

## On a two-variables fractional partial differential inclusion via Riemann-Liouville derivative

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**Abstract.** We investigate the existence of solution for a two-variables fractional partial differential inclusion via Riemann-Liouville derivative. Also, we provide an example to illustrate our main result.

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### 1. Introduction

There are many published works about fractional partial differential equations by using the notions delay or time-fractional (see for example, [1], [2], [9] and [10]). It is interesting to work on two variables fractional partial differential equations (see for example, [3], [4], [6] and [11]).

Let  $\theta = (0, 0)$  and  $\alpha = (\alpha_1, \alpha_2)$  where  $0 < \alpha_1, \alpha_2 \leq 1$ . Also, put  $J_a \times J_b = [0, a] \times [0, b]$  where  $a$  and  $b$  are positive constants. The Riemann-Liouville fractional partial integral of  $u \in L^1(J_a \times J_b)$  is defined by

$$(I_{\theta}^{\alpha} u)(x, y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1} (y-t)^{\alpha_2-1} u(s, t) dt ds$$

whenever the integral exists (see for example, [14], [15] and [16]). The Riemann-Liouville partial derivative of fractional order  $\alpha$  for a function  $u \in L^1(J_a \times J_b)$  is defined by

$$(D_{\theta}^{\alpha} u)(x, y) = D_{xy}^2 (I_{\theta}^{1-\alpha} u)(x, y) = \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y \frac{(x-s)^{-\alpha_1} (y-t)^{-\alpha_2}}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)} dt ds$$

(see for more details (see for example, [14], [15] and [16]). Note that,

$$(I_{\theta}^{\alpha} I_{\theta}^{\beta} u)(x, y) = (I_{\theta}^{\alpha+\beta} u)(x, y)$$

whenever  $\beta = (\beta_1, \beta_2) > 0$  ([16]).

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Let  $(X, d)$  be a metric space,  $\mathcal{P}(X)$  the class of all nonempty subsets of  $X$ ,  $\mathcal{P}_{cl}(X)$  the class of all closed subsets of  $X$ ,  $\mathcal{P}_{bd}(X)$  the class of all bounded subsets of  $X$ ,  $\mathcal{P}_{cp}(X)$  the class of all compact subsets of  $X$  and  $\mathcal{P}_{cv}(X)$  the class of all convex subsets of  $X$ . A multi-valued map  $F : J_a \times J_b \rightarrow \mathcal{P}_{cl}(\mathbb{R})$  is measurable whenever the function  $(x, y) \mapsto d(w, F(x, y)) = \inf\{\|w - v\| : v \in F(x, y)\}$  is measurable for all  $w \in \mathbb{R}$ , where  $J_a \times J_b = [0, a] \times [0, b]$  ([12]). Also, the Pompeiu-Hausdorff metric  $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow [0, \infty)$  is defined by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where  $d(A, b) = \inf_{a \in A} d(a, b)$  ([7]). Then  $(\mathcal{P}_{cl, bd}(X), H_d)$  is a metric space and  $(\mathcal{P}_{cl}(X), H_d)$  is a generalized metric space ([7]). Recall that a multifunction  $F : X \rightarrow \mathcal{P}(X)$  is said to be a contraction if there exists  $k \in (0, 1)$  such that  $H_d(F(u), F(v)) \leq kd(u, v)$  for all  $u, v \in X$  ([13]). An element  $u \in X$  is called endpoint of the multifunction  $F : X \rightarrow \mathcal{P}(X)$  whenever  $Fu = \{u\}$  ([8]). We say that the multifunction  $F$  has an approximate endpoint property whenever  $\inf_{u \in X} \sup_{w \in Fu} d(u, w) = 0$  ([8]). A real-valued function  $f$  on  $\mathbb{R}$  is called upper semi-continuous whenever  $\limsup_{n \rightarrow \infty} f(\lambda_n) \leq f(\lambda)$  for all sequence  $\{\lambda_n\}_{n \geq 1}$  with  $\lambda_n \rightarrow \lambda$ . In this paper, by using main idea of [5], [6] and [17], we investigate the existence of solutions for the two-variables fractional partial differential inclusion

