

ON A LORENTZIAN PARA-SASAKIAN MANIFOLD WITH RESPECT TO THE QUARTER-SYMMETRIC METRIC CONNECTION

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Abstract. In this paper, we study certain curvature conditions satisfying by the conharmonic curvature tensor in a Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection.

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1. Introduction

In 1989, K. Matsumoto [17] introduced the notion of Lorentzian para-Sasakian manifolds. I. Mihai and R. Rosca [19] introduced the same notion independently and obtained several results. Lorentzian para-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [18]; U. C. De, K. Matsumoto and A. A. Shaikh [3] and many others such as ([20],[23]-[25]).

A linear connection $\bar{\nabla}$ in a Riemannian manifold M is said to be a quarter-symmetric connection [8] if the torsion tensor T of the connection $\bar{\nabla}$

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y]$$

satisfies

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where η is a 1-form and ϕ is a (1, 1) tensor field. If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y, Z) = 0$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields of the manifold M , then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection. If we put $\phi X = X$ and $\phi Y = Y$, then the quarter-symmetric metric connection reduces

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to the semi-symmetric metric connection [7]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection. A quarter-symmetric metric connection have been studied by various authors ([1], [5], [9]-[11], [14]-[16]).

A relation between the quarter-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ in an n -dimensional Lorentzian para-Sasakian manifold M is given by [21]

$$(1.1) \quad \bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y.$$

T. Takahashi [26] introduced the notion of locally ϕ -symmetry on a Sasakian manifold and obtained few interesting properties. U. C. De, A.A. Shaikh and S. Biswas [6] generalized the notion of ϕ -symmetric manifolds to ϕ -recurrent manifolds in the context of Sasakian manifold. Venkatesha and C.S.Bagewadi [27] studied concircular ϕ -recurrent LP -Sasakian manifolds which generalizes the notion of locally concircular ϕ -symmetric LP -Sasakian manifolds and obtain some interesting results. Recently, U. C. De and Pradip Manjhi have studied ϕ -Weyl semisymmetric and ϕ -projectively semisymmetric generalized Sasakian space forms and give some illustrative examples [4].

Motivated by the above studies, in this paper we study certain curvature conditions satisfying by the conharmonic curvature tensor in a Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection.

The paper is organized as follows: In Section 2, we give a brief introduction of a Lorentzian para-Sasakian manifold. In Section 3, we deduce the relation between the curvature tensor of Lorentzian para-Sasakian manifolds with respect to the quarter-symmetric metric connection and the Levi-Civita connection. Sections 4 and 5 are devoted to study conharmonically flat and ϕ -conharmonically flat Lorentzian para-Sasakian manifolds with respect to the quarter-symmetric metric connection, respectively. Section 6 deals with the study of ϕ -conharmonically semi-symmetric η -Einstein Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection. In Section 7, we study Lorentzian para-Sasakian manifolds satisfying the condition $\bar{C}(\xi, X) \cdot \bar{S} = 0$. Lorentzian para-Sasakian manifold whose curvature tensor of manifold is covariant constant with respect to the quarter-symmetric metric connection and manifold if recurrent with a Levi-Civita connection is studied in Section 8.

2. Preliminaries

A differentiable manifold of dimension n is called a Lorentzian para-Sasakian manifold, if it admits a $(1, 1)$ -tensor field ϕ , a contravariant vector field ξ , a 1-form η and a Lorentzian metric g which satisfy

$$(2.1) \quad \phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1,$$

$$(2.2) \quad g(X, \xi) = \eta(X), \quad \phi\xi = 0, \quad \eta(\phi X) = 0,$$

$$(2.3) \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

$$(2.4) \quad (\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

$$(2.5) \quad \nabla_X \xi = \phi X,$$

where ∇ denotes the covariant differentiation with respect to the Lorentzian metric g .

If we put

$$(2.6) \quad \Phi(X, Y) = g(\phi X, Y)$$

for all vector fields X and Y , then the tensor field $\Phi(X, Y)$ is a symmetric $(0, 2)$ tensor field [17]. Also since the 1-form η is closed in an LP -Sasakian manifold, we have [3]

$$(2.7) \quad (\nabla_X \eta)(Y) = \Phi(X, Y), \quad \Phi(X, \xi) = 0$$

for all vector fields $X, Y \in \chi(M)$.

