Novi Sad J. Math.
Vol. XX, No. Y, 20ZZ, ??-??

# FACTORIZATION OF OPERATORS WITH $G_\alpha^\alpha(\mathbb{R}_+^d)$ AND $g_\alpha^\alpha(\mathbb{R}_+^d)$ KERNELS

**Smiljana Jakšić[1] and Snježana Maksimović[2]**

**Abstract.** The aim of this paper is to prove that any linear operator with kernel in the spaces $G_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha \geq 1$ and $g_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha > 1$ is a composition of two operators in the same space.

*AMS Mathematics Subject Classification* (2010): 47B34, 45P05

*Key words and phrases:* Laguerre functions, Ultradistributions over $[0,\infty)^d$, Kernel theorems, Factorization algebra

## 1. Introduction

The test space $\mathcal{S}(\mathbb{R}_+)$ for the space of tempered distributions supported by $[0,\infty)$ is studied in [3], [9] and [12]; recently, the space $\mathcal{S}(\mathbb{R}_+^d)$ is examined in [5]. In [6] $G$-type spaces, $G_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha \geq 1$ and their dual spaces, i.e. the spaces of ultradistributions of Roumier type over $[0,\infty)^d$, are characterized in terms of their Foirier-Laguerre coefficients; cf. Duran [4] for the one-dimensional case. Actually, the results of [4] is extended and the full topological characterization is given, in all dimensions, as well as applications to pseudo-differential operators with radial symbols.

In this paper we introduce $g$-type spaces, $g_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha > 1$ and their dual spaces i.e. the spaces of ultradistributions of Beurling type over $[0,\infty)^d$. We give the kernel theorem for $g_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha > 1$.

For any topological vector space $\mathcal{B}$, the set of continuous linear functionals on $\mathcal{B}$, denoted by $\mathcal{M} = \mathfrak{L}(\mathcal{B})$, is a factorization algebra (also the term a decomposition algebra can be used). This means that any operator $T$ in $\mathcal{M}$ is a composition of two operators $T_1, T_2$ in $\mathcal{M}$ since we can choose $T_1$ as the identity operator and $T_2 = T$. If $\mathcal{B}$ is a Hilbert space, then it follows from spectral decomposition that the set of compact operators on $\mathcal{B}$ is a factorization algebra, where the factorization property are obtained by straight-forward applications of the spectral theorem.

An interesting subclass of linear and continuous operators on an $L^2$ space concerns the set of all linear operators whose kernels belong to the Schwartz space (see e.g. [1], [7],[11]). Similar facts hold true for the set of operators with kernels in Gelfand-Shilov spaces (cf. [10]) and Pilipović spaces (cf. [2]).

In this paper we consider the case when $\mathcal{M}$ is the set of all linear operators

[1]Faculty of Forestry, University of Belgrade, Kneza Višeslava 1, 11000, Belgrade, Serbia, e-mail: smiljana.jaksic@sfb.bg.ac.rs
[2]Faculty of Electrical Engineering, University of Banja Luka, Patre 5, 78000 Banja Luka, Bosnia and Herzegovina,
e-mail: snjezana.maksimovic@etfbl.net

with kernels in $G^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$ and $g^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha > 1$ spaces. We prove that any such $\mathcal{M}$ is a factorization algebra. Note that the identity operator does not belong to these operator classes.

## 2. Preliminaries

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of positive integers, integers, real and complex numbers, respectively; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0,\infty)$, $\mathbb{R}^d_+ = (0,\infty)^d$ and $\overline{\mathbb{R}^d_+} = [0,\infty)^d$. We use the standard multi-index notation. Let $x = (x_1,\dots,x_d) \in \mathbb{R}^d$, $k = (k_1,\dots,k_d) \in \mathbb{N}^d_0$. Then $|x| = \sqrt{x_1^2+\ldots+x_d^2}$, $|k| = k_1+\ldots+k_d$, $k! = k_1!\cdots k_d!$, $x^k = \prod_{i=1}^d x_i^{k_i}$, $D^k = \prod_{i=1}^d \partial^{k_i}/\partial x_i^{k_i}$. Furthermore, if $x,\gamma \in \overline{\mathbb{R}^d_+}$ we also use $x^\gamma = \prod_{j=1}^d x_j^{\gamma_j}$. In this case, if $x_j = 0$ and $\gamma_j = 0$, we use the convention $0^0 = 1$.

