

## HYBRID COUPLED FIXED POINT THEOREMS FOR MAPS UNDER (CLR<sub>g</sub>) PROPERTY IN FUZZY METRIC SPACES

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### Abstract

In this paper, we introduce (CLR<sub>g</sub>) property for a hybrid pair of maps in fuzzy metric spaces and utilize the same to prove two unique common coupled fixed point theorems for two hybrid pairs of maps satisfying  $\psi + \phi$  contractive condition in fuzzy metric spaces.

*AMS Mathematics Subject Classification* (2010): 54H25, 47H10.

*Key words and phrases*: Hybrid pair; (CLR<sub>g</sub>) property;  $w$ -compatible maps;coupled fixed points.

## 1 Introduction

The concept of fuzzy sets was initiated by Zadeh [28] in 1965 which has inspired the fuzzification of almost all existing Mathematics. With similar quest, George and Veeramani [9] and Kramosil and Michalek [14] have introduced the concept of fuzzy topological spaces induced by fuzzy metrics which was required to be slightly manipulated to become Hausdorff. Thereafter, many authors proved fixed and common fixed point theorems in fuzzy metric spaces (e. g.[7, 8, 10, 11, 15, 17, 19, 22, 23, 26, 27]).

Now, we present the required preliminaries.

**Definition 1.1.** ([20]). A binary operation  $*$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is a continuous  $t$ -norm if it satisfies the following conditions:

1.  $*$  is associative and commutative,
2.  $*$  is continuous,
3.  $a * 1 = a$  for all  $a \in [0, 1]$ ,
4.  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$ , for each  $a, b, c, d \in [0, 1]$ .

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Two natural examples of a continuous  $t$ -norm are  $a * b = ab$  and  $a * b = \min\{a, b\}$ .

**Definition 1.2.** ([9]). A 3-tuple  $(X, M, *)$  is called a *fuzzy metric space* if  $X$  is an arbitrary (non-empty) set,  $*$  is a continuous  $t$ -norm and  $M$  is a fuzzy set on  $X^2 \times (0, \infty)$  satisfying the following conditions (for each  $x, y, z \in X$  and  $t, s > 0$ ):

1.  $M(x, y, t) > 0$ ,
2.  $M(x, y, t) = 1$  if and only if  $x = y$ ,
3.  $M(x, y, t) = M(y, x, t)$ ,
4.  $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ ,
5.  $M(x, y, \cdot) : (0, \infty) \longrightarrow [0, 1]$  is continuous.

Let  $(X, M, *)$  be a fuzzy metric space. For  $t > 0$ , the *open ball*  $B(x, r, t)$  with center  $x \in X$  and radius  $0 < r < 1$  is defined by  $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ . Let  $(X, M, *)$  be a fuzzy metric space and  $\tau$  the collection of all subsets  $A \subset X$  with  $x \in A$  if and only if there exist  $t > 0$  and  $0 < r < 1$  such that  $B(x, r, t) \subset A$ . Then  $\tau$  forms a topology on  $X$  induced by the fuzzy metric  $M$ . This topology is Hausdorff as well as first countable.

A sequence  $\{x_n\}$  in  $X$  converges to  $x$  if and only if  $M(x_n, x, t) \rightarrow 1$  as  $n \rightarrow \infty$ , for each  $t > 0$  and the same (sequence) is called a Cauchy sequence in the sense of [10] if  $\lim_{n \rightarrow \infty} M(x_n, x_{n+p}, t_n) = 1$ , for all  $t > 0$  and each positive integer  $p$ . The fuzzy metric space  $(X, M, *)$  is said to be *complete* if every Cauchy sequence in it is convergent. A subset  $A$  of  $X$  is said to be  $F$ -bounded if there exist  $t > 0$  and  $0 < r < 1$  such that  $M(x, y, t) > 1 - r$  for all  $x, y \in A$ .

**Example 1.3.** Let  $X = (-\infty, \infty)$ . Put  $a * b = ab$  for all  $a, b \in [0, 1]$ . For each  $t \in (0, \infty)$ , define  $M(x, y, t) = \frac{t}{t + |x - y|}$  for all  $x, y \in X$ .

**Example 1.4.** Let  $X = [0, 1]$  and  $a * b = ab$  for all  $a, b \in [0, 1]$  and let  $M$  be the fuzzy set on  $X \times X \times (0, \infty)$  defined by

$$M(x, y, t) = e^{-\frac{|x-y|}{t}}$$

for all  $t \geq 0$ . Then  $(X, M, *)$  is a fuzzy metric space.

**Example 1.5.** Let  $X = [0, 1]$  and  $a * b = ab$  for all  $a, b \in [0, 1]$  and let  $M$  be the fuzzy set on  $X \times X \times (0, \infty)$  defined by

$$M(x, y, t) = \left( \frac{t}{t+1} \right)^{|x-y|}$$

for all  $t \geq 0$ . Then  $(X, M, *)$  is a fuzzy metric space.

**Lemma 1.6.** ([10]) Let  $(X, M, *)$  be a fuzzy metric space. Then  $M(x, y, t)$  is non-decreasing with respect to  $t$ , for all  $x, y$  in  $X$ .

**Definition 1.7.** Let  $(X, M, *)$  be a fuzzy metric space. Then  $M$  is said to be continuous on  $X^2 \times (0, \infty)$  if  $\lim_{n \rightarrow \infty} M(x_n, y_n, t_n) = M(x, y, t)$ , whenever a sequence  $\{(x_n, y_n, t_n)\}$  in  $X^2 \times (0, \infty)$  converges to a point  $(x, y, t) \in X^2 \times (0, \infty)$ . i.e.  $\lim_{n \rightarrow \infty} M(x_n, x, t) = \lim_{n \rightarrow \infty} M(y_n, y, t) = 1$  and  $\lim_{n \rightarrow \infty} M(x, y, t_n) = M(x, y, t)$ .

**Lemma 1.8.** ([18]). Let  $(X, M, *)$  be a fuzzy metric space. Then  $M$  is a continuous function on  $X^2 \times (0, \infty)$ .

Recently, Aamri and Moutawakil [1] introduced the concept of the property (E.A.) and proved common fixed point theorems under strict contractive condition. Thereafter, Sintunavarat and Kumam [24] introduced the new notion namely: Common Limit Range property (in short CLRg). For some more references of this kind, one can be referred to [5, 6, 13, 21]. Very recently, Khan and Sumitra [3] extended (CLRg)property for coupled maps (also see [25]) as follows

**Definition 1.9.** Let  $(X, M, *)$  be a fuzzy metric space. Two maps  $F : X \times X \rightarrow X$  and  $f : X \rightarrow X$  are said to satisfy (CLRg)property if there exist sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} F(x_n, y_n) = \lim_{n \rightarrow \infty} f x_n = f(p)$  and  $\lim_{n \rightarrow \infty} F(y_n, x_n) = \lim_{n \rightarrow \infty} f y_n = f(q)$  for some  $p, q \in X$ .