$$(1.1) \quad (D_\theta^\alpha u)(x, y) \in F(x, y, u(x, y)),$$

with the partial integral boundary value conditions

$$(1.2) \quad (I_\theta^{1-\alpha} u)(x, 0) = \lambda_1 \varphi(x), \quad (I_\theta^{1-\alpha} u)(0, y) = \lambda_2 \gamma(y),$$

where  $D_\theta^\alpha$  denotes the Riemann-Liouville fractional partial derivative of order  $\alpha$ ,  $(x, y) \in J_a \times J_b$ ,  $0 < \alpha_i \leq 1$ ,  $\lambda_i \in \mathbb{R}^+$  ( $i = 1, 2$ ) and  $F : J_a \times J_b \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$  is a compact valued multi-valued map. Here, the functions  $\varphi : J_a \rightarrow \mathbb{R}$  and  $\gamma : J_b \rightarrow \mathbb{R}$  are absolutely continuous with  $\varphi(0) = \gamma(0) = 0$ . We need next endpoint result.

**Theorem 1.1.** ([8]) *Suppose that  $(X, d)$  is a complete metric space,  $\psi : [0, \infty) \rightarrow [0, \infty)$  is an upper semi-continuous function such that  $\psi(t) < t$  and  $\liminf_{t \rightarrow \infty} (t - \psi(t)) > 0$  for all  $t > 0$  and  $T : X \rightarrow CB(X)$  is a multifunction such that  $H_d(Tx, Ty) \leq \psi(d(x, y))$  for all  $x, y \in X$ . Then  $T$  has a unique endpoint if and only if  $T$  has approximate endpoint property.*

## 2. Main results

Now, we are ready to state and prove our main results. First, we give next key result.

**Lemma 2.1.** *Let  $f \in L(J_a \times J_b)$  and  $\alpha = (\alpha_1, \alpha_2) \in (0, 1] \times (0, 1]$ . Then the continuous function  $u_0 \in L(J_a \times J_b)$  is a solution for the fractional partial differential equation*

$$(2.1) \quad D_\theta^\alpha u(x, y) = f(x, y)$$

with boundary conditions  $(I_\theta^{1-\alpha}u)(x, 0) = \lambda_1\varphi(x)$  and  $(I_\theta^{1-\alpha}u)(0, y) = \lambda_2\gamma(y)$  if and only if  $u_0$  is a solution for the fractional integral equation

$$(2.2) \quad u(x, y) = \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)} (I_\theta^{\alpha_2} \dot{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)} (I_\theta^{\alpha_1} \dot{\varphi})(x) + (I_\theta^\alpha f)(x, y).$$

*Proof.* Let  $u_0$  be a solution for the fractional partial differential equation (2.1). Then, we have  $D_{xy}^2(I_\theta^{1-\alpha}u_0)(x, y) = f(x, y)$  and so

$$(I_\theta^{1-\alpha}u_0)(x, y) - (I_\theta^{1-\alpha}u_0)(x, 0) - (I_\theta^{1-\alpha}u_0)(0, y) + (I_\theta^{1-\alpha}u_0)(0, 0) = (I_\theta^1 f)(x, y).$$

By using the boundary conditions, we get

$$(I_\theta^{1-\alpha}u_0)(x, y) - \lambda_1\varphi(x) - \lambda_2\gamma(y) + \lambda_1\varphi(0) = (I_\theta^1 f)(x, y)$$

and so  $(I_\theta^{1-\alpha}u_0)(x, y) - (I_\theta^1 f)(x, y) = \lambda_1\varphi(x) + \lambda_2\gamma(y)$ . Since  $(I_\theta^0 u_0)(x, y) = u_0(x, y)$ , we obtain  $(I_\theta^{1-\alpha}(u_0(x, y) - (I_\theta^1 f)(x, y)))(x, y) = \lambda_1\varphi(x) + \lambda_2\gamma(y)$ . On the other hand, we have

$$I_\theta^\alpha [I_\theta^{1-\alpha}(u_0(x, y) - (I_\theta^1 f)(x, y))](x, y) = I_\theta^\alpha (\lambda_1\varphi(x) + \lambda_2\gamma(y))$$

and so

$$I_\theta^1 (u_0(x, y) - (I_\theta^1 f)(x, y)) = I_\theta^\alpha (\lambda_1\varphi(x) + \lambda_2\gamma(y)) = (I_\theta^\alpha p)(x, y). \quad (2.3)$$

