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A CLASS OF SOME THIRD-METACYCLIC 2-GROUPS

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Abstract. Third-metacyclic finite 2-groups are groups with a non-metacyclic second-maximal subgroup and all its third-maximal subgroups being metacyclic. Among these groups we are looking for all of those whose non-metacyclic subgroups, including group itself, are generated by involutions.

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1. Introduction

The aim of this article is to determine all third-metacyclic 2-groups whose all non-metacyclic subgroups are generated by involutions. The property of a group G that all non-metacyclic subgroups of G are generated by involutions, we denote by $\mathcal{S}$ for brevity. We begin with some basic definitions.

Definition 1.1. Let G be a finite group. A subgroup M_1 is *maximal* in G, $M_1 <_{max} G$, if for subgroup H in G from $M_1 \leq H < G$ follows $H = M_1$.
A subgroup M_2 is *second-maximal* subgroup of G if M_2 is a maximal subgroup of any maximal subgroup of G. Generally, subgroup M_n is *n-maximal subgroup* of G if M_n is a maximal subgroup of any $(n-1)$-maximal subgroup of G.

Definition 1.2. A group G is *metacyclic*, $G \in \mathcal{MC}$, if it possesses a cyclic normal subgroup $N \trianglelefteq G$ such that the factor-group G/N is also cyclic.

Definition 1.3. A group G is *n-metacyclic*, $G \in \mathcal{MC}_n$, if it possesses a non-metacyclic $(n-1)$-maximal subgroup and all its n-maximal subgroups are metacyclic. Specially, a group G is *third-metacyclic*, $G \in \mathcal{MC}_3$, if it possesses a non-metacyclic second-maximal subgroup and all its third-maximal subgroups are metacyclic. Obviously, if G is a p-group then $|G| \geq p^{n+2}$, because all groups of order p^2 are metacyclic.

Definition 1.4. Let G be a p-group. The group $\Omega_i(G)$ is

$$\Omega_i(G) = \langle x \in G \mid x^{p^i} = 1 \rangle, \ i \in \mathbb{N}.$$

Obviously $\Omega_i(G)$ char G. Therefore, 2-group G is generated by involutions exactly when $\Omega_1(G) = G$.

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Lemma 1.5. *Let G be a 2-group and $G \in \mathcal{MC}_n$. Then there exists maximal subgroup M in G, $M <_{max} G$, and $M \in \mathcal{MC}_{n-1}$.*

Proof. Because $G \in \mathcal{MC}_n$ there exists $H \leq G$ such that $|G : H| = 2^{n-1}$ and $H \notin \mathcal{MC}$, while each subgroup $K < G$, for which is $|G : K| \geq 2^n$, is metacyclic. Then there exists a maximal subgroup $M <_{max} G$ where $H \leq M$ and $|M : H| = 2^{n-2}$. If $K < M$ and $|M : K| = 2^{n-1}$ it follows $|G : K| = 2^n$, i.e. $K \in \mathcal{MC}$, then by Definition 1.3 is $M \in \mathcal{MC}_{n-1}$. □

Lemma 1.6. *Let G be a 2-group generated by involutions, i.e. $\Omega_1(G) = G$. Then G is an extension by involution of any of its maximal subgroups. Specially, if $G \in \mathcal{MC}_n$, then G is an extension by involution of any of its maximal subgroups $M \in \mathcal{MC}_{n-1}$.*

Proof. Let $M <_{max} G$ be any maximal subgroup of group G. We state that there exists an involution $t \in G \setminus M$, such $G = \langle M, t\rangle = M \cdot \langle t\rangle$. That is, if such involution did not exist it would follow that all involutions from G are in M, i.e. $\Omega_1(G) = G \leq M$, and that is a contradiction with the assumption $M <_{max} G$.

Specially, if $G \in \mathcal{MC}_n$, by Lemma 1.6 follows that there exists subgroup $M \in \mathcal{MC}_{n-1}$ of group G, so that G is an extension of M by involution. □

Lemma 1.7. *Let G be a non-metacyclic 2-group where every non-metacyclic subgroup $H \leq G$ is generated by involutions, i.e. $\Omega_1(H) = H$. Then, every non-metacyclic subgroup $H \leq G$ and every non-metacyclic factor-group H/N, where $N \trianglelefteq H$, are also groups with that property.*

Proof. The first part of our claim is obvious. For $H \leq G$, $H \notin \mathcal{MC}$, is by our assumption $H = \langle i_1, ..., i_n\rangle$, where $i_j^2 = 1$, for $j \in \{1, ..., n\}$.

For $N \trianglelefteq H$ and $H/N \notin \mathcal{MC}$ is $\overline{H} = H/N = \langle \overline{i_1}, ..., \overline{i_n}\rangle$, while $\overline{i_j} = i_j N$ and $\overline{i_j}^2 = N$, i.e. $\Omega_1(\overline{H}) = \overline{H}$. □

Lemma 1.8. *Let G be a 2-group with property $\mathcal{S}$. If $d(Z(G)) \geq 3$ then G is an elementary abelian group. For $d(Z(G)) = 2$ is $Z(G) \cong E_4$, and for $d(Z(G)) = 1$ $Z(G)$ is a cyclic group.*