Bhaskar and Lakshmikantham [4] introduced the concept of coupled fixed points. On the analogous lines Lakshmikantham and Ćirić [16] defined the common coupled fixed points. Later Xin-Qi Hu [12] defined the common fixed points for maps  $F : X \times X \rightarrow X$  and  $f : X \rightarrow X$ . Abbas et al.[2] introduced the  $w$ -compatible pair of maps.

**Definition 1.10.** Let  $F : X \times X \rightarrow X$  and  $f : X \rightarrow X$ .

- (i)([4]). An element  $(x, y) \in X \times X$  is called a coupled fixed point of  $F$  if  $F(x, y) = x$  and  $F(y, x) = y$ .
- (ii)([16]). An element  $(x, y) \in X \times X$  is called a common coupled fixed point of  $F$  and  $f$  if  $F(x, y) = f x = x$  and  $F(y, x) = f y = y$ .
- (iii)([12]). A point  $x \in X$  is called a common fixed point of  $F$  and  $f$  if  $F(x, x) = x = f x$ .
- (iv)([2]).  $F$  and  $f$  are said to be  $w$ -compatible if  $f(F(x, y)) = F(fx, fy)$  and  $f(F(y, x)) = F(fy, fx)$  whenever  $fx = F(x, y)$  and  $fy = F(y, x)$  for all  $x, y \in X$ .

From now on,  $CB(X)$  denotes the set of all non-empty closed and bounded subsets of  $X$ . For  $A, B \in CB(X)$  and for every  $t > 0$ , we write

$$\delta_M(A, B, t) = \inf\{M(a, b, t) : a \in A, b \in B\}.$$

Draft version November 26, 2015

If  $A$  consists of a single point  $a$ , we write  $\delta_M(A, B, t) = \delta_M(a, B, t)$ . If  $B$  also consists of a single point  $b$ , we write  $\delta_M(A, B, t) = \delta_M(a, b, t) = M(a, b, t)$ .

It follows immediately from the definition that

$$\delta_M(A, B, t) = \delta_M(B, A, t) \geq 0,$$

$$\delta_M(A, B, t) = 1 \iff A = B = \{a \text{ singleton}\},$$

for all  $A, B \in CB(X)$ .

**Definition 1.11.** A sequence  $\{A_n\}$  in  $CB(X)$  is said to be convergent to a set  $A \in CB(X)$  if  $\lim_{n \rightarrow \infty} \delta_M(A_n, A, t) = 1$  for all  $t > 0$ .

Also, one can prove the following:

**Lemma 1.12.** Let  $\{A_n\}$  and  $\{B_n\}$  be sequences in  $CB(X)$  converging to  $A$  and  $B$  in  $CB(X)$  respectively. Then  $\lim_{n \rightarrow \infty} \delta_M(A_n, B_n, t) = \delta_M(A, B, t)$  for all  $t > 0$ .

In this paper, we give a new definition and utilize the same to prove two common fixed point theorems for two hybrid pairs of maps in the next section.

## 2 Main results

Firstly, we give the following definition.

**Definition 2.1.** Let  $(X, M, *)$  be a fuzzy metric space. The hybrid pair of mappings  $F : X \times X \rightarrow CB(X)$  and  $S : X \rightarrow X$  is said to have Common Limit Range property ( in short CLRg) with respect to  $S$  if there exist sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} M(Sx_n, Sa, t) = 1, \quad \lim_{n \rightarrow \infty} \delta_M(F(x_n, y_n), A, t) = 1,$$

$$\lim_{n \rightarrow \infty} M(Sy_n, Sb, t) = 1, \quad \lim_{n \rightarrow \infty} \delta_M(F(y_n, x_n), B, t) = 1,$$

for some  $a, b \in X$ ,  $Sa \in A \in CB(X)$  and  $Sb \in B \in CB(X)$ .

Let  $\Psi$  be the class of monotonically increasing continuous functions  $\psi : [0, 1] \rightarrow [0, 1]$  and  $\Phi$  the class of monotonically increasing continuous functions  $\phi : [0, 1] \rightarrow [0, 1]$  such that  $\phi(t) > t$  for  $0 < t < 1$ .

In what follows,  $(X, M, *)$  stands for a fuzzy metric space,  $F, G : X \times X \rightarrow CB(X)$  and  $S, T : X \rightarrow X$  besides

$$m_{u, v}^{x, y} = \min \left\{ \begin{array}{l} M(Sx, Tu, t), M(Sy, Tv, t), \delta_M(Sx, F(x, y), t) \\ \delta_M(Sy, F(y, x), t), \delta_M(Tu, G(u, v), t), \delta_M(Tv, G(v, u), t) \\ \delta_M(Sx, G(u, v), t), \delta_M(Sy, G(v, u), t), \\ \delta_M(Tu, F(x, y), t), \delta_M(Tv, F(y, x), t) \end{array} \right\}.$$

Now, we are equipped to prove our main result as follows.

**Theorem 2.2.** *Let  $(X, M, *)$  be a fuzzy metric space. If  $F, G : X \times X \rightarrow CB(X)$  and  $S, T : X \rightarrow X$  are maps which satisfy the following conditions:*

- (2.2.1) *the pairs  $(F, S)$  and  $(G, T)$  satisfy (CLRg) property with respect to  $S$  and  $T$  respectively,*
- (2.2.2) *the pairs  $(F, S)$  and  $(G, T)$  are  $w$ -compatible,*
- (2.2.3)  $\psi(\delta_M(F(x, y), G(u, v), t)) \geq \psi(m_{u,v}^{x,y}) + \phi(m_{u,v}^{x,y})$   
*for all  $x, y, u, v \in X, t > 0$ , where  $\psi \in \Psi, \phi \in \Phi$ .*

*Then there exists a unique  $x \in X$  such that  $F(x, x) = \{Sx\} = \{x\} = \{Tx\} = G(x, x)$ .*

*Proof.* . Since the pairs  $(F, S)$  and  $(G, T)$  satisfy the (CLRg) property with respect to  $S$  and  $T$  respectively, therefore there exist sequences  $\{x_n\}, \{y_n\}, \{u_n\}$  and  $\{v_n\}$  in  $X$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} M(Sx_n, Sa, t) &= 1, & \lim_{n \rightarrow \infty} \delta_M(F(x_n, y_n), A, t) &= 1, \\ \lim_{n \rightarrow \infty} M(Sy_n, Sb, t) &= 1, & \lim_{n \rightarrow \infty} \delta_M(F(y_n, x_n), B, t) &= 1, \\ \lim_{n \rightarrow \infty} M(Tu_n, Ta', t) &= 1, & \lim_{n \rightarrow \infty} \delta_M(G(u_n, v_n), P, t) &= 1, \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} M(Tv_n, Tb', t) = 1, \quad \lim_{n \rightarrow \infty} \delta_M(G(v_n, u_n), Q, t) = 1,$$

for some  $a, b, a', b' \in X$  and  $Sa \in A \in CB(X), Sb \in B \in CB(X), Ta' \in P \in CB(X), Tb' \in Q \in CB(X)$ .  
 Suppose  $0 < \min \{\delta_M(A, P, t), \delta_M(B, Q, t)\} < 1$  for some  $t > 0$ . Consider,

$$(2.1) \quad \psi(\delta_M(F(x_n, y_n), G(u_n, v_n), t)) \geq \psi(m_{u_n, v_n}^{x_n, y_n}) + \phi(m_{u_n, v_n}^{x_n, y_n})$$

wherein

$$m_{u_n, v_n}^{x_n, y_n} = \min \left\{ \begin{array}{l} M(Sx_n, Tu_n, t), M(Sy_n, Tv_n, t), \delta_M(Sx_n, F(x_n, y_n), t) \\ \delta_M(Sy_n, F(y_n, x_n), t), \delta_M(Tu_n, G(u_n, v_n), t), \delta_M(Tv_n, G(v_n, u_n), t) \\ \delta_M(Sx_n, G(u_n, v_n), t), \delta_M(Sy_n, G(v_n, u_n), t), \\ \delta_M(Tu_n, F(x_n, y_n), t), \delta_M(Tv_n, F(y_n, x_n), t) \end{array} \right\} \text{ and}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} m_{u_n, v_n}^{x_n, y_n} &= \min \left\{ \begin{array}{l} M(Sa, Ta', t), M(Sb, Tb', t), 1, 1, 1, 1 \\ \delta_M(Sa, P, t), \delta_M(Sb, Q, t), \delta_M(Ta', A, t), \delta_M(Tb', B, t) \end{array} \right\} \\ &\geq \min \{\delta_M(A, P, t), \delta_M(B, Q, t)\}. \end{aligned}$$

On letting  $n \rightarrow \infty$  in (2.1), we get

$$\psi(\delta_M(A, P, t)) \geq \psi(\min \{\delta_M(A, P, t), \delta_M(B, Q, t)\}) + \phi(\min \{\delta_M(A, P, t), \delta_M(B, Q, t)\}).$$

Similarly, we can also show that

$$\psi(\delta_M(B, Q, t)) \geq \psi(\min \{\delta_M(A, P, t), \delta_M(B, Q, t)\}) + \phi(\min \{\delta_M(A, P, t), \delta_M(B, Q, t)\}).$$

Thus, in all we have

$$\psi \left( \min \left\{ \begin{array}{l} \delta_M(A, P, t), \\ \delta_M(B, Q, t) \end{array} \right\} \right) \geq \psi \left( \min \left\{ \begin{array}{l} \delta_M(A, P, t), \\ \delta_M(B, Q, t) \end{array} \right\} \right) + \phi \left( \min \left\{ \begin{array}{l} \delta_M(A, P, t), \\ \delta_M(B, Q, t) \end{array} \right\} \right)$$

which in turn yields that

$$0 \geq \phi(\min \{\delta_M(A, P, t), \delta_M(B, Q, t)\}) > \min \{\delta_M(A, P, t), \delta_M(B, Q, t)\},$$

a contradiction. Hence for all  $t > 0$ , we have

$$\min \{\delta_M(A, P, t), \delta_M(B, Q, t)\} = 1$$

so that  $A = P = \{a \text{ singleton}\}$  and  $B = Q = \{a \text{ singleton}\}$ .

Since  $Sa \in A$  and  $Sb \in B$  we have

$$(2.2) \quad \begin{aligned} A = P &= \{Sa\} = \{Ta'\} \\ B = Q &= \{Sb\} = \{Tb'\} \end{aligned}$$

Now suppose that  $0 < M(Sa, Sb, t) < 1$  for some  $t > 0$ .

Consider

$$(2.3) \quad \psi(\delta_M(F(y_n, x_n), G(u_n, v_n), t)) \geq \psi(m_{u_n, v_n}^{y_n, x_n}) + \phi(m_{u_n, v_n}^{y_n, x_n})$$

$$m_{u_n, v_n}^{y_n, x_n} = \min \left\{ \begin{array}{l} M(Sy_n, Tu_n, t), M(Sx_n, Tv_n, t), \delta_M(Sy_n, F(y_n, x_n), t) \\ \delta_M(Sx_n, F(x_n, y_n), t), \delta_M(Tu_n, G(u_n, v_n), t), \delta_M(Tv_n, G(v_n, u_n), t) \\ \delta_M(Sy_n, G(u_n, v_n), t), \delta_M(Sx_n, G(v_n, u_n), t), \\ \delta_M(Tu_n, F(y_n, x_n), t), \delta_M(Tv_n, F(x_n, y_n), t) \end{array} \right.$$

$$\begin{aligned} \lim_{n \rightarrow \infty} m_{u_n, v_n}^{y_n, x_n} &= \min \left\{ \begin{array}{l} M(Sb, Ta', t), M(Sa, Tb', t), 1, 1, 1, 1 \\ \delta_M(Sb, P, t), \delta_M(Sa, Q, t), \delta_M(Ta', A, t), \delta_M(Tb', B, t) \end{array} \right\} \\ &= M(Sb, Sa, t), \text{ from (2.2)} \end{aligned}$$

Letting  $n \rightarrow \infty$  in (2.3), we get

$$\begin{aligned} \psi(M(Sb, Sa, t)) &\geq \psi(M(Sb, Sa, t)) + \phi(M(Sb, Sa, t)) \\ 0 &\geq \phi(M(Sb, Sa, t)) > M(Sb, Sa, t). \end{aligned}$$

It is a contradiction. Hence  $M(Sa, Sb, t) = 1$  for all  $t > 0$  so that  $Sa = Sb$ .

Thus

$$(2.4) \quad Ta' = Sa = Sb = Tb'$$

Suppose that  $0 < \min \{\delta_M(F(a, b), Sa, t), \delta_M(F(b, a), Sb, t)\} < 1$  for some  $t > 0$ .

Consider,

$$(2.5) \quad \psi(\delta_M(F(a, b), G(u_n, v_n), t)) \geq \psi(m_{u_n, v_n}^{a, b}) + \phi(m_{u_n, v_n}^{a, b})$$

wherein

$$m_{u_n, v_n}^{a, b} = \min \left\{ \begin{array}{l} M(Sa, Tu_n, t), M(Sb, Tv_n, t), \delta_M(Sa, F(a, b), t) \\ \delta_M(Sb, F(b, a), t), \delta_M(Tu_n, G(u_n, v_n), t), \delta_M(Tv_n, G(v_n, u_n), t) \\ \delta_M(Sa, G(u_n, v_n), t), \delta_M(Sb, G(v_n, u_n), t), \\ \delta_M(Tu_n, F(a, b), t), \delta_M(Tv_n, F(b, a), t) \end{array} \right\}.$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} m_{u_n, v_n}^{a, b} &= \min \left\{ \begin{array}{l} M(Sa, Ta', t), M(Sb, Tb', t), \delta_M(Sa, F(a, b), t) \\ \delta_M(Sb, F(b, a), t), \delta_M(Ta', P, t), \delta_M(Tb', Q, t) \\ \delta_M(Sa, P, t), \delta_M(Sb, Q, t), \\ \delta_M(Ta', F(a, b), t), \delta_M(Tb', F(b, a), t) \end{array} \right\} \\ &\geq \min \left\{ \begin{array}{l} 1, 1, \delta_M(Sa, F(a, b), t), \delta_M(Sb, F(b, a), t), 1, 1, 1, 1 \\ \delta_M(Sa, F(a, b), t), \delta_M(Sb, F(b, a), t) \end{array} \right\} \\ &= \min \{ \delta_M(Sa, F(a, b), t), \delta_M(Sb, F(b, a), t) \} \end{aligned}$$

Letting  $n \rightarrow \infty$  in (2.5), we get

$$\psi(\delta_M(Sa, F(a, b), t)) \geq \psi \left( \min \left\{ \begin{array}{l} \delta_M(Sa, F(a, b), t), \\ \delta_M(Sb, F(b, a), t) \end{array} \right\} \right) + \phi \left( \min \left\{ \begin{array}{l} \delta_M(Sa, F(a, b), t), \\ \delta_M(Sb, F(b, a), t) \end{array} \right\} \right)$$

Similarly we can show that

$$\psi(\delta_M(Sb, F(b, a), t)) \geq \psi \left( \min \left\{ \begin{array}{l} \delta_M(Sa, F(a, b), t), \\ \delta_M(Sb, F(b, a), t) \end{array} \right\} \right) + \phi \left( \min \left\{ \begin{array}{l} \delta_M(Sa, F(a, b), t), \\ \delta_M(Sb, F(b, a), t) \end{array} \right\} \right)$$

Thus, we have

$$\begin{aligned} \psi \left( \min \left\{ \begin{array}{l} \delta_M(Sa, F(a, b), t), \\ \delta_M(Sb, F(b, a), t) \end{array} \right\} \right) &\geq \psi \left( \min \left\{ \begin{array}{l} \delta_M(Sa, F(a, b), t), \\ \delta_M(Sb, F(b, a), t) \end{array} \right\} \right) \\ &\quad + \phi \left( \min \left\{ \begin{array}{l} \delta_M(Sa, F(a, b), t), \\ \delta_M(Sb, F(b, a), t) \end{array} \right\} \right). \end{aligned}$$

which in turn yields that

$$0 \geq \phi \left( \min \left\{ \begin{array}{l} \delta_M(Sa, F(a, b), t), \\ \delta_M(Sb, F(b, a), t) \end{array} \right\} \right) > \min \left\{ \begin{array}{l} \delta_M(Sa, F(a, b), t), \\ \delta_M(Sb, F(b, a), t) \end{array} \right\},$$

a contradiction. Hence for every  $t > 0$ , we have

$$\min \{ \delta_M(Sa, F(a, b), t), \delta_M(Sb, F(b, a), t) \} = 1$$

so that

$$(2.6) \quad F(a, b) = \{Sa\} \quad \text{and} \quad F(b, a) = \{Sb\}.$$

Similarly by taking  $x = x_n, y = y_n, u = a', v = b'$  and  $x = y_n, y = x_n, u = b', v = a'$  in (2.2.3) and letting  $n \rightarrow \infty$ , we can show that

$$(2.7) \quad G(a', b') = \{Ta'\} \quad \text{and} \quad G(b', a') = \{Tb'\}.$$

Let  $x = Sa$ . Then from (2.4),  $Sa = Sb = Ta' = Tb' = x$ .

Since  $(F, S)$  and  $(G, T)$  are  $w$ -compatible, from (2.6),(2.7), it follows that

$$(2.8) \quad Sx = SSa = SF(a, b) = F(Sa, Sb) = F(x, x).$$

and

$$(2.9) \quad Tx = TTa = TG(a', b') = G(Ta', Tb') = G(x, x).$$

Suppose  $0 < M(Sx, x, t) < 1$  for some  $t > 0$ .

Consider

$$\begin{aligned} \psi(M(Sx, x, t)) &= \psi(M(F(x, x), G(a', b'), t)) \text{ from (2.7) and (2.8),} \\ &= \psi(\delta_M(F(x, x), G(a', b'), t)), \text{ since } F(x, x) = \{Sx\} \text{ and} \\ &\hspace{15em} G(a', b') = \{Ta'\} = \{x\} \\ &= \psi\left(m_{a', b'}^{x, x}\right) + \phi\left(m_{a', b'}^{x, x}\right) \end{aligned}$$

$$\begin{aligned} m_{a', b'}^{x, x} &= \min \left\{ \begin{array}{l} M(Sx, Ta', t), M(Sx, Tb', t), \delta_M(Sx, F(x, x), t) \\ \delta_M(Sx, F(x, x), t), \delta_M(Ta', G(a', b'), t), \delta_M(Tb', G(b', a'), t) \\ \delta_M(Sx, G(a', b'), t), \delta_M(Sx, G(b', a'), t), \\ \delta_M(Ta', F(x, x), t), \delta_M(Tb', F(x, x), t) \end{array} \right\} \\ &= \min \left\{ \begin{array}{l} M(Sx, x, t), M(Sx, x, t), 1, 1, 1, 1, M(Sx, x, t) \\ M(Sx, x, t), M(x, Sx, t), M(x, Sx, t) \end{array} \right\} \\ &= M(Sx, x, t). \end{aligned}$$

Thus

$$\begin{aligned} \psi(M(Sx, x, t)) &\geq \psi(M(Sx, x, t)) + \phi(M(Sx, x, t)). \\ 0 &\geq \phi(M(Sx, x, t)) > M(Sx, x, t), \end{aligned}$$

a contradiction. Hence  $M(Sx, x, t) = 1$  for every  $t > 0$  so that  $Sx = x$ . Similarly we can show that  $Tx = x$ . Thus from (2.8) and (2.9), we have

$$F(x, x) = \{Sx\} = \{x\} = \{Tx\} = G(x, x).$$

Hence  $(x, x)$  is a common coupled fixed point of  $F, G, S$  and  $T$ .

Uniqueness of  $x$  follows easily from (2.2.3).  $\square$

One can prove the following in the similar lines of Theorem 2.2.

**Theorem 2.3.** *Let  $(X, M, *)$  be a fuzzy metric space. If  $F, G : X \times X \rightarrow CB(X)$  and  $S, T : X \rightarrow X$  are maps which satisfy the following conditions:*

(2.3.1) *the pairs  $(F, S)$  and  $(G, T)$  satisfy (CLRg) property with respect to  $S$  and  $T$  respectively,*

(2.3.2) *the pairs  $(F, S)$  and  $(G, T)$  are  $w$ -compatible,*

(2.3.3)  $\delta_M(F(x, y), G(u, v), kt) \geq m_{u, v}^{x, y}$  for all  $x, y, u, v \in X$ ,  
 $t > 0$ , where  $k \in (0, 1)$  and  $\lim_{t \rightarrow \infty} M(x, y, t) = 1$  for all  $x, y \in X$ .

*Then there exists a unique  $x \in X$  such that  $F(x, x) = \{Sx\} = \{x\} = \{Tx\} = G(x, x)$ .*

**Theorem 2.4.** *Let  $(X, M, *)$  be a fuzzy metric space,  $F, G : X \times X \rightarrow X$  and  $S, T : X \rightarrow X$  be mappings satisfying*



(2.4.1) the pairs  $(F, S)$  and  $(G, T)$  satisfy (CLRg) property with respect to  $S$  and  $T$  respectively,

(2.4.2) the pairs  $(F, S)$  and  $(G, T)$  are  $w$ -compatible,

(2.4.3)  $M(F(x, y), G(u, v), kt) \geq m_{u, v}^{x, y}$   
for all  $x, y, u, v \in X, t > 0$ , where  $k \in (0, 1)$  and

$$m_{u, v}^{x, y} = \min \left\{ \begin{array}{l} M(Sx, Tu, t), M(Sy, Tv, t), M(Sx, F(x, y), t) \\ M(Sy, F(y, x), t), M(Tu, G(u, v), t), M(Tv, G(v, u), t) \\ M(Sx, G(u, v), t), M(Sy, G(v, u), t), \\ M(Tu, F(x, y), t), M(Tv, F(y, x), t) \end{array} \right\},$$

(2.4.4)  $\lim_{t \rightarrow \infty} M(x, y, t) = 1$  for all  $x, y \in X$ .

Then there exists  $x \in X$  such that  $F(x, x) = Sx = x = Tx = G(x, x)$ .

Now, we give two examples to illustrate Theorem 2.4.

**Example 2.5.** Let  $X = [0, 1]$  and  $a * b = ab$  for all  $a, b \in [0, 1]$  and let  $M$  be the fuzzy set on  $X \times X \times (0, \infty)$  defined by

$$M(x, y, t) = e^{-\frac{|x-y|}{t}}$$

for all  $t \geq 0$ . Then  $(X, M, *)$  is a fuzzy metric space.

Define  $F, G : X \times X \rightarrow X$  and  $S, T : X \rightarrow X$  by  $F(x, y) = \frac{x+y}{8}, G(x, y) =$

$\frac{x+y}{16}, Sx = \frac{x}{2}$  and  $Tx = \frac{x}{4}$ . Then

$$|\frac{x+y}{8} - \frac{u+v}{16}| = \frac{1}{16}|2x - u + 2y - v| \leq \frac{1}{2} \max\{\frac{|2x-u|}{4}, \frac{|2y-v|}{4}\}. \quad \text{Now,}$$

$$\begin{aligned} M(F(x, y), G(u, v), \frac{1}{2}t) &= e^{-\frac{|\frac{x+y}{8} - \frac{u+v}{16}|}{\frac{1}{2}t}} \\ &\geq e^{-\frac{\frac{1}{2} \max\{\frac{|2x-u|}{4}, \frac{|2y-v|}{4}\}}{\frac{1}{2}t}} \\ &= e^{-\frac{\max\{\frac{|2x-u|}{4}, \frac{|2y-v|}{4}\}}{t}} \\ &\geq \min\left\{e^{-\frac{|2x-u|}{4t}}, e^{-\frac{|2y-v|}{4t}}\right\} \\ &= \min\{M(Sx, Tu, t), M(Sy, Tv, t)\} \\ &\geq m_{u, v}^{x, y}. \end{aligned}$$

Also  $(F, S)$  and  $(G, T)$  satisfy (CLRg) property with respect to  $S$  and  $T$  respectively with sequences  $\{x_n\} = \{\frac{1}{\sqrt{n}}\}, \{y_n\} = \{\frac{1}{n}\}, \{u_n\} = \{\frac{1}{n^2}\}$  and  $\{v_n\} = \{\frac{1}{n}\}$  respectively. Clearly, the pairs  $(F, S)$  and  $(G, T)$  are  $w$ -compatible. Clearly  $(0, 0)$  is the unique common fixed point of  $F, G, S$  and  $T$ .

**Example 2.6.** Let  $X = [0, 1]$  and  $a * b = ab$  for all  $a, b \in [0, 1]$  and let  $M$  be the fuzzy set on  $X \times X \times (0, \infty)$  defined by

$$M(x, y, t) = \left(\frac{t}{t+1}\right)^{|x-y|}$$

for all  $t \geq 0$ . Then  $(X, M, *)$  is a fuzzy metric space.

Define  $F, G : X \times X \rightarrow X$  and  $S, T : X \rightarrow X$  by  $F(x, y) = \frac{x^2+y^2}{16}$ ,  $G(x, y) = \frac{x+y}{16}$ ,  $Sx = \frac{x^2}{4}$  and  $Tx = \frac{x}{4}$ .

We have  $\frac{\frac{t}{2}}{\frac{t}{2}+1} \geq \left(\frac{t}{t+1}\right)^{\frac{1}{2}}$  for all  $t \geq 0$ . Now,

$$\begin{aligned} M(F(x, y), G(u, v), \frac{1}{2}t) &= \left(\frac{\frac{t}{2}}{\frac{t}{2}+1}\right)^{\left|\frac{x^2+y^2}{16} - \frac{u+v}{16}\right|} \\ &\geq \left(\frac{t}{t+1}\right)^{\left|\frac{x^2-u+y^2-v}{8}\right|} \\ &\geq \left(\frac{t}{t+1}\right)^{\frac{|x^2-u|+|y^2-v|}{8}} \\ &= \left(\frac{t}{t+1}\right)^{\frac{|fx-gu|+|fy-gv|}{2}} \\ &\geq \left(\frac{t}{t+1}\right)^{\max\{|fx-gu|, |fy-gv|\}} \\ &\geq \min\left\{\left(\frac{t}{t+1}\right)^{|fx-gu|}, \left(\frac{t}{t+1}\right)^{|fy-gv|}\right\} \\ &= \min\{M(fx, gu, t), M(fy, gv, t)\} \\ &\geq m_{u,v}^{x,y}. \end{aligned}$$

Also  $(F, S)$  and  $(G, T)$  satisfy (CLRg) property with respect to  $S$  and  $T$  respectively with sequences  $\{x_n\} = \{\frac{1}{\sqrt{n}}\}$ ,  $\{y_n\} = \{\frac{1}{n}\}$ ,  $\{u_n\} = \{\frac{1}{n^2}\}$  and  $\{v_n\} = \{\frac{1}{n}\}$  respectively. Clearly, the pairs  $(F, S)$  and  $(G, T)$  are  $w$ -compatible. Clearly  $(0, 0)$  is the unique common fixed point of  $F, G, S$  and  $T$ .

*Remark 2.7.* Recently, Sumitra et al.[25] proved a unique coupled common fixed point theorem for four self mappings (see Theorem 3.2 of [25]). Inherently they used the condition  $\lim_{t \rightarrow \infty} M(x, y, t) = 1$  for all  $x, y \in X$  in the proof of Theorem 3.2. Moreover, the condition  $a * b \geq ab, \forall a, b \in [0, 1]$  is redundant. Our Theorem 2.3 with  $H$ -type  $t$ -norm is a generalization and extension of Theorem 3.2 of [25].

**Theorem 2.8.** *Let  $(X, M, *)$  be a fuzzy metric space. If  $F, G : X \times X \rightarrow CB(X)$  and  $S, T : X \rightarrow X$  are maps which satisfy the following conditions:*

- (2.8.1) *the pairs  $(F, S)$  and  $(G, T)$  satisfy (CLRg) property with respect to  $S$  and  $T$  respectively,*
- (2.8.2) *the pairs  $(F, S)$  and  $(G, T)$  are  $w$ -compatible,*
- (2.8.3)  *$\delta_M(F(x, y), G(u, v), t) \geq \phi(m_{u,v}^{x,y})$  for all  $x, y, u, v \in X$ ,  $t > 0$ , where  $\phi : [0, 1] \rightarrow [0, 1]$  is continuous, monotonically increasing and  $\phi(t) > t$  for  $0 < t < 1$ .*

*Then there exists a unique  $x \in X$  such that  $F(x, x) = \{Sx\} = \{x\} = \{Tx\} = G(x, x)$ .*

Finally we prove the following.

**Theorem 2.9.** Let  $(X, M, *)$  be a fuzzy metric space. If  $F, G : X \times X \rightarrow CB(X)$  and  $S, T : X \rightarrow X$  are maps which satisfy the following conditions:

(2.9.1)(a) the pair  $(F, S)$  satisfies (CLRg) property with respect to  $S$  and  $F(X \times X \subseteq T(X)$ ,

or

(2.9.1)(b) the pair  $(G, T)$  satisfies (CLRg) property with respect to  $T$  and  $G(X \times X \subseteq S(X)$ ,

(2.9.2) the pairs  $(F, S)$  and  $(G, T)$  are  $w$ -compatible,

(2.9.3)  $\psi(\delta_M(F(x, y), G(u, v), t)) \geq \psi(m_{u, v}^{x, y}) + \phi(m_{u, v}^{x, y})$   
for all  $x, y, u, v \in X, t > 0$ , where  $\psi \in \Psi, \phi \in \Phi$ .

Then there exists a unique  $x \in X$  such that  $F(x, x) = \{Sx\} = \{x\} = \{Tx\} = G(x, x)$ .

*Proof.* Suppose (2.9.1)(a) holds.

Then there exist sequences  $\{x_n\}, \{y_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} M(Sx_n, Sa, t) = 1, \quad \lim_{n \rightarrow \infty} \delta_M(F(x_n, y_n), A, t) = 1,$$

$$\lim_{n \rightarrow \infty} M(Sy_n, Sb, t) = 1, \quad \lim_{n \rightarrow \infty} \delta_M(F(y_n, x_n), B, t) = 1$$

for some  $a, b \in X$  and  $Sa \in A \in CB(X), Sb \in B \in CB(X)$ .

Since  $F(x_n, y_n) \subseteq F(X \times X) \subseteq T(X)$ , there exist  $\alpha_n \in F(x_n, y_n)$  and  $u_n \in X$  such that  $\alpha_n = Tu_n$  for all  $n$ .

Also  $M(Tu_n, Sa, t) = M(\alpha_n, Sa, t) \geq \delta_M(F(x_n, y_n), A, t) \rightarrow 1$  as  $n \rightarrow \infty$ .

Hence  $\lim_{n \rightarrow \infty} M(Tu_n, Sa, t) = 1$ .

Similarly there exists  $v_n \in X$  such that  $\lim_{n \rightarrow \infty} M(Tv_n, Sb, t) = 1$ .

Let  $\lim_{n \rightarrow \infty} G(u_n, v_n) = P$  and  $\lim_{n \rightarrow \infty} G(v_n, u_n) = Q$ .

Suppose  $0 < \min\{\delta_M(A, P, t), \delta_M(B, Q, t)\} < 1$  for some  $t > 0$ .

Consider

$$(2.10) \quad \psi(\delta_M(F(x_n, y_n), G(u_n, v_n), t)) \geq \psi(m_{u_n, v_n}^{x_n, y_n}) + \phi(m_{u_n, v_n}^{x_n, y_n})$$

$$m_{u_n, v_n}^{x_n, y_n} = \min \left\{ \begin{array}{l} M(Sx_n, Tu_n, t), M(Sy_n, Tv_n, t), \delta_M(Sx_n, F(x_n, y_n), t) \\ \delta_M(Sy_n, F(y_n, x_n), t), \delta_M(Tu_n, G(u_n, v_n), t), \delta_M(Tv_n, G(v_n, u_n), t) \\ \delta_M(Sx_n, G(u_n, v_n), t), \delta_M(Sy_n, G(v_n, u_n), t), \\ \delta_M(Tu_n, F(x_n, y_n), t), \delta_M(Tv_n, F(y_n, x_n), t) \end{array} \right\}.$$

$$\begin{aligned} \lim_{n \rightarrow \infty} m_{u_n, v_n}^{x_n, y_n} &= \min \left\{ 1, 1, 1, 1, \delta_M(Sa, P, t), \delta_M(Sb, Q, t), \right\} \\ &\geq \min \{ \delta_M(A, P, t), \delta_M(B, Q, t) \} \end{aligned}$$

Letting  $n \rightarrow \infty$  in (2.10), we get

$$\psi(\delta_M(A, P, t)) \geq \psi(\min \{ \delta_M(A, P, t), \delta_M(B, Q, t) \}) + \phi(\min \{ \delta_M(A, P, t), \delta_M(B, Q, t) \}).$$

Similarly we can show that

$$\psi(\delta_M(B, Q, t)) \geq \psi(\min\{\delta_M(A, P, t), \delta_M(B, Q, t)\}) + \phi(\min\{\delta_M(A, P, t), \delta_M(B, Q, t)\}).$$

Thus we have

$$\psi\left(\min\left\{\begin{array}{c} \delta_M(A, P, t), \\ \delta_M(B, Q, t) \end{array}\right\}\right) \geq \psi\left(\min\left\{\begin{array}{c} \delta_M(A, P, t), \\ \delta_M(B, Q, t) \end{array}\right\}\right) + \phi\left(\min\left\{\begin{array}{c} \delta_M(A, P, t), \\ \delta_M(B, Q, t) \end{array}\right\}\right)$$

which in turn yields that

$$0 \geq \phi(\min\{\delta_M(A, P, t), \delta_M(B, Q, t)\}) > \min\{\delta_M(A, P, t), \delta_M(B, Q, t)\}.$$

It is a contradiction. Hence for all  $t > 0$ , we have

$$\min\{\delta_M(A, P, t), \delta_M(B, Q, t)\} = 1.$$

Hence  $A = P = \{a \text{ singleton}\}$  and  $B = Q = \{a \text{ singleton}\}$ .

Since  $Sa \in A$  and  $Sb \in B$  we have  $A = P = \{Sa\}$  and  $B = Q = \{Sb\}$ .

Thus  $\lim_{n \rightarrow \infty} G(u_n, v_n) = \{Sa\}$  and  $\lim_{n \rightarrow \infty} G(v_n, u_n) = \{Sb\}$ .

Now by taking  $x = x_n$ ,  $y = y_n$ ,  $u = v_n$ ,  $v = u_n$  in (2.9.3) and letting  $n \rightarrow \infty$ , we can show that

$$(2.11) \quad Sa = Sb.$$

Taking  $x = a$ ,  $y = b$ ,  $u = u_n$ ,  $v = v_n$  and  $x = b$ ,  $y = a$ ,  $u = v_n$ ,  $v = u_n$  in (2.9.3) and letting  $n \rightarrow \infty$ , we can show that  $F(a, b) = \{Sa\}$  and

$$F(b, a) = \{Sb\}.$$

Since  $\{Sa\} = F(a, b) \subseteq F(X \times X) \subseteq T(X)$ , there exists  $a' \in X$  such that  $Sa = Ta'$ .

Since  $\{Sb\} = F(b, a) \subseteq F(X \times X) \subseteq T(X)$ , there exists  $b' \in X$  such that  $Sb = Tb'$ .

From (2.11), we have  $Ta' = Sa = Sb = Tb'$ .

Now taking  $x = x_n$ ,  $y = y_n$ ,  $u = a'$ ,  $v = b'$  and  $x = y_n$ ,  $y = x_n$ ,  $u = b'$ ,  $v = a'$  in (2.9.3) and letting  $n \rightarrow \infty$ , we can show that  $G(a', b') = \{Ta'\}$  and  $G(b', a') = \{Tb'\}$ .

The rest of proof follows as in Theorem 2.2.

Similarly we can prove Theorem 2.9 if (2.9.1)(b) holds.  $\square$

### 3 An Application

As an application of Theorem 2.4, we prove a theorem on the existence and uniqueness of the solution of a Fredholm nonlinear integral equation. To accomplish this purpose, we consider the following integral equation:

$$(3.1) \quad x(p) = \int_a^b (K_1(p, q) + K_2(p, q)) [f(q, x(q)) + g(q, x(q))] dq + h(p),$$

for all  $p \in I = [a, b]$ ,  $K_1, K_2 \in C(I \times I, R)$  and  $h \in C(I, R)$ .

Let  $\Theta$  be the set of all functions  $\theta : R^+ \rightarrow R^+$  satisfying the following conditions:

- (i $_{\theta}$ )  $\theta$  is non-decreasing,
- (ii $_{\theta}$ )  $\theta(p) \leq p$ .

We also do require the functions  $K_1, K_2, f$ , and  $g$  to satisfy the following conditions:

**Assumption (3.1)**

- (i)  $K_1(p, q) \geq 0$  and  $K_2(p, q) \leq 0$  for all  $p, q \in I$ ,
- (ii) there exist positive numbers  $\lambda, \mu$  and  $\theta \in \Theta$  such that for all  $x, y \in C(I, R)$  with  $x \geq y$ , the following conditions hold:

$$(3.2) \quad 0 \leq f(q, x) - f(q, y) \leq \lambda\theta(x - y) - \mu\theta(x - y)$$

$$(3.3) \quad \lambda\theta(x - y) - \mu\theta(x - y) \leq g(q, x) - g(q, y) \leq 0,$$

(iii)

$$(3.4) \quad \max\{\lambda, \mu\} \sup_{p \in I} \int_a^b [K_1(p, q) - K_2(p, q)]dq \leq \frac{1}{4}.$$

Now, we are equipped to prove the following theorem:

**Theorem 3.1.** *Consider the integral equation (3.1) with  $K_1, K_2 \in C(I \times I, R)$  and  $h \in C(I, R)$ . If all the conditions embodied in the Assumption (3.1) are satisfied, then the integral equation (3.1) has a unique solution in  $C(I, R)$ .*

*Proof.* It is well known that  $X = C(I, R)$  is a complete metric space with respect to the sup metric

$$d(x, y) = \sup_{p \in I} |x(p) - y(p)|.$$

It is straight forward to check that  $(X, M, *)$  is a fuzzy metric space if we define

$$M(x, y, t) = e^{-\frac{d(x,y)}{t}}, \text{ for all } x, y \in C(I, R) \text{ and } t > 0,$$

wherein  $*$  is defined by  $x * y = xy$  (for all  $x, y \in I$ ). Now, define a mapping  $F : X \times X \rightarrow X$  by

$$\begin{aligned} F(x, y)(p) &= \int_a^b K_1(p, q)[f(q, x(q)) + g(q, y(q))]dq \\ &\quad + \int_a^b K_2(p, q)[f(q, y(q)) + g(q, x(q))]dq + h(p), \end{aligned}$$

for all  $p \in I$ . On using (3.2) and (3.3), we have (for  $x, y, u, v \in X$ )

$$\begin{aligned}
 (3.5) \quad & F(x, y)(p) - F(u, v)(p) \\
 &= \int_a^b K_1(p, q) [f(q, x(q)) + g(q, y(q))] dq \\
 &\quad + \int_a^b K_2(p, q) [f(q, y(q)) + g(q, x(q))] dq \\
 &\quad - \int_a^b K_1(p, q) [f(q, u(q)) + g(q, v(q))] dq \\
 &\quad - \int_a^b K_2(p, q) [f(q, v(q)) + g(q, u(q))] dq \\
 &= \int_a^b K_1(p, q) [(f(q, x(q)) - f(q, u(q))) - (g(q, v(q)) - g(q, y(q)))] dq \\
 &\quad - \int_a^b K_2(p, q) [(f(q, v(q)) - f(q, y(q))) - (g(q, x(q)) - g(q, u(q)))] dq \\
 &\leq \int_a^b K_1(p, q) [\lambda\theta(x(q) - u(q)) + \mu\theta(v(q) - y(q))] dq \\
 &\quad - \int_a^b K_2(p, q) [\lambda\theta(v(q) - y(q)) + \mu\theta(x(q) - u(q))] dq.
 \end{aligned}$$

As the function  $\theta$  is non-decreasing, we have

$$\begin{aligned}
 \theta(x(q) - u(q)) &\leq \theta\left(\sup_{q \in I} |x(q) - u(q)|\right) = \theta(d(x, u)), \\
 \theta(v(q) - y(q)) &\leq \theta\left(\sup_{q \in I} |v(q) - y(q)|\right) = \theta(d(y, v)).
 \end{aligned}$$

Appealing to (3.5) and making use of the fact that  $K_2(p, q) \leq 0$ , we obtain

$$\begin{aligned}
 & |F(x, y)(p) - F(u, v)(p)| \\
 & \leq \int_a^b K_1(p, q) [\lambda\theta(d(x, u)) + \mu\theta(d(y, v))] dq \\
 & \quad - \int_a^b K_2(p, q) [\lambda\theta(d(y, v)) + \mu\theta(d(x, u))] dq, \\
 & \leq \int_a^b K_1(p, q) [\max\{\lambda, \mu\}\theta(d(x, u)) + \max\{\lambda, \mu\}\theta(d(y, v))] dq \\
 & \quad - \int_a^b K_2(p, q) [\max\{\lambda, \mu\}\theta(d(y, v)) + \max\{\lambda, \mu\}\theta(d(x, u))] dq.
 \end{aligned}$$

Now, taking the supremum with respect to  $p$  and making use of (3.4), we get

$$\begin{aligned}
 (3.6) \quad & d(F(x, y), F(u, v)) \\
 & \leq \max\{\lambda, \mu\} \sup_{p \in I} \int_a^b (K_1(p, q) - K_2(p, q)) dq \cdot [\theta(d(x, u)) + \theta(d(y, v))] \\
 & \leq \frac{\theta(d(x, u)) + \theta(d(y, v))}{4}.
 \end{aligned}$$

Since  $\theta$  is non-decreasing, we have

$$\begin{aligned}
 \theta(d(x, u)) & \leq \theta(\max\{d(x, u), d(y, v)\}), \\
 \theta(d(y, v)) & \leq \theta(\max\{d(x, u), d(y, v)\}),
 \end{aligned}$$

which implies (due to  $(ii_\theta)$ ) that

$$\begin{aligned}
 \frac{\theta(d(x, u)) + \theta(d(y, v))}{2} & \leq \theta(\max\{d(x, u), d(y, v)\}) \\
 & \leq \max\{d(x, u), d(y, v)\},
 \end{aligned}$$

so that (owing to (3.6)), we have

$$(3.7) \quad d(F(x, y), F(u, v)) \leq \frac{1}{2} \max\{d(x, u), d(y, v)\}.$$

Now, on making use of (3.7), it follows that

$$\begin{aligned}
 & M(F(x, y), F(u, v), \frac{t}{2}) \\
 &= e^{-\frac{d(F(x,y), F(u,v))}{\frac{t}{2}}} \\
 &\geq e^{-\frac{\frac{1}{2} \max\{d(x,u), d(y,v)\}}{\frac{t}{2}}} \\
 &= e^{-\frac{\max\{d(x,u), d(y,v)\}}{t}} \\
 &\geq \min \left\{ e^{-\frac{d(x,u)}{t}}, e^{-\frac{d(y,v)}{t}} \right\} \\
 &= \min \{M(Sx, Tu, t), M(Sy, Tv, t)\} \\
 &\geq m_{u,v}^{x,y}.
 \end{aligned}$$

Thus the involved contractive condition of Theorem 2.4 is satisfied if we set  $F = G$  and  $Sx = Tx = x$ , Also, it is straight forward to notice that all the hypotheses of Theorem 2.4 are satisfied and henceforth  $F$  has a coupled fixed point  $(x, x) \in X^2$  which also remains the solution of the integral equation (3.1). □

### Acknowledgement

All the authors are grateful to the learned referees for their critical reviews which let to several improvements.

### References

- [1] Aamri,M. and Moutawakil,D.El., Some new common fixed point theorems under strict contractive conditions, Jour. Math. Anal. Appl., 27(2002),181-188.
- [2] Abbas,M., Ali Khan,M. and Radenovic,S., Common coupled fixed point theorems in cone metric spaces for w-compatible mappings,Appl.Math. Comput,217(2010),195-202.
- [3] Alamgir Khan,M. and Sumitra,CLRg property for coupled fixed point theorems in fuzzy metric spaces, Int.J.Appl.Phy. Math., 2(5)(2012),355-358.
- [4] Bhaskar,T.G. and Lakshmikantham,V., Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis, 65(7) (2006), 1379 - 1393.
- [5] Chauhan,S., Sintunavarat,W. and Kumam,P., Common fixed point theorems for weakly compatible mappings in fuzzy metric spaces using (JCLR) property, J. Appl.Math.,3(9)(2012),976-982.

Draft version November 26, 2015



- [6] Chauhan,S., Alamgir Khan,M. and Sintunavarat,W. ,Common fixed point theorems in fuzzy metric spaces satisfying  $\phi$ -contractive condition with common limit range property,Abstract Applied Analysis,(2013),Article ID 735217(2013),14 pages.
- [7] Efe,H., Alaca,C. and Yildiz,C., Fuzzy multi-metric spaces, J. Comput. Anal. Appl.,10(2008),367-375.
- [8] Fang,J.X., On fixed point theorems in fuzzy metric spaces, Fuzzy Set. System. ,46(1992),107-113.
- [9] George,A. and Veeramani,P., On some result in fuzzy metric space, Fuzzy Set. System.,64(1994),395-399.
- [10] Grabiec,M., Fixed points in fuzzy metric spaces, Fuzzy Set. System., 27(1988),385-389.
- [11] Gregori,V. and Sapena,A., On fixed-point theorem in fuzzy metric spaces, Fuzzy Set. System., 125(2002), 245-252.
- [12] Hu,Xin-Qi.,Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces, Fixed point theory Appl., 2011(2011),Article ID 363716,14 pages.
- [13] Jain,M.,Tas,K., Kumar,S. and Gupta,N., Coupled fixed point theorems for a pair of weakly compatible maps along with CLRg property in fuzzy metric spaces, J.Appl.Math.,(2012), Article ID 961210(2012),14 pages.
- [14] Kramosil,I. and Michalek,J., Fuzzy metric and statistical metric spaces, Kybernetica ,11(1975),326-334.
- [15] Kumar,S., Fixed points theorems weakly compatible maps under E.A. property in fuzzy metric spaces, J. Appl. Math. & Informatics, 29(1-2)(2011),395 - 405.
- [16] Lakshmikantham,V. and Ćirić,Lj., Coupled fixed point theorems for non-linear contractions in partially ordered metric spaces, Nonlinear Analysis: Theory, Method. Appl., 70(12)(2009), 4341-4349.
- [17] Mihet, D., A Banach contraction theorem in fuzzy metric spaces, Fuzzy Set. System.,144(2004),431-439.
- [18] Rodríguez-López,J. and Romaguera,S., The Hausdorff fuzzy metric on compact sets, Fuzzy Set. System. ,147(2004),273-283.
- [19] Saadati,R. and Sedghi,S., A common fixed point theorem for R-weakly commuting maps in fuzzy metric spaces, 6th Iranian Conference on Fuzzy Systems,(2006),387-391.
- [20] Schweizer,B. and Sklar,A., Statistical metric spaces, Pacific J. Math., 10(1960),314-334.

- [21] Sedghi,S. and Shobe,N., Fixed point theorems in  $\mathcal{M}$ -fuzzy metric spaces with property(E), *Advan. fuzzy Math.*,1(1)(2006),55-65.
- [22] Sedghi,S., Rao,K.P.R. and Shobe,N.,A general Common fixed point theorem for multi maps satisfying an implicit relation on fuzzy metric spaces,*Filomat*, 22(1)(2008),1-11.
- [23] Sedghi,S., Shobe,N. and Aliouche,A., A common fixed point theorem in fuzzy metric spaces for weakly compatible mappings in fuzzy metric spaces, *Gen. Math.*, 18(3)(2010), 3-12.
- [24] Sintunavarat,W. and Kumam,P.,Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, *Jour. Appl. Math.*,2011(2011),Article ID 637958,14 pages.
- [25] Sumitra and Masmali,I., Coupled fixed point theorems with CLRg property in fuzzy metric spaces, *IJRRAS*,15(3)(2013),322-329.
- [26] Vasuki,R., Common fixed points for R-weakly commuting maps in fuzzy metric spaces, *Indian J. Pure Appl. Math.*,30(1999), 419-423.
- [27] Wadhwa,K., Dubey,H. and Jain,R., Impact of "E.A. Like" property on common fixed point theorems in fuzzy metric spaces, *Jour. Advanced Studies Topology*, 3(1)(2012),52-59.
- [28] Zadeh,L.A., Fuzzy sets, *Inform. and Control*, 8(1965),338-353.