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Turning Monoidal Categories into Strict Ones

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Abstract. It is well-known that every monoidal category is equivalent to a strict one. We show that for categories of sets with additional structure (which we define quite formally below) it is not even necessary to change the category: The same category has a different (but isomorphic) tensor product, with which it is a strict monoidal category. The result applies to ordinary (bi)modules, where it shows that one can choose a realization of the tensor product for each pair of modules in such a way that tensor products are strictly associative. Perhaps more surprisingly, the result also applies to such nontrivially nonstrict categories as the category of modules over a quasibialgebra.

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1. Introduction

It is well-known that the tensor product of vector spaces is not associative on the nose, but it is so up to isomorphism. In fact strict associativity is not to be expected, since the tensor product is, by its abstract definition by a universal property, only defined up to isomorphism; also, if one chooses one of the standard constructions of the tensor product, it turns out not to be associative. At any rate, the relevant associativity isomorphism is so obvious and so well-behaved that there is no harm in identifying tensor products of more than two factors that only differ in the way parentheses are used to build them from binary tensor products.

The well-behavedness of the associativity isomorphisms is axiomatized in the definition of a monoidal category, which requires that the two ways of switching parentheses from the left to the right in a fourfold tensor product by using the

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associativity isomorphisms for three factors should be the same (this is Mac Lane's famous pentagon axiom). The statement that there is no harm in identifying is of course sloppy, and takes its exact form in Mac Lane's coherence theorem [2], which says (also sloppily) that all diagrams one can build up from associativity isomorphisms commute.

A very convenient and conceptual way of expressing the coherence theorem is the following: Every monoidal category is monoidally equivalent to a strict monoidal category (i.e., one in which the associativity isomorphism *is* the identity after all). Essentially this says that any reasonable fact about monoidal categories that holds in the strict case should also hold in the general case. But if one is interested in a concrete category $\mathcal{C}$, the theorem forces one to abandon $\mathcal{C}$ and replace it with a new, quite abstractly defined category $\mathcal{C}^{\mathrm{str}}$, which is then strict. If $\mathcal{C}$ is, say, the category of vector spaces, then $\mathcal{C}^{\mathrm{str}}$ is strict, but on the other hand it does not contain any vector spaces.

Our main result says that to get a strict category, one may very often keep the category, and just has to redefine the tensor product, namely to replace the tensor product of each pair of objects with an isomorphic copy (an operation which is surely to be considered irrelevant). In particular, one can define the tensor product of vector spaces or bimodules (i.e., one can choose a solution for each of the relevant universal problems for each pair of vector spaces or bimodules) in such a way that the tensor product is associative, without any need of identifying or keeping track of associativity isomorphisms. In fact the result holds much more generally for any category whose objects are sets with additional structure, and whose morphisms are the structure preserving maps (we give a formal definition of such categories below). In particular, it holds for the category of modules over a quasibialgebra H: One can choose an object $V \odot W \cong V \otimes W \in {}_H\mathcal{M}$ for all $V, W \in {}_H\mathcal{M}$ in such a way that $U \odot (V \odot W) = (U \odot V) \odot W$ for all $U, V, W \in {}_H\mathcal{M}$.

The result for bimodules tells us that we are perhaps more justified than we thought when we (as is customary) identify iterated tensor products of (bi)modules. After all, one has just to choose the right realization of the tensor product, then the things we want to identify are really the same. The result for modules over quasibialgebras and similar manifestly nonstrict categories should perhaps be read as a warning rather than a convenience: One really has to take the definitions (say, that the underlying functor from modules to vector spaces is not coherent) seriously and should not be led to think that the category of modules over a quasibialgebra is nonstrict in any more essential way than the category of vector spaces is.