Moreover, the curvature tensor R , the Ricci tensor S and the Ricci operator Q in a Lorentzian para-Sasakian manifold M with respect to the Levi-Civita connection satisfy the following equations [24]:

$$(2.8) \quad \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y),$$

$$(2.9) \quad R(\xi, X)Y = -R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X,$$

$$(2.10) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$(2.11) \quad R(\xi, X)\xi = -R(X, \xi)\xi = X + \eta(X)\xi,$$

$$(2.12) \quad S(X, \xi) = (n - 1)\eta(X), \quad Q\xi = (n - 1)\xi,$$

$$(2.13) \quad S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y)$$

for all vector fields $X, Y \in \chi(M)$.

Definition 2.1. A Lorentzian para-Sasakian manifold M is said to be an η -Einstein manifold if its Ricci tensor S of type $(0, 2)$ satisfies

$$(2.14) \quad S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where a and b are smooth functions on M . In particular, if $b = 0$, then an η -Einstein manifold is an Einstein manifold.

Contracting (2.14), we have

$$(2.15) \quad r = na - b.$$

On the other hand, putting $X = Y = \xi$ in (2.14) and using (2.1), (2.2) and (2.12), we also have

$$(2.16) \quad -(n-1) = -a + b.$$

Hence it follows from (2.15) and (2.16) that

$$a = \frac{r}{n-1} - 1, \quad b = \frac{r}{n-1} - n.$$

So the Ricci tensor S of an η -Einstein Lorentzian para-Sasakian manifold is given by

$$(2.17) \quad S(X, Y) = \left(\frac{r}{n-1} - 1\right)g(X, Y) + \left(\frac{r}{n-1} - n\right)\eta(X)\eta(Y).$$

3. Curvature tensor of a Lorentzian para-Sasakian manifolds with respect to the quarter-symmetric metric connection

If R and \bar{R} , respectively, are the curvature tensors of the Levi-Civita connection ∇ and the quarter-symmetric metric connections $\bar{\nabla}$ in a Lorentzian para-Sasakian manifold M . Then we have [2]

$$(3.1) \quad \begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &\quad + g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi, \end{aligned}$$

$$(3.2) \quad \bar{R}(\xi, Y)Z = -\bar{R}(Y, \xi)Z = -2\eta(Z)Y - 2\eta(X)\eta(Y)\xi,$$

$$(3.3) \quad \bar{R}(X, Y)\xi = 2\eta(Y)X - \eta(X)Y,$$

$$(3.4) \quad \bar{S}(Y, Z) = S(Y, Z) - g(Y, Z) - n\eta(Y)\eta(Z),$$

$$(3.5) \quad \bar{Q}Y = QY - Y - n\eta(Y)\xi, \quad \bar{Q}\xi = 2(n-1)\xi,$$

$$(3.6) \quad \bar{S}(Y, \xi) = 2(n-1)\eta(Y), \quad \bar{S}(\xi, \xi) = -2(n-1),$$

$$(3.7) \quad \bar{S}(\phi Y, \phi Z) = S(Y, Z) - g(Y, Z) - (n-2)\eta(Y)\eta(Z)$$

for all vector fields $X, Y, Z \in \chi(M)$.

4. Conharmonically flat Lorentzian para-Sasakian manifolds with respect to the quarter-symmetric metric connection

As a special subgroup of the conformal transformation group, Ishii [13] introduced the notion of conharmonic transformation under which a harmonic function transform into a harmonic function. The conharmonic curvature tensor C of type $(1, 3)$ in a Lorentzian para-Sasakian manifold M of dimension n is defined by ([12],[13])

$$(4.1) \quad C(X, Y)Z = R(X, Y)Z - \frac{1}{(n-2)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]$$

for all vector fields $X, Y, Z \in \chi(M)$, which is invariant under conharmonic transformation.

Analogous to the equation (4.1), we define the conharmonic curvature tensor \bar{C} in a Lorentzian para-Sasakian manifold M with respect to the quarter symmetric metric connection $\bar{\nabla}$ by

$$(4.2) \quad \bar{C}(X, Y)Z = \bar{R}(X, Y)Z - \frac{1}{(n-2)}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y + g(Y, Z)\bar{Q}X - g(X, Z)\bar{Q}Y],$$

where \bar{R} , \bar{S} and \bar{Q} are the Riemannian curvature tensor, the Ricci tensor and the Ricci operator with respect to the connection $\bar{\nabla}$, respectively. A manifold whose conharmonic curvature tensor vanishes at every point of the manifold is called conharmonically flat manifold.