But, we have

$$\begin{aligned} (I_\theta^\alpha p)(x, y) &= \frac{\lambda_1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1} (y-t)^{\alpha_2-1} \varphi(s) dt ds \\ &+ \frac{\lambda_2}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1} (y-t)^{\alpha_2-1} \gamma(t) dt ds. \end{aligned}$$

Since  $\varphi(x) = \int_0^x \dot{\varphi}(s) ds$  and  $\gamma(y) = \int_0^y \dot{\gamma}(t) dt$ , we obtain

$$\begin{aligned} (I_\theta^\alpha p)(x, y) &= \frac{\lambda_1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1} (y-t)^{\alpha_2-1} \left( \int_0^s \dot{\varphi}(\tau) d\tau \right) dt ds \\ &+ \frac{\lambda_2}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1} (y-t)^{\alpha_2-1} \left( \int_0^t \dot{\gamma}(\tau) d\tau \right) dt ds \\ &= \frac{\lambda_1 y^{\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2+1)} \int_0^x (x-s)^{\alpha_1-1} \left( \int_0^s \dot{\varphi}(\tau) d\tau \right) ds \\ &+ \frac{\lambda_2 x^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2)} \int_0^y (y-t)^{\alpha_2-1} \left( \int_0^t \dot{\gamma}(\tau) d\tau \right) dt \\ &= \frac{\lambda_1 y^{\alpha_2}}{\Gamma(\alpha_2+1)} \int_0^x \left( \frac{1}{\Gamma(\alpha_1)} \int_0^s (s-\tau)^{\alpha_1-1} \dot{\varphi}(\tau) d\tau \right) ds \\ &+ \frac{\lambda_2 x^{\alpha_1}}{\Gamma(\alpha_1+1)} \int_0^y \left( \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-\tau)^{\alpha_2-1} \dot{\gamma}(\tau) d\tau \right) dt \\ &= \frac{\lambda_1 y^{\alpha_2}}{\Gamma(\alpha_2+1)} \int_0^x (I_\theta^{\alpha_1} \dot{\varphi})(s) ds + \frac{\lambda_2 x^{\alpha_1}}{\Gamma(\alpha_1+1)} \int_0^y (I_\theta^{\alpha_2} \dot{\gamma})(t) dt. \end{aligned}$$

Since  $(I_\theta^{\alpha_1}\dot{\varphi})(x) \in L(J_a)$  and  $(I_\theta^{\alpha_2}\dot{\gamma})(y) \in L(J_b)$ , the functions  $\int_0^x (I_\theta^{\alpha_1}\dot{\varphi})(s)ds$  and  $\int_0^y (I_\theta^{\alpha_2}\dot{\gamma})(t)dt$  are absolutely continuous and so there exists  $D_{xy}^2(I_\theta^\alpha p)(x, y)$  for almost all  $(x, y) \in J_a \times J_b$ . By applying the operator  $D_{xy}^2$  on both sides of (2.3), we get  $D_{xy}^2 \left[ I_\theta^1 \left( u_0(x, y) - (I_\theta^\alpha f)(x, y) \right) \right] = D_{xy}^2 \left[ (I_\theta^\alpha p)(x, y) \right]$ . Thus,