2. Definitions and conventions

We refer largely to [1] for the basic definitions and facts on monoidal categories. We use the following conventions: A monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ consists of a category $\mathcal{C}$, a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $I \in \mathcal{C}$, and isomorphisms

$$\alpha_{XYZ}\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z),$$
$$\lambda_X\colon I \otimes X \to X, \text{ and } \rho_X\colon X \otimes I \to X$$

functorial in $X, Y, Z \in \mathcal{C}$, called the associativity and left and right unit constraints. These are required to fulfill the following coherence conditions:

$$(W \otimes \alpha_{XYZ})\alpha_{W,X\otimes Y,Z}(\alpha_{WXY} \otimes Z) = \alpha_{W,X,Y\otimes Z}\alpha_{W\otimes X,Y,Z},$$

usually depicted as a pentagonal diagram, in which both sides are morphisms

$$((W \otimes X) \otimes Y) \otimes Z \to W \otimes (X \otimes (Y \otimes Z)),$$

and $(X \otimes \lambda_Y)\alpha_{XIY} = \rho_X \otimes Y$, usually depicted as a triangular diagram. The coherence conditions imply commutativity of "all" diagrams that one can build from the constraints, in particular $(X \otimes \rho_Y)\alpha_{XYI} = \rho_{X\otimes Y}$, and $\lambda_{X\otimes Y}\alpha_{IXY} = \lambda_X \otimes Y$.

A monoidal category is strict if all three constraints are identities.

A monoidal functor

$$(F, \xi, \xi_0)\colon (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho) \to (\mathcal{C}', \otimes', I', \alpha', \lambda', \rho')$$

consists of a functor $F\colon \mathcal{C} \to \mathcal{C}'$, a natural isomorphism

$$\xi_{XY}\colon F(X) \otimes' F(Y) \to F(X \otimes Y)$$

and an isomorphism $\xi_0\colon I' \to F(I)$ making the hexagons

$$\begin{tikzcd}
(F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[r, "\xi\otimes\mathrm{id}"] \arrow[d, "\alpha'"'] & F(X \otimes Y) \otimes' F(Z) \arrow[r, "\xi"] & F((X \otimes Y) \otimes Z)) \arrow[d, "F(\alpha)"] \\
F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[r, "\mathrm{id}\otimes\xi"] & F(X) \otimes' F(Y \otimes Z) \arrow[r, "\xi"] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

and the squares

$$\begin{tikzcd}
I' \otimes F(X) \arrow[r, "\xi_0\otimes\mathrm{id}"] \arrow[d, "\lambda'"] & F(I) \otimes' F(X) \arrow[d, "\xi"] \\
F(X) & F(I \otimes X) \arrow[l, "F(\lambda)"']
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
F(X) \otimes I' \arrow[r, "\mathrm{id}\otimes\xi_0"] \arrow[d, "\rho'"] & F(X) \otimes' F(I) \arrow[d, "\xi"] \\
F(X) & F(X \otimes I) \arrow[l, "F(\rho)"']
\end{tikzcd}$$

commute. A monoidal functor is called strict if both ξ and ξ_0 are identities (this may well happen if the categories are not strict). Without giving the relevant formulas, we note that the composition of monoidal functors has again the structure of a monoidal functor.

A functorial morphism $\phi\colon F \to F'$ between two monoidal functors (F, ξ, ξ_0) and (F', ξ', ξ'_0) is said to be monoidal if

$$\phi_{X\otimes Y}\xi_{XY} = \xi'_{XY}(\phi_X \otimes \phi_Y)\colon F(X) \otimes' F(Y) \to F'(X \otimes Y)$$

and $\phi_I\xi_0 = \xi'_0\phi_I$. We note that if (F, ξ, ξ_0) is a monoidal functor, F' is a functor, and $\phi\colon F \to F'$ is a functorial isomorphism, then there is a unique monoidal functor structure (F', ξ', ξ'_0) such that ϕ is a monoidal transformation.

