New York Journal of Mathematics

New York J. Math. **1** (1994) 26–38.

Projective Hilbert $\mathbb{A}(\mathbb{D})$-Modules

Jon F. Carlson, Douglas N. Clark, Ciprian Foias, and J.P. Williams

Jim Williams died in 1983. The surviving three authors are pleased to dedicate this paper to his memory.

Abstract. Let $\mathcal{C}$ denote the category of Hilbert modules which are similar to contractive Hilbert modules. It is proved that if $H_0, H \in \mathcal{C}$ and if H_1 is similar to an isometric Hilbert module, then the sequence

$$0 \to H_0 \to H \to H_1 \to 0$$

splits. Thus the isometric Hilbert modules are projective in $\mathcal{C}$. It follows that $\mathrm{Ext}^n_{\mathcal{C}}(K, H) = 0$, whenever $n > 1$, for $H, K \in \mathcal{C}$. In addition, it is proved that (Hilbert modules similar to) unitary Hilbert modules are projective in the category $\mathcal{H}$ of all Hilbert modules. Connections with the conjecture that $\mathcal{C}$ is a *proper* subset of $\mathcal{H}$ are discussed.

1. Introduction.

A few years ago, Douglas and Paulsen [2] introduced the notion of a Hilbert module as a Hilbert space together with the action of a function algebra $\mathbb{A}$. The category $\mathcal{H}$ of Hilbert $\mathbb{A}$-modules is a natural setting for numerous questions in operator theory. Some of these can be expressed in terms of homological constructions such as extensions and the extension groups, $\mathrm{Ext}^1_{\mathcal{H}}(-,-)$. However any attempt to apply standard homological algebra methods to the category of Hilbert modules immediately encounters some obstacles. One of the most formidable of the difficulties is that the categories may not have enough projective and injective objects. For example, in the case of the disk algebra $\mathbb{A} = \mathbb{A}(\mathbb{D})$ it was not previously known if there were any projective Hilbert $\mathbb{A}(\mathbb{D})$-modules. As we shall see the lack of projective modules seems intimately involved in some basic questions of operator theory. But also it has an effect on even the definitions of some of the standard homological constructions.

Douglas and Paulsen in [2, Chapter 4] avoid projective modules by introducing hypo-projective Hilbert modules. They succeed in characterizing the hypo-projectives and in using them to give a new proof of the lifting theorem, but they

Received June 9, 1994.

Mathematics Subject Classification. Primary: 47D, Secondary: 47C.

Key words and phrases. Hilbert module, projective, lifting theorem, polynomially bounded operator, cramped Hilbert module.

The first and third authors were partially supported by NSF grants.

ISSN 1076-9803/94

do not produce projective modules, for any choice of the function algebra $\mathbb{A}$ except for $\mathbb{A} = \mathbb{C}(X)$, where every Hilbert module is projective.

One of the main purposes of this paper is to report some progress on the problem of the existence of projective objects in the category of Hilbert $\mathbb{A}(\mathbb{D})$-modules. We show in Section 4 that if the operator of multiplication by z has a unitary action on a Hilbert module H then H is both a projective and an injective module. This answers Problem 4.6 of [2]. Moreover if we restrict ourselves to the subcategory of Hilbert $\mathbb{A}(\mathbb{D})$-modules for which the action of z is similar to a contraction then any isometric Hilbert module is projective. Its dual which is coisometric is injective. The result settles several questions about the subcategory. For one thing, the functional model [7] for any completely nonunitary contractive Hilbert module is a projective resolution in the subcategory. This is called a Šilov resolution in [2] and the method is adaptable to get a projective resolution for any object in the subcategory.

All of this appears to be connected to the well known question of the existence of Hilbert modules not similar to contractive ones [2, Problem 2.4], which, in turn, is equivalent to the question whether a polynomially bounded operator must be similar to a contraction [3] . Indeed, it is a consequence of our results that if isometric Hilbert modules are not projective then both the above questions have negative answers.

In Section 2 we introduce some notation and some standard results which will be needed later. Included is some homological machinery. Although the techniques are well known some tedious care must be taken to avoid the peculiar pitfalls of the categories in question. In Section 3 we consider only Hilbert modules similar to contractions and show the projectivity in the isometric case. Some easy consequences are derived. Section 4, contains the proof of general projectivity of unitary Hilbert modules, and derives consequences for $\mathrm{Ext}_{\mathcal{H}}$ groups in the larger category. In particular we show the impossibility of getting a Hilbert module not similar to a contraction from certain types of extensions. We end in Section 5 with some remarks and discussion on the problem of constructing a polynomially bounded operator which is not similar to a contraction.

2. Notation and Preliminaries.

Suppose that $\mathbb{A} = \mathbb{A}(\mathbb{D})$ is the disk algebra; that is, the set of all $f(z)$ analytic in the unit disk $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$, with the sup norm, $||f|| = \sup_{|z|<1} |f(z)|$. A *Hilbert module over* $\mathbb{A}$ is a Hilbert space H, which is equipped with the structure of an $\mathbb{A}$-module in such a way that the multiplication $(a, f) \to af$ from $\mathbb{A} \times H$ to H is continuous in both variables; see Douglas and Paulsen [2] for more details.