Proof. At first let $d(Z(G)) \geq 3$, i.e. there exist $K \leq Z(G)$ and involutions $a, b, c \in G$ such as $K = \langle a, b, c\rangle \cong E_8$. We claim that G is an elementary abelian group. Let us assume opposite, i.e. that there exists some element $d \in G \setminus K$, $|\langle d\rangle| = 4$ and note $H = \langle K, d\rangle = \langle a, b, c, d\rangle$. Then we have either $d^2 \in \langle a, b, c\rangle$ or $d^2 \notin \langle a, b, c\rangle$. If $d^2 \in \langle a, b, c\rangle$, we can assume, without loss of generality, that $d^2 = a$. But then we have $\langle b, c, d\rangle \cong E_4 \times Z_4$ is a non-metacyclic subgroup, but $\Omega_1(\langle b, c, d\rangle) = \langle b, c, d^2\rangle \cong E_8$, i.e. non-metacyclic subgroup $\langle b, c, d\rangle$ of group the G is not generated by involutions, which is against our assumption.

In the case $d^2 \notin \langle a, b, c\rangle$ we have $H \cong E_8 \times Z_4$, which is a non-metacyclic group, but also $\Omega_1(H) = \langle a, b, c, d^2\rangle \cong E_{16}$, again a contradiction. Therefore, $d^2 = 1$, i.e. G is elementary abelian group.

Now, $d(Z(G)) = 2$. Because of $\Omega_1(G) = G$ is $G = \langle a_1, ..., a_k \mid a_i^2 = 1, \forall i = 1, ..., k\rangle$. If $a_i \in Z(G)$, $\forall i = 1, ..., k$, then G would again be an elementary abelian group. Thus, there exists $a_j \equiv a$ such as $a \in G \setminus Z(G)$. Let us assume $Z(G) = \langle z_1\rangle \times \langle z_2\rangle$, where $z_i^2 \neq 1$, at least for one $i \in \{1, 2\}$. We have now $\langle a, Z(G)\rangle = \langle a, z_1, z_2\rangle = \langle a\rangle \times \langle z_1\rangle \times \langle z_2\rangle \geq \langle a\rangle \times \langle z_1'\rangle \times \langle z_2'\rangle$, where $\langle z_1'\rangle \leq \langle z_1\rangle$ and $\langle z_2'\rangle \leq \langle z_2\rangle$. We can assume, without loss of generality, that $|z_1'| = 2$ and $|z_2'| = 4$. Now, it follows $\langle a, Z(G)\rangle \geq \langle a\rangle \times \langle z_1'\rangle \times \langle z_2'\rangle \equiv H \cong E_4 \times Z_4 \notin \mathcal{MC}$, but $\Omega_1(H) = \langle a, z_1', z_2'^2\rangle < H$, a contradiction. Therefore, in $\langle z_1\rangle$ and $\langle z_2\rangle$ there are no elements of order 4. Thus $Z(G) \cong E_4$.
For $d(Z(G)) = 1$, $Z(G)$ is a cyclic group. □

2. A class of some third-metacyclic 2-groups

Now we turn to solving the main problem of this article. According to Lemma 1.6 and Lemma 1.7 we reduce theorems from [2] and [1] where all $\mathcal{MC}_1$ and $\mathcal{MC}_2$ groups are classified, in the way that we extract only $\mathcal{MC}_1$-groups with property $\mathcal{S}$ from [2], and then from [1] only extensions of those $\mathcal{MC}_1$-groups by involutions. We get the following results:

Theorem 2.1. *Let $G \in \mathcal{MC}_1$ be a group with property $\mathcal{S}$. Then $G \cong Q_8$ or $G \cong Q_8 * Z_4 \cong D_8 * Z_4$.*

Theorem 2.2. *Let $G \in \mathcal{MC}_2$ be a group with property $\mathcal{S}$. Then G is isomorphic to one of the following groups:*
a) E_{16}
b) $D_8 \times Z_2$
c) $D_{16} * Z_4 = \langle a, b, c \mid a^8 = b^2 = c^4 = 1, a^4 = c^2, a^b = a^{-1}, a^c = a, b^c = b\rangle$.

According to Lemma 1.6, having extended by involutions $\mathcal{MC}_2$-groups from Theorem 2.2, we get all $\mathcal{MC}_3$-groups with property $\mathcal{S}$. Before stating the main result, we introduce additional necessary definitions and known results.

Definition 2.3. A finite 2-group G is called *quasi-dihedral* if it possesses an abelian maximal subgroup A of exponent strictly larger than 2, i.e. $exp(A) > 2$, and an involution that is not in A, i.e. $t \in G\setminus A$, such that t inverts each element in A.

Definition 2.4. Let G be a $p-$group. Then the set of all elements of order k in G is

$$O_k(G) = \{g \in G \mid |g| = k\}, \text{where } k \mid |G|.$$

The group $\langle O_k(G)\rangle$ generated by the set $O_k(G)$ is characteristic in G, $\langle O_k(G)\rangle$char G.

Definition 2.5. Let G be a $p-$group. If $G = M \cdot N$, where $M, N \leq G$ such that $M \cap N = [M, N] \cong Z_p$, then we say that G is *second-direct product* of M and N. We denote $G = M \times_2 N$.