A monoidal category equivalence is by definition a monoidal functor (F, ξ, ξ_0) in which the functor F is an equivalence of categories. The definition is justified since for any monoidal equivalence there is an inverse monoidal equivalence [3]. More precisely, if (F, ξ, ξ_0) is a monoidal equivalence, and if G is a quasi-inverse equivalence, with isomorphisms $\eta\colon Id \to GF$ and $\epsilon\colon FG \to Id$ that are also adjunction morphisms making F a left adjoint of G, then there is a unique monoidal functor structure (G, ζ, ζ_0) such that η and ϵ are monoidal isomorphisms (where the identity functors are strict monoidal functors, and the compositions are given the canonical monoidal functor structures we skipped explaining above).

Knowing this, it makes sense to say two monoidal categories are (monoidally) equivalent, if there exists a monoidal category equivalence between them. Our main result relies on the following form of Mac Lane's coherence theorem, which can be found in Kassel's book [1] (The present paper's title is stolen from the title of section XI.5 there.): Any monoidal category $\mathcal{C}$ is monoidally equivalent to a strict monoidal category $\mathcal{C}^{\mathrm{str}}$.

Suppose that $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{C}, \odot, I^{\odot}, \alpha^{\odot}, \lambda^{\odot}, \rho^{\odot})$ are two monoidal category structures on the same underlying category. When there exists a monoidal equivalence of the form (Id, ξ, ξ_0) between the two structures, we shall say that they are equivalent monoidal category structures. Observe this says that for any two $X, Y \in \mathcal{C}$ the two tensor products $X \otimes Y$ and $X \odot Y$ are isomorphic, and functorially so; in addition to this there are coherence conditions, of course.

3. Making the unit constraints strict

In this section we show that, by replacing some tensor products with suitably chosen isomorphic copies, one can always make the unit constraints of any monoidal category strict. This is based on the following observation:

Lemma 3.1. *Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category.*

Suppose we are given a map $\odot\colon \operatorname{Ob}\mathcal{C} \times \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{C}$, a family of isomorphisms $\xi_{XY}\colon X \otimes Y \to X \odot Y$, an object $I^{\odot}$ of $\mathcal{C}$, and an isomophism $\xi_0\colon I \to I^{\odot}$.

Then there is a unique monoidal category structure $(\odot, I^{\odot}, \alpha^{\odot}, \lambda^{\odot}, \rho^{\odot})$ on $\mathcal{C}$ such that

$$(Id, \xi, \xi_0)\colon (\mathcal{C}, \odot, I^{\odot}, \alpha^{\odot}, \lambda^{\odot}, \rho^{\odot}) \to (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$$

is a monoidal functor.

Proof. There is a unique way of extending $\odot$ to a functor $\odot\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that the ξ_{XY} are the components of a functorial isomorphism. To come up with the new monoidal category structure, we now merely have to define the new associativity and unit constraints by the relevant coherence diagrams

$$\begin{tikzcd}
(X \odot Y) \odot Z \arrow[r, "\alpha^{\odot}"] & X \odot (Y \odot Z) \\
(X \odot Y) \otimes Z \arrow[u, "\xi"] & X \otimes (Y \odot Z) \arrow[u, "\xi"'] \\
(X \otimes Y) \otimes Z \arrow[u, "\xi \otimes Z"] \arrow[r, "\alpha"] & X \otimes (Y \otimes Z) \arrow[u, "X \otimes \xi"']
\end{tikzcd}$$

$$\begin{tikzcd}
I^{\odot} \odot X \arrow[r, "\lambda^{\odot}"] & X \\
I^{\odot} \otimes X \arrow[u, "\xi"] & I \otimes X \arrow[l, "\xi_0 \otimes X"] \arrow[u, "\lambda"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \odot I^{\odot} \arrow[r, "\rho^{\odot}"] & X \\
X \otimes I^{\odot} \arrow[u, "\xi"] & X \otimes I \arrow[l, "X \otimes \xi_0"] \arrow[u, "\rho"']
\end{tikzcd}$$