A *Hilbert module map* between Hilbert modules H and K is a linear function $L : H \to K$ which is continuous and which commutes with the action of $\mathbb{A}$ ($L(af) = aL(f)$ for $a \in \mathbb{A}$ and $f \in H$). We denote by $\mathcal{H}$ the category of Hilbert modules over $\mathbb{A}$, with Hilbert-module maps.

The operator of multiplication by z must be bounded on the Hilbert module H. Indeed, multiplication by any $a \in \mathbb{A}$ must be bounded on H. The subcategory $\mathcal{C}$ of $\mathcal{H}$ is defined to be the subcategory of all Hilbert modules H over $\mathbb{A}$ with the property that multiplication by z on H is similar to a contraction ($||LzL^{-1}f|| \leq ||f||$ for some bounded invertible Hilbert space operator $L : H \to H'$). We shall call such a Hilbert module a *cramped* Hilbert module. The statement that $\mathcal{C}$ is *full in*

$\mathcal{H}$ means that, if H, $K \in \mathcal{C}$ then the set of homomorphisms from H to K in $\mathcal{C}$ is the same as in $\mathcal{H}$. Thus

$$\mathrm{Hom}_{\mathcal{C}}(H, K) = \mathrm{Hom}_{\mathcal{H}}(H, K)$$

for all H, $K \in \mathcal{C}$.

In [1] , the first two authors studied the Ext^1 functor in the category $\mathcal{H}$. Briefly, an element in $\mathrm{Ext}_{\mathcal{H}}(K, H)$ is an equivalence class of exact sequences

$$E: \quad 0 \longrightarrow H \xrightarrow{\alpha} J \xrightarrow{\beta} K \longrightarrow 0$$

of Hilbert modules and maps. A sequence

$$E': \quad 0 \longrightarrow H' \xrightarrow{\alpha} J' \xrightarrow{\beta} K' \longrightarrow 0$$

is *equivalent* to E if there exists $\psi : J \to J'$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r, "\alpha"] \arrow[d, equal] & J \arrow[r, "\beta"] \arrow[d, "\psi"'] & K \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H' \arrow[r] & J' \arrow[r] & K' \arrow[r] & 0
\end{tikzcd}$$

commutes. It can be seen that ψ is an isomorphism. It is one-to-one and onto and hence is invertible. Additionally, if we write the orthogonal decomposition $J = \alpha(H) \oplus \alpha(H)^{\perp}$ of the Hilbert space J, then the operator $T : J \to J$ of multiplication by z, has the decomposition

$$T = \begin{bmatrix} T_0 & V \\ 0 & T_1 \end{bmatrix} \tag{2.1}$$

where T_0 and T_1 are the operators of multiplication by z on $\alpha(H)$ and $\alpha(H)^{\perp}$, respectively.

In [1], the following theorem was proved.

Theorem 2.2. *Suppose that H and K are Hilbert modules over $\mathbb{A}$. Then*

$$\mathrm{Ext}_{\mathcal{H}}(K, H) = \mathfrak{A}/\mathfrak{B},$$

where $\mathfrak{A} = \mathfrak{A}(K, H)$ is the set of all continuous, bilinear functions $\sigma : \mathbb{A} \times K \to H$ satisfying the condition

$$a\sigma(b, k) + \sigma(a, bk) = \sigma(ab, k),$$

for $a, b \in \mathbb{A}, k \in K$, and $\mathfrak{B}$ is the subspace of $\mathfrak{A}$ consisting of those σ which have the form

$$\sigma(a, h) = aL(h) - L(ah)$$

for all $a \in \mathbb{A}$ and $h \in K$ and for some bounded, linear (Hilbert space) operator $L : K \to H$.

In the representation (2.1) of the operator of multiplication by z on the middle term of an extension, the operator $V : K \to H$ is, in fact, $\sigma(z, \cdot)$.

One of our main concerns in this paper is with the projectives of the functor $\mathrm{Ext}_{\mathcal{C}}$ of extensions which take place in the category $\mathcal{C}$. The first thing we need to verify is that $\mathrm{Ext}_{\mathcal{C}}$ is a functor. For this we need the following.

Lemma 2.3. *Pushouts and pullbacks exist in the category* $\mathcal{C}$.