Proposition 2.6. *Let H be a normal elementary abelian subgroup of 2-group G and let $g \in G$ and $g^2 \in H$. Then*

$$|C_H(g)|^2 \geq |H|.$$

Proof. Because $g^2 \in H$ and H is abelian we have $x^{g^2} = x$, for any $x \in H$. Thus $(xx^g)^g = x^g x^{g^2} = x^g x = xx^g$, $\forall x \in H$, i.e. $xx^g \in C_H(g)$. Now, for $x, y \in H$, we have $xx^g = yy^g \Leftrightarrow xy = x^g y^g = (xy)^g \Leftrightarrow xy \in C_H(g) \Leftrightarrow xy^{-1} \in C_H(g) \Leftrightarrow C_H(g)x = C_H(g)y$. Therefore, $xx^g \neq yy^g \Leftrightarrow C_H(g)x \neq C_H(g)y$, and so $|C_H(g)| \geq |H : C_H(g)| \Rightarrow |C_H(g)|^2 \geq |H|$. □

Proposition 2.7. *For the elementary abelian group G of order p^n, $G \cong E_{p^n}$, the number of subgroups of order p is*

$$\frac{p^n - 1}{p - 1} = p^{n-1} + p^{n-2} + ... + p + 1.$$

Proof. If G is the elementary abelian group of order p^n, $G \cong E_{p^n}$, every non-identity element generates a subgroup of order p containing $p - 1$ non-identity elements. Since any two of these subgroups are either equal or disjoint, the number of such subgroups is $\frac{p^n-1}{p-1} = p^{n-1} + p^{n-2} + ... + p + 1$. □

Theorem 2.8. *If G is a non-abelian $p-$group, possessing an abelian maximal subgroup, then*

$$|G| = p \cdot |G'| \cdot |Z(G)|.$$

Proof. Let A be a maximal subgroup of G which is abelian, and $g \in G \setminus A$. The mapping $\varphi : A \to A$, $\varphi(a) = [a, g]$, is homomorphism with $Im\varphi = G', Ker\varphi = Z(G)$, and thus $A \setminus Z(G) \cong G'$. Therefore $|A| = |G| : p = |Z(G)| \cdot |G'|$ which yields the above formula. □

Now we state the main theorem of this article. From Theorem 2.2 we can see that all $\mathcal{MC}_2$ groups with property $\mathcal{S}$ are of order 16 or 32, so if we extend those groups by involution we will get all $\mathcal{MC}_3$ groups with property $\mathcal{S}$ of order 32 and 64, respectively.
In representing groups by generator order and commutators, we will omit, for brevity, those commutators of generators which equal 1 (that is for the pairs of commuting generators).

Theorem 2.9. *Let $G \in \mathcal{MC}_3$ be a group with property $\mathcal{S}$. Then G is one of the following 7 groups:*

a) of order 32, (extensions of E_{16} and $D_8 \times Z_2$)

$G_1 = \langle a, b, c, d, e \mid a^2 = b^2 = c^2 = d^2 = e^2 = 1\rangle \cong E_{32}$;
$G_2 = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, a^b = a^{-1}, a^d = a^{-1}, b^d = bc\rangle =$
$= (\langle a\rangle \times \langle bd\rangle) \cdot \langle d\rangle \cong (Z_4 \times Z_4) \cdot Z_2$, *quasi-dihedral group;*
$G_3 = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, a^b = a^{-1}, a^d = a^{-1}, b^d = ab\rangle =$
$= \langle bd, d \mid (bd)^8 = d^2 = 1, (bd)^d = (bd)^{-1}\rangle \times \langle c \mid c^2 = 1\rangle \cong D_{16} \times Z_2 \cong$
$\cong (Z_8 \times Z_2) \cdot Z_2$, *quasi-dihedral group;*

$G_4 = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, a^b = a^{-1}, c^d = a^2c\rangle =$
$= \langle a, b \mid a^4 = b^2 = 1, a^b = a^{-1}\rangle * \langle cd, d \mid (cd)^4 = d^2 = 1, (cd)^d = (cd)^{-1}\rangle \cong$
$\cong D_8 * D_8$, *where* $(cd)^2 = a^2$;
$G_5 = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, a^b = a^{-1}, b^d = abc, c^d = a^2c\rangle =$
$= \langle bd, c \mid (bd)^8 = c^2 = 1, (bd)^c = (bd)^5\rangle \cdot \langle d \mid d^2 = 1\rangle \cong M_{16} \cdot Z_2 \cong D_{16} \times_2 E_4$;

b) of order 64 *(extensions of* $D_{16} * Z_4$*)*

$G_6 = \langle x, y, z, e \mid x^8 = 1, z^2 = x^4, x^y = x^{-1}, x^e = x^{-1}, y^e = xy\rangle =$
$= \langle ye, e \mid (ye)^{16} = e^2 = 1, (ye)^e = (ye)^{-1}\rangle * \langle z \mid z^4 = 1\rangle \cong D_{32} * Z_4$;
$G_7 = \langle x, y, z, e \mid x^8 = 1, z^2 = x^4, x^y = x^{-1}, x^e = x^{-1}, y^e = xy, z^e = x^4z =$
$= z^{-1}\rangle = \langle ye, x^2z \mid (ye)^{16} = (x^2z)^2 = 1, (ye)^{x^2z} = (ye)^9\rangle \cdot \langle e \mid e^2 = 1\rangle \cong$
$\cong M_{32} \cdot Z_2$ *or written differently (with* $e \equiv u$*):*
$G_7 = \langle x^8, u, x, y \mid x^{16} = u^2 = y^2 = 1, x^y = x^{-1}, u^x = x^8u, u^y = u\rangle =$
$= \langle x, y \mid x^{16} = y^2 = 1, x^y = x^{-1}\rangle \times_2 \langle x^8, u \mid (x^8)^2 = u^2 = 1, (x^8)^u = x^8\rangle \cong$
$\cong D_{32} \times_2 E_4$.