(Actually it is a little tedious to verify coherence of the new constraints, but we shall nevertheless omit it here.) $\square$

To twist away the nonstrictness of unit constraints, it just remains to find a suitable choice to replace tensor products with:

Theorem 3.2. *Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category, and $I^\odot$ any object of $\mathcal{C}$ isomorphic to I.*

Then there is an equivalent monoidal category structure $(\odot, I^\odot, \alpha^\odot, \lambda^\odot, \rho^\odot)$ on $\mathcal{C}$ in which $\lambda^\odot$ and $\rho^\odot$ are identities. Moreover, if α is the identity, and $I^\odot$ is not in the image of $\otimes$, then $\alpha^\odot$ is also the identity, hence the new monoidal category structure is strict.

Proof. We use Lemma 3.1 with the following special choices: We let

$$X \odot Y := \begin{cases} X \otimes Y & \text{if } X \neq I^\odot \neq Y \\ Y & \text{if } X = I^\odot \\ X & \text{if } Y = I^\odot \end{cases}$$

and define $\xi_{XY} \colon X \otimes Y \to X \odot Y$ to be the identity if $X \neq I^\odot \neq Y$, while we put $\xi_{X,I^\odot} = \rho_X(X \otimes \xi_0^{-1})$ and $\xi_{I^\odot,Y} = \lambda_Y(\xi_0^{-1} \otimes Y)$; this is actually well-defined since $\lambda_I = \rho_I$, and hence

$$\begin{aligned} \rho_{I^\odot}(I^\odot \otimes \xi_0^{-1})(\xi_0 \otimes \xi_0) &= \rho_{I^\odot}(\xi_0 \otimes I) = \xi_0 \rho_I \\ &= \xi_0 \lambda_I = \lambda_{I^\odot}(I \otimes \xi_0) = \lambda_{I^\odot}(\xi_0^{-1} \otimes I^\odot)(\xi_0 \otimes \xi_0). \end{aligned}$$

Observe that with the above choices $\lambda^\odot$ and $\rho^\odot$ are identities by definition. This implies that $\alpha^\odot_{XYZ} = \mathrm{id}$ whenever $I^\odot \in \{X, Y, Z\}$. If $I^\odot$ is not in the image of $\otimes$, then $\alpha^\odot_{XYZ} = \alpha_{XYZ}$ whenever $I^\odot \notin \{X, Y, Z\}$, so if $\alpha = \mathrm{id}$, then $\alpha^\odot = \mathrm{id}$. $\square$

4. Making the whole category strict

While we could remove nonstrictness of the unit constraints in *any* monoidal category in the preceding section, this does not appear to be possible for the associativity constraints. We shall show, however, that it is possible for a large natural class of categories, which we now define:

Definition 4.1. A category of structured sets is a category $\mathcal{C}$ for which there exists a faithful functor $\mathcal{U} \colon \mathcal{C} \to \underline{Sets}$ such that

1. $\mathcal{U}$ generates isomorphisms, that is, for all objects $X \in \mathcal{C}$, sets S and bijections $f \colon \mathcal{U}(X) \to S$ there exist an object $Y \in \mathcal{C}$ with $\mathcal{U}(Y) = S$ and an isomorphism $g \colon X \to Y$ with $\mathcal{U}(g) = f$.
2. If X, Y are objects of $\mathcal{C}$, and $g \colon X \to Y$ is an isomorphism, then $\mathcal{U}(X) = \mathcal{U}(Y)$ and $\mathcal{U}(g) = \mathrm{id}$ imply $X = Y$ and $g = \mathrm{id}$.