Let us assume that the manifold M with respect to the quarter-symmetric metric connection is conharmonically flat, that is, $\bar{C} = 0$. Then from (4.2), we have

$$(4.3) \quad \bar{R}(X, Y)Z = \frac{1}{(n-2)}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y + g(Y, Z)\bar{Q}X - g(X, Z)\bar{Q}Y].$$

By putting $Y = Z = \xi$ in (4.3) and then using (3.3), (3.6) and (2.2), we find

$$(4.4) \quad \bar{Q}X = -2X - 2n\eta(X)\xi.$$

Similarly we have

$$(4.5) \quad \bar{Q}Y = -2X - 2n\eta(Y)\xi.$$

Now from (4.3)-(4.5), we have

$$(4.6) \quad \bar{R}(X, Y)Z = \frac{1}{(n-2)}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y - 2g(Y, Z)X + 2g(X, Z)Y]$$

$$-2ng(Y, Z)\eta(X)\xi + 2ng(X, Z)\eta(Y)\xi].$$

Putting $X = \xi$ and using (3.2), (3.6), (2.1) and (2.2), we get

$$\bar{S}(Y, Z) = -2(n-1)g(Y, Z) - 4(n-1)\eta(Y)\eta(Z)$$

which by using (3.4) becomes

$$(4.7) \quad S(Y, Z) = -(2n-3)g(Y, Z) - (3n-4)\eta(Y)\eta(Z).$$

Hence contracting (4.7), we obtain

$$(4.8) \quad r = -2(n-1)(n-2).$$

Thus we have the following theorem:

Theorem 4.1. *An n -dimensional conharmonically flat Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection is an η -Einstein manifold with the scalar curvature $r = -2(n-1)(n-2)$.*

5. ϕ -conharmonically flat Lorentzian para-Sasakian manifolds with respect to the quarter-symmetric metric connection

Definition 5.1 ([22]). An n -dimensional ($n > 3$) Lorentzian para-Sasakian manifold satisfying the condition

$$(5.1) \quad \phi^2 C(\phi X, \phi Y)\phi Z = 0$$

is called ϕ -conharmonically flat.

Analogous to the equation (5.1), we define an n -dimensional Lorentzian para-Sasakian manifold is said to be ϕ -conharmonically flat with respect to the quarter-symmetric metric connection if it satisfies

$$(5.2) \quad \phi^2 \bar{C}(\phi X, \phi Y)\phi Z = 0$$

for any vector fields $X, Y, Z \in \chi(M)$.

Assuming that the manifold is ϕ -conharmonically flat with respect to the quarter-symmetric metric connection. Then from (5.2), we have

$$(5.3) \quad g(\bar{C}(\phi X, \phi Y)\phi Z, \phi W) = 0$$

for any $X, Y, Z, W \in \chi(M)$. Using (4.2) in (5.3), we have

$$(5.4) \quad g(\bar{R}(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{(n-2)} [g(\phi Y, \phi Z)\bar{S}(\phi X, \phi W)$$

$$-g(\phi X, \phi Z)\bar{S}(\phi Y, \phi W) + g(\phi X, \phi W)\bar{S}(\phi Y, \phi Z) - g(\phi Y, \phi W)\bar{S}(\phi X, \phi Z)].$$

Now in view of (3.1) and (3.4), (5.4) becomes

$$(5.5) \quad \begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi W) &= \frac{1}{(n-2)} [g(\phi Y, \phi Z)S(\phi X, \phi W) \\ &-g(\phi X, \phi Z)S(\phi Y, \phi W) + g(\phi X, \phi W)S(\phi Y, \phi Z) - g(\phi Y, \phi W)S(\phi X, \phi Z)]. \\ &+ \frac{1}{(n-2)} [-g(\phi Y, \phi Z)g(\phi X, \phi W) + g(\phi X, \phi Z)g(\phi Y, \phi W) \\ &-g(\phi X, \phi W)g(\phi Y, \phi Z) + g(\phi Y, \phi W)g(\phi X, \phi Z)]. \end{aligned}$$

Let $\{e_1, e_2, \dots, e_{n-1}\}$ be a local orthonormal basis of vector fields in M . Using that $\{\phi e_1, \phi e_2, \dots, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (5.5) and sum up with respect to i , then

$$(5.6) \quad \begin{aligned} \sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) &= \frac{1}{(n-2)} \sum_{i=1}^{n-1} [g(\phi Y, \phi Z)S(\phi e_i, \phi e_i) \\ &-g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) + g(\phi e_i, \phi e_i)S(\phi Y, \phi Z) - g(\phi Y, \phi e_i)S(\phi e_i, \phi Z)]. \\ &+ \frac{1}{(n-2)} \sum_{i=1}^{n-1} [-g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) + g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) \\ &-g(\phi e_i, \phi e_i)g(\phi Y, \phi Z) + g(\phi Y, \phi e_i)g(\phi e_i, \phi Z)]. \end{aligned}$$