Proof. We refer to [4] for a more complete description of pushouts and pullbacks. We need to recall that if we have a pullback diagram

$$\begin{tikzcd} & H_0 \arrow[d, "\alpha_0"] \\ H_1 \arrow[r, "\alpha_1"'] & K \end{tikzcd}$$

then the pullback J (with maps $\gamma_0 : J \to H_0$ and $\gamma_1 : J \to H_1$, satisfying $\alpha_0\sigma_0 = \alpha_1\sigma_1$) is isomorphic to the following subset of $H_0 \oplus H_1$

$$J \cong \{(h_0, h_1) \in H_0 \oplus H_1 | \alpha_0(h_0) = \sigma_1(h_1)\} \subseteq H_0 \oplus H_1.$$

Now if the operators T_i of multiplication by z on H_i are both similar to contractions $(i = 0, 1)$ then so is the operator T of multiplication by z on $H_0 \oplus H_1$. As a result, the same is true for the restriction of T to any Hilbert submodule of $H_0 \oplus H_1$. This proves the existence of pullbacks in $\mathcal{C}$. For pushouts, it is only necessary to take duals $\overline{H}_0, \overline{H}_1$ of the Hilbert modules H_0, H_1. For the dual of the pushout, consider the dual of the diagram

$$\begin{tikzcd} K \arrow[r, "\alpha_0"] \arrow[d, "\alpha_1"'] & H_0 \\ H_1 & \end{tikzcd}$$

which is the pullback diagram

$$\begin{tikzcd} & \overline{H}_0 \arrow[d, "\overline{\alpha}_0"] \\ \overline{H}_1 \arrow[r, "\overline{\alpha}_1"'] & \overline{K} \end{tikzcd}$$

and observe that $\overline{H}_i$ is in $\mathcal{C}$ if and only if $H_i \in \mathcal{C}$, $k = 0, 1$. $\square$

Lemma 2.3 yields the functoriality of $\mathrm{Ext}_{\mathcal{C}}$. Indeed, if $\mathrm{cls}(E) \in \mathrm{Ext}_{\mathcal{C}}(K, H)$ is the class of an exact sequence

$$E: \quad 0 \longrightarrow H \xrightarrow{\alpha} J \xrightarrow{\beta} K \longrightarrow 0$$

and if $\zeta : H \to H'$ is a homomorphism of Hilbert modules with $H' \in \mathcal{C}$, then let ζE be the sequence obtained from the pushout

$$\begin{tikzcd} E : 0 \arrow[r] & H \arrow[r, "\alpha"] \arrow[d, "\zeta"'] & J \arrow[r, "\beta"] \arrow[d, "\zeta'"] & K \arrow[r] \arrow[d, equal] & 0 \\ \zeta E : 0 \arrow[r] & H' \arrow[r, "\alpha'"] & J' \arrow[r, "\beta'"] & K \arrow[r] & 0 \end{tikzcd}$$

(J' being the pushout of the diagram of H, H', J, α and ζ). Likewise, if $\gamma : K' \to K$, then $E\gamma$ is obtained by taking the pullback along γ of E. Then we get homomorphisms

$$\zeta_* : \mathrm{Ext}_{\mathcal{C}}(K, H) \to \mathrm{Ext}_{\mathcal{C}}(K, H')$$

and

$$\gamma* : \mathrm{Ext}_{\mathcal{C}}(K, H) \to \mathrm{Ext}_{\mathcal{C}}(K', H)$$

where $\zeta_*(cls(E)) = cls(\zeta E), \gamma^*(cls(E)) = cls(E\gamma)$.

Theorem 2.4. $\mathrm{Ext}_{\mathcal{C}}(\cdot,\cdot)$ *is a bifunctor on the category* $\mathcal{C}$ *to the category of* $\mathbb{A}$ *modules.*

To finish the proof, one needs only to observe that

$$\zeta(E\gamma) \text{ is in the same class as } (\zeta E)\gamma.$$

Also, we may note that $\mathrm{Ext}_{\mathcal{C}}$ and $\mathrm{Ext}_{\mathcal{H}}$ are naturally $\mathbb{A}$ modules (although not Hilbert modules over $\mathbb{A}$). The module action of $\mathbb{A}$ on $\mathrm{Ext}_{\mathcal{C}}(K, H)$ is given by the diagram for ζE with $\zeta = L_a : H \to H$, and

$$a \cdot cls(E) = cls(\zeta E)$$

where $\zeta : H \to H$ is multiplication by $a \in \mathbb{A}$. It is easy to check that $cls(\zeta E) = cls(E\zeta')$ where ζ' is left multiplication by a on K.

Following Theorem 2.4, we may use standard techniques to prove the following.

Proposition 2.5. *If* $E : 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is an exact sequence of objects in* $\mathcal{C}$ *and if* D *is in* $\mathcal{C}$*, then we have exact sequences*

$$0 \to \mathrm{Hom}_{\mathcal{C}}(D, A) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathcal{C}}(D, B) \xrightarrow{\beta_*} \mathrm{Hom}_{\mathcal{C}}(D, C) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathcal{C}}(C, D)$$
$$\xrightarrow{\beta^*} \mathrm{Ext}^1_{\mathcal{C}}(B, D) \xrightarrow{\alpha^*} \mathrm{Ext}^1_{\mathcal{C}}(A, D).$$

We conclude this section with the construction of an exact sequence we will need later. The term *isometric* [resp., *unitary*] Hilbert module, below, will refer to a Hilbert module H such that the operator of multiplication by z on H is an isometry [respectively, a unitary operator].

Theorem 2.6. *Suppose that* K *is an object in* $\mathcal{C}$*. Then there exist isometric Hilbert modules* H *and* J *and maps* $\alpha : H \to J$, $\beta : J \to K$ *such that*

$$0 \longrightarrow H \xrightarrow{\alpha} J \xrightarrow{\beta} K \longrightarrow 0$$

is exact.