$$\begin{aligned} & u_0(x, y) - (I_\theta^\alpha f)(x, y) \\ &= D_{xy}^2 \left[ \frac{\lambda_1 y^{\alpha_2}}{\Gamma(\alpha_2 + 1)} \int_0^x (I_\theta^{\alpha_1}\dot{\varphi})(s)ds + \frac{\lambda_2 x^{\alpha_1}}{\Gamma(\alpha_1 + 1)} \int_0^y (I_\theta^{\alpha_2}\dot{\gamma})(t)dt \right] \\ &= D_x \left[ \frac{\lambda_1 \alpha_2 y^{\alpha_2-1}}{\Gamma(\alpha_2 + 1)} \int_0^x (I_\theta^{\alpha_1}\dot{\varphi})(s)ds + \frac{\lambda_2 x^{\alpha_1}}{\Gamma(\alpha_1 + 1)} (I_\theta^{\alpha_2}\dot{\gamma})(y) \right] \\ &= \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)} (I_\theta^{\alpha_1}\dot{\varphi})(x) + \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)} (I_\theta^{\alpha_2}\dot{\gamma})(y). \end{aligned}$$

Hence,  $u_0(x, y) = \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)} (I_\theta^{\alpha_1}\dot{\varphi})(x) + \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)} (I_\theta^{\alpha_2}\dot{\gamma})(y) + (I_\theta^\alpha f)(x, y)$ . This shows that  $u_0$  is a solution for the fractional integral equation (2.2). Now, let  $u_0$  be a solution for the fractional integral equation (2.2). Then,  $(I_\theta^{1-\alpha} u_0)(x, y) = I_\theta^{1-\alpha} \left[ \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)} (I_\theta^{\alpha_1}\dot{\varphi})(x) + \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)} (I_\theta^{\alpha_2}\dot{\gamma})(y) \right] (x, y) + (I_\theta^1 f)(x, y)$ . On the other hand by using  $\mathcal{B}(z, w) = \int_0^1 (1-x)^{w-1} x^{z-1} dx = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$ , we get

$$\begin{aligned} & I_\theta^{1-\alpha} \left[ \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)} (I_\theta^{\alpha_2}\dot{\gamma})(y) \right] (x, y) \\ &= \lambda_2 \int_0^x \int_0^y \frac{(x-s)^{-\alpha_1} (y-t)^{-\alpha_2}}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)} \left( \frac{s^{\alpha_1-1}}{\Gamma(\alpha_1)} (I_\theta^{\alpha_2}\dot{\gamma})(t) \right) dt ds \\ &= \frac{\lambda_2}{\Gamma(\alpha_1)\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)} \int_0^x \int_0^y (x-s)^{-\alpha_1} s^{\alpha_1-1} (y-t)^{-\alpha_2} (I_\theta^{\alpha_2}\dot{\gamma})(t) dt ds \\ &= \frac{\lambda_2}{\Gamma(\alpha_1)\Gamma(1-\alpha_1)} \int_0^x (x-s)^{-\alpha_1} s^{\alpha_1-1} ds \left( \frac{1}{\Gamma(1-\alpha_2)} \int_0^y (y-t)^{-\alpha_2} (I_\theta^{\alpha_2}\dot{\gamma})(t) dt \right) \\ &= \frac{\lambda_2 \mathcal{B}(\alpha_1, 1-\alpha_1)}{\Gamma(\alpha_1)\Gamma(1-\alpha_1)} \left( I_\theta^{1-\alpha_2} (I_\theta^{\alpha_2}\dot{\gamma})(t) \right) (y) \\ &= \frac{\lambda_2}{\Gamma(1)} (I_\theta^1 \dot{\gamma})(y) = \lambda_2 (\gamma(y) - \gamma(0)) = \lambda_2 \gamma(y) \end{aligned}$$

and similarly  $I_\theta^{1-\alpha} \left[ \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)} (I_\theta^{\alpha_1}\dot{\varphi})(x) \right] (x, y) = \lambda_1 \varphi(x)$ . Thus,

$$(2.3) \quad (I_\theta^{1-\alpha} u_0)(x, y) = \lambda_2 \gamma(y) + \lambda_1 \varphi(x) + (I_\theta^1 f)(x, y).$$