For $j \in \mathbb{N}_0$ and $\gamma > -1$, the $j$-th Laguerre polynomial of order $\gamma$ is defined by

$$L^\gamma_j(x) = \frac{x^{-\gamma}e^x}{j!}\frac{d^j}{dx^j}(e^{-x}x^{\gamma+j}), \quad x \geq 0.$$

For $\gamma = (\gamma_1,\dots,\gamma_d) \in \mathbb{R}^d$ such that $\gamma_j > -1$, $j = 1,\dots,d$ and $n \in \mathbb{N}^d_0$, the $d$-dimensional $n$-th Laguerre polynomial of order $\gamma$ is defined by $L^\gamma_n(x) = L^{\gamma_1}_{n_1}(x_1)\dots L^{\gamma_d}_{n_d}(x_d)$. For $\gamma = 0$, we write $L_n(x)$ instead of $L^0_n(x)$.

The $j$-th Laguerre function (of order 0) is defined by $l_j(x) = L_j(x)e^{-x/2}$, $x \geq 0$, $j \in \mathbb{N}_0$ and in a $d$-dimensional case we have $l_n(x) = l_{n_1}(x_1)\dots l_{n_d}(x_d)$, $x \in \overline{\mathbb{R}^d_+}$, $n \in \mathbb{N}^d_0$. The Laguerre functions form an orthonormal basis for $L^2(\mathbb{R}^d_+)$. Also, they have a special role for the characterisation of the spaces $G^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$ and $g^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha > 1$ considered below.

### 2.1. Basic spaces

We denote by $\mathbb{R}^d_+$ the set $(0,\infty)^d$ and by $\overline{\mathbb{R}^d_+}$ its closure, i.e. $[0,\infty)^d$. The space $\mathcal{S}(\mathbb{R}^d_+)$ consists of all $f \in \mathcal{C}^\infty(\mathbb{R}^d_+)$ such that all derivatives $D^p f$, $p \in \mathbb{N}^d_0$, extend to continuous functions on $\overline{\mathbb{R}^d_+}$ and

$$\sup_{x\in\mathbb{R}^d_+} x^k|D^p f(x)| < \infty \ , \forall k,p \in \mathbb{N}^d_0.$$

Let $A > 0$. We denote by $G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ the space of all $f \in \mathcal{S}(\mathbb{R}^d_+)$ for which

$$\sup_{p,k\in\mathbb{N}^d_0} \frac{\|t^{(p+k)/2}D^p f(t)\|_2}{A^{|p+k|}k^{(\alpha/2)k}p^{(\alpha/2)p}} < \infty.$$

With the following seminorms

$$\sigma_{A,j}(f) = \sup_{p,k\in\mathbb{N}^d_0} \frac{\|t^{(p+k)/2}D^p f(t)\|_{L^2(\mathbb{R}^d_+)}}{A^{|p+k|}k^{(\alpha/2)k}p^{(\alpha/2)p}} + \sup_{\substack{|p|\leq j\\|k|\leq j}} \sup_{t\in\mathbb{R}^d_+} |t^k D^p f(t)|, \ j \in \mathbb{N}_0,$$

one easily verifies that it becomes an $(F)$-space.

Define $G^\alpha_\alpha(\mathbb{R}^d_+) = \varinjlim_{A\to\infty} G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ and $g^\alpha_\alpha(\mathbb{R}^d_+) = \varprojlim_{A\to 0} G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$. Clearly, $G^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$ and $g^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha > 1$ are continuously injected into $\mathcal{S}(\mathbb{R}^d_+)$ i.e.

$$g^\alpha_\alpha(\mathbb{R}^d_+) \hookrightarrow G^\alpha_\alpha(\mathbb{R}^d_+) \hookrightarrow \mathcal{S}(\mathbb{R}^d_+).$$

We immediately have

$$(G^\alpha_\alpha(\mathbb{R}^d_+))' = \bigcup_{A\in\mathbb{N}} (G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+))' \tag{2.1}$$

in set theoretical sense (see [8, (1.2), p.34]). Since the sequence $G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$, when $A \to \infty$, is reduced, we have

$$(g^\alpha_\alpha(\mathbb{R}^d_+))' = \bigcap_{A\in\mathbb{N}} (G^{\alpha,1/A}_{\alpha,1/A}(\mathbb{R}^d_+))' \tag{2.2}$$

in the set theoretical sense (see [8, (1.1), p.33]).