The theorem is certainly not new and is nothing more than a restatement of the orthogonal splitting of a contraction into its unitary and completely nonunitary parts as well as of the unitary equivalence of the latter to its functional model. The sequence is called a Šilov resolution in [2] where even more general things are done. The point is that if K is a contractive Hilbert module and if K' is a unitary submodule, then K' is invariant under the operator of multiplication by z on K and also under its adjoint. Thus $K = K_0 \oplus K_1$, with K_0 unitary and K_1 completely nonunitary, and we have a sequence

$$0 \longrightarrow H \longrightarrow K_0 \oplus J' \longrightarrow K_0 \oplus K_1 \longrightarrow 0$$

where the subsequence $0 \to H \to J' \to K_1 \to 0$ is the functional model [7] of K_1 (see also [2]).

3. Extensions of isometric modules.

The main purpose of this section is to establish the existence of projective objects in the category $\mathcal{C}$ of cramped Hilbert $\mathbb{A}(\mathbb{D})$-modules. For notation, we let $\hat{\oplus}$ denote the direct sum operation for Hilbert spaces. So in any short exact sequence $0 \to H \to J \to K \to 0$ of Hilbert modules the middle term $J \cong H\hat{\oplus}K$ though it is not necessarily the direct sum of H and K as Hilbert modules.

Theorem 3.1. *Let H_0 and H_1 be Hilbert modules and denote*

$$T_0 f = zf \qquad \text{for } f \in H_0$$

and

$$T_1 f = zf \qquad \text{for } f \in H_1,$$

and let

$$Tf = zf \qquad \text{on } H_0\hat{\oplus}H_1$$

be defined by

$$T(f, f_1) = (T_0 f + \sigma(z, f_1), T_1 f_1).$$

Under these definitions, suppose H_1 is similar to an isometric Hilbert module and $H_0\hat{\oplus}H_1$ is cramped. Then the sequence

$$E : 0 \longrightarrow H_0 \longrightarrow H_0\hat{\oplus}H_1 \longrightarrow H_1 \longrightarrow 0$$

splits, i.e. $cls(E) = 0$.

Proof. Suppose that $\|ZTZ^{-1}\| \leq 1$, where $Z : H_0\hat{\oplus}H_1 \to J$. Give J the structure of a contractive Hilbert module with $zf = ZTZ^{-1}f$, for $f \in J$. The commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_0 & \longrightarrow & H_0\hat{\oplus}H_1 & \longrightarrow & H_1 & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle Z}\big\downarrow & & \| & & \\
0 & \longrightarrow & H_0 & \longrightarrow & J & \longrightarrow & H_1 & \longrightarrow & 0
\end{array}$$

shows that it suffices to prove the theorem in case T is contractive.

Since $H_0\hat{\oplus}H_1$ is contractive if and only if the operator matrix

$$\begin{pmatrix} T_0 & \sigma(z, \cdot) \\ 0 & T_1 \end{pmatrix}$$

is a contraction, we can apply [6] to conclude that $\sigma(z, f)$ has the form $\sigma(z, f) = (I - T_0 T_0^*)^{\frac{1}{2}} L (I - T_1^* T_1)^{\frac{1}{2}}$, where $\|L\| \leq 1$.

Consider the isometric dilation U_+ of T_1 acting on $\mathbf{H} = H_1 \oplus D \oplus D \oplus \cdots$ ($D = \overline{(I - T_1^* T_1)^{\frac{1}{2}} H_1}$) by

$$U_+(h_0, h_1, \cdots) = (T_1 h_0, (I - T_1^* T_1)^{\frac{1}{2}} h_0, h_1, h_2, \cdots)$$

By the Lifting Theorem [7, Proposition II.2.2], if T_1 is similar to an isometry V, say $T_1 = XVX^{-1}$, there is an operator $M : H_1 \to \mathbf{H}$ satisfying $MT_1 = U_+ M$ and having the form

$$Mh = (h, W_0 h, W_1 h, \cdots).$$

Indeed in the notation of [7, §II.2.2], let Y be the dilation of X to $\mathbf{H}$ such that $YV = U_+Y$, $||Y|| = ||X||$ then $M = YX^{-1}$ and this accounts for the fact that M compresses to the identity on H_1. Each $W_j : H_1 \to H_1$ is bounded and

$$||f||^2 + \sum_{j=0}^{\infty} ||W_j f||^2 \le ||X||^2 ||X^{-1}||^2 ||f||^2, f \in H_1 \ . \tag{3.2}$$

We have

$$\begin{aligned} U_+Mh &= (T_1h, (I - T_1^*T_1)^{\frac{1}{2}}h, W_0h, W_1h \cdots) \\ MT_1h &= (T_1h, W_0T_1h, W_1T_1h, W_2T_1h, \cdots) \end{aligned}$$

and it follows that

$$\begin{aligned} W_jT_1 &= W_{j-1} \qquad j = 1, 2, \cdots \\ W_0T_1 &= (I - T_1^*T_1)^{\frac{1}{2}}. \end{aligned}$$

