g_1 - g_2$ must have this property on any compact $\mathbb{K}' \subset \mathbb{J}$, since translation by a multiple of p will always take us outside \mathbb{K} . Therefore, $g_1 - g_2 = 0$, which yields the result. \square

4. Composition and classical results

We first recall some results from the classical setting [15] which will, due to their purely algebraic nature, straightforwardly extend to the generalized case.

Theorem 14. *Let $F : X \rightarrow X$. The following properties are equivalent:*

- (i) *For every $u \in P_{p,k}(X)$, $F \circ u \in P_{p,k}(X)$.*
- (ii) *For all $x \in X$, $F(e^{i k \cdot p} x) = e^{i k \cdot p} F(x)$.*

Remark 15. When $k \cdot p = \pi$, this means that F is an odd function. Note that every linear map F satisfies both of these properties.

For given $F : \mathbb{R}^n \times X \rightarrow X$, we define Nemytskii's superposition operator by

$$\mathcal{N}_F(\varphi)(\cdot) := F(\cdot, \varphi(\cdot)), \quad \varphi \in P_{p,k}(X).$$

Then we have

Theorem 16. *Let $F : \mathbb{R}^n \times X \rightarrow X$. The following properties are equivalent:*

- (i) *For every $u \in P_{p,k}(X)$, $\mathcal{N}_F(u) \in P_{p,k}(X)$;*
- (ii) *$\forall (t, x) \in \mathbb{R}^n \times X$, $F(t + p, e^{i k \cdot p} x) = e^{i k \cdot p} F(t, x)$.*

Corollary 17. *Let $F : \mathbb{R}^n \times X \rightarrow X$ satisfy one of the equivalent properties of the preceding Theorem, with $k \cdot p$ a multiple of $\frac{2\pi}{m}$, $m \in \mathbb{N}^*$. Then F is mp -periodic w.r.t. the first variable.*

Corollary 18. *Consider $k \cdot p = \pi$ and $F : \mathbb{R}^n \times X \rightarrow X$ satisfying either of*

- (i) *F is (p, k) -Bloch-periodic with respect to the first variable and even w.r.t. the second;*
- (ii) *F is p -periodic with respect to the first variable and odd with respect to the second.*

Then, for every function $u \in P_{p,k}(X)$, $\mathcal{N}_F(u) \in P_{p,k}(X)$.

Proposition 19. *If $f \in C^1(\mathbb{R}^n, X)$ is (p, k) -Bloch-periodic, then $f' \in P_{p,k}(L(\mathbb{R}^n, X))$.*

Proof. The proof is immediate, using that f' is defined to be the unique map such that $f(x + h) - f(x) = f'(x)h + o(h)$. From this it is obvious that Bloch periodicity of f entails that of f' . \square

Corollary 20. *Let $F \in C^1(\mathbb{R}^n \times X, X)$ satisfy the equivalent properties (i) and (ii) of Theorem 16. Then for every function $u \in C^1(\mathbb{R}^n, X)$ which is (p, k) -Bloch-periodic, the function $\Phi : \mathbb{R}^n \rightarrow X$ defined by $\Phi(\cdot) = F(\cdot, u(\cdot))$ (which is (p, k) -Bloch-periodic) is differentiable, and Φ' is (p, k) -Bloch-periodic.*

Now we give a last result which we have proved in the classical case [15]:

Corollary 21. *If $F \in C^1(\mathbb{R}^n \times X, X)$ satisfies the equivalent properties (i) and (ii) of Theorem 16, so does the partial derivative (with respect to the first variable $t \in \mathbb{R}^n$) D_1F of F . Moreover the partial derivative (with respect to the second variable $x \in X$) D_2F of F verifies:*

$$\forall (t, x) \in \mathbb{R}^n \times X, \quad D_2F(t + \omega, e^{i k \cdot p} x) = D_2F(t, x).$$

Since Colombeau generalized functions are infinitely differentiable (as well as a whole, as in each component), these results also hold in the generalized case, *i.e.*, for Bloch periodic generalized functions $u \in \mathcal{G}P_{p,k}\mathcal{E}(X)$. In forthcoming developments, we will use these rather elementary results in the study of more elaborate integro-differential problems, where Bloch periodic generalized functions as initial data yield solutions with the same symmetry.

5. Summary and outlook

We have defined the notion of Bloch-periodic and asymptotically Bloch-periodic generalized functions, and provided some very basic examples for illustration. But this work clearly calls for further investigations. First, we would like to work out in detail more classical results that extend to the generalized case, and find the most appropriate sheaves of functions relevant for the given applications. In particular, we would like to as far as possible in generalizing our recent results concerning Bloch-periodic solutions to the integro-differential equation

$$u'(t) = Au(t) + \int_{-\infty}^t a(t-s) Au(s) ds + f(t, Cu(t)).$$

To that end, we plan to extend the systematic study and use of convolution operators in this setting.

We also intend to extend our previous results from Colombeau type algebras of generalized functions, for example the study of values in generalized points, to the case of Bloch periodic generalized functions. Finally, it would also be interesting to consider Bloch periodicity with generalized parameters (p, k) , *i.e.*, consider the possibility of one or both of these to be vectors whose components are generalized numbers.

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