It should be obvious that the axioms of a category of structured sets do capture some properties of what one might naïvely have called a category of structured sets: The idea is that a set S is given the structure of an object of $\mathcal{C}$ by exhibiting $X \in \mathcal{C}$ with $\mathcal{U}(X) = S$, so $\mathcal{U}$ plays the rôle of an underlying functor. Axiom (1) says that any set in bijection with the underlying set of an object of $\mathcal{C}$ is itself the underlying set of an isomorphic object of $\mathcal{C}$. The structure can be transported along the bijection, as it were, so that the bijection is the underlying map of an isomorphism. As the reader will verify easily, Axiom (2) implies that Y and g in Axiom (1) are unique. In particular, all categories of algebraic structures are categories of structured sets, but also the category of topological spaces, of topological groups, of manifolds with an action of some fixed group. On the other hand, the homotopy category of topological spaces, or a derived category do not seem to be categories of structured sets (at least not in any obvious way). Also, if $\mathcal{C}$ is a category of structured sets, it is not clear whether the same is true for the opposite category.

For later use, we note:

Remark 4.2. Let $\mathcal{C}$ be a category of structured sets, with an "underlying" functor $\mathcal{U}$ as in Definition 4.1.

Let $T\colon \underline{Sets} \to \underline{Sets}$ be an automorphism of the category of sets (not merely an auto-equivalence!). Then $\mathcal{U}' := T\mathcal{U}\colon \mathcal{C} \to \underline{Sets}$ also satisfies the requirements in Definition 4.1.

Now we come to our main result:

Theorem 4.3. *Let $\mathcal{C}$ be a category of structured sets. Then for each monoidal category structure on $\mathcal{C}$ there is an equivalent strict monoidal category structure with the same unit object.*

Proof. Recall from [1] that there is a strict monoidal category $\mathcal{C}^{\text{str}}$ monoidally equivalent to $\mathcal{C}$. We denote the tensor product in $\mathcal{C}^{\text{str}}$ by $*$, and the unit object by $\emptyset$; the unit object in the concrete construction of $\mathcal{C}^{\text{str}}$ in [1] really is the empty sequence, but this is irrelevant, and we would rather have $\emptyset$ understood as a mere notation for the unit object. We fix a monoidal equivalence $(F, \varphi, \varphi_0)\colon \mathcal{C}^{\text{str}} \to \mathcal{C}$, a quasi-inverse equivalence $G\colon \mathcal{C} \to \mathcal{C}^{\text{str}}$, and an isomorphism $\epsilon\colon FG \to Id$.

We construct a new category equivalence $R\colon \mathcal{C}^{\text{str}} \to \mathcal{C}$, which we write $R(V) = [V]$, as follows: We fix a functor $\mathcal{U}\colon \mathcal{C} \to \underline{Sets}$ satisfying the conditions in Definition 4.1. For any $V \in \mathcal{C}^{\text{str}}$ there is by assumption a unique pair $([V], \chi_V)$ in which $[V] =: RV$ is an object of $\mathcal{C}$ satisfying $\mathcal{U}[V] = \{V\} \times \mathcal{U}FV$, and $\chi_V\colon [V] \to FV$ is an isomorphism in $\mathcal{C}$ such that $\mathcal{U}(\chi_V)\colon \{V\} \times \mathcal{U}FV \to \mathcal{U}FV$ is the (bijective) projection onto the second factor. We make R a functor in the unique way such that $\chi\colon R \to F$ is functorial.

As an immediate consequence R, being isomorphic to an equivalence, is itself an equivalence of categories. In addition, R is injective on objects. This depends on the properties of ordered pairs and the cartesian product of sets: For a set S one can decide wether or not S is a cartesian product of two other sets, and if it is, then there are unique sets S_1 and S_2 such that $S = S_1 \times S_2$, and each element of S has a unique first component in S_1 and second component in S_2. Note that we have to take the set-theoretic definition of the cartesian product rather serious to ensure this, in particular we need to remember $S_1 \times (S_2 \times S_3) \neq (S_1 \times S_2) \times S_3$.

We now construct a specific quasi-inverse equivalence L for R as follows:

The functor $L\colon \mathcal