Define

$$Bf = \sum_{n=0}^{\infty} T_0^n (I - T_0T_0^*)^{\frac{1}{2}} LW_n f$$

for $f \in H_1$. We will prove that the sum converges in the weak operator topology on H_1 to a bounded operator B. Once we prove this, we will have, for $f \in H_1$,

$$\begin{aligned} (BT_1 - T_0B)f &= \sum_{n=0}^{\infty} T_0^n (I - T_0T_0^*)^{\frac{1}{2}} LW_nT_1 f - \sum_{n=0}^{\infty} T_0^{n+1} (I - T_0T_0^*)^{\frac{1}{2}} LW_n f \\ &= (I - T_0T_0^*)^{\frac{1}{2}} L (I - T_1^*T_1)^{\frac{1}{2}} f + \sum_{n=1}^{\infty} T_0^n (I - T_0T_0^*)^{\frac{1}{2}} LW_{n-1} f \\ &\qquad - \sum_{n=0}^{\infty} T_0^{n+1} (I - T_0T_0^*)^{\frac{1}{2}} LW_n f \\ &= (I - T_0T_0^*)^{\frac{1}{2}} L (I - T_1^*T_1)^{\frac{1}{2}} f = \sigma(z, f) \end{aligned}$$

and it remains to show that the series defining B is weakly convergent.

For weak convergence in the definition of B, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} |\langle T_0^n (I - T_0T_0^*)^{\frac{1}{2}} LW_n f, g\rangle| = \sum_{n=0}^{\infty} |\langle LW_n f, (I - T_0T_0^*)^{\frac{1}{2}} T_0^{*n} g\rangle| \\ &\le \sum_{n=0}^{\infty} ||W_n f|| \, ||\big(I - T_0T_0^*\big)^{\frac{1}{2}} T_0^{*n} g|| \le \Big(\sum_{n=0}^{\infty} ||W_n f||^2\Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{\infty} ||(I - T_0T_0^*)^{\frac{1}{2}} T_0^{*n} g||^2\Big)^{\frac{1}{2}} \\ &\le ||X|| \, ||X^{-1}|| \, ||f|| \Big(\lim_{N\to\infty} \sum_{n=0}^{N} ||\big(I - T_0T_0^*\big)^{\frac{1}{2}} T_0^{*n} g||^2\Big)^{\frac{1}{2}} \\ &= ||X|| \, ||X^{-1}|| \, ||f|| \Big(\lim_{N\to\infty} \sum_{n=0}^{N} \big[||T_0^{*n} g||^2 - ||T_0^{*n+1} g||^2\big]\Big)^{\frac{1}{2}} \\ &= ||X|| \, ||X^{-1}|| \, ||f|| \Big(\lim_{N\to\infty} \big[||g||^2 - ||T_0^{*N+1} g||^2\big]\Big)^{\frac{1}{2}} \le ||X|| \, ||X^{-1}|| \, ||f|| \, ||g||. \end{aligned}$$

By [1, Theorem 2.2.2], this completes the proof that E splits. $\square$

Corollary 3.3. *In the category $\mathcal{C}$ any isometric Hilbert module is projective.*

Proof. Suppose K is an isometric Hilbert module. The theorem shows that if $0 \to H \to J \to K \to 0$ is a short exact sequence in $\mathcal{C}$ then it splits. This condition is well known to be equivalent to the definition of projectivity for K. $\square$

Corollary 3.4. *The category $\mathcal{C}$ has enough projective objects and enough injective objects. That is, every cramped Hilbert module is a homomorphic image of a projective (in $\mathcal{C}$) Hilbert module, and is also isomorphic to a closed submodule of an injective Hilbert module. In particular the exact sequence, derived from the functional model in Theorem* 2.6*, is a projective resolution in $\mathcal{C}$ of the Hilbert module K.*

Proof. In view of Corollary 3.3, the last statement is obvious, and the existence of enough projective in $\mathcal{C}$ is clear. For injectives, we need only note that the dual of a projective Hilbert module is injective, and hence any coisometric Hilbert module is an injective object in $\mathcal{C}$. Thus the embedding of any cramped Hilbert module U into an injective Hilbert module is achieved by taking the dual of the sequence in Theorem 2.6 for $K = U^*$. $\square$

Corollary 3.5. *For any cramped Hilbert modules H and K,* $\mathrm{Ext}^n_{\mathcal{C}}(K,H) = 0$ *whenever $n \geq 2$.*

Proof. It was shown in the last corollary that every cramped Hilbert module K has a projective resolution

$$0 \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \longrightarrow K \longrightarrow 0$$

Hence $\mathrm{Ext}^n_{\mathcal{C}}$ can be defined as the n^{th}-derived functor of $\mathrm{Hom}_{\mathcal{C}}$.