Let $\alpha \geq 1$ and $a > 1$. We define $s^{\alpha,a}$ as the space of all complex sequences $\{a_n\}_{n\in\mathbb{N}^d_0}$ for which $\|\{a_n\}_{n\in\mathbb{N}^d_0}\|_{s^{\alpha,a}} = \sup_{n\in\mathbb{N}^d_0} |a_n| a^{|n|^{1/\alpha}} < \infty$. With this norm $s^{\alpha,a}$ becomes a $(B)$-space. We define $s^\alpha = \varinjlim_{a\to 1^+} s^{\alpha,a}$ (resp. $\sigma^\alpha = \varprojlim_{a\to\infty} \sigma^{\alpha,a}$). In particular, $s^\alpha$ is a $(DFN)$-space (resp. $\sigma^\alpha$ is a $(FN)$-space). The strong dual $(s^\alpha)'$ of $s^\alpha$ is an $(FN)$-space of all complex valued sequences $\{b_n\}_{n\in\mathbb{N}^d_0}$ such that, for each $a > 1$, $\|\{b_n\}_{n\in\mathbb{N}^d_0}\|_{(s^{\alpha,a})'} = \sum_{n\in\mathbb{N}^d_0} |b_n| a^{-|n|^{1/\alpha}} < \infty$ (resp. the strong dual $(\sigma^\alpha)'$ of $\sigma^\alpha$ is a $(DFN)$-space of all complex valued sequences $\{b_n\}_{n\in\mathbb{N}^d_0}$ such that there exists $a > 1$ such that $\|\{b_n\}_{n\in\mathbb{N}^d_0}\|_{(s^{\alpha,a})'} < \infty$).

**Theorem 2.1.** *([6, Theorem 5.7.]) Let $\alpha \geq 1$. For $f \in L^2(\mathbb{R}^d_+)$ let $a_n = \int_{\mathbb{R}^d_+} f(t) l_n(t) dt$, $n \in \mathbb{N}^d_0$. Then $f \in G^\alpha_\alpha(\mathbb{R}^d_+)$ if and only if there exist $c > 0$ and $a > 1$ such that $|a_n| \leq c a^{-|n|^{1/\alpha}}$.*

With small modifications of the arguments (by the use of the closed graph theorem for an $F$-space), we have the next theorem

**Theorem 2.2.** *Let $\alpha > 1$. For $f \in L^2(\mathbb{R}^d_+)$ let $a_n = \int_{\mathbb{R}^d_+} f(t) l_n(t) dt$, $n \in \mathbb{N}^d_0$. Then $f \in g^\alpha_\alpha(\mathbb{R}^d_+)$ if and only if for every $a > 1$ there exists $c > 0$ such that $|a_n| \leq c a^{-|n|^{1/\alpha}}$.*

**Theorem 2.3.** *([6, Theorem 6.1]) Let $\alpha \geq 1$. The mapping $\iota : G^\alpha_\alpha(\mathbb{R}^d_+) \to s^\alpha$, $\iota(f) = \{\langle f, l_n\rangle\}_{n\in\mathbb{N}^d_0}$, is a topological isomorphism between $G^\alpha_\alpha(\mathbb{R}^d_+)$ and $s^\alpha$.*

*For each $f \in G^\alpha_\alpha(\mathbb{R}^d_+)$, $\sum_{n\in\mathbb{N}^d_0} \langle f, l_n\rangle l_n$ is summable to $f$ in $G^\alpha_\alpha(\mathbb{R}^d_+)$.*

In a similar way, we prove the next theorem.