By applying the operator  $D_{xy}^2$  on both sides of (2.3), we obtain

$$D_{xy}^2 \left[ (I_\theta^{1-\alpha} u_0)(x, y) \right] = D_{xy}^2 \left[ \lambda_2 \gamma(y) + \lambda_1 \varphi(x) + (I_\theta^1 f)(x, y) \right]$$

and so  $(D_\theta^\alpha u_0)(x, y) = f(x, y)$ . By using (2.3), we get

$$(I_\theta^{1-\alpha} u_0)(x, 0) = \lambda_2 \gamma(0) + \lambda_1 \varphi(x) + (I_\theta^1 f)(x, 0) = \lambda_1 \varphi(x)$$

and  $(I_\theta^{1-\alpha} u_0)(0, y) = \lambda_2 \gamma(y) + \lambda_1 \varphi(0) + (I_\theta^1 f)(0, y) = \lambda_2 \gamma(y)$ . This completes the proof.  $\square$

Consider the Banach space  $X = C(J_a \times J_b, \mathbb{R})$  endowed with the norm  $\|u\| = \sup_{(x,y) \in J_a \times J_b} |u(t)|$ . For  $u \in X$ , define the set of selections of  $F$  by

$$S_{F,u} := \{v \in L^1(J_a \times J_b, \mathbb{R}) : v(x, y) \in F(x, y, u(x, y)) \text{ for almost all } (x, y) \in J_a \times J_b\}.$$

It has been proved that  $S_{F,u} \neq \emptyset$  for all  $u \in C(J_a \times J_b, X)$  ([11]). We say that  $u \in X$  is a solution for the boundary value problem (1.1)-(1.2) whenever it satisfies the boundary value conditions (1.2) and also there is a function  $v \in L^1(J_a \times J_b, \mathbb{R})$  such that  $v(x, y) \in F(x, y, u(x, y))$  for all  $(x, y) \in J_a \times J_b$  and

$$u(x, y) = \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I_\theta^{\alpha_2} \dot{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_\theta^{\alpha_1} \dot{\varphi})(x) \\ + \frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1 - 1} (y-t)^{\alpha_2 - 1} v(s, t) dt ds$$

for almost all  $(x, y) \in J_a \times J_b$ . Define the multifunction  $\mathcal{N} : X \rightarrow \mathcal{P}(X)$  by

$$\mathcal{N}(u) = \left\{ h \in X : h(x, y) = \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I_\theta^{\alpha_2} \dot{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_\theta^{\alpha_1} \dot{\varphi})(x) \right. \\ \left. + \frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1 - 1} (y-t)^{\alpha_2 - 1} v(s, t) dt ds \text{ for all } (x, y) \in J_a \times J_b \right\}.$$

Here, we provide our main result.

**Theorem 2.2.** *Suppose that  $\psi : [0, \infty) \rightarrow [0, \infty)$  is a nondecreasing upper semi-continuous map such that  $\liminf_{t \rightarrow \infty} (t - \psi(t)) > 0$  and  $\psi(t) < t$  for all  $t > 0$ ,  $F : J_a \times J_b \times \mathbb{R} \rightarrow \mathcal{P}_{cp}(\mathbb{R})$  is an integrable bounded multifunction such that  $F(\cdot, \cdot, u) : J_a \times J_b \rightarrow \mathcal{P}_{cp}(\mathbb{R})$  is measurable for all  $u \in \mathbb{R}$ . Assume that there exists  $m \in C(J_a \times J_b, [0, \infty))$  such that*

$$H_d(F(x, y, u) - F(x, y, u')) \leq \frac{1}{\Lambda_1} m(x, y) \psi(|u - u'|)$$

for all  $(x, y) \in J_a \times J_b$  and  $u, u' \in \mathbb{R}$ , where  $\Lambda_1 = \|m\| \left\{ \frac{a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} \right\}$ . If the multifunction  $\mathcal{N}$  has the approximate endpoint property, then the fractional partial differential inclusion problem (1.1)-(1.2) has a solution.