For if we apply the functor $\mathrm{Hom}_{\mathcal{C}}(\ ,H)$ to the projective resolution (P_*, ∂) we get a complex

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P_0,H) \xrightarrow{\alpha_1^*} \mathrm{Hom}_{\mathcal{C}}(P_1,H) \xrightarrow{\alpha_2^*}$$

whose homology is the $\mathrm{Ext}_{\mathcal{C}}$ functor,

$$\mathrm{Ext}^n_{\mathcal{C}}(K,H) = H^n(\mathrm{Hom}_{\mathcal{C}}(P_*,H)) = \ker\partial^*_{n+1}/\mathrm{Image}\partial^*_n$$

in the traditional definition [4]. But because $P_n = 0$ for $n > 2$ we have that $\mathrm{Ext}^n_{\mathcal{C}} = 0$ for $n \geq 2$. $\square$

4. Projectives in the category $\mathcal{H}$.

In this section we return to the category $\mathcal{H}$ of all Hilbert $\mathbb{A}(\mathbb{D})$-modules. Our first result shows that the category does have projective objects. This gives an affirmative answer to [2, Problem 4.6], though it is still an open question whether there are enough projectives.

Theorem 4.1. *If K is a unitary Hilbert module, then*

$$0 \longrightarrow H \longrightarrow J \longrightarrow K \longrightarrow 0$$

splits. Hence any unitary Hilbert module is both projective and injective.

Proof. As usual we write $J = H\hat{\oplus}K$ with

$$a(z)(f,g) = (af + \sigma(a,g), ag)$$

for $(f,g) \in J$, $a \in \mathbb{A}$.

By [1] , $\sigma(a,f)$ satisfies

$$\sigma(z^{n+1}, f) = \sum_{j=0}^{n} z^j \sigma(z, z^{n-j} f) = \sum_{j=0}^{n} z^j \sigma(z, U^{n-j} f)$$

where $U : K \to K$ represents multiplication by z in K. Since U is unitary, we have, setting $g = U^{n-1} f$,

$$\sigma(z^{n+1}, U^{*n-1} g) = \sum_{j=0}^{n} z^j \sigma(z, U^{*j+1} g), \qquad g \in K, \tag{4.2}$$

so,

$$|| \sum_{j=0}^{n} z^j \sigma(z, U^{*j+1} g)|| \leq C||g||, \qquad g \in K,\ n = 1, 2, \cdots ,$$

with an appropriate constant C. Now suppose lim is a translation invariant Banach limit on H. We may conclude that $L : K \to H$, defined by

$$Lg = \lim_n \sum_{j=0}^{n} z^j \sigma(z, U^{*j+1} g), \qquad g \in K$$

exists in the weak operator topology.

We compute in the weak topology (of H)

$$\begin{aligned} Lzf - zLf &= \lim_n \sum_{j=0}^{n} \left[z^j \sigma(z, U^{*j} f) - z^{j+1} \sigma(z, U^{*j+1} f) \right] \\ &= \sigma(z, f) - \lim_n z^{n+1} \sigma(z, U^{*n+1} f) \\ &= \sigma(z, f) + \lim_n \big[\sum_{j=0}^{n} z^j \sigma(z, U^{*j} f) - \sum_{j=0}^{n+1} z^j \sigma(z, U^{*j} f) \big] \end{aligned}$$

and, by translation invariance of the Banach limit, we have

$$Lzf - zLf = \sigma(z, f)$$

and this proves that the sequence splits.

The condition implies the projectivity of K. The injectivity is a consequence of the fact that K is the dual of K^* which is also unitary and hence also projective.

The theorem provides us with some information about the functor $\mathrm{Ext}^1_{\mathcal{H}}$ for certain modules. For example, we can show the following. $\square$

Corollary 4.2. *Let H and K be cramped Hilbert modules and suppose that either H is an isometric Hilbert module or K is coisometric. Then* $\mathrm{Ext}^1_{\mathcal{C}}(K,H) = \mathrm{Ext}^1_{\mathcal{H}}(K,H)$ *and, in particular, the middle term, J, of any exact sequence*

$$0 \longrightarrow H \longrightarrow J \longrightarrow K \longrightarrow 0$$

is cramped (an object in $\mathcal{C}$).

Proof. It is clear from the Wold decomposition of an isometry that H is isomorphic to a submodule of a unitary Hilbert module U. So let $\theta : H \to U$ be the inclusion with $\theta(H)$ closed in U. Let $W \cong U/\theta(H)$ be the quotient. If J is any extension of K by H then we have a commutative diagram

$$\begin{tikzcd} E: 0 \arrow[r] & H \arrow[r] \arrow[d, equal] & J \arrow[r] \arrow[d, "\sigma"] & K \arrow[r] \arrow[d, "\tau"] & 0 \\ 0 \arrow[r] & H \arrow[r, "\theta"] & U \arrow[r, "\mu"] & W \arrow[r] & 0 \end{tikzcd}$$

with exact rows. The map σ exists because U is injective and the homomorphism τ is induced by σ on the quotients. Now notice that J is the pullback of the diagram

$$U \xrightarrow{\ \mu\ } W \xleftarrow{\ \tau\ } K$$

and hence is a cramped module by Lemma 2.3. The case that K is a coisometric Hilbert module follows similarly by duality. $\square$