It can be easily verify that [22]

$$(5.7) \quad \sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z),$$

$$(5.8) \quad \sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r + n - 1,$$

$$(5.9) \quad \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) = S(\phi Y, \phi Z),$$

$$(5.10) \quad \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1,$$

$$(5.11) \quad \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = g(\phi Y, \phi Z).$$

By virtue of (5.7)-(5.11), the equation (5.6) can be written as

$$(5.12) \quad S(\phi Y, \phi Z) + g(\phi Y, \phi Z) = \frac{1}{(n-2)} [(r+n-1)g(\phi Y, \phi Z) \\ + (n-3)S(\phi Y, \phi Z)] - 2g(\phi Y, \phi Z),$$

from which it follows that

$$(5.13) \quad S(\phi Y, \phi Z) = (r-2n+5)g(\phi Y, \phi Z).$$

By using (2.3) and (2.13) in (5.13), we get

$$(5.14) \quad S(Y, Z) = (r-2n+5)g(Y, Z) + (r-3n+6)\eta(Y)\eta(Z).$$

Hence contracting (5.14), we obtain

$$(5.15) \quad r = \frac{2(n-1)(n-3)}{n-2}.$$

Thus we have the following theorem:

Theorem 5.2. *Let M be an n -dimensional ($n > 3$), ϕ -conharmonically flat Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection. Then M is an η -Einstein manifold with the scalar curvature $r = \frac{2(n-1)(n-3)}{n-2}$.*

6. ϕ -conharmonically semi-symmetric η -Einstein Lorentzian para-Sasakian manifolds with respect to the quarter-symmetric metric connection

Definition 6.1. An η -Einstein Lorentzian para-Sasakian manifold (M^n, g) , $n > 1$ is said to be ϕ -conharmonically semisymmetric with respect to the quarter-symmetric metric connection if

$$\bar{C}(X, Y) \cdot \phi = 0$$

on M for all $X, Y \in \chi(M)$.

Let M be an n -dimensional ϕ -conharmonically semi-symmetric η -Einstein Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection. Therefore $\bar{C}(X, Y) \cdot \phi = 0$ turns into

$$(6.1) \quad (\bar{C}(X, Y) \cdot \phi)Z = \bar{C}(X, Y)\phi Z - \phi\bar{C}(X, Y)Z = 0$$

for any vector fields $X, Y, Z \in \chi(M)$. From (4.2), we have

$$(6.2) \quad \bar{C}(X, Y)\phi Z = \bar{R}(X, Y)\phi Z - \frac{1}{(n-2)} [\bar{S}(Y, \phi Z)X - \bar{S}(X, \phi Z)Y \\ + g(Y, \phi Z)\bar{Q}X - g(X, \phi Z)\bar{Q}Y].$$

By using (3.1), (3.4) and (3.5), the last equation takes the form

$$(6.3) \quad \begin{aligned} \bar{C}(X, Y)\phi Z &= R(X, Y)\phi Z + \eta(X)g(Y, \phi Z)\xi - \eta(Y)g(X, \phi Z)\xi \\ &\quad - \frac{1}{(n-2)}[S(Y, \phi Z)X - g(Y, \phi Z)X - S(X, \phi Z)Y + g(X, \phi Z)Y \\ &\quad + g(Y, \phi Z)(QX - 2X - n\eta(X)\xi) - g(X, \phi Z)(QY - 2Y - n\eta(Y)\xi)]. \end{aligned}$$

Also we have

$$(6.4) \quad \begin{aligned} \phi\bar{C}(X, Y)Z &= \phi R(X, Y)Z + \eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X \\ &\quad - \frac{1}{(n-2)}[S(Y, Z)\phi X - S(X, Z)\phi Y - g(Y, Z)\phi X + g(X, Z)\phi Y \\ &\quad - n\eta(Y)\eta(Z)\phi X + n\eta(X)\eta(Z)\phi Y + g(Y, Z)(\phi QX - \phi X) - g(X, Z)(\phi QY - \phi Y)]. \end{aligned}$$