*Proof.* First, we prove that the multifunction  $\mathcal{N}$  has at least one endpoint. Let  $u \in X$ . Since the multivalued map  $(x, y) \mapsto F(x, y, u(x, y))$  is measurable and is closed-value, it has measurable selection and so  $S_{F,u}$  is nonempty. Let  $\{p_n\}_{n \geq 1}$  be a sequence in  $\mathcal{N}(u)$  with  $p_n \rightarrow p$ . For each  $n$ , choose  $v_n \in S_{F,u_n}$  such that

$$p_n(x, y) = \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I_{\theta}^{\alpha_2} \dot{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_{\theta}^{\alpha_1} \dot{\varphi})(x) \\ + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1 - 1} (y-t)^{\alpha_2 - 1} v_n(s, t) dt ds$$

for all  $(x, y) \in J_a \times J_b$ . Since the operator  $F$  is compact, the sequence  $\{v_n\}_{n \geq 1}$  has a subsequence converging to some  $v \in L^1(J_a \times J_b)$ . We denote this subsequence again by  $\{v_n\}_{n \geq 1}$ . It is easy to see that  $v \in S_{F,u}$  and

$$p(x, y) = \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I_{\theta}^{\alpha_2} \dot{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_{\theta}^{\alpha_1} \dot{\varphi})(x) \\ + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1 - 1} (y-t)^{\alpha_2 - 1} v(s, t) dt ds$$

for all  $(x, y) \in J_a \times J_b$ . This shows that  $p \in \mathcal{N}(u)$  and so  $\mathcal{N}(u)$  is closed. Note that,  $\mathcal{N}(u)$  is bounded because  $F$  has a compact values. Now, we show that  $H_d(\mathcal{N}(u), \mathcal{N}(w)) \leq \psi(\|u - w\|)$  for all  $u, w \in X$ . Let  $u, w \in X$  and  $h_1 \in \mathcal{N}(w)$ . Choose  $v_1 \in S_{F,w}$  such that

$$h_1(x, y) = \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I_{\theta}^{\alpha_2} \dot{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_{\theta}^{\alpha_1} \dot{\varphi})(x) \\ + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1 - 1} (y-t)^{\alpha_2 - 1} v_1(s, t) dt ds$$

for almost all  $(x, y) \in J_a \times J_b$ . By using the hypothesis, we have

$$H_d(F(x, y, u(x, y)) - F(x, y, w(x, y))) \leq \frac{1}{\Lambda_1} m(x, y) \psi(|u(x, y) - w(x, y)|)$$

and so we can choose  $z \in F(x, y, u(x, y))$  such that

$$|v_1(x, y) - z| \leq \frac{1}{\Lambda_1} m(x, y) \psi(|u(x, y) - w(x, y)|).$$

Define the multivalued map  $U : J_a \times J_b \rightarrow \mathcal{P}(\mathbb{R})$  by

$$U(x, y) = \{z \in \mathbb{R} : |v_1(x, y) - z| \leq \frac{1}{\Lambda_1} m(x, y) \psi(|u(x, y) - w(x, y)|)\}.$$

Since  $v_1$  and  $\eta = \frac{1}{\Lambda_1} m \psi(|u - w|)$  are measurable, the multifunction  $U(\cdot, \cdot) \cap F(\cdot, \cdot, u(\cdot, \cdot))$  is measurable. Hence, there exists  $v_2(x, y) \in F(x, y, u(x, y))$  such

that  $|v_1(x, y) - v_2(x, y)| \leq \frac{1}{\Lambda_1} m(x, y) \psi(|u(x, y) - w(x, y)|)$ . Now, consider the element  $h_2 \in \mathcal{N}(u)$  defined by

$$h_2(x, y) = \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I_\theta^{\alpha_2} \dot{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_\theta^{\alpha_1} \dot{\varphi})(x) \\ + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1 - 1} (y-t)^{\alpha_2 - 1} v_2(s, t) dt ds$$