5. Polynomially bounded operators.

Some of the results of this paper may have a bearing on the well known problem of the existence of Hilbert modules over $\mathbb{A}(\mathbb{D})$ not similar to contractive Hilbert modules; i.e. of Hilbert modules in $\mathcal{H}$ but not in $\mathcal{C}$. Phrased another way, this problem asks: if T is a polynomially bounded operator on a Hilbert space J,

$$||p(T)x|| \leq K||p||_\infty ||x||,$$

for p a polynomial and $x \in J$, is T similar to a contraction? See [3, Problem 6]. Indeed, Theorem 3.1 had its origin in an attempt, by the third and fourth authors, to construct a counter example to the conjecture in the form

$$T = \begin{pmatrix} T_0 & X \\ 0 & T_1 \end{pmatrix} \tag{5.1}$$

where $T_0 : H \to H$ and $T_1 : K \to K$ are contraction operators.

The operator T defined by (5.1) is polynomially bounded if and only if the Hilbert space $J = H \hat{\oplus} K$ is a Hilbert module under the multiplication (by z) defined by

$$z(f,g) = T(f,g) = (T_0 f + Xg, T_1 g).$$

If multiplication by z on H and K is given, respectively, by the action of T_0 and T_1, then we have an exact sequence of Hilbert modules

$$0 \longrightarrow H \xrightarrow{\ \alpha\ } J \xrightarrow{\ \beta\ } K \longrightarrow 0 \tag{5.2}$$

where $\alpha(h) = (h, 0)$ and $\beta(h, k) = k$.

It is a consequence of Theorem 3.1, that *if* T_1 *is an isometry* [*or* T_0^* *is an isometry*] *then* T *is similar to a contraction if and only* (5.2) *splits.* For sufficiency, note that the splitting of (5.2) implies that T is similar to the diagonal operator $T_0 \oplus T_1$ on J; substitution of the bracketed clause comes from taking the adjoint of T, and noticing that T^* is unitarily equivalent to

$$\begin{pmatrix} T_1^* & X^* \\ 0 & T_0^* \end{pmatrix}.$$

Evidence suggests that a good place to start looking for a counterexample to the conjecture would be a polynomially bounded operator (5.1), where $T_0^* = T_1$ is the unilateral shift and X is a Hankel matrix.

We conclude by indicating some results which give necessary and sufficient conditions for an operator T of the form (5.1), where $T_0^* = T_1$ is the unilateral shift S, to be polynomially bounded or similar to a contraction. The first of these conditions is an L^∞ condition, while the second is roughly its $\mathbb{L}^2$ analogue. This is further evidence that an example of a non-cramped Hilbert module may lie here.

Let $X : \mathbb{H}^2 \to \mathbb{H}^2$ be a bounded operator, let S be the shift on $\mathbb{H}^2$ and, for $n = 0, 1, 2, \cdots$, let

$$A_n = \sum_{j=0}^{n-1} S^{*(n-1-j)} X S^j.$$

Theorem 5.3. *Let* T *denote the operator*

$$T = \begin{pmatrix} S^* & X \\ 0 & S \end{pmatrix}$$

on $\mathbb{H}^2 \oplus \mathbb{H}^2$*. Then*

a) T *is polynomially bounded if and only if, for each* $\xi, \eta \in \mathbb{H}^2$ *there is a constant* K *such that*

$$\sup_{|z|<1} |\sum_{n=1}^{\infty} a_n \langle A_n \xi, \eta \rangle z^n| \le K || \sum a_n z^n ||_\infty. \tag{5.4}$$

b) T *is similar to a contraction if and only if there exists* $f \in \mathbb{H}^2$ *such that the operator* W *defined on polynomials by*

$$W z^n = S^*(A_{n+1} \cdot 1 - S^{*n} f) \qquad\qquad n = 0, 1, \cdots$$

extends to a bounded linear operator on $\mathbb{H}^2$*.*

Proof. (a) The operator A_n is the (1, 2) entry in the matrix for T^n and, by virtue of the Banach-Steinhaus theorem, the condition in (5.4) implies, with a suitable constant M,

$$|| \sum_1^n a_n A_n || \le M ||p||_\infty \tag{5.5}$$

for any polynomial p, which implies T is polynomially bounded.

Conversely, polynomial boundedness of T, that is (5.5), implies, by the Hahn-Banach and Riesz representation theorems, the existence of a measure $\mu = \mu_{\xi,\eta}$ on $|z| = 1$ such that

$$\langle A_{n+1}\xi, \eta\rangle = \int e^{-in\theta} d\mu(\theta) \qquad n = 0, 1, \cdots .$$

Thus

$$\sum_1^{\infty} a_n \langle A_n\xi, \eta\rangle z^n = \int \left(\sum a_n z^n e^{in\theta}\right) e^{i\theta} d\mu(\theta)$$

and (5.4) follows.

(b) For a bounded operator B on $\mathbb{H}^2$ denote $\triangle(B) = BS - S^*B$. If a bounded operator W, as in b, exists, then it follows directly that $S^*XS = \triangle(W)$. We must show that this implies $X = \triangle(B)$, for some B.