**Theorem 2.4.** *Let $\alpha > 1$. The mapping $\iota : g^\alpha_\alpha(\mathbb{R}^d_+) \to \sigma^\alpha$, $\iota(f) = \{\langle f, l_n\rangle\}_{n\in\mathbb{N}^d_0}$, is a topological isomorphism between $g^\alpha_\alpha(\mathbb{R}^d_+)$ and $\sigma^\alpha$.*

*For each $f \in g^\alpha_\alpha(\mathbb{R}^d_+)$, $\sum_{n\in\mathbb{N}^d_0} \langle f, l_n\rangle l_n$ is summable to $f$ in $g^\alpha_\alpha(\mathbb{R}^d_+)$.*

The last four results are crucial and we will often tacitly apply them throughout the rest of this article.

## 3. $G$- and $g$- kernels

Firstly, we state the kernel theorems:

**Theorem 3.1.** *([6, Theorem 6.4.]) Let $\alpha \geq 1$. We have the following canonical isomorphism:*

$$(G^\alpha_\alpha(\mathbb{R}^{d_1}_+))'\hat{\otimes}(G^\alpha_\alpha(\mathbb{R}^{d_2}_+))' \cong (G^\alpha_\alpha(\mathbb{R}^{d_1+d_2}_+))' \cong \mathcal{L}(G^\alpha_\alpha(\mathbb{R}^{d_1}_+),(G^\alpha_\alpha(\mathbb{R}^{d_2}_+))'). \tag{3.1}$$

**Theorem 3.2.** *Let $\alpha > 1$. We have the following canonical isomorphism:*

$$(g^\alpha_\alpha(\mathbb{R}^{d_1}_+))'\hat{\otimes}(g^\alpha_\alpha(\mathbb{R}^{d_2}_+))' \cong (g^\alpha_\alpha(\mathbb{R}^{d_1+d_2}_+))' \cong \mathcal{L}(g^\alpha_\alpha(\mathbb{R}^{d_1}_+),(g^\alpha_\alpha(\mathbb{R}^{d_2}_+))'). \tag{3.2}$$

*Proof.* The proof for $(g^\alpha_\alpha(\mathbb{R}^d_+))'$ can be obtained it the same way as for $(G^\alpha_\alpha(\mathbb{R}^d_+))'$ in Theorem 3.1. □

The isomorphisms (3.1) and (3.2) call for some comment. In what follows we use the convention that if $T$ is a linear and continuous operator from $G^1_1(\mathbb{R}^{d_1}_+)$ to $(G^1_1(\mathbb{R}^{d_2}_+))'$, and $g \in (G^1_1(\mathbb{R}^{d_0}_+))'$, then $T \otimes g$ is the linear and continuous operator from $G^1_1(\mathbb{R}^{d_1}_+)$ to $(G^1_1(\mathbb{R}^{d_2+d_0}_+))'$, given by

$$(T \otimes g) : f \mapsto (Tf) \otimes g.$$

The following theorem is the main result of this paper.

**Theorem 3.3.** *Let $T$ be a linear and continuous operator from $G^1_1(\mathbb{R}^{d_1}_+)$ to $(G^1_1(\mathbb{R}^{d_2}_+))'$ with the kernel $K$, and let $d_0 \geq \min(d_1, d_2)$. Then the following is true:*

- *(R) If $\alpha \geq 1$ and $K \in G^\alpha_\alpha(\mathbb{R}^{d_2+d_1}_+)$, then there are operators $T_1$ and $T_2$ with kernels $K_1 \in G^\alpha_\alpha(\mathbb{R}^{d_0+d_1}_+)$ and $K_2 \in G^\alpha_\alpha(\mathbb{R}^{d_2+d_0}_+)$ respectively such that $T = T_2 \circ T_1$.*
- *(B) If $\alpha > 1$ and $K \in g^\alpha_\alpha(\mathbb{R}^{d_2+d_1}_+)$, then there are operators $T_1$ and $T_2$ with kernels $K_1 \in g^\alpha_\alpha(\mathbb{R}^{d_0+d_1}_+)$ and $K_2 \in g^\alpha_\alpha(\mathbb{R}^{d_2+d_0}_+)$ respectively such that $T = T_2 \circ T_1$.*

*Remark* 3.4. Let $\alpha \geq 1$. An operator with kernel in $G^\alpha_\alpha(\mathbb{R}^{2d}_+)$ is sometimes called a regularizing operator with respect to $G^\alpha_\alpha(\mathbb{R}^d_+)$, because it extends uniquely to a continuous mapping from $(G^\alpha_\alpha(\mathbb{R}^d_+))'$, into $G^\alpha_\alpha(\mathbb{R}^d_+)$.

Let $\alpha > 1$. An operator with kernel in $g^\alpha_\alpha(\mathbb{R}^{2d}_+)$ is sometimes called a regularizing operator with respect to $g^\alpha_\alpha(\mathbb{R}^d_+)$.

*Proof.* First we assume that $d_0 = d_1$, and start to prove (R). Let $l_{d,n}(x)$ be the Laguerre function on $\mathbb{R}^d_+$ of order $n \in \mathbb{N}^d$. Then $K$ posses the expansion

$$K(x,y) = \sum_{n\in\mathbb{N}^{d_2}} \sum_{k\in\mathbb{N}^{d_1}} a_{n,k} l_{d_2,n}(x) l_{d_1,k}(y), \tag{3.3}$$

where the coefficients $a_{n,k}$ satisfies

$$\sup_{n,k} |a_{n,k} e^{A(|n|^{1/\alpha}+|k|^{1/\alpha})}| < \infty, \tag{3.4}$$

for some $A > 0$. Let $z \in \mathbb{R}_+^{d_1}$ and

$$\begin{aligned} K_{0,1}(z,y) &= \sum_{n,k\in\mathbb{N}^{d_1}} c_{n,k} l_{d_1,n}(z) l_{d_1,k}(y) \\ K_{0,2}(x,z) &= \sum_{n\in\mathbb{N}^{d_2}} \sum_{k\in\mathbb{N}^{d_1}} b_{n,k} l_{d_2,n}(x) l_{d_1,k}(z), \end{aligned} \tag{3.5}$$

where

$$c_{n,k} = \chi_{n,k} e^{-\frac{A}{2}|n|^{1/\alpha}} \quad \text{and} \quad b_{n,k} = a_{n,k} e^{\frac{A}{2}|k|^{1/\alpha}}$$

and $\chi_{n,k}$ is the Kronecker delta. Then we have

$$\int K_{0,2}(x,z) K_{0,1}(z,y)\,dz = \sum_{n\in\mathbb{N}^{d_2}} \sum_{k\in\mathbb{N}^{d_1}} a_{n,k} l_{d_2,n}(x) l_{d_1,k}(y) = K(x,y).$$

Hence, if $T_j$ is the operator with kernel $K_{0,j}$, $j = 1,2$, then $T = T_2 \circ T_1$. Furthermore,

$$\sup_{n,k} |b_{n,k} e^{\frac{A}{2}(|n|^{1/\alpha}+|k|^{1/\alpha})}| \le \sup_{n,k} |a_{n,k} e^{A(|n|^{1/\alpha}+|k|^{1/\alpha})}| < \infty$$

and

$$\sup_{n,k} |c_{n,k} e^{\frac{A}{4}(|n|^{1/\alpha}+|k|^{1/\alpha})}| = \sup_{n} |e^{-\frac{A}{2}|n|^{1/\alpha}} e^{\frac{A}{2}|n|^{1/\alpha}}| < \infty.$$

This implies that $K_{0,1} \in G_\alpha^\alpha(\mathbb{R}_+^{d_1+d_1})$ and $K_{0,2} \in G_\alpha^\alpha(\mathbb{R}_+^{d_2+d_1})$ (see Theorem 2.1). If we put $K_1 = K_{0,1}$ and $K_2 = K_{0,2}$, we proved (R) in the case when $d_0 = d_1$.