for all  $(x, y) \in J_a \times J_b$ . Put  $\sup_{(x,y) \in J_a \times J_b} |m(x, y)| = \|m\|$ . Then, we have

$$|h_1(x, y) - h_2(x, y)| \leq \left| \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I_\theta^{\alpha_2} \dot{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_\theta^{\alpha_1} \dot{\varphi})(x) \right. \\ \left. + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1 - 1} (y-t)^{\alpha_2 - 1} v_1(s, t) dt ds \right. \\ \left. - \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I_\theta^{\alpha_2} \dot{\gamma})(y) - \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_\theta^{\alpha_1} \dot{\varphi})(x) \right. \\ \left. - \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1 - 1} (y-t)^{\alpha_2 - 1} v_2(s, t) dt ds \right| \\ \leq \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1 - 1} (y-t)^{\alpha_2 - 1} |v_1(s, t) - v_2(s, t)| dt ds \\ \leq \frac{1}{\Lambda_1} \|m\| \psi(\|u - w\|) \left\{ \frac{x^{\alpha_1} y^{\alpha_2}}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} \right\} \\ = \frac{\Lambda_1}{\Lambda_1} \psi(\|u - w\|) = \psi(\|u - w\|)$$

and so  $\|h_1 - h_2\| = \sup_{(x,y) \in J_a \times J_b} |h_1(x, y) - h_2(x, y)| = \psi(\|u - w\|)$ . Hence,  $H_d(\mathcal{N}(u), \mathcal{N}(w)) \leq \psi(\|u - w\|)$  for all  $u, w \in X$ . Since the multifunction  $\mathcal{N}$  has approximate endpoint property by using Theorem 1.1, there exists  $u^* \in X$  such that  $\mathcal{N}(u^*) = \{u^*\}$ . It is easy to check that  $u^*$  is a solution for the fractional partial differential inclusion problem (1.1)-(1.2).  $\square$

For illustration of our main result, we give the following example.

**Example 2.3.** Consider the fractional partial differential inclusion

$$D_\theta^\alpha u(x, y) \in \left[ 0, \frac{0.03xy |\sin u(x, y)|}{1 + |\sin u(x, y)|} \right]$$

with boundary value conditions  $(I_\theta^{1-\alpha} u)(x, 0) = 0.1(e^x - 1)$  and  $(I_\theta^{1-\alpha} u)(0, y) = 0.01y^2$ , where  $(x, y) \in [0, 1] \times [0, 1]$ . Let  $\alpha = (\alpha_1, \alpha_2)$  with  $\alpha_1, \alpha_2 \in (0, 1]$ ,  $\lambda_1 = 0.1$  and  $\lambda_2 = 0.01$ . Define the multifunction  $F : [0, 1] \times [0, 1] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$  by  $F(x, y, z) = \left[ 0, \frac{0.03xy |\sin z(t)|}{1 + |\sin z(t)|} \right]$ . If  $m : [0, 1] \times [0, 1] \rightarrow [0, \infty)$  is defined by

$m(x, y) = \frac{3}{100}xy$ , then  $\|m\| = \frac{3}{100}$ . Consider the map  $\psi(t) = \frac{t}{2}$ . It is clear that  $\psi$  is nondecreasing, upper semi-continuous on  $(0, 1]$ ,  $\liminf_{t \rightarrow \infty} (t - \psi(t)) > 0$  and  $\psi(t) < t$  for all  $t > 0$ . Since  $\Gamma(\alpha_i + 1) < \frac{1}{2}$ , we get

$$\Lambda_1 = \|m\| \left\{ \frac{a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} \right\} = \frac{3}{100} \left( \frac{1}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} \right) < 0.12.$$

One can easily check that

$$H_d(F(x, y, u_1) - F(x, y, u_2)) \leq \frac{1}{\Lambda_1} m(x, y) \psi(|u_1 - u_2|).$$