Denote

$$D(Y, A) = S(Y^* + A^*)(I - SS^*) + A^*S^*.$$

Another direct computation shows that

$$S^*Y = \triangle(A) \quad \text{implies } Y^* = \triangle(D(Y, A)) \tag{5.6}$$

Thus from $S^*XS = \triangle(W)$, it follows that

$$S^*X^* = (XS)^* = \triangle(D(XS, -W))$$

and then, again from (5.6), it follows that

$$X = \triangle(D(X^*, -D(XS, -W))).$$

For the converse statement, if T is similar to a contraction, then $X = S^*Y - YS$, for some bounded $Y : \mathbb{H}^2 \to \mathbb{H}^2$, by Theorem 3.1. But then $X = WS^* - SW + 1 \otimes g - f \otimes 1$. We omit the remaining details, which use this W and this f. $\square$

We conclude with a corollary of Theorem 5.3 in which the operator T acts on $\mathbb{L}^2$ and X is replaced by a Hankel operator from $\mathbb{H}^2$ to $\mathbb{H}^{2\perp} = \mathbb{L}^2 \ominus \mathbb{H}^2$. We believe this variant of Theorem 5.3 may be helpful in the search for a noncramped Hilbert module.

Corollary 5.7. *Let T denote the operator*

$$T = \begin{pmatrix} S' & X \\ 0 & S \end{pmatrix}$$

on $\mathbb{L}^2 = \mathbb{H}^{2\perp} \oplus \mathbb{H}^2$, where S' is the compression of multiplication by z on $\mathbb{L}^2$ to $\mathbb{H}^{2\perp}$ and X is the Hankel operator

$$Xh = (I - P_{\mathbb{H}^2})(\psi h), \qquad h \in \mathbb{H}^2$$

with symbol $\psi \in \mathbb{L}^\infty(\mathbb{T})$. Then

a) *T is polynomially bounded if and only if*

$$\operatorname{dist}_{\mathbb{H}^\infty}(\psi\overline{\varphi}, \mathbb{H}^\infty) \le M||\varphi||_\infty \qquad \varphi \in \mathbb{A}. \tag{5.8}$$

b) *T is similar to a contraction if and only if there exists $\theta \in \mathbb{H}^{2\perp}$ such that*

$$\operatorname{dist}_{\mathbb{H}^2}(\psi\varphi' + \varphi\theta, \mathbb{H}^2) \le M||\varphi||_2 \qquad \varphi \in \mathbb{H}^2, \varphi(0) = 0. \tag{5.9}$$

Proof. In this case, we have

$$A_n = nS'^{n-1}X = nXS^{n-1} \qquad n = 1, 2, \cdots$$

and for a polynomial $\varphi = \sum a_n z^n$ we have

$$\Big(\sum a_n A_n\Big) h = \big(I - P_{\mathbb{H}^2}\big)[(\psi\varphi')h] \qquad h \in \mathbb{H}^2.$$

Condition (5.5) means that the Hankel operator with symbol $\psi\varphi'$ is of norm $\leq M||\varphi||_\infty$, or equivalently, that (5.8) holds. This proves (a).

For (b), we deduce from Theorem 3.1 that T is similar to a contraction if and only if there exists an operator $L : \mathbb{H}^2 \to \mathbb{H}^{2\perp}$ satisfying

$$LS - S'L = X \tag{5.10}$$

whence (by direct computation)

$$LS^n - S'^n L = A_n.$$

Therefore, with $\theta = L1$, we have

$$Lz^n = A_n 1 + S'^n\theta$$

so that for a polynomial $\varphi = \sum a_n z^n$, we have

$$L\varphi = (I - P_{\mathbb{H}^2})(\psi\varphi' + \varphi\theta). \tag{5.11}$$

Thus the operator L satisfying (5.10) exists if and only if the operator defined by (5.11) is bounded, that is, if and only if (5.9) holds. $\square$

References

1. J. F. Carlson and D. N. Clark, *Cohomology and extensions of Hilbert modules*, J. Functional Analysis (to appear).
2. R. G. Douglas and V. I. Paulsen, *Hilbert Modules over Function Algebras*, Longman Scientific & Technical, New York, 1989.
3. P. R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76** (1970), 887–933.
4. A. Ya. Helemskii, *The Homology of Banach and Topological Algebras*, Kluwer Academic Publishers, Dordrecht, 1989.
5. P. J. Hilton and U. Stammbach, *A Course in Homological Algebra*, Springer-Verlag, New York, 1970.
6. B. Sz-Nagy and C. Foias, *Forme triangulaire d'une contraction et factorisation de la fonction characteŕistique*, Acta Sci. Math. (Szeged) **28** (1967), 201–212.
7. ——, *Harmonic Analysis of Operators on Hilbert space*, North Holland, Amsterdam-London, 1970.

Department of Mathematics, University of Georgia, Athens, GA 30602
e-mail: jfc@sloth.math.uga.edu

Department of Mathematics, University of Georgia, Athens, GA 30602
e-mail: dnc@joe.math.uga.edu

Department of Mathematics, Indiana University, Bloomington, IN 47405
e-mail: foias@ucs.indiana.edu

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX