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ON THE PRODUCT OF SMOOTH FUZZY TOPOLOGICAL SPACES

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Abstract. In 1994 R. Srivastava defined the concepts of basis and subbasis, for a smooth fuzzy topology, described a way to obtain a topology from a basis and also discussed the product spaces. We point out that a topology obtained from a basis or a subbasis given in that paper is not well defined. So we redefine the concept of basis and subbasis for a smooth fuzzy topology in a natural manner so that every smooth fuzzy topology becomes a basis as well as a subbasis of itself. We also define and discuss product of smooth fuzzy topological spaces using the new definition of basis introduced in this paper.

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1. Introduction

The concept of fuzzy topology on a set X was introduced by Chang [2] as a collection τ of fuzzy sets of X satisfying certain conditions and he called each member of τ as a fuzzy open set. To overcome the lack of fuzziness in Chang's approach A. P. Šostak[9] developed a theory by declaring fuzzy topology as a function $\mathcal{T}$ from the collection of all fuzzy subsets of X to $[0, 1]$ satisfying some properties. A. A. Ramadan [8] gave a similar definition of fuzzy topology on a fuzzy set in Šostak's sense under the name of "smooth fuzzy topological spaces". R. Srivastava, W.Peeters, C. K. Park, W. K. Min, M. H. Kim, C. Kalaivani and R. Roopkumar [10, 6, 7, 3, 4] are some of the others who studied the concept of fuzzy topological spaces in Šostak's sense.

R. Srivastava [10] introduced the concept of basis, subbasis, product topology and separation axioms in Šostak sense; according to this theory, a basis (called a base in [10]) is defined as a collection of fuzzy sets of a set X satisfying a condition which depends on a fuzzy topology on X (see Definition 2.3) and a subbasis of $(X, \mathcal{T})$ is also defined in the same manner. According to these definitions a basis or a subbasis is defined only if a fuzzy topology is available on X; but the definitions for a basis and a subbasis which we are going to give shortly do not need any fuzzy topological structure on X. In this paper, we define basis and subbasis as functions from the collection of all fuzzy subsets of

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a set X to $[0,1]$ in contrast with the one given by R. Srivastava as collections of fuzzy subsets of X.

In Section 2, we give the basic definitions and results from the literature which we need to develop our theory. In Section 3, we first point out that the extension to obtain a fuzzy topology from a basis given in Theorem 2.5 of [10] is not well defined. In the same section, we give a new definition of basis for smooth fuzzy topology and prove certain results analogous to the results in classical theory. In Section 4, we point out that the product of fuzzy spaces defined in [10] is not well defined; we define the product of two smooth fuzzy topological spaces in more natural way and prove certain results. Finally, we give some concluding remarks in Section 5.

2. Preliminary Definitions and Results

First let us fix the notations. For any set X, a function $\mu : X \to [0,1]$ is called a fuzzy subset of X. By I and I^X we denote the closed interval $[0,1]$ and the set of all fuzzy subsets of X. As usual by 0_X and 1_X we denote the fuzzy sets $0_X(x) = 0$ and $1_X(x) = 1$ for all $x \in X$. The union $\vee_{\lambda \in J}\mu_\lambda$ and intersection $\wedge_{\lambda \in J}\mu_\lambda$ of a collection $\{\mu_\lambda : \lambda \in J\}$ of fuzzy sets of X, where J is an arbitrary indexing set, are defined as follows:

$$\left(\vee_{\lambda \in J}\mu_\lambda\right)(x) = \sup_{\lambda \in J} \mu_\lambda(x) \quad \text{and} \quad \left(\wedge_{\lambda \in J}\mu_\lambda\right)(x) = \inf_{\lambda \in J} \mu_\lambda(x).$$

If A and B are two fuzzy subsets of X such that $A \geq B$, then the complement $A - B$ of B in A is defined as $(A - B)(x) = A(x) - B(x)$ for all $x \in X$. Now we give some definitions and results from the literature.

Definition 2.1. [9, 8] Let μ be a fuzzy subset of a nonempty set X and let $\Im_\mu = \{A \in I^X / A \leq \mu\}$. Let $\mathcal{T} : \Im_\mu \to [0,1]$ be a mapping satisfying the following conditions:

i. $\mathcal{T}(\mu) = 1$

ii. $\mathcal{T}(0_X) = 1$

iii. $\mathcal{T}(A \wedge B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$ for any two fuzzy subsets A, B of μ

iv. $\mathcal{T}(\vee A_\lambda) \geq \wedge \mathcal{T}(A_\lambda)$ for any collection $\{A_\lambda\}_{\lambda \in \Lambda}$, $A_\lambda \in \Im_\mu$.

Then $\mathcal{T}$ is called a smooth fuzzy topology on μ and the pair $(\mu, \mathcal{T})$ is called a smooth fuzzy topological space; for any $A \in \Im_\mu$, $\mathcal{T}(A)$ is called the degree of openness of the fuzzy set A.

Let $\mathcal{C} : \Im_\mu \to [0,1]$ be the mapping defined by $\mathcal{C}(A) = \mathcal{T}(\mu - A)$. Then $\mathcal{C}(A)$ is called the degree of closedness of the fuzzy set A.

If $\mathcal{T}$ and $\mathcal{T}'$ are two smooth fuzzy topologies on a given set X and if $\mathcal{T}' \geq \mathcal{T}$, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$ or equivalently $\mathcal{T}$ is coarser than $\mathcal{T}'$. Strict finer and strict coarser can be defined accordingly. We say that $\mathcal{T}$ is comparable with $\mathcal{T}'$ if either $\mathcal{T}' \geq \mathcal{T}$ or $\mathcal{T} \geq \mathcal{T}'$.

Definition 2.2. [1] Let μ be a fuzzy subset of X and ν be a fuzzy subset of Y. The product $\mu \times \nu$ of μ and ν is defined as a fuzzy subset of $X \times Y$, by

$$(\mu \times \nu)(x, y) = \mu(x) \wedge \nu(y),$$

for each $(x, y) \in X \times Y$.

In what follows in this section, a fuzzy topology means a smooth fuzzy topology.

Definition 2.3. [10] Let $(X, \mathcal{T})$ be a fuzzy topological space. Then a family $\mathscr{B} = \{B \in I^X : \mathcal{T}(B) > 0\}$ is called a base of $(X, \mathcal{T})$ if for all $U \in I^X$ with $\mathcal{T}(U) > 0$ and for all fuzzy points $x_\alpha \in U$, there exists a $B \in \mathscr{B}$ such that $x_\alpha \in B \subseteq U$.

Definition 2.4. [10] A family $\mathscr{S} \subseteq I^X$ of a fuzzy topological space is called a subbase of $(X, \mathcal{T})$ if the family $\mathscr{B}_{\mathscr{S}}$ of finite intersections of members of $\mathscr{S}$ is a base of $(X, \mathcal{T})$.

Theorem 2.5. *[10] Let $\mathscr{S} \subseteq I^X$, contain 0_X and 1_X. Let $\mathcal{T}$ be any map from $\mathscr{S}$ to $[0, 1]$ such that $\mathcal{T}(0_X) = \mathcal{T}(1_X) = 1$ and $\mathcal{T}(U) > 0$, for all $U \in \mathscr{S}$. Then the extension $\mathcal{T}_{\mathscr{S}} : I^X \to [0, 1]$ given as follows: for each $U \in I^X$,*

$$\mathcal{T}_{\mathscr{S}}(U) = \begin{cases} \mathcal{T}(U_1) \wedge \mathcal{T}(U_2) & \textit{if } \ U = U_1 \cap U_2 \ \textit{where } U_l, U_2 \in \mathscr{S} \\ \sup \mathcal{T}(W_i) & \textit{if } \ U = \cup W_i \ \textit{where each } W_i \in \mathscr{B}_{\mathscr{S}} \\ 0 & \textit{otherwise} \end{cases}$$

defines a gradation of openness on X.

Definition 2.6. [10] Let $\{(X_i, \mathcal{T}_i)/i \in \mathscr{A}\}$ be a family of fuzzy topological spaces and $P_i : X = \prod_{i \in \mathscr{A}} X_i \to X_i$ denote the i^{th} projection map. Consider the family

$$\mathscr{S} = \{P_i^{-1}(U_i) : \mathcal{T}_i(U_i) > 0, i \in \mathscr{A}\}$$

where $P_i^{-1}(U_i)(\mathbf{x}) = U(P_i(\mathbf{x}))$ for all $\mathbf{x} \in X$. Let $\mathscr{B}_{\mathscr{S}}$ be the collection of all finite intersections of members of $\mathscr{S}$. Define $\mathcal{T}$ from I^X to I by

$$\mathcal{T}(U) = \begin{cases} \mathcal{T}_i(V) & \text{if } U = P_i^{-1}(V) \\ \mathcal{T}(U_1) \wedge \mathcal{T}(U_2) & \text{if } U = U_1 \cap U_2 \text{ where } U_l, U_2 \in \mathscr{S} \\ \sup \mathcal{T}(W_i) & \text{if } U = \cup W_i \text{ where each } W_i \in \mathscr{B}_{\mathscr{S}} \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathcal{T}$ is called the product of $\mathcal{T}_i$'s and $(X, \mathcal{T})$ is called the product of the fuzzy topological spaces $\{(X_i, \mathcal{T}_i)/i \in \mathscr{A}\}$.

Results and definitions used in this paper which are not stated explicitly here are taken from [5].

3. Basis for Smooth Fuzzy Topological Spaces

Now we justify that the extension described in Theorem 3.3 of [10] (see Theorem 2.5) is not well-defined.

Let U_1 and U_2 be fuzzy subsets of a set X such that $0_X < U_1 < U_2 < 1_X$. Let $\mathscr{S} = \{0_X, 1_X, U_1, U_2\}$. Define $t : \mathscr{S} \to [0,1]$ as follows:

$$t(0_X) = t(1_X) = t(U_1) = 1 \ \text{ and } \ t(U_2) = \frac{1}{2}.$$

Then $\mathscr{S}$ and t satisfy all the requirements of Theorem 3.3 of [10]; now, since $U_1 = U_1 \wedge U_2$, according to the suggested extension, we must have $t(U_1) = t(U_1) \wedge t(U_2) = \frac{1}{2}$; which leads to a contradiction. This shows that the suggested extension is not well defined. Now we define the concept of basis and obtain a topology from it.

Definition 3.1. Let μ be fuzzy subset of a nonempty set X. Let

$$\mathcal{B} : \mathfrak{I}_\mu \to [0,1]$$

be a function satisfying the following conditions:

i. Given $x \in X$ and $\epsilon > 0$ there exists $A \le \mu$ such that

$$A(x) = \mu(x) \text{ and } \mathcal{B}(A) \ge 1 - \epsilon.$$

ii. For $x \in X$, A, B in $\mathfrak{I}_\mu$ and $\epsilon > 0$, there exists $C \in \mathfrak{I}_\mu$ such that $C(x) = A(x) \wedge B(x)$, $C \le A \wedge B$ and $\mathcal{B}(C) \ge (\mathcal{B}(A) \wedge \mathcal{B}(B)) - \epsilon$.

then $\mathcal{B}$ is called a basis for a smooth fuzzy topology on μ.

Definition 3.2. Let A be a fuzzy subset of a set X. A collection $\{A_\lambda\}_{\lambda\in\Lambda}$ of non-zero fuzzy subsets of A is called an inner cover for A if for each $x \in X$ there exists some $\lambda \in \Lambda$ such that $A_\lambda(x) = A(x)$ and $A_\lambda \le A$ for all $\lambda \in \Lambda$.

We note that $\vee A_\lambda = A$ in this case.

Definition 3.3. Let μ be fuzzy subset of a nonempty set X and let $\mathcal{B}$ be a basis for a smooth fuzzy topology on μ. Define the smooth fuzzy topology $\mathcal{T} : \mathfrak{I}_\mu \to [0,1]$ generated by $\mathcal{B}$ as follows:

$$\mathcal{T}(A) = \begin{cases} 1 & \text{if } A = 0_X \\ \sup\limits_{\Lambda\in\Gamma}\{\inf\limits_{A_\lambda \in C_\Lambda}\{\mathcal{B}(A_\lambda)\}\} & \text{if } A \neq 0_X \end{cases}$$

where $\{C_\Lambda\}_{\Lambda\in\Gamma}$ is the collection of all inner covers $C_\Lambda = \{A_\lambda\}_{\lambda\in\Lambda}$ of A.

In the above definition, for $A \neq 0_X$ we find $\mathcal{T}(A)$ as follows. For any inner cover $C_\Lambda = \{A_\lambda\}_{\lambda\in\Lambda}$ of A we compute $\mathcal{B}(A_\lambda)$ for all $A_\lambda \in C_\Lambda$ and find the infimum of them. After finding the infimum with respect to all covers of A, we find the supremum of them and declare it as $\mathcal{T}(A)$.

Example 3.4. Let $X = (0, 1)$ and χ_A denote the characteristic function of A. Define $\mathcal{B} : I^X \to [0, 1]$ by

$$\mathcal{B}(A) = \begin{cases} 1 & \text{if } A = \chi_X \\ q & \text{if } A = \chi_{(q,1)}, \text{where } q \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathcal{B}$ is a basis for a smooth fuzzy topology $\mathcal{T}$ on X.

We note that $\mathcal{B}(\chi_{(a,1)}) = 0$, whereas $\mathcal{T}(\chi_{(a,1)}) = 1$ for all irrational a in $(0, 1)$. So, $\mathcal{T}$ is not an extension of $\mathcal{B}$.

Theorem 3.5. *Let μ be a fuzzy subset of a set X and $\mathcal{B}$ be a basis. If $\mathcal{T}$ is as defined in Definition 3.3, then $\mathcal{T}$ is a smooth fuzzy topology on μ.*

Proof. As the collection $\{A\}$ is an inner cover for A and since $\mathcal{B}$ takes the value in $[0, 1]$, $\mathcal{T}$ is well defined. Now we prove that $\mathcal{T}(\mu) = 1$. For each $x \in X$ and $\epsilon > 0$ let $A_{x,\epsilon} \leq \mu$ be such that $A_{x,\epsilon}(x) = \mu(x)$ and $\mathcal{B}(A_{x,\epsilon}) \geq 1 - \epsilon$. The collection $\mathfrak{C}_\epsilon = \{A_{x,\epsilon}\}_{x \in X}$ is then an inner cover for μ and $\inf\limits_{x\in X}\{\mathcal{B}(A_{x,\epsilon})\} \geq 1-\epsilon$. Thus for each $\epsilon > 0$ there exists an inner cover $\mathfrak{C}_\epsilon = \{A_{x,\epsilon}\}_{x\in X}$ of μ such that

$$\inf_{x\in X}\{\mathcal{B}(A_{x,\epsilon})\} \geq 1 - \epsilon.$$

This implies that,

$$\mathcal{T}(\mu) \geq \sup_{\mathfrak{C}_\epsilon}\{\inf_{x\in X}\{\mathcal{B}(A_{x,\epsilon})\}\} \geq 1,$$

and hence $\mathcal{T}(\mu) = 1$. From the definition of $\mathcal{T}$, we have $\mathcal{T}(0_X) = 1$.

Now we prove that $\mathcal{T}(A \wedge B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$ for any two fuzzy subsets A, B in $\mathfrak{I}_\mu$. If $A \wedge B = 0_X$, then there is nothing to prove. If $A \wedge B \neq 0_X$, then let $C = A \wedge B$. For $\epsilon > 0$, let $\{A_\lambda\}_{\lambda\in\Lambda_1}$ and $\{B_\gamma\}_{\gamma\in\Lambda_2}$, be inner covers such that

$$\inf_{\lambda\in\Lambda_1}\{\mathcal{B}(A_\lambda)\} \geq \mathcal{T}(A) - \frac{\epsilon}{2} \text{ and } \inf_{\gamma\in\Lambda_2}\{\mathcal{B}(B_\gamma)\} \geq \mathcal{T}(B) - \frac{\epsilon}{2}.$$

Let $C_{\lambda,\gamma} = A_\lambda \wedge B_\gamma$ for $\lambda \in \Lambda_1$ and $\gamma \in \Lambda_2$. Let Λ denote the set of all pairs (λ, γ) for which $C_{\lambda,\gamma} \neq 0_X$. Since $A \wedge B \neq 0_X$ there is an $x \in X$ such that $A(x) \wedge B(x) \neq 0$; then by the definition of inner cover there exist A_{λ_0} and B_{γ_0} in the corresponding inner covers such that $A_{\lambda_0}(x) = A(x)$ and $B_{\gamma_0}(x) = B(x)$; this implies that $A_{\lambda_0}(x) \wedge B_{\gamma_0}(x) \neq 0$ and hence $(\lambda_0, \gamma_0) \in \Lambda$; thus we have $\Lambda \neq \emptyset$.

Now, for $(\lambda, \gamma) \in \Lambda$ and for each $x \in X$, let $D_{\lambda,\gamma,x}$ be such that $D_{\lambda,\gamma,x}(x) = A_\lambda(x) \wedge B_\gamma(x) = C_{\lambda,\gamma}(x)$, $D_{\lambda,\gamma,x} \leq C_{\lambda,\gamma}$ and

$$\mathcal{B}(D_{\lambda,\gamma,x}) \geq (\mathcal{B}(A_\lambda) \wedge \mathcal{B}(B_\gamma)) - \frac{\epsilon}{2}.$$

Thus by the construction of $D_{\lambda,\gamma,x}$, the collection $\{D_{\lambda,\gamma,x}\}_{x\in X}$ is an inner cover for $C_{\lambda,\gamma}$ and hence the collection $\{D_{\lambda,\gamma,x}\}_{\lambda,\gamma,x}$ is an inner cover for C.

Now,

$$\begin{aligned}
\inf_{\substack{x\in X\\(\lambda,\gamma)\in\Lambda}} \{\mathcal{B}(D_{\lambda,\gamma,x})\} &\geq \inf_{(\lambda,\gamma)\in\Lambda}\left\{\mathcal{B}(A_\lambda)\wedge\mathcal{B}(B_\gamma)-\frac{\epsilon}{2}\right\}\\
&= \inf_{(\lambda,\gamma)\in\Lambda}\{\mathcal{B}(A_\lambda)\wedge\mathcal{B}(B_\gamma)\}-\frac{\epsilon}{2}\\
&\geq \left\{\inf_{(\lambda,\gamma)\in\Lambda}\{\mathcal{B}(A_\lambda)\}\wedge\inf_{(\lambda,\gamma)\in\Lambda}\{\mathcal{B}(B_\gamma)\}\right\}-\frac{\epsilon}{2}\\
&\geq \left\{\inf_{\lambda\in\Lambda_1}\{\mathcal{B}(A_\lambda)\}\wedge\inf_{\gamma\in\Lambda_2}\{\mathcal{B}(B_\gamma)\}\right\}-\frac{\epsilon}{2}\\
&\geq \left(\mathcal{T}(A)-\frac{\epsilon}{2}\right)\wedge\left(\mathcal{T}(B)-\frac{\epsilon}{2}\right)-\frac{\epsilon}{2}\\
&= (\mathcal{T}(A)\wedge\mathcal{T}(B))-\frac{\epsilon}{2}-\frac{\epsilon}{2}\\
&= (\mathcal{T}(A)\wedge\mathcal{T}(B))-\epsilon
\end{aligned}$$

But,

$$\begin{aligned}
\mathcal{T}(A\wedge B) &= \mathcal{T}(C)\\
&\geq \inf_{\substack{x\in X\\(\lambda,\gamma)\in\Lambda}}\{\mathcal{B}(D_{\lambda,\gamma,x})\}\\
&\geq (\mathcal{T}(A)\wedge\mathcal{T}(B))-\epsilon
\end{aligned}$$

This is true for every $\epsilon > 0$ and hence

$$\mathcal{T}(A\wedge B)\geq(\mathcal{T}(A)\wedge\mathcal{T}(B))$$

for any two subsets A, B of X.

Now we prove that $\mathcal{T}(\bigvee_{\lambda\in\Lambda} A_\lambda) \geq \bigwedge_{\lambda\in\Lambda}\mathcal{T}(A_\lambda)$ for any collection $\{A_\lambda\}_{\lambda\in\Lambda}$, $A_\lambda\in\mathfrak{I}_\mu$.

For each $\epsilon > 0$ and for each A_λ, let $\{A_{\lambda,\gamma}\}_{\gamma\in\Gamma_\lambda}$ be an inner cover for A_λ such that

$$\inf_{\gamma\in\Gamma_\lambda}\{\mathcal{B}(A_{\lambda,\gamma})\}\geq\mathcal{T}(A_\lambda)-\epsilon.$$

Since $\{A_{\lambda,\gamma}\}_{\gamma\in\Gamma_\lambda}$ is an inner cover for A_λ, we have $\{A_{\lambda,\gamma}\}_{\lambda\in\Lambda,\gamma\in\Gamma_\lambda}$ is an inner cover for $\bigvee_{\lambda\in\Lambda} A_\lambda$. Now,

$$\begin{aligned}
\mathcal{T}(\bigvee_{\lambda\in\Lambda}A_\lambda) &\geq \inf_{\lambda\in\Lambda,\gamma\in\Gamma_\lambda}\{\mathcal{B}(A_{\lambda,\gamma})\}\\
&= \inf_{\lambda\in\Lambda}\left\{\inf_{\gamma\in\Gamma_\lambda}\{\mathcal{B}(A_{\lambda,\gamma})\}\right\}\\
&\geq \inf_{\lambda\in\Lambda}\{\mathcal{T}(A_\lambda)-\epsilon\}\\
&= \inf_{\lambda\in\Lambda}\{\mathcal{T}(A_\lambda)\}-\epsilon
\end{aligned}$$

Since this is true for every $\epsilon > 0$, we get

$$\mathcal{T}(\vee A_\lambda)\geq\inf_\lambda\{\mathcal{T}(A_\lambda)\}.$$

Thus $\mathcal{T}(\underset{\lambda\in\Lambda}{\vee} A_\lambda) = \underset{\lambda\in\Lambda}{\wedge}\mathcal{T}(A_\lambda)$ for any collection $\{A_\lambda\}_{\lambda\in\Lambda}$, $A_\lambda \in \mathfrak{I}_\mu$. Hence $\mathcal{T}$ is a smooth fuzzy topology on μ. □

From the definition of $\mathcal{T}$ constructed from $\mathcal{B}$, it is clear that the smooth fuzzy topology generated by a basis is unique.

Theorem 3.6. *Let $(\mu, \mathcal{T})$ be a smooth fuzzy topological space, then $\mathcal{T}$ is a basis for a smooth fuzzy topology on μ and the smooth fuzzy topology generated by $\mathcal{T}$ is itself.*

Proof. We take $\mathcal{B} = \mathcal{T}$ and prove that $\mathcal{B}$ is a basis for the smooth fuzzy topology $\mathcal{T}$ on μ. For any $x \in X$ and $\epsilon > 0$, taking $A = \mu$, we have $A(x) = \mu(x)$, $A \le \mu$ and

$$\mathcal{B}(A) = \mathcal{T}(A) = \mathcal{T}(\mu) \ge 1 - \epsilon.$$

Let $x \in X$ and A, B in $\mathfrak{I}_\mu$ and $\epsilon > 0$. Taking $C = A \wedge B$, we have $C(x) = A(x) \wedge B(x)$ and $C \le A \wedge B$. Now,

$$\mathcal{B}(C) = \mathcal{T}(C) = \mathcal{T}(A \wedge B) \ge \mathcal{T}(A) \wedge \mathcal{T}(B) = (\mathcal{B}(A) \wedge \mathcal{B}(B)) - \epsilon.$$

Thus $\mathcal{B}$ is a basis for a smooth fuzzy topology on μ.

Now we prove that the smooth fuzzy topology generated by $\mathcal{B}$ is $\mathcal{T}$. Let $\mathcal{T}'$ be the smooth fuzzy topology generated by $\mathcal{B}$. Let $E \in \mathfrak{I}_\mu$. Since $\mathcal{T}'$ is the smooth fuzzy topology generated by $\mathcal{B}$, we have

$$\mathcal{T}'(E) = \sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}(E_\lambda)\}\right\},$$

where $\{C_\Lambda\}_{\Lambda\in\Gamma}$ is the collection of all possible inner covers $\{E_\lambda\}_{\lambda\in\Lambda}$ of E. Since E itself is an inner cover for E, we have

$$\mathcal{T}'(E) \ge \mathcal{B}(E) = \mathcal{T}(E).$$

Thus $\mathcal{T}' \ge \mathcal{T}$. But by our assumption, $\mathcal{T}$ is a smooth fuzzy topology on μ. Thus for any inner cover $\{E_\lambda\}_{\lambda\in\Lambda}$ of E we have,

$$\mathcal{T}(E) = \mathcal{T}(\underset{\lambda}{\vee} E_\lambda) \ge \wedge\mathcal{T}(E_\lambda).$$

This implies that

$$\mathcal{T}(E) \ge \sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}(E_\lambda)\}\right\} = \sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}(E_\lambda)\}\right\} = \mathcal{T}'(E).$$

This implies that $\mathcal{T} \ge \mathcal{T}'$ and hence $\mathcal{T} = \mathcal{T}'$. □

Definition 3.7. A function $\mathcal{B} : \mathfrak{I}_\mu \to I$ is said to be a basis for the smooth topology $\mathcal{T}$ on μ if $\mathcal{B}$ is a basis for a smooth fuzzy topology on μ and the topology generated by $\mathcal{B}$ is $\mathcal{T}$.

Theorem 3.8. *Let $(\mu, \mathcal{T})$ be a smooth fuzzy topological space. Let $\mathcal{B} : \mathfrak{I}_\mu \to [0, 1]$ be a function satisfying* *i.* $\mathcal{T}(A) \geq \mathcal{B}(A)$ *for all* $A \in \mathfrak{I}_\mu$

ii. if $A \in \mathfrak{I}_\mu$, $x \in X$ *and* $\epsilon > 0$, *then there exists* $B \in \mathfrak{I}_\mu$ *such that* $B \leq A$, $B(x) = A(x)$ *and* $\mathcal{B}(B) \geq \mathcal{T}(A) - \epsilon$.

Then $\mathcal{B}$ *is a basis for the smooth fuzzy topology* $\mathcal{T}$ *on* μ.

Proof. By a similar argument as in the proof of Theorem 3.6 we can verify that $\mathcal{B}$ is a basis. Now we prove that the smooth fuzzy topology generated by $\mathcal{B}$ is $\mathcal{T}$. Let $\mathcal{T}'$ be the smooth fuzzy topology generated by $\mathcal{B}$.

Let $E \in \mathfrak{I}_\mu$ and let $\{E_\lambda\}_{\lambda\in\Lambda}$ be an inner cover for E. Then for all $x \in X$ and for each E_λ, by (ii.) there exists $E_{\lambda,x} \in \mathfrak{I}_\mu$ such that

$$E_{\lambda,x}(x) = E_\lambda(x),\ E_{\lambda,x} \leq E_\lambda \ \text{ and } \ \mathcal{B}(E_{\lambda,x}) \geq \mathcal{T}(E_\lambda) - \epsilon.$$

Then the collection $\{E_{\lambda,x}\}_{x\in X}$ is an inner cover for E_λ and therefore the collection $\{E_{\lambda,x}\}_{\lambda\in\Lambda,x\in X}$ is an inner cover for E. Thus for any given inner cover $\{E_\lambda\}_{\lambda\in\Lambda}$ of E there exists an inner cover $\{E_{\lambda,x}\}_{\lambda\in\Lambda,x\in X}$ of E such that

$$\mathcal{B}(E_{\lambda,x}) \geq \mathcal{T}(E_\lambda) - \epsilon,$$

for all $\lambda \in \Lambda$, $x \in X$. This implies,

$$\inf_{\lambda\in\Lambda,x\in X}\{\mathcal{B}(E_{\lambda,x})\} \geq \inf_{\lambda\in\Lambda}\{\mathcal{T}(E_\lambda) - \epsilon\} = \inf_{\lambda\in\Lambda}\{\mathcal{T}(E_\lambda)\} - \epsilon.$$

Since this is true for every inner cover $C_\Lambda = \{E_\lambda\}_{\lambda\in\Lambda}$, we have

$$\sup_{\Lambda\in\Gamma}\left\{\inf_{\substack{E_\lambda\in C_\Lambda\\ x\in X}}\{\mathcal{B}(E_{\lambda,x})\}\right\} \geq \sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{T}(E_\lambda)\}\right\} - \epsilon,$$

where $\{C_\Lambda\}_{\Lambda\in\Gamma}$ is the collection of all possible inner covers $\{E_\lambda\}_{\lambda\in\Lambda}$ of E. But by definition of $\mathcal{T}'$ we have $\mathcal{T}'(E) \geq \sup_{\Lambda\in\Gamma}\left\{\inf_{\substack{E_\lambda\in C_\Lambda\\ x\in X}}\{\mathcal{B}(E_{\lambda,x})\}\right\}$. This implies,

$$\begin{aligned}\mathcal{T}'(E) &\geq \sup_{\Lambda\in\Gamma}\left\{\inf_{\substack{E_\lambda\in C_\Lambda\\ x\in X}}\{\mathcal{B}(E_{\lambda,x})\}\right\}\\ &\geq \sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{T}(E_\lambda)\}\right\} - \epsilon\\ &= \mathcal{T}(E) - \epsilon\end{aligned}$$

Since this is true for every $\epsilon > 0$, we have $\mathcal{T}'(E) \geq \mathcal{T}(E)$ and therefore $\mathcal{T}' \geq \mathcal{T}$.

Now, let $E \in \mathfrak{I}_\mu$ and $C_\Lambda = \{E_\lambda\}_{\lambda\in\Lambda}$ be an inner cover for E. Then by (i.) we have, $\mathcal{T}(E_\lambda) \geq \mathcal{B}(E_\lambda)$ for all $\lambda \in \Lambda$. This implies that

$$\inf_{E_\lambda\in C_\Lambda}\{\mathcal{T}(E_\lambda)\} \geq \inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}(E_\lambda)\}.$$

Since this is true for every inner cover $C_\Lambda = \{E_\lambda\}_{\lambda\in\Lambda}$, we have

$$\sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{T}(E_\lambda)\}\right\} \geq \sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}(E_\lambda)\}\right\}.$$

This implies that $\mathcal{T} \geq \mathcal{T}'$ and hence $\mathcal{T} = \mathcal{T}'$. □

Theorem 3.9. *If $\mathcal{B}$ is a basis for the smooth fuzzy topological space $(\mu, \mathcal{T})$, then*

i. $\mathcal{T}(A) \geq \mathcal{B}(A)$ *for all* $A \in \mathfrak{I}_\mu$.

ii. if $x \in X$, $A \in \mathfrak{I}_\mu$ *and* $\epsilon > 0$, *then there exists* $B \in \mathfrak{I}_\mu$ *such that* $B \leq A$, $B(x) = A(x)$ *and* $\mathcal{B}(B) \geq \mathcal{T}(A) - \epsilon$.

Proof. Let $A \in \mathfrak{I}_\mu$. Since $\{A\}$ is an inner cover for A, (i.) follows. To prove (ii.), let $x \in X$, $A \in \mathfrak{I}_\mu$ and $\epsilon > 0$. Then there exists an inner cover $C_\Lambda = \{A_\lambda\}_{\lambda\in\Lambda}$ such that $\inf\limits_{A_\lambda\in C_\Lambda}\{\mathcal{B}(A_\lambda)\} \geq \mathcal{T}(A) - \epsilon$. Since $C_\Lambda = \{A_\lambda\}_{\lambda\in\Lambda}$ is an inner cover for A, there exists $A_{\lambda_0} \in \{A_\lambda\}_{\lambda\in\Lambda}$ such that $A_{\lambda_0}(x) = A(x)$. Since $A_{\lambda_0} \in \{A_\lambda\}_{\lambda\in\Lambda}$, we have $A_{\lambda_0} \leq A$ and $\mathcal{B}(A_{\lambda_0}) \geq \mathcal{T}(A) - \epsilon$. □

Theorem 3.8 and Theorem 3.9 together give a characterization for a function $\mathcal{B} : \mathfrak{I}_\mu \to [0, 1]$ to be a basis for a smooth fuzzy topology.

Theorem 3.10. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the smooth fuzzy topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on μ. Then the following conditions are equivalent.*

i. $\mathcal{T}'$ *is finer than* $\mathcal{T}$.

ii. if $B \in \mathfrak{I}_\mu$, $x \in X$ *and* $\epsilon > 0$ *there exists* $B' \in \mathfrak{I}_\mu$ *such that* $B' \leq B$, $B'(x) = B(x)$ *and* $\mathcal{B}'(B') \geq \mathcal{B}(B) - \epsilon$.

Proof. First we assume that $\mathcal{T}'$ is finer than $\mathcal{T}$. Let $B \in \mathfrak{I}_\mu$, $x \in X$ and $\epsilon > 0$. Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$, by Theorem 3.9, there exists $B' \in \mathfrak{I}_\mu$ such that

$$B'(x) = B(x),\ B' \leq B \text{ and } \mathcal{B}'(B') \geq \mathcal{T}'(B) - \epsilon.$$

By our assumption, we have $\mathcal{T}'(B) \geq \mathcal{T}(B)$, and hence we obtain that

$$\mathcal{B}'(B') \geq \mathcal{T}'(B) - \epsilon \geq \mathcal{T}(B) - \epsilon \geq \mathcal{B}(B) - \epsilon.$$

Conversely, we assume that (ii.) holds. Let $E \in \mathfrak{I}_\mu$ and $\epsilon > 0$. Let $\{C_\Lambda\}_{\Lambda\in\Gamma}$ be the collection of all possible inner covers $\{E_\lambda\}_{\lambda\in\Lambda}$ of E. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, there exists an inner cover $C_\Lambda = \{E_\lambda\}_{\lambda\in\Lambda}$ such that

$$\inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}(E_\lambda)\} \geq \mathcal{T}(E) - \epsilon.$$

Then by (ii.) for each E_λ and for each $x \in X$ there exists $E_{\lambda,x} \in \mathfrak{I}_\mu$ such that $E_{\lambda,x}(x) = E_\lambda(x)$, $E_{\lambda,x} \leq E_\lambda$ and

$$\mathcal{B}'(E_{\lambda,x}) \geq \mathcal{B}(E_\lambda) - \epsilon.$$

Then $\{E_{\lambda,x}\}_{x\in X}$ is an inner cover for E_λ and therefore $\{E_{\lambda,x}\}_{\lambda\in\Lambda,x\in X}$ is an inner cover for E. Thus for any given inner cover $C_\Lambda = \{E_\lambda\}_{\lambda\in\Lambda}$, there exists an inner cover $\{E_{\lambda,x}\}_{\lambda\in\Lambda,x\in X}$ of E such that

$$\mathcal{B}'(E_{\lambda,x}) \geq \mathcal{B}(E_\lambda) - \epsilon,$$

for all $\lambda \in \Lambda$, $x \in X$. This implies that,

$$\inf_{x\in X}\{\mathcal{B}'(E_{\lambda,x})\} \geq \mathcal{B}(E_\lambda) - \epsilon$$

for all $\lambda \in \Lambda$ and hence

$$\inf_{\substack{E_\lambda\in C_\Lambda\\ x\in X}}\{\mathcal{B}'(E_{\lambda,x})\} \geq \inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}(E_\lambda) - \epsilon\} \geq \inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}(E_\lambda)\} - \epsilon.$$

Since this is true for every inner cover $C_\Lambda = \{E_\lambda\}_{\lambda\in\Lambda}$, we have

$$\sup_{\Lambda\in\Gamma}\left\{\inf_{\substack{E_\lambda\in C_\Lambda\\ x\in X}}\{\mathcal{B}'(E_{\lambda,x})\}\right\} \geq \sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}(E_\lambda)\}\right\} - \epsilon.$$

But by definition of $\mathcal{T}'$ we have

$$\begin{aligned}\mathcal{T}'(E) &\geq \sup_{\Lambda\in\Gamma}\left\{\inf_{\substack{E_\lambda\in C_\Lambda\\ x\in X}}\{\mathcal{B}'(E_{\lambda,x})\}\right\}\\ &\geq \sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}(E_\lambda)\}\right\} - \epsilon\\ &= \mathcal{T}(E) - \epsilon.\end{aligned}$$

As this is true for every $\epsilon > 0$ we have, $\mathcal{T}'(E) \geq \mathcal{T}(E)$, and hence $\mathcal{T}' \geq \mathcal{T}$. □

Theorem 3.11. *If $\{\mathcal{T}_\delta\}_{\delta\in\Delta}$ is a family of smooth fuzzy topologies on μ, then $\inf_{\delta\in\Delta}\{\mathcal{T}_\delta\}$ is also a smooth fuzzy topology on μ.*

Proof. Let $\mathcal{T} = \inf_{\delta\in\Delta}\{\mathcal{T}_\delta\}$. Clearly $\mathcal{T}(\mu) = 1$ and $\mathcal{T}(0_X) = 1$.

Let A and B be in $\mathfrak{I}_\mu$. Then

$$\begin{aligned}\mathcal{T}(A\wedge B) &= \inf_{\delta\in\Delta}\{\mathcal{T}_\delta(A\wedge B)\}\\ &\geq \inf_{\delta\in\Delta}\{\mathcal{T}_\delta(A)\wedge\mathcal{T}_\delta(B)\}\\ &= \inf_{\delta\in\Delta}\{\mathcal{T}_\delta(A)\}\wedge\inf_{\delta\in\Delta}\{\mathcal{T}_\delta(B)\}\\ &= \mathcal{T}(A)\wedge\mathcal{T}(B).\end{aligned}$$

Let $\{A_\lambda\}_{\lambda\in\Lambda}$, $A_\lambda \in \mathfrak{I}_\mu$ be any collection of fuzzy subsets of X. Then

$$\begin{aligned}\mathcal{T}(\bigvee_{\lambda\in\Lambda} A_\lambda) &= \inf_{\delta\in\Delta}\{\mathcal{T}_\delta(\bigvee_{\lambda\in\Lambda} A_\lambda)\}\\ &\geq \inf_{\delta\in\Delta}\{\bigwedge_{\lambda\in\Lambda}\{\mathcal{T}_\delta(A_\lambda)\}\}\\ &= \bigwedge_{\lambda\in\Lambda}\{\inf_{\delta\in\Delta}\{\mathcal{T}_\delta(A_\lambda)\}\}\\ &= \bigwedge_{\lambda\in\Lambda}\mathcal{T}(A_\lambda).\end{aligned}$$

Thus $\mathcal{T}$ is a smooth fuzzy topology on μ. □

As in the case of crisp topology, if $\{\mathcal{T}_\delta\}_{\delta\in\Delta}$ is a family of smooth fuzzy topologies on μ, then $\sup_{\delta\in\Delta}\{\mathcal{T}_\delta\}$ need not be a smooth fuzzy topology on μ.

4. Product of Smooth Fuzzy Topological Spaces

In [10] the product of smooth fuzzy topological spaces is defined and discussed, and in [4] the definition is modified to get that product of closed sets is closed. However, the definitions of the product topology given in both papers [10] and [4] are not well defined. Indeed, we justify that Definition 3.3 of [10] (see Definition 2.6) defining the product is not well-defined.

Let $X_1 = \mathbb{R}$ and $X_2 = \mathbb{R}$. Let U_1 and U_2 be defined by $U_1(x) = \frac{1}{2}$ for all $x \in X_1$ and $U_2(x) = \frac{1}{2}$ for all $x \in X_2$. Let $\mathcal{T}_1$ be a function from I^{X_1} to I defined by $\mathcal{T}_1(0_{X_1}) = \mathcal{T}_1(1_{X_1}) = 1$, $\mathcal{T}_1(U_1) = \frac{1}{4}$ and $\mathcal{T}_1(A) = 0$ for all other $A \in I^{X_1}$. Let $\mathcal{T}_2$ be a function from I^{X_2} to I defined by $\mathcal{T}_1(0_{X_2}) = \mathcal{T}_1(1_{X_2}) = 1$, $\mathcal{T}_2(U_2) = \frac{1}{2}$ and $\mathcal{T}_2(A) = 0$ for all other $A \in I^{X_2}$. Clearly $\mathcal{T}_1$ and $\mathcal{T}_2$ are gradation of openness on X_1 and X_2 respectively. Let

$$\mathscr{S} = \{P_1^{-1}(U) : \mathcal{T}_1(U) > 0\} \cup \{P_2^{-1}(U) : \mathcal{T}_2(U) > 0\}.$$

Let $X = X_1 \times X_2$ and let $V(x_1, x_2) = \frac{1}{2}$ for all $(x_1, x_2) \in X$. Clearly it follows that

$$V = P_1^{-1}(U_1) = P_2^{-1}(U_2).$$

This shows the Definition 3.3 of [10] is not well defined.

So we define the concept of product fuzzy topology in a new manner as follows.

Definition 4.1. Let μ and ν be fuzzy subsets of X and Y respectively. Let $(\mu, \mathcal{T}_\mu)$ and $(\nu, \mathcal{T}_\nu)$ be two smooth fuzzy topological spaces. A basis $\mathcal{B}$ for the smooth fuzzy topology on $\mu \times \nu$ is defined as a function $\mathcal{B} : \mathfrak{I}_{\mu\times\nu} \to [0, 1]$ as follows:

> Let $E \in \mathfrak{I}_{\mu\times\nu}$. If E cannot be written as $A \times B$ for any $A \in \mathfrak{I}_\mu$ and $B \in \mathfrak{I}_\nu$, then define $\mathcal{B}(E) = 0$. Otherwise define
>
> $$\mathcal{B}(E) = \sup_{\lambda\in\Lambda}\{\mathcal{T}_\mu(A_\lambda) \wedge \mathcal{T}_\nu(B_\lambda)\}$$
>
> where $\{A_\lambda \times B_\lambda\}_{\lambda\in\Lambda}$ is the collection of all possible ways of writing E as $E = A_\lambda \times B_\lambda$, where $A_\lambda \in \mathfrak{I}_\mu$, $B_\lambda \in \mathfrak{I}_\nu$.

The smooth fuzzy topology generated by $\mathcal{B}$ is called the product topology on $\mu \times \nu$.

Theorem 4.2. *Let $(\mu, \mathcal{T}_\mu)$ and $(\nu, \mathcal{T}_\nu)$ be two smooth fuzzy topological spaces. Let $\mathcal{B} : \mathfrak{I}_{\mu\times\nu} \to [0, 1]$ be the function defined in Definition 4.1. Then $\mathcal{B}$ is a basis for the smooth fuzzy topology on $\mu \times \nu$.*

Proof. Since $\mathcal{T}_\mu(\mu) = \mathcal{T}_\nu(\nu) = 1$, we have $\mathcal{B}(\mu \times \nu) = 1$ and hence (i.) of Definition 3.1 follows. Let $(x, y) \in X$, A, B in $\mathfrak{I}_{\mu\times\nu}$ and $\epsilon > 0$. Suppose any one of A and B, say A, cannot be written as $A_1 \times A_2$ for any $A_1 \in \mathfrak{I}_\mu$ and $A_2 \in \mathfrak{I}_\nu$, then $\mathcal{B}(A) = 0$, and hence (ii.) of Definition 3.1 follows in this case. Otherwise by definition of $\mathcal{B}$, there exists $A_1, B_1 \in \mathfrak{I}_\mu$, and $A_2, B_2 \in \mathfrak{I}_\nu$ such that $A = A_1 \times A_2$, $B = B_1 \times B_2$,

$$\mathcal{T}_\mu(A_1) \wedge \mathcal{T}_\nu(A_2) \geq \mathcal{B}(A) - \epsilon \text{ and } \mathcal{T}_\mu(B_1) \wedge \mathcal{T}_\nu(B_2) \geq \mathcal{B}(B) - \epsilon.$$

Let $C = A \wedge B$. Clearly $C(x, y) = A(x, y) \wedge B(x, y)$ and $C \leq A \wedge B$. Now,

$$\begin{aligned}
\mathcal{B}(C) &= \mathcal{B}(A \wedge B) \\
&= \mathcal{B}((A_1 \times A_2) \wedge (B_1 \times B_2)) \\
&= \mathcal{B}((A_1 \wedge B_1) \times (A_2 \wedge B_2)) \\
&\geq \mathcal{T}_\mu(A_1 \wedge B_1) \wedge \mathcal{T}_\nu(A_2 \wedge B_2) \\
&\geq (\mathcal{T}_\mu(A_1) \wedge \mathcal{T}_\mu(B_1)) \wedge (\mathcal{T}_\nu(A_2) \wedge \mathcal{T}_\nu(B_2)) \\
&= \{\mathcal{T}_\mu(A_1) \wedge \mathcal{T}_\nu(A_2)\} \wedge \{\mathcal{T}_\mu(B_1) \wedge \mathcal{T}_\nu(B_2)\} \\
&\geq (\mathcal{B}(A) - \epsilon) \wedge (\mathcal{B}(B) - \epsilon) \\
&= (\mathcal{B}(A) \wedge \mathcal{B}(B)) - \epsilon
\end{aligned}$$

and hence (ii.) of Definition 3.1 follows in this case also. Thus $\mathcal{B}$ is a basis. □

Theorem 4.3. *Let $(\mu, \mathcal{T}_\mu)$ and $(\nu, \mathcal{T}_\nu)$ be two smooth fuzzy topological spaces. Let $\mathcal{B}_\mu, \mathcal{B}_\nu$ be bases for the smooth fuzzy topologies $\mathcal{T}_\mu, \mathcal{T}_\nu$ respectively. Define a function $\mathcal{B}_{\mu\times\nu} : \mathfrak{I}_{\mu\times\nu} \to [0, 1]$ as follows:*

> *Let $E \in \mathfrak{I}_{\mu\times\nu}$. If E cannot be written as $A \times B$ for any $A \in \mathfrak{I}_\mu$ and $B \in \mathfrak{I}_\nu$, then define $\mathcal{B}_{\mu\times\nu}(E) = 0$. Otherwise define*
>
> $$\mathcal{B}_{\mu\times\nu}(E) = \sup_{\lambda\in\Lambda}\{\mathcal{B}_\mu(A_\lambda) \wedge \mathcal{B}_\nu(B_\lambda)\}$$
>
> *where $\{A_\lambda \times B_\lambda\}_{\lambda\in\Lambda}$ is the collection of all possible ways of writing E as $E = A_\lambda \times B_\lambda$, where $A_\lambda \in \mathfrak{I}_\mu$, $B_\lambda \in \mathfrak{I}_\nu$.*

then $\mathcal{B}_{\mu\times\nu}$ is a basis for the product topology on $\mu \times \nu$.

Proof. Let $(x, y) \in X \times Y$ and $\epsilon > 0$. Since $\mathcal{B}_\mu$ and $\mathcal{B}_\nu$ are bases for the smooth fuzzy topologies $\mathcal{T}_\mu$ and $\mathcal{T}_\nu$, there exists $A \in \mathfrak{I}_\mu$ and $B \in \mathfrak{I}_\nu$ such that $A(x) = \mu(x)$, $B(y) = \nu(y)$ with

$$\mathcal{B}_\mu(A) \geq 1 - \epsilon \text{ and } \mathcal{B}_\nu(B) \geq 1 - \epsilon.$$

Taking $E = A \times B$, we have

$$\mathcal{B}_{\mu\times\nu}(E) \geq \mathcal{B}_\mu(A) \wedge \mathcal{B}_\nu(B) \geq 1 - \epsilon.$$

Thus (i.) of Definition 3.1 follows.

Let $(x, y) \in X$, A, B in $\Im_{\mu\times\nu}$ and $\epsilon > 0$. Suppose any one of A and B, say A, cannot be written as $A_1 \times A_2$ for any $A_1 \in \Im_\mu$ and $A_2 \in \Im_\nu$, then $\mathcal{B}_{\mu\times\nu}(A) = 0$, and hence (ii.) of Definition 3.1 follows in this case. Otherwise by definition of $\mathcal{B}_{\mu\times\nu}$, there exists $A_1, B_1 \in \Im_\mu$, and $A_2, B_2 \in \Im_\nu$ such that $A = A_1 \times A_2$, $B = B_1 \times B_2$,

$$\mathcal{B}_\mu(A_1) \wedge \mathcal{B}_\nu(A_2) \geq \mathcal{B}_{\mu\times\nu}(A) - \frac{\epsilon}{2} \text{ and } \mathcal{B}_\mu(B_1) \wedge \mathcal{B}_\nu(B_2) \geq \mathcal{B}_{\mu\times\nu}(B) - \frac{\epsilon}{2}.$$

Now since $x \in X$ and A_1, B_1 are in $\Im_\mu$, there exists $C_1 \in \Im_\mu$ such that $C_1(x) = (A_1 \wedge B_1)(x)$, $C_1 \leq A_1 \wedge B_1$ and

$$\mathcal{B}_\mu(C_1) \geq (\mathcal{B}_\mu(A_1) \wedge \mathcal{B}_\mu(B_1)) - \frac{\epsilon}{2}.$$

Analogously, there exists $C_2 \in \Im_\nu$ such that $C_2(y) = (A_2 \wedge B_2)(y)$, $C_2 \leq A_2 \wedge B_2$ and

$$\mathcal{B}_\nu(C_2) \geq (\mathcal{B}_\nu(A_2) \wedge \mathcal{B}_\nu(B_2)) - \frac{\epsilon}{2}.$$

Let $C = C_1 \times C_2$. Then,

$$\begin{aligned} C(x, y) &= (C_1 \times C_2)(x, y) \\ &= C_1(x) \wedge C_2(y) \\ &= (A_1 \wedge B_1)(x) \wedge (A_2 \wedge B_2)(y) \\ &= (A_1(x) \wedge B_1(x)) \wedge (A_2(y) \wedge B_2(y)) \\ &= (A_1(x) \wedge A_2(y)) \wedge (B_1(x) \wedge B_2(y)) \\ &= (A_1(x) \times A_2(y)) \wedge (B_1(x) \times B_2(y)) \\ &= A(x, y) \wedge B(x, y) \end{aligned}$$

and similarly it follows that $C \leq A \wedge B$. Now,

$$\begin{aligned} \mathcal{B}_{\mu\times\nu}(C) &= \mathcal{B}_{\mu\times\nu}(C_1 \times C_2) \\ &\geq \mathcal{B}_\mu(C_1) \wedge \mathcal{B}_\nu(C_2) \\ &\geq \left\{(\mathcal{B}_\mu(A_1) \wedge \mathcal{B}_\mu(B_1)) - \frac{\epsilon}{2}\right\} \wedge \left\{(\mathcal{B}_\nu(A_2) \wedge \mathcal{B}_\nu(B_2)) - \frac{\epsilon}{2}\right\} \\ &= \{(\mathcal{B}_\mu(A_1) \wedge \mathcal{B}_\mu(B_1)) \wedge (\mathcal{B}_\nu(A_2) \wedge \mathcal{B}_\nu(B_2))\} - \frac{\epsilon}{2} \\ &= \{(\mathcal{B}_\mu(A_1) \wedge \mathcal{B}_\nu(A_2)) \wedge (\mathcal{B}_\mu(B_1) \wedge \mathcal{B}_\nu(B_2))\} - \frac{\epsilon}{2} \\ &\geq \left(\mathcal{B}_{\mu\times\nu}(A) - \frac{\epsilon}{2}\right) \wedge \left(\mathcal{B}_{\mu\times\nu}(B) - \frac{\epsilon}{2}\right) - \frac{\epsilon}{2} \\ &= (\mathcal{B}_{\mu\times\nu}(A) \wedge \mathcal{B}_{\mu\times\nu}(B)) - \frac{\epsilon}{2} - \frac{\epsilon}{2} \\ &= (\mathcal{B}_{\mu\times\nu}(A) \wedge \mathcal{B}_{\mu\times\nu}(B)) - \epsilon. \end{aligned}$$

Thus (ii.) of Definition 3.1 follows in this case also. Hence $\mathcal{B}_{\mu\times\nu}$ is a basis for a smooth fuzzy topology on $\mu \times \nu$. Let $\mathcal{T}$ be the smooth fuzzy topology generated by $\mathcal{B}_{\mu\times\nu}$, $\mathcal{T}_{\mu\times\nu}$ be the product topology on $\mu \times \nu$ and $\mathcal{B}^p_{\mu\times\nu}$ be the basis for $\mathcal{T}_{\mu\times\nu}$ as described in Definition 4.1. Now we prove that $\mathcal{T}_{\mu\times\nu} = \mathcal{T}$. Let $E \in \mathfrak{I}_{\mu\times\nu}$, then

$$\mathcal{T}(E) = \sup_{\Lambda\in\Gamma}\left\{\inf_{E_\lambda\in C_\Lambda}\left\{\mathcal{B}^p_{\mu\times\nu}(E_\lambda)\right\}\right\},$$

where $\{C_\Lambda\}_{\Lambda\in\Gamma}$ is the collection of all possible inner covers $C_\Lambda = \{E_\lambda\}_{\lambda\in\Lambda}$ of E. We divide the collection $\{C_\Lambda\}_{\Lambda\in\Gamma}$, say $\mathfrak{C}$, into two subcollections $\mathfrak{C}'$ and $\mathfrak{C}''$ where $\mathfrak{C}'$ is the collection all possible inner covers $\{E_\lambda\}_{\lambda\in\Lambda}$ of E so that for all $\lambda \in \Lambda$, E_λ is of the form $A_\lambda \times B_\lambda$ for at least one $A_\lambda \in \mathfrak{I}_\mu$ and one $B_\lambda \in \mathfrak{I}_\nu$, and $\mathfrak{C}''$ is the complement of $\mathfrak{C}'$ in $\mathfrak{C}$. If an inner cover $C_\Lambda = \{E_\lambda\}_{\lambda\in\Lambda}$ of E is in $\mathfrak{C}''$, then for at least one $\lambda_0 \in \Lambda$, E_{λ_0} is not of the form $A \times B$ for any $A \in \mathfrak{I}_\mu$ and $B \in \mathfrak{I}_\nu$; hence $\mathcal{B}^p_{\mu\times\nu}(E_{\lambda_0}) = 0$ and therefore

$$\inf_{E_\lambda\in C_\Lambda}\left\{\mathcal{B}^p_{\mu\times\nu}(E_\lambda)\right\} = 0 = \inf_{E_\lambda\in C_\Lambda}\left\{\mathcal{B}_{\mu\times\nu}(E_\lambda)\right\}.$$

If $\mathfrak{C}' = \emptyset$, then $\mathcal{T}_{\mu\times\nu}(E) = \mathcal{T}(E) = 0$ and hence we consider the case $\mathfrak{C}' \neq \emptyset$. Now

$$\begin{aligned}
\mathcal{T}_{\mu\times\nu}(E) &= \sup_{\mathfrak{C}}\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}^p_{\mu\times\nu}(E_\lambda)\}\} \\
&= \sup_{\mathfrak{C}'}\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}^p_{\mu\times\nu}(E_\lambda)\}\} \\
&= \sup_{\mathfrak{C}'}\{\inf_{E_\lambda\in C_\Lambda}\{\sup_{E_\lambda=A_\lambda\times B_\lambda}\{\inf\{\mathcal{T}_\mu(A_\lambda),\mathcal{T}_\nu(B_\lambda)\}\}\}\} \\
&\geq \sup_{\mathfrak{C}'}\{\inf_{E_\lambda\in C_\Lambda}\{\sup_{E_\lambda=A_\lambda\times B_\lambda}\{\inf\{\mathcal{B}_\mu(A_\lambda),\mathcal{B}_\nu(B_\lambda)\}\}\}\} \\
&= \sup_{\mathfrak{C}'}\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}_{\mu\times\nu}(E_\lambda)\}\} \\
&= \sup_{\mathfrak{C}}\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}_{\mu\times\nu}(E_\lambda)\}\} \\
&= \mathcal{T}(E)
\end{aligned}$$

This implies that, $\mathcal{T}_{\mu\times\nu} \geq \mathcal{T}$.

To prove the reverse inequality, let $E \in \mathfrak{I}_{\mu\times\nu}$ and $\mathfrak{C}, \mathfrak{C}', \mathfrak{C}''$ be as above. Let $C_\Lambda = \{E_\lambda\}_{\lambda\in\Lambda}$ be an inner cover for E. As above it is enough to consider the case $\mathfrak{C}' \neq \emptyset$. Now let $C_\Lambda \in \mathfrak{C}'$. Then for all $\lambda \in \Lambda$, we have $E_\lambda = A \times B$ for at least one $A \in \mathfrak{I}_\mu$ and one $B \in \mathfrak{I}_\nu$. Fix a $\lambda \in \Lambda$. Let $\mathfrak{B}_\lambda$ denote the set of all pairs (A, B) such that $E_\lambda = A \times B$. Let $(A, B) \in \mathfrak{B}_\lambda$ and $\epsilon > 0$. Since $\mathcal{B}_\mu$, $\mathcal{B}_\nu$ are bases for $\mathcal{T}_\mu$, $\mathcal{T}_\nu$, by Theorem 3.9, for any $x \in X$ and $y \in Y$, there exists $A_x \in \mathfrak{I}_\mu$ and $B_y \in \mathfrak{I}_\nu$ such that $A_x(x) = A(x)$, $A_x \leq A$ and $B_y(y) = B(y)$, $B_y \leq B$ with

$$\mathcal{B}_\mu(A_x) + \epsilon \geq \mathcal{T}_\mu(A) \text{ and } \mathcal{B}_\nu(B_y) + \epsilon \geq \mathcal{T}_\nu(B).$$

Clearly $\{A_x\}_{x\in X}$ is an inner cover for A and $\{B_y\}_{y\in Y}$ is an inner cover for B. Then the collection $\{A_x \times B_y\}_{x\in X,\, y\in Y}$ is an inner cover for $A \times B = E_\lambda$. Thus for any pair $(A, B) \in \mathfrak{B}_\lambda$, we have an inner cover $\{A_x \times B_y\}_{x\in X,\, y\in Y}$ for E_λ such that

$$\mathcal{B}_\mu(A_x) + \epsilon \geq \mathcal{T}_\mu(A) \text{ and } \mathcal{B}_\nu(B_y) + \epsilon \geq \mathcal{T}_\nu(B),$$

for all $x \in X$, $y \in X$. Now,

$$\begin{aligned}
\mathcal{T}_{\mu\times\nu}(E) &= \sup_{\mathfrak{C}}\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{B}^p_{\mu\times\nu}(E_\lambda)\}\} \\
&= \sup_{\mathfrak{C}'}\{\inf_{E_\lambda\in C_\Lambda}\{\sup_{(A,B)\in\mathfrak{B}_\lambda}\{\mathcal{T}_\mu(A)\wedge\mathcal{T}_\nu(B)\}\}\} \\
&\le \sup_{\mathfrak{C}'}\{\inf_{E_\lambda\in C_\Lambda}\{\sup_{(A,B)\in\mathfrak{B}_\lambda}\{\inf_{(x,y)\in X\times Y}\{\mathcal{B}_\mu(A_x)\wedge\mathcal{B}_\nu(B_y)\}+\epsilon\}\}\} \\
&\le \sup_{\mathfrak{C}'}\{\inf_{E_\lambda\in C_\Lambda}\{\sup_{(A,B)\in\mathfrak{B}_\lambda}\{\inf_{(x,y)\in X\times Y}\{\mathcal{B}_{\mu\times\nu}(A_x\times B_y)\}\}\}\}+\epsilon \\
&\le \sup_{\mathfrak{C}'}\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{T}(E_\lambda)\}\}+\epsilon \\
&= \sup_{\mathfrak{C}}\{\inf_{E_\lambda\in C_\Lambda}\{\mathcal{T}(E_\lambda)\}\}+\epsilon \\
&= \mathcal{T}(E)+\epsilon.
\end{aligned}$$

Since this is true for every $\epsilon > 0$ we have, $\mathcal{T}_{\mu\times\nu}(E) \le \mathcal{T}(E)$, and hence $\mathcal{T}_{\mu\times\nu} \le \mathcal{T}$. Therefore both the smooth fuzzy topologies are the same. □

Now we discuss, the closedness of product of fuzzy subsets in the product of smooth fuzzy topologies. When $\mu = 1$ the smooth fuzzy topological space $(\mu, \mathcal{T}_\mu)$ on X is usually denoted by $(X, \mathcal{T}_X)$.

Theorem 4.4. *If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are two smooth fuzzy topological spaces, then $\mathcal{C}_{X\times Y}(A \times B) \ge \mathcal{C}_X(A) \wedge \mathcal{C}_Y(B)$ for all fuzzy subsets A and B of X and Y.*

Proof. We denote the smooth fuzzy product topology on $X \times Y$ by $\mathcal{T}_{X\times Y}$. From Definition 4.1 we have, $\mathcal{T}_{X\times Y}(A \times B) \ge \mathcal{T}_X(A) \wedge \mathcal{T}_Y(B)$. Now we claim that,

$$1_{X\times Y} - (A\times B) = ((1_X - A)\times 1_Y)\vee(1_X\times(1_Y - B)).$$

For,

$((1_X - A)\times 1_Y)\vee(1_X\times(1_Y - B))(x, y)$

$$\begin{aligned}
&= ((1_X - A)\times 1_Y)(x,y)\vee(1_X\times(1_Y - B))(x,y) \\
&= ((1_X - A)(x)\wedge 1_Y(y))\vee(1_X(x)\wedge(1_Y - B)(y)) \\
&= (1_X - A)(x)\vee(1_Y - B)(y) \\
&= (1 - A(x))\vee(1 - B(y)) \\
&= 1 - (A(x)\wedge B(y)) \\
&= 1 - (A\times B)(x,y) \\
&= (1_{X\times Y} - (A\times B))(x,y).
\end{aligned}$$

Now,

$$\begin{aligned}
\mathcal{C}_{X\times Y}(A\times B) &= \mathcal{T}_{X\times Y}(1_{X\times Y}-(A\times B))\\
&= \mathcal{T}_{X\times Y}(((1_X-A)\times 1_Y)\vee(1_X\times(1_Y-B)))\\
&\geq \mathcal{T}_{X\times Y}((1_X-A)\times 1_Y)\wedge\mathcal{T}_{X\times Y}(1_X\times(1_Y-B))\\
&\geq (\mathcal{T}_X(1_X-A)\wedge\mathcal{T}_Y(1_Y))\wedge(\mathcal{T}_X(1_X)\wedge\mathcal{T}_Y(1_Y-B))\\
&= \mathcal{T}_X(1_X-A)\wedge\mathcal{T}_Y(1_Y-B)\\
&= \mathcal{C}_X(A)\wedge\mathcal{C}_Y(B).
\end{aligned}$$

Thus $\mathcal{C}_{X\times Y}(A\times B)\geq\mathcal{C}_X(A)\wedge\mathcal{C}_Y(B)$. □

The identity $(\mu\times\nu)-(A\times B)=((\mu-A)\times\nu)\vee(\mu\times(\nu-B))$ is the fuzzy version of the crisp identity $(A\times B)^c=(A^c\times Y)\cup(X\times B^c)$. When $\mu=\nu=1$, the identity becomes

$$1_{X\times Y}-(A\times B)=((1_X-A)\times 1_Y)\vee(1_X\times(1_Y-B)),$$

which we proved above. But this identity is not true in general. For example, on $\mathbb{R}$ define $\mu(x)=\frac{1}{2}$, $\nu(x)=1$, $A(x)=\frac{1}{4}$ and $B(x)=\frac{1}{2}$. This is a situation where the smooth fuzzy theory defined on $\mathfrak{I}_\mu$ differs with the theory defined on I^X.

We note that Theorem 4.4 is not valid in general as seen in the following example.

Example 4.5. Let $X_1=X_2=\mathbb{R}$. Define $\mu:X_1\to[0,1]$ as $\mu(x)=\frac{1}{8}$ for all $x\in X_1$ and $\nu:X_2\to[0,1]$ as $\nu(x)=\frac{1}{4}$ for all $x\in X_2$. Define $\mathcal{T}_\mu:\mathfrak{I}_\mu\to[0,1]$ as follows

$$\mathcal{T}_\mu(A)=\begin{cases}1 & \text{if } A=\mu \ \text{ or } \ A=0_{X_1}\\ \frac{1}{2} & \text{if } A(x)=\frac{1}{10} \ \text{ for all } x\in X_1\\ \frac{1}{8} & \text{if } A(x)=\frac{1}{40} \ \text{ for all } x\in X_1\\ 0 & \text{otherwise}\end{cases}$$

Define $\mathcal{T}_\nu:\mathfrak{I}_\nu\to[0,1]$ as follows

$$\mathcal{T}_\nu(B)=\begin{cases}1 & \text{if } B=\nu \ \text{ or } \ B=0_{X_2}\\ \frac{1}{4} & \text{if } B(x)=\frac{1}{6} \ \text{ for all } x\in X_2\\ \frac{1}{8} & \text{if } B(x)=\frac{1}{12} \ \text{ for all } x\in X_2\\ 0 & \text{otherwise}\end{cases}$$

Clearly $\mathcal{T}_\mu$ and $\mathcal{T}_\nu$ are smooth fuzzy topologies on μ and ν respectively. Now, consider the fuzzy subsets $A(x)=\frac{1}{10}$ for all $x\in X_1$ and $B(x)=\frac{1}{12}$ for all $x\in X_2$ of X_1 and X_2 respectively. Then,

$$\mathcal{C}_\mu(A)=\mathcal{T}_\mu(\mu-A)=\frac{1}{8}$$

and

$$\mathcal{C}_\nu(B) = \mathcal{T}_\nu(\nu - B) = \frac{1}{4}.$$

Let $C = \mu \times \nu - A \times B$, then $C(x_1, x_2) = \frac{1}{24}$ for all (x_1, x_2) in $X_1 \times X_2$. Now consider

$$\mathcal{C}_{\mu\times\nu}(A \times B) = \mathcal{T}_{\mu\times\nu}(\mu \times \nu - A \times B) = \mathcal{T}_{\mu\times\nu}(C) = 0.$$

Thus $\mathcal{C}_{\mu\times\nu}(A \times B) \ngeq \mathcal{C}_\mu(A) \wedge \mathcal{C}_\nu(B)$, in this case.

Definition 4.6. Let μ be fuzzy subset of a set X. Let $\mathcal{S} : \mathfrak{I}_\mu \to [0,1]$ be a function satisfying the following condition:

Given $x \in X$ and $\epsilon > 0$ there exists $A \leq \mu$ such that

$$A(x) = \mu(x) \text{ and } \mathcal{S}(A) \geq 1 - \epsilon.$$

Then $\mathcal{S}$ is called a subbasis for a smooth fuzzy topology on μ.

We now present a way to get a basis from a subbasis, from which a smooth fuzzy topology can be generated.

Theorem 4.7. *Let μ be a fuzzy subset of X. Let $\mathcal{S} : \mathfrak{I}_\mu \to [0,1]$ be a subbasis for a smooth fuzzy topology on μ. Define $\mathcal{B} : \mathfrak{I}_\mu \to [0,1]$ as*

$$\mathcal{B}(A) = \sup_{D\in\mathfrak{D}}\{\inf_{i\in I_D}\{\mathcal{S}(A_i)\}\}$$

where $\mathfrak{D}$ is the family of all possible finite collections $D = \{A_i\}_{i\in I_D}$ of members of $\mathfrak{I}_\mu$ such that $A = \underset{i\in I_D}{\wedge} A_i$. Then the $\mathcal{B}$ is a basis for a smooth fuzzy topology on μ.

Proof. Since every $A \in \mathfrak{I}_\mu$ can be represented as the intersection of members of the collection consisting of A alone, and as $0 \leq \mathcal{S}(A) \leq 1$, $\mathcal{B}$ is well defined. As $\mathcal{B}$ clearly satisfies (i) of Definition 3.1, we prove (ii) of Definition 3.1 only.

Let $x \in X$, A, B in $\mathfrak{I}_\mu$ and $\epsilon > 0$. Then by definition of $\mathcal{B}$ there exist collections $\{A_i\}_{i=1,2,\ldots,n}$ and $\{B_j\}_{j=1,2,\ldots,m}$ such that

$$A = \overset{n}{\underset{i=1}{\wedge}} A_i, \quad B = \overset{m}{\underset{j=1}{\wedge}} B_j$$

with

$$\inf_i\{\mathcal{B}(A_i)\} \geq \mathcal{B}(A) - \epsilon \quad \text{and} \quad \inf_j\{\mathcal{B}(B_j)\} \geq \mathcal{B}(B) - \epsilon.$$

Then

$$A \wedge B = (\overset{n}{\underset{i=1}{\wedge}} A_i) \wedge (\overset{m}{\underset{j=1}{\wedge}} B_j).$$

Now define a collection of fuzzy subsets $C_k \leq \mu$, for $k = 1, 2, \ldots, n+m$, as

$$C_k = \begin{cases} A_k & \text{if } k \leq n \\ B_{k-n} & \text{if } k > n \end{cases}.$$

If we let $C = \bigwedge_{k=1}^{n+m} C_k$, then $C(x) = (A \wedge B)(x)$. By definition of $\mathcal{B}$, we have

$$\mathcal{B}(C) = \sup_{D \in \mathfrak{D}} \{ \inf_{i \in I_D} \{\mathcal{S}(E_i)\}\}$$

where $\mathfrak{D}$ is the family of all possible finite collections $D = \{E_i\}_{i \in I_D}$ of members of $\mathfrak{I}_\mu$ such that $C = \bigwedge_{i \in I_D} E_i$. Now,

$$\begin{aligned} \mathcal{B}(C) &= \sup_{D \in \mathfrak{D}} \{ \inf_{i \in I_D} \{\mathcal{S}(E_i)\}\} \\ &\geq \inf_k \{\mathcal{S}(C_k)\} \\ &= \inf_i \{\mathcal{S}(A_i)\} \wedge \inf_j \{\mathcal{S}(B_j)\} \\ &\geq (\mathcal{B}(A) - \epsilon) \wedge (\mathcal{B}(B) - \epsilon) \\ &= (\mathcal{B}(A) \wedge \mathcal{B}(B)) - \epsilon \end{aligned}$$

Thus, $\mathcal{B}(C) \geq (\mathcal{B}(A) \wedge \mathcal{B}(B)) - \epsilon$ and hence, $\mathcal{B}$ is a basis. □

5. Conclusion

We pointed out some mistakes in the theory of generating a smooth fuzzy topology from a basis and in the definition of product topology developed in [10]. By redefining basis, smooth fuzzy topology generated by a basis, product topology and subbasis, we rectify the mistakes in the existing theory of fuzzy topology generated by a basis and product of fuzzy topology introduced in [10].

The definitions of basis and subbasis given in this paper are more natural and resembles many interesting properties of crisp topology. Therefore many results available in the crisp topology may be extended without much difficulty using the definition given here. So the theory developed here will serve as a core to many theories in future.

References

[1] Azad, A. K., Fuzzy hausdorff spaces and fuzzy perfect mappings. J. Math. Anal. Appl. 82(2) (1981), 297-305.

[2] Chang, C. L., Fuzzy topological spaces. J. Math. Anal. Appl. 24(1) (1968), 182-190.

[3] Kalaivani, C., Fuzzy Proper Functions. Ph. D. Thesis. Alagappa University, Karaikudi - 630 004, India, (2012).

[4] Kalaivani, C., Roopkumar, R., Fuzzy Perfect Mappings and Q-Compactness in Smooth Fuzzy Topological Spaces. Fuzzy Inf. Eng. (6) (2014), 115-131.

[5] Munkres, J. R., Topology. New Jersey: Pearson Education Inc (2000).

[6] Park, C. K., Min, W. K., Kim, M. H., Weak smooth α-structure of smooth topological spaces. Commun. Korean Math. Soc. 19(1) (2004), 143-153.

[7] Peeters, W., Subspaces of smooth fuzzy topologies and initial smooth fuzzy structures. Fuzzy sets and Systems. 104(3) (1999), 423-433.

[8] Ramadan, A. A., Fath Alla, M. A., Abbas, S. E., Smooth Fuzzy topology on fuzzy sets. J. Fuzzy Math. 10(1) (2002), 59-68.

[9] Šostak, A. P., On a fuzzy topological structure. Rend. Circ. Matem. Palermo Ser. II, 11 (1985), 89-103.

[10] Srivastava, R., On separation axioms on newly defined fuzzy topological spaces. Fuzzy Sets and Systems. 62(3) (1994), 341-346.