By using (6.3) and (6.4), (6.1) takes the form

$$(6.5) \quad \begin{aligned} g(Y, \phi Z)X - g(X, \phi Z)Y + \eta(X)g(Y, \phi Z)\xi - \eta(Y)g(X, \phi Z)\xi - g(Y, Z)\phi X \\ + g(X, Z)\phi Y - \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X \\ - \frac{1}{(n-2)}[S(Y, \phi Z)X - g(Y, \phi Z)X - S(X, \phi Z)Y + g(X, \phi Z)Y \\ + g(Y, \phi Z)(QX - 2X - n\eta(X)\xi) - g(X, \phi Z)(QY - 2Y - n\eta(Y)\xi)] \\ + \frac{1}{(n-2)}[S(Y, Z)\phi X - S(X, Z)\phi Y - g(Y, Z)\phi X + g(X, Z)\phi Y \\ - n\eta(Y)\eta(Z)\phi X + n\eta(X)\eta(Z)\phi Y + g(Y, Z)(\phi QX - \phi X) - g(X, Z)(\phi QY - \phi Y)] = 0. \end{aligned}$$

Taking $Y = \xi$ and then using (2.1), (2.2), (2.12) and (3.6), (6.5) reduces to

$$S(X, \phi Z)\xi + (2n - 4)g(X, \phi Z)\xi + n\eta(Z)\phi X = 0$$

which in view of (2.17) becomes

$$\left(\frac{r}{n-1} + 2n - 5\right)g(X, \phi Z)\xi + n\eta(Z)\phi X = 0.$$

Now considering Z to be orthogonal to ξ , then $\eta(Z) = 0$ and $g(X, \phi Z) \neq 0$, which implies that

$$r = -(n-1)(2n-5).$$

Thus we can state the following theorem:

Theorem 6.2. *For an n -dimensional ϕ -conharmonically semi-symmetric η -Einstein Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection, the scalar curvature $r = -(n-1)(2n-5)$.*

7. Lorentzian para-Sasakian manifold satisfying

$$\bar{C}(\xi, X) \cdot \bar{S} = 0$$

Let us consider a Lorentzian para-Sasakian manifold satisfying $\bar{C}(\xi, X) \cdot \bar{S} = 0$. Then we have

$$(7.1) \quad \bar{S}(\bar{C}(\xi, X)Y, Z) + \bar{S}(Y, \bar{C}(\xi, X)Z) = 0.$$

In view of (4.2), we have

$$(7.2) \quad \bar{C}(\xi, X)Y = -\frac{1}{(n-2)}[-\eta(Y)X + (2n-4)\eta(X)\eta(Y)\xi + S(X, Y)\xi \\ -\eta(Y)QX + (2n-3)g(X, Y)\xi].$$

Making use of (7.2), (7.1) takes the form

$$(7.3) \quad \bar{S}(\bar{C}(\xi, X)Y, Z) = -\frac{1}{(n-2)}[\eta(Y)g(X, Z) + 2(n-1)(2n-3)g(X, Y)\eta(Z) \\ + (4(n-1)(n-2) + n(n-1) + n)\eta(X)\eta(Y)\eta(Z) + 2(n-1)\eta(Z)S(X, Y) - \eta(Y)S(QX, Z)].$$

Similarly we have

$$(7.4) \quad \bar{S}(Y, \bar{C}(\xi, X)Z) = -\frac{1}{(n-2)}[\eta(Z)g(X, Y) + 2(n-1)(2n-3)g(X, Z)\eta(Y) \\ + (4(n-1)(n-2) + n(n-1) + n)\eta(X)\eta(Y)\eta(Z) + 2(n-1)\eta(Y)S(X, Z) - \eta(Z)S(QX, Y)].$$

Using (7.3) and (7.4) in (7.1), we have

$$(7.5) \quad 2(4(n-1)(n-2) + n(n-1) + n)\eta(X)\eta(Y)\eta(Z) + \eta(Y)g(X, Z) \\ + \eta(Z)g(X, Y) + 2(n-1)\eta(Z)S(X, Y) + 2(n-1)\eta(Y)S(X, Z) \\ + 2(n-1)(2n-3)g(X, Y)\eta(Z) + 2(n-1)(2n-3)g(X, Z)\eta(Y) \\ - \eta(Y)S(QX, Z) - \eta(Z)S(QX, Y) = 0.$$