In order to prove (B) assume that $K \in g_\alpha^\alpha(\mathbb{R}_+^{d_2+d_1})$ and let $a_{n,k}$ be the same as the above. Then (3.4) holds for any $A > 0$, which implies that if $N \ge 0$ is an integer, then

$$\Sigma_N = \sup\{|k| : |a_{n,k}| \ge e^{-2(N+1)(|n|^{1/\alpha}+|k|^{1/\alpha})} \ \text{for some} \ \ n \in \mathbb{N}^{d_2}\} \tag{3.6}$$

is finite. Let $I_1 = \{k \in \mathbb{N}^{d_1} : |k| \le \Sigma_1 + 1\}$ and define inductively

$$I_j = \{k \in \mathbb{N}^{d_1} \setminus I_{j-1} : |k| \le \Sigma_j + j\},\ j \ge 2.$$

Then

$$I_j \cap I_k = \emptyset \ \text{ when } j \ne k, \quad \text{and} \quad \mathbb{N}^{d_1} = \bigcup_{j\ge1} I_j.$$

Let $K_{0,1}$ and $K_{0,2}$ be as in (3.5) where

$$c_{n_1,k} = \chi_{n_1,k} e^{-j|k|^{1/\alpha}} \quad \text{and} \quad b_{n_2,k} = a_{n_2,k} e^{j|k|^{1/\alpha}},$$

$n_1 \in \mathbb{N}^{d_1}$, $n_2 \in \mathbb{N}^{d_2}$ and $k \in I_j$. If $T_j$ is the operator with kernel $K_{0,j}$, $j = 1, 2$, then it follows that $T = T_2 \circ T_1$. Moreover, if $A > 0$ we have

$$\sup_{n,k} |b_{n,k} e^{A(|n|^{1/\alpha}+|k|^{1/\alpha})}| \le J_1 + J_2,$$

where

$$J_1 = \sup_{j \le A+1} \sup_n \sup_{k \in I_j} |b_{n,k} e^{A(|n|^{1/\alpha}+|k|^{1/\alpha})}| \tag{3.7}$$

and

$$J_2 = \sup_{j > A+1} \sup_n \sup_{k \in I_j} |b_{n,k} e^{A(|n|^{1/\alpha}+|k|^{1/\alpha})}|. \tag{3.8}$$

We will prove that $J_1$ and $J_2$ are finite. Since in (3.7) we have the finite numbers of $k$, from (3.4) and the definition of $b_{n,k}$ we obtain that $J_1$ is finite.

For $J_2$ we have

$$\begin{aligned} J_2 &= \sup_{j > A+1} \sup_n \sup_{k \in I_j} |a_{n,k} e^{A|n|^{1/\alpha}+(A+j)|k|^{1/\alpha}}| \\ &\le \sup_{j > A+1} \sup_n \sup_{k \in I_j} |e^{-2j(|n|^{1/\alpha}+|k|^{1/\alpha})} e^{A|n|^{1/\alpha}+(A+j)|k|^{1/\alpha}}| < \infty, \end{aligned}$$

where the first inequality follows from (3.6). Hence,

$$\sup_{n,k} |b_{n,k} e^{A(|n|^{1/\alpha}+|k|^{1/\alpha})}| < \infty,$$

which implies that $K_{0,2} \in g^\alpha_\alpha(\mathbb{R}^{d_2+d_1}_+)$.

In a similar way if we replace $b_{n,k}$ with $c_{n,k}$ in the definition of $J_1$ and $J_2$) we obtain

$$\sup_{n,k} |c_{n,k} e^{A(|n|^{1/\alpha}+|k|^{1/\alpha})}| < \infty,$$

which implies $K_{0,1} \in g^\alpha_\alpha(\mathbb{R}^{d_1+d_1}_+)$ and (B) follows in the case $d_0 = d_1$.

Next, assume that $d_0 > d_1$ and let $d = d_0 - d_1 \ge 1$. Then we set

$$K_1(z, y) = K_{0,1}(z_1, y) l_{d,0}(z_2) \quad \text{and} \quad K_2(x, z) = K_{0,2}(x, z_1) l_{d,0}(z_2),$$

where $z_1 \in \mathbb{R}^{d_1}_+$, $z_2 \in \mathbb{R}^d_+$ and hence, $z = (z_1, z_2) \in \mathbb{R}^{d_0}_+$. Next, we obtain

$$\int_{\mathbb{R}^{d_0}_+} K_2(x, z) K_1(z, y) dz = \int_{\mathbb{R}^{d_1}_+} K_{0,2}(x, z_1) K_{0,1}(z_1, y) dz_1 = K(x, y).$$

In this case the assertion (R) and (B) follows from the equivalences

$$K_1 \in G^\alpha_\alpha(\mathbb{R}^{d_0+d_1}_+) \ (g^\alpha_\alpha(\mathbb{R}^{d_0+d_1}_+)) \iff K_{0,1} \in G^\alpha_\alpha(\mathbb{R}^{d_1+d_1}_+) \ (g^\alpha_\alpha(\mathbb{R}^{d_1+d_1}_+))$$

and

$$K_2 \in G^\alpha_\alpha(\mathbb{R}^{d_2+d_0}_+) \ (g^\alpha_\alpha(\mathbb{R}^{d_2+d_0}_+)) \iff K_{0,1} \in G^\alpha_\alpha(\mathbb{R}^{d_2+d_1}_+) \ (g^\alpha_\alpha(\mathbb{R}^{d_2+d_1}_+)).$$

It remains to prove the result in the case $d_0 \geq d_2$. The rules of $d_1$ and $d_2$ are interchanged when taking the adjoint opeartors. Hence, the result follows from the first part of the proof in combination with the facts that $G^\alpha_\alpha$ and $g^\alpha_\alpha$ are invariant under pullbacks of bijective linear transformations i.e. $(x,y) \mapsto F(x,y)$ belongs to $G^\alpha_\alpha(\mathbb{R}^{d_1}_+ \times \mathbb{R}^{d_2}_+)$ if and only if $(y,x) \mapsto \overline{F(x,y)}$ belongs to $G^\alpha_\alpha(\mathbb{R}^{d_2}_+ \times \mathbb{R}^{d_1}_+)$. The proof is complete. □

## Acknowledgement

This work is supported by the Project 19/6-020/961-38/15 of the Republic of Srpska Ministry of Science and Technology.

## References


[1] R. Beals, *Characterization of pseudodifferential operators and applications*, Duke Math. J. 44 (1977), 45-57.

[2] Y. Chen, M. Signahl, J. Toft, *Factorizations and singular value estimates of operators with Gelfand-Shilov and Pilipović kernels*, arXiv:1511.06257

[3] A. J. Duran, *Laguerre expansions of Tempered Distributions and Generalized Functions*, Journal of Mathematical Analysis and Applications 150(1990), 166-180.

[4] A. J. Duran, *Laguerre expansions of Gel'fand-Shilov spaces*, J. Approx. Theory 74 (1993), 280-300.

[5] S. Jakšić, B. Prangoski, *Extension theorem of Whitney type for $\mathcal{S}(\mathbb{R}^d_+)$ by the use of the Kernel Theorem*, Publ. Inst. Math. Beograd, 99 (113) (2016), 59-65.

[6] S. Jakšić, S. Pilipović, B. Prangoski, *G-type spaces of ultradistributions over $\mathbb{R}^d_+$ and the Weyl pseudo-differential operators with radial symbols*, arXiv:1602.04017.

[7] Czeslaw Klis, *A simple proof of the identity $\mathcal{S} \circ \mathcal{S} = \mathcal{S}$* , in: Proc. of the Conference on Convergence and Generalized Functions, Katowice 1983, Preprint of the Institute of Math., Polish Academy of Sci., pp. 71-74.

[8] H. Komatsu, *Ultradistributions, I: Structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 20 (1973), 25-105.

[9] S. Pilipović, *On the Laguerre expansions of generalized functions*, C. R. Math. Rep. Acad. Sci. Canada 11(1) (1989), 23-27.

[10] J. Toft, A. Khrennikov, B. Nilsson, S. Nordebo, *Decompositiond of Gelfand-Shilov kernels into kernels of similar class*, Journal of Mathematical Analysis and Applications,396 (1) (2012), 315-322.

[11] J. Voigt, *Factorization in some Fréchet algebras of differentiable functions*, Studia Math. 77 (1984), 333-348.

[12] A. I. Zayed, *Laguerre series as boundary values*, SIAM J. Math. Anal. 13(2) (1982), 263-279.