Proof. According to Lemma 1.6, all $\mathcal{MC}_3$-groups generated by involutions are obtained by extending three $\mathcal{MC}_2$-groups from Theorem 2.2 with involution.

A. Extension of E_{16} :
Let $G = \langle a, b, c, d, e\rangle$ and $H = \langle a, b, c, d\rangle \cong E_{16}$ be a maximal subgroup in G, $e \in G \setminus H$, $e^2 = 1$. If $G' = 1 \Rightarrow G \cong E_{32}$. For $G' \neq 1$ follows $Z(G) \leq H$ (otherwise, $G = H \cdot Z(G)$ is an abelian group), so is $Z(G) = C_H(e)$. According to Proposition 2.6, it follows $|C_H(e)|^2 \geq |H| = 16$, i.e. $|C_H(e)| \in \{4, 8, 16\}$. We know $C_H(e) \leq Z(G)$, thus for $|C_H(e)| \in \{8, 16\}$ is $d(Z(G)) \geq 3$, and according to Lemma 1.8, it follows that $G \cong E_{32}$. Therefore remains the case $|C_H(e)| = 4$, so we can take, without loss of generality, $Z(G) = C_H(e) = \langle a, b\rangle \cong E_4$. Group H is an abelian maximal subgroup of group G, so $|G| = 2 \cdot |G'| \cdot |Z(G)| \Rightarrow |G'| = 4$. We have $\overline{G} = G/Z(G) = \langle \overline{c}, \overline{d}, \overline{e}\rangle$, $\langle \overline{c}, \overline{d}\rangle = \overline{H}$, thus according to Proposition 2.6, it follows $|C_{\overline{H}}(\overline{e})|^2 \geq |\overline{H}| = 4 \Rightarrow |C_{\overline{H}}(\overline{e})| \in \{2, 4\}$.

<u>Case 1:</u> $|C_{\overline{H}}(\overline{e})| = 2$

We can assume, without loss of generality, $C_{\overline{H}}(\overline{e}) = \langle \overline{c}\rangle$, so we have $\overline{c}^{\overline{e}} = \overline{c}$ i $\overline{d}^{\overline{e}} = \overline{c}\overline{d}$, i.e. for originals, without loss of generality, we have $c^e = ac$ and $d^e = a^\gamma b^\delta cd$, $(\gamma, \delta) \neq (0, 0)$. From $d^{e^2} = d^1 = d$ and $d^{e^2} = (d^e)^e = (a^\gamma b^\delta cd)^e = ad$ we get a contradiction, so this case does not apply.

<u>Case 2:</u> $|C_{\overline{H}}(\overline{e})| = 4$

It follows that $C_{\overline{H}}(\overline{e}) = \overline{H}$, so $\overline{c}^{\overline{e}} = \overline{c}$ and $\overline{d}^{\overline{e}} = \overline{d}$. For originals we have $c^e = a^\alpha b^\beta c$, $d^e = a^\gamma b^\delta d$, where (α, β), $(\gamma, \delta) \neq (0, 0)$, because $Z(G) = \langle a, b\rangle$. Because a, b and ab are interchangeable without loss of generality we have: $c^e = ac$, $d^e = ad$ or $d^e = bd$. From $c^e = ac$, $d^e = ad$ follows $G' = \langle a\rangle$, which is a contradiction with $|G'| = 4$. Therefore, $c^e = ac$, $d^e = bd$. Now $(de)^2 = de^2d^e = dbd = b$, thus $M = \langle a, c, de \mid a^2 = c^2 = (de)^4 = 1, a^{de} = a, c^{de} = ac\rangle$, where we denote $a \equiv x, c \equiv y$, $de \equiv z$, $M = \langle x, y, z \mid x^2 = y^2 = z^4 = 1, x^z = x, y^z = xy\rangle = \langle x, y\rangle \cdot \langle z\rangle \cong E_4 \cdot Z_4$. Group M is a non-metacyclic subgroup of G, but $\Omega_1(M) = \langle x, y, z^2\rangle \neq M$, a contradiction with our theorem assumption. Therefore, only extension of group $H \cong E_{16}$ with property $\mathcal{S}$ is group $G \equiv G_1 \cong E_{32}$.

B. Extension of $D_8 \times Z_2$:
We denote $H \equiv D_8 \times Z_2 = \langle a, b, c \mid a^4 = b^2 = c^2 = 1, a^b = a^{-1} \rangle$. Then $\mho_1(H) = \Phi(H) = \langle a^2 \rangle \cong Z_2,\ Z(H) = \langle a^2, c \rangle \cong E_4$. For the set $O_4(H) = \{x \in H \mid |x| = 4\} = \{a, a^3, ac, a^3c\}$ is $\langle O_4(H) \rangle = \langle a, c \rangle \cong Z_4 \times Z_2$. Let $G = \langle H, d \rangle,\ |d| = 2$, be an extension of the group H by involution d. Since $\mho_1(H) = \Phi(H) = \langle a^2 \rangle$ and $Z(H) = \langle a^2, c \rangle$ are characteristic in G, it is possible to interchange: element a with any element of order 4 in $O_4(H) = \{a, a^3, ac, a^3c\}$; element b with any involution which is not in $Z(H)$, thus with elements from $O_2(H) \setminus Z(H) = \{b, ab, a^2b, a^3b, bc, abc, a^2bc, a^3bc\}$; element c with any central involution which is not in $\mho_1(H)$, thus with elements from $Z(H) \setminus \mho_1(H) = \{c, a^2c\}$. Now we have $a^d \in \{a, a^3, ac, a^3c\} - 4$ cases, $c^d \in \{c, a^2c\} - 2$ cases, and $b^d = a^\alpha bc^\gamma,\ \alpha \in \{0, 1, 2, 3\} - 4$ cases, $\gamma \in \{0, 1\} - 2$ cases, so for a group G constructed as $G = \langle H, d \rangle$ we have $4 \cdot 2 \cdot 4 \cdot 2 = 64$ cases altogether. Now, we observe each particular case. If $a^d = a,\ c^d = c$ we would have non-metacyclic subgroup $\langle a, c, d \rangle \cong Z_4 \times E_4$ which is not generated by involutions, a contradiction. If we take $a^d = a^3,\ c^d = a^2c$ it follows, by replacing element a with element ac, $(ac)^d = a^3a^2c = ac,\ c^d = (ac)^2c$, so this case leads to case $a^d = a, c^d = a^2c$. If we take $a^d = ac,\ c^d = a^2c$, by replacing element a with element ac, we have $(ac)^d = aca^2c = a^3$, but from $a^{d^2} = a$ and $a^{d^2} = (ac)^d = a^3$ we get a contradiction. For $a^d = a^3c,\ c^d = c$, by replacing element c with element a^2c we get $a^d = a^3c = a(a^2c),\ (a^2c)^d = a^2c^2c = a^2c$, so this case leads to case $a^d = ac,\ c^d = c$ which leads to a contradiction. For $a^d = ac,\ c^d = c$ the group G contains the subgroup $\langle a, c, d \mid a^4 = c^2 = d^2 = 1, c^a = c, c^d = c, d^a = cd \rangle = \langle c, d \rangle \cdot \langle a \rangle \cong E_4 \cdot Z_4$, a contradiction.
Therefore, we have two cases: 1.) $a^d = a,\ c^d = a^2c$,
2.) $a^d = a^3,\ c^d = c$,
where $b^d = a^\alpha bc^\gamma,\ \alpha \in \{0, 1, 2, 3\},\ \gamma \in \{0, 1\}$. In case 1.) we have $b^{d^2} = b$ and $b^{d^2} = (b^d)^d = (a^\alpha bc^\gamma)^d = a^\alpha a^\alpha bc^\gamma a^{2\gamma} c^\gamma = a^{2\alpha+2\gamma}b$. Thus, $2(\alpha + \gamma) \equiv 0 (\mathrm{mod}\ 4) \Rightarrow (\alpha+\gamma) \equiv 0 (\mathrm{mod}\ \ 2) \Rightarrow (\alpha, \gamma) \in \{(0, 0), (1, 1), (3, 1), (2, 0)\}$. Now we replace in case 1.) element b with involution bc, and in case 2.) element b with involution $a^\beta b$, for $\beta \in \{0, 1, 2, 3\}$:
1.) From $(bc)^d = a^\alpha bc^\gamma a^2c = a^{\alpha+2}(bc)c^\gamma$ follows: $(\alpha, \gamma) \in \{(0, 0), (1, 1)\}$, because α can be replaced with $\alpha + 2$.
2.) From $(a^\beta b)^d = a^{3\beta}a^\alpha bc^\gamma = a^{2\beta}a^\alpha(a^\beta b)c^\gamma$ for $\beta \in \{1, 3\}$ follows $(a^\beta b)^d = = a^{\alpha+2}(a^\beta b)c^\gamma$, so we have $(\alpha, \gamma) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. In case 2.) for $(\alpha, \gamma) = (0, 0)$ we get $b^d = b$, so $\langle a^2, b, c, d \rangle \cong E_{16}$, and we have already solved this case.
Also in case 2.) for $(\alpha, \gamma) = (1, 1)$ we have $b^d = abc$, so by replacing element a with element ac we get $(ac)^d = a^3c = a^3c^3 = (ac)^3,\ b^d = ac \cdot b$, so this case leads to case $(\alpha, \gamma) = (1, 0)$. Therefore we have 4 groups:

$$G_2 = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, a^b = a^{-1}, a^d = a^{-1}, b^d = bc \rangle = \\ = (\langle a \rangle \times \langle bd \rangle) \cdot \langle d \rangle \cong (Z_4 \times Z_4) \cdot Z_2;$$

$G_3 = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, a^b = a^{-1}, a^d = a^{-1}, b^d = ab\rangle =$
$= \langle bd, d \mid (bd)^8 = d^2 = 1, (bd)^d = (bd)^{-1}\rangle \times \langle c \mid c^2 = 1\rangle \cong D_{16} \times Z_2 \cong$
$\cong (Z_8 \times Z_2) \cdot Z_2$, quasi-dihedral group;
$G_4 = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, a^b = a^{-1}, c^d = a^2c\rangle =$
$= \langle a, b \mid a^4 = b^2 = 1, a^b = a^{-1}\rangle * \langle cd, d \mid (cd)^4 = d^2 = 1, (cd)^d =$
$(cd)^{-1}\rangle \cong D_8 * D_8$, where $(cd)^2 = a^2$;
$G_5 = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, a^b = a^{-1}, b^d = abc, c^d = a^2c\rangle =$
$= \langle bd, c \mid (bd)^8 = c^2 = 1, (bd)^c = (bd)^5\rangle \cdot \langle d \mid d^2 = 1\rangle \cong M_{16} \cdot Z_2 \cong$
$\cong D_{16} \times_2 E_4$.

C. Extension of $D_{16} * Z_4$:

We denote $H \equiv D_{16} * Z_4 = \langle x, y, z \mid x^8 = y^2 = 1, z^2 = x^4, x^y = x^{-1}\rangle$. A $\mathcal{MC}_2$-group H is extension of $\mathcal{MC}_1$-group $D_8 * Z_4$ by an involution. Now we extend group H by involution to $\mathcal{MC}_3$-group G, i.e. for $e \in G \setminus H$, $|e| = 2$ we have:

$$G = \langle H, e \mid (H), e^2 = 1, x^e = x^\gamma, y^e = x^\delta y, z^e = x^{4\varepsilon} z\rangle,$$

where (H) denotes relations in H, $\gamma \in \{1, 3, 5, 7\}, \delta \in \{0, ..., 7\}, \varepsilon \in \{0, 1\}$. We have $Z(H) = \langle z\rangle \cong Z_4$ char $H \lhd G$. From $y^{e^2} = y$ and $y^{e^2} = (x^\delta y)^e = x^{\gamma\delta}x^\delta y$ follows $\delta(\gamma + 1) \equiv 0(\bmod 8)$. For $\gamma = 1$ is $2\delta \equiv 0(\bmod 8)$, i.e. $\delta \in \{0, 4\}$. For $\gamma = 3$ is $4\delta \equiv 0(\bmod 8)$, i.e. $\delta \in \{0, 2, 4, 6\}$. For $\gamma = 5$ is $6\delta \equiv 0(\bmod 8)$, i.e. $\delta \in \{0, 4\}$. For $\gamma = 7$ is $8\delta \equiv 0(\bmod 8)$, i.e. $\delta \in \{0, ..., 7\}$. We have 32 possibilities. Involution y can be replaced with any involution $x^\alpha y$, $\alpha \in \{0, ..., 7\}$. We have $(x^\alpha y)^e = x^{\alpha\gamma}x^\delta y = x^{\alpha\gamma+\delta-\alpha}x^\alpha y = x^{\delta'}x^\alpha y$, so it follows that δ must be transferred into new $\delta' = \alpha(\gamma-1)+\delta$, $\alpha \in \{0, ..., 7\}$, i.e. for $\gamma = 1$ is $\delta' \in \{0, 4\}$; for $\gamma = 3$ is $\delta' = 0$; for $\gamma = 5$ is $\delta' = 0$; for $\gamma = 7$ is $\delta' \in \{0, 1\}$. For $\varepsilon \in \{0, 1\}$ we have 12 possible cases. For the sake of brevity we again denote $\delta' \equiv \delta$. Now, for $\delta = 0$ is $y^e = y$, but for $\varepsilon = 0$ we have $\langle z, y, e\rangle \cong Z_4 \times E_4$, a contradiction. Thus, for $\delta = 0$ must be $\varepsilon = 1$. In the case $\gamma = 1, \delta = 4, \varepsilon = 0$ we get the group $\langle x^2, z, e\rangle = \langle x^2\rangle \times \langle x^2 z, e\rangle \cong Z_4 \times E_4$, so this case also leads to a contradiction. For $\gamma \in \{1, 3, 5, 7\}$ and $\delta = 0, \varepsilon = 1$ we get group $\langle x^4, y, e\rangle \cong E_8$, a contradiction again. Furthermore, a group in which we have $x^e = x^{-1}, y^e = y, z^e = x^4 z$ does not stand, because if we observe this group as a factor-group over $\langle x^4\rangle$ we get: $\overline{x}^{\overline{e}} = \overline{x}$, $\overline{y}^{\overline{e}} = \overline{y}$, $\overline{z}^{\overline{e}} = \overline{z}$, i.e. $\langle \overline{x}, \overline{z}, \overline{e}\rangle \cong Z_4 \times E_4$, and this is a non-metacyclic group not generated with involutions, so according to Lemma 1.6, the original group is also not generated with involutions. It is a contradiction. The group in which we have $x^e = x, y^e = x^4 y, z^e = x^4 z$ also does not stand, because its factor-group over $\langle x^4\rangle$ is again group $\langle \overline{x}, \overline{z}, \overline{e}\rangle \cong Z_4 \times E_4$. Therefore we have 2 groups:

$G_6 = \langle x, y, z, e \mid x^8 = 1, z^2 = x^4, x^y = x^{-1}, x^e = x^{-1}, y^e = xy\rangle = \langle ye, e \mid$
$(ye)^{16} = e^2 = 1, (ye)^e = (ye)^{-1}\rangle * \langle z \mid z^4 = 1\rangle \cong D_{32} * Z_4$;
$G_7 = \langle x, y, z, e \mid x^8 = 1, z^2 = x^4, x^y = x^{-1}, x^e = x^{-1}, y^e = xy, z^e = x^4 z =$
$z^{-1}\rangle = \langle ye, x^2 z \mid (ye)^{16} = (x^2 z)^2 = 1, (ye)^{x^2 z} = (ye)^9\rangle \cdot \langle e \mid e^2 = 1\rangle \cong$
$\cong M_{32} \cdot Z_2 \cong D_{32} \times_2 E_4$. □

This ends proof of Theorem 2.9. Now it remains to verify that all the established groups $G_1 - G_5$ and G_6, G_7 are not mutually isomorphic.

Remark 2.10. (Maximal subgroups of the $\mathcal{MC}_3$ 2-groups with property $\mathcal{S}$)

Having observed all maximal subgroups of the groups established in Theorem 2.9, we verify that each of those groups is a $\mathcal{MC}_3$ 2-group with property $\mathcal{S}$ and also prove that the groups established in Theorem 2.9 are not mutually isomorphic. Specifically, those maximal subgroups are either metacyclic $(D_{16}, SD_{16}, M_{16}, D_{32}, SD_{32}, M_{32}, Z_{16}\times Z_2, Z_8\times Z_2, Z_4\times Z_4)$ or non-metacyclic $(E_{16}, D_8\times Z_2, D_8 * Z_4, D_{16} * Z_4, D_{16}\times Z_2)$. Here is a list of all determined maximal subgroups:

a) of order 32

$G_1 = \langle a, b, c, d\rangle \cong E_{32}$

According to Proposition 2.7 group G_1 has 31 maximal subgroups, and all these subgroups are isomorphic to E_{16}.

$G_2 = \langle a, b, c, d | a^4 = b^2 = c^2 = d^2 = 1, a^b = a^{-1}, a^d = a^{-1}, b^d = bc\rangle \cong (Z_4 \times Z_4)\cdot Z_2$

$\Phi(G_2) = \langle a^2, c\rangle \cong E_4$ holds. Therefore, $G_2/\Phi(G_2) \cong E_8$ and according to Proposition 2.7 it follows that G_2 possesses 7 maximal subgroups which are:

$H_1 = \langle a, b, c\rangle \cong \langle a, b\rangle \times \langle c\rangle \cong D_8 \times Z_2,$
$H_2 = \langle a, c, d\rangle \cong \langle a, d\rangle \times \langle c\rangle \cong D_8 \times Z_2,$
$H_3 = \langle a, c, bd\rangle \cong \langle a\rangle \times \langle bd\rangle \cong Z_4 \times Z_4,$
$H_4 = \langle b, d, c\rangle \cong \langle bd, d\rangle \times \langle c\rangle \cong D_8 \times Z_2,$
$H_5 = \langle b, ad, c\rangle \cong \langle bad, ad\rangle \times \langle c\rangle \cong D_8 \times Z_2,$
$H_6 = \langle ab, d, c\rangle \cong \langle abd, d\rangle \times \langle c\rangle \cong D_8 \times Z_2,$
$H_7 = \langle ab, ad, c\rangle \cong \langle abad, ad\rangle \times \langle c\rangle \cong D_8 \times Z_2.$

$$G_3 = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, a^b = a^{-1}, a^d = a^{-1}, b^d = ab\rangle \cong$$
$$\cong (Z_8 \times Z_2)\cdot Z_2$$

$\Phi(G_3) = \langle a\rangle \cong Z_4$ holds. Therefore, $G_3/\Phi(G_3) \cong E_8$, so it follows that G_3 possesses 7 maximal subgroups which are:

$H_1 = \langle a, b, c\rangle \cong \langle a, b\rangle \times \langle c\rangle \cong D_8 \times Z_2,$
$H_2 = \langle a, b, d\rangle \cong \langle bd, d\rangle \cong D_{16},$
$H_3 = \langle a, c, d\rangle \cong \langle a, d\rangle \times \langle c\rangle \cong D_8 \times Z_2,$
$H_4 = \langle a, bc, b\rangle \cong \langle bcd, d\rangle \cong D_{16},$
$H_5 = \langle a, bd, c\rangle \cong \langle bd\rangle \times \langle c\rangle \cong Z_8 \times Z_2,$
$H_6 = \langle a, cd, b\rangle \cong \langle bcd, cd\rangle \cong D_{16},$
$H_7 = \langle a, bd, cd\rangle \cong \langle bd, cd\rangle \cong D_{16}.$

$G_4 = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, a^b = a^{-1}, c^d = a^2c\rangle \cong D_8 * D_8$

$\Phi(G_4) = \langle a^2\rangle \cong Z_2$ holds. Therefore, $G_4/\Phi(G_4) \cong E_{16}$, so it follows that G_4 possesses 15 maximal subgroups which are:

$H_1 = \langle a, b, c\rangle \cong \langle a, b\rangle \times \langle c\rangle \cong D_8 \times Z_2$,
$H_2 = \langle a, b, d\rangle \cong \langle a, b\rangle \times \langle d\rangle \cong D_8 \times Z_2$,
$H_3 = \langle a, c, d\rangle \cong \langle ac, d\rangle * \langle a\rangle \cong D_8 * Z_4$,
$H_4 = \langle b, c, d\rangle \cong \langle cd, d\rangle \times \langle b\rangle \cong D_8 \times Z_2$,
$H_5 = \langle a, b, cd\rangle \cong \langle a, b\rangle * \langle cd\rangle \cong D_8 * Z_4$,
$H_6 = \langle a, c, bd\rangle \cong \langle a, bd\rangle * \langle ac\rangle \cong D_8 * Z_4$,
$H_7 = \langle a, d, bc\rangle \cong \langle a, bc\rangle * \langle ad\rangle \cong D_8 * Z_4$,
$H_8 = \langle b, c, ad\rangle \cong \langle ad, b\rangle \times \langle bc\rangle \cong D_8 \times Z_2$,
$H_9 = \langle b, d, ac\rangle \cong \langle ac, b\rangle \times \langle bd\rangle \cong D_8 \times Z_2$,
$H_{10} = \langle c, d, ab\rangle \cong \langle cd, d\rangle \times \langle ab\rangle \cong D_8 \times Z_2$,
$H_{11} = \langle a, bc, bd\rangle \cong \langle a, bc\rangle \times \langle acd\rangle \cong D_8 \times Z_2$,
$H_{12} = \langle b, ac, ad\rangle \cong \langle ac, b\rangle * \langle bcd\rangle \cong D_8 * Z_4$,
$H_{13} = \langle c, ab, ad\rangle \cong \langle ad, c\rangle \times \langle abc\rangle \cong D_8 \times Z_2$,
$H_{14} = \langle ab, ac, d\rangle \cong \langle ac, d\rangle \times \langle abd\rangle \cong D_8 \times Z_2$,
$H_{15} = \langle bc, cd, a\rangle \cong \langle cd, bc\rangle \times \langle acd\rangle \cong D_8 \times Z_2$.

$G_5 = \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^2 = 1, a^b = a^{-1}, b^d = abc, c^d = a^2c\rangle \cong$
$\cong M_{16} \cdot Z_2 \cong D_{16} \times_2 E_4$

$\Phi(G_5) = \langle ac\rangle \cong Z_4$ holds. Therefore, $G_5/\Phi(G_5) \cong E_8$, so it follows that G_5 possesses 7 maximal subgroups which are:

$H_1 = \langle ac, a, b\rangle \cong \langle ac, a\rangle \times \langle b\rangle \cong D_8 \times Z_2$,
$H_2 = \langle ac, a, d\rangle \cong \langle ac, d\rangle * \langle a\rangle \cong D_8 * Z_4$,
$H_3 = \langle ac, b, d\rangle \cong \langle bd, d\rangle \cong D_{16}$,
$H_4 = \langle ac, ab, d\rangle \cong \langle abd, d\rangle \cong D_{16}$,
$H_5 = \langle ac, ad, b\rangle \cong \langle bad, ad\rangle \cong SD_{16}$,
$H_6 = \langle ac, bd, a\rangle \cong \langle bd, c\rangle \cong M_{16}$,
$H_7 = \langle ac, ab, ad\rangle \cong \langle abad, ad\rangle \cong SD_{16}$.

b) of order 64

$G_6 = \langle x, y, z, e \mid x^8 = 1, x^4 = z^2, x^y = x^{-1}, x^e = x^{-1}, y^e = xy, z^e = z\rangle \cong$
$\cong D_{32} * Z_4$

$\Phi(G_6) = \langle x\rangle \cong Z_8$ holds. Therefore, $G_6/\Phi(G_6) \cong E_8$, so it follows that G_6 possesses 7 maximal subgroups which are:

$H_1 = \langle x, y, z\rangle \cong \langle x, y\rangle * \langle z\rangle \cong D_{16} * Z_4$,
$H_2 = \langle x, y, e\rangle \cong \langle ye, e\rangle \cong D_{32}$,
$H_3 = \langle x, z, e\rangle \cong \langle x, e\rangle * \langle z\rangle \cong D_{16} * Z_4$,
$H_4 = \langle x, yz, e\rangle \cong \langle yze, e\rangle \cong SD_{32}$,
$H_5 = \langle x, ye, z\rangle \cong \langle ye\rangle \times \langle x^2z\rangle \cong Z_{16} \times Z_2$,
$H_6 = \langle x, ze, y\rangle \cong \langle yze, e\rangle \cong SD_{32}$,
$H_7 = \langle x, ye, ze\rangle \cong \langle ye, z^2e\rangle \cong D_{32}$.

$G_7 = \langle x, y, z, e \mid x^8 = 1, x^4 = z^2, x^y = x^{-1}, x^e = x^{-1}, y^e = xy, z^e = z^{-1} \rangle \cong$
$\cong M_{32} \cdot Z_2 \cong D_{32} \times_2 E_4$

$\Phi(G_7) = \langle x \rangle \cong Z_8$ holds. Therefore, $G_7/\Phi(G_7) \cong E_8$, so it follows that G_7 possesses 7 maximal subgroups which are:

$H_1 = \langle x, y, z \rangle \cong \langle x, y \rangle * \langle z \rangle \cong D_{16} * Z_4$,
$H_2 = \langle x, y, e \rangle \cong \langle ye, e \rangle \cong D_{32}$,
$H_3 = \langle x, z, e \rangle \cong \langle x, e \rangle \times \langle x^2 z \rangle \cong D_{16} \times Z_2$,
$H_4 = \langle x, yz, e \rangle \cong \langle yze, e \rangle \cong SD_{32}$,
$H_5 = \langle x, ye, z \rangle \cong \langle ye, x^2 z \rangle \cong M_{32}$,
$H_6 = \langle x, ze, y \rangle \cong \langle yze, ze \rangle \cong D_{32}$,
$H_7 = \langle x, ye, ze \rangle \cong \langle ye, ze \rangle \cong SD_{32}$.

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