Let λ be the eigenvalue of the endomorphism Q corresponding to an eigenvector X . Then

$$(7.6) \quad QX = \lambda X.$$

By using (7.6), (7.5) takes the form

$$(7.7) \quad 2(4(n-1)(n-2) + n(n-1) + n)\eta(X)\eta(Y)\eta(Z) + \eta(Y)g(X, Z) \\ + \eta(Z)g(X, Y) + 2(n-1)\lambda\eta(Z)g(X, Y) + 2(n-1)\lambda\eta(Y)g(X, Z) \\ + 2(n-1)(2n-3)g(X, Y)\eta(Z) + 2(n-1)(2n-3)g(X, Z)\eta(Y) \\ - \lambda^2\eta(Y)g(X, Z) - \lambda^2\eta(Z)g(X, Y) = 0$$

which on putting $Z = \xi$ reduces to

$$(7.8) \quad [\lambda^2 - 2(n-1)\lambda - 2(n-1)(2n-3) - 1]g(X, Y) - [\lambda^2 - 2(n-1)\lambda - 2(n-1)(2n-3) - 1 + 2(4(n-1)(n-2) + n(n-1) + n)]\eta(X)\eta(Y) = 0.$$

By replcing $Y = \xi$ in (7.8), we get

$$[\lambda^2 - 2(n-1)\lambda - 2(n-1)(2n-3) - 1 + 4(n-1)(n-2) + n(n-1) + n]\eta(X) = 0.$$

This gives

$$\lambda^2 - 2(n-1)\lambda + (n-1)^2 = 0, \quad \eta(X) \neq 0.$$

Hence we can state the following:

Theorem 7.1. *If an n -dimensional Lorentzian para-Sasakian manifold satisfies $\bar{C}(\xi, X) \cdot \bar{S} = 0$, then the non-zero eigenvalues of the symmetric endomorphism Q of the tangent space corresponding to S are congruent, such as $(n-1)$.*

8. A Lorentzian para-Sasakian manifold whose curvature tensor of manifold is covariant constant with respect to the quarter-symmetric metric connection and M is recurrent with respect to the Levi-Civita connection

Definition 8.1 ([2]). A Lorentzian para-Sasakian manifold with respect to the Levi-Civita connection is called the recurrent, if its curvature tensor R satisfies the condition

$$(8.1) \quad (\nabla_W R)(X, Y)Z = A(W)R(X, Y)Z,$$

where A is the 1-form

Analogous to the equation (8.1), a Lorentzian para-Sasakian manifold with respect to the quarter symmetric metric connection $\bar{\nabla}$ is called the recurrent, if its curvature tensor \bar{R} satisfies the condition

$$(8.2) \quad (\bar{\nabla}_W \bar{R})(X, Y)Z = A(W)\bar{R}(X, Y)Z,$$

where \bar{R} is the curvature tensor with respect to the connection $\bar{\nabla}$.

Theorem 8.2. *If an n -dimensional Lorentzian para-Sasakian manifold whose curvature tensor of manifold is covariant constant with respect to the quarter-symmetric metric connection and the manifold is recurrent with respect to the Levi-Civita connection and the associated 1-form A is equal to the associated 1-form η , then the scalar curvatre r vanishes by providing the trace of ϕ is zero.*

Proof. From (1.1), (2.6) and (2.8), we have

$$(8.3) \quad (\bar{\nabla}_W R)(X, Y)Z = \bar{\nabla}_W R(X, Y)Z - R(\bar{\nabla}_W X, Y)Z - R(X, \bar{\nabla}_W Y)Z$$

$$-R(X, Y)\bar{\nabla}_W Z = (\nabla_W R)(X, Y)Z + 2\eta(W)[g(Y, \phi Z)X - g(X, \phi Z)Y].$$

Suppose that $(\bar{\nabla}_W R)(X, Y)Z = 0$, then from (8.2) it follows that

$$(8.4) \quad (\nabla_W R)(X, Y)Z + 2\eta(W)[g(Y, \phi Z)X - g(X, \phi Z)Y] = 0$$

which on using (8.1) becomes

$$(8.5) \quad A(W)R(X, Y)Z + 2\eta(W)[g(Y, \phi Z)X - g(X, \phi Z)Y] = 0.$$

Now contracting X in (8.5), we get

$$(8.6) \quad A(W)S(Y, Z) + 2(n-1)g(Y, \phi Z)\eta(W) = 0.$$

Suppose the associated 1-form A is equal to the associated 1-form η , then from the last equation, we get

$$(8.7) \quad S(Y, Z) = -2(n-1)g(Y, \phi Z), \quad \eta(W) \neq 0.$$

Hence contracting (8.7), we get

$$(8.8) \quad r = -2(n-1)\psi, \quad \text{where } \psi = \text{trace}\phi$$

which completes the proof of the theorem. \square

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