Put  $X = C_{\mathbb{R}}([0, 1] \times [0, 1])$ . Define  $\mathcal{N} : X \rightarrow \mathcal{P}(X)$  by

$$\mathcal{N}(u) = \{h \in X : \text{there exists } v \in S_{F,u} \text{ such that}$$

$$h(x, y) = w(x, y) \text{ for all } (x, y) \in [0, 1] \times [0, 1]\},$$

where

$$w(x, y) = \frac{0.01x^{\alpha_1-1}}{\Gamma(\alpha_1)} (I_{\theta}^{\alpha_2} \dot{\gamma})(y) + \frac{0.1y^{\alpha_2-1}}{\Gamma(\alpha_2)} (I_{\theta}^{\alpha_1} \dot{\varphi})(x) \\ + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1} (y-t)^{\alpha_2-1} v(s, t) dt ds.$$

Since  $\sup_{u \in \mathcal{N}(0)} \|u\| = 0$ ,  $\inf_{u \in X} \sup_{s \in \mathcal{N}(u)} \|u - s\| = 0$  and so  $\mathcal{N}$  has the approximate endpoint property. Now by using Theorem 2.2, we conclude that the above fractional partial differential inclusion problem has a solution.

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### References

- [1] S. Abbas, D. Baleanu, M. Benchohra, *Global attractivity for fractional order delay partial integro-differential equations*, Adv. Diff. Equ. (2012) 2012:62.
- [2] S. Abbas, M. Benchohra, *Darboux problem for perturbed partial differential equations of fractional order with finite delay*, Nonlinear Anal. Hybrid Syst. 3 (2009) 597–604.
- [3] S. Abbas, M. Benchohra, *Fractional order partial hyperbolic differential equations involving Caputo derivative*, Stud. Univ. Babeş-Bolyai Math. 57 No. 4 (2012), 469–479.
- [4] S. Abbas, M. Benchohra, *Partial hyperbolic differential equations with finite delay involving the Caputo fractional derivative*, Commun. Math. Anal. 7 No. 2 (2009), 62–72.



- [5] R. P. Agarwal, D. Baleanu, V. Hedayati, Sh. Rezapour, *Two fractional derivative inclusion problems via integral boundary condition*, Appl. Math. Comput. 257 (2015), 205–212.
- [6] B. Ahmad, S. K. Ntouyas, J. Tariboon, *A study of mixed Hadamard and RiemannLiouville fractional integro-differential inclusions via endpoint theory*, Appl. Math. Lett. **52** (2016), 9–14.
- [7] S. M. A. Aleomraninejad, Sh. Rezapour, N. Shahzad, *On generalizations of the Suzuki's method*, Appl. Math. Lett. 24 (2011), 1037–1040.
- [8] A. Amini-Harandi, *Endpoints of set-valued contractions in metric spaces*, Non-linear Anal. 72 (2010), 132–134.
- [9] D. Baleanu, Sh. Rezapour, S. Etemad, A. Alsaedi *On a time-fractional integro-differential equation via three-point boundary value conditions*, Math. Problems Engin. (2015), Article ID 785738, 12 pages.
- [10] M. Benchohra, M. Hellal, *Perturbed partial functional fractional order differential equations with infinite delay*, J. Adv. Res. Dyn. Control Syst. 5 No. 2 (2013), 1–15.
- [11] M. Benchohra, J. Henderson, F. Z. Mostefai, *Weak solutions for hyperbolic partial fractional differential inclusions in Banach spaces*, Computer Math. with Appl. 64 (2012), 3101–3107.
- [12] K. Deimling, *Multi-valued differential equations*, Walter de Gruyter, Berlin 1992.
- [13] H. Covitz, S. Nadler, *Multivalued contraction mappings in generalized metric spaces*, Israel, J. Math. 8 (1970), 5–11.
- [14] S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley 1993.
- [15] I. Podlubny, *Fractional differential equations*, Academic Press 1999.
- [16] G. Samko, A. Kilbas, O. Marichev, *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach 1993.
- [17] A.N. Vityuk, A.V. Golushkov, *Existence of solutions of systems of partial differential equations of fractional order*, Nonlinear Oscil. 7 No. 3 (2004), 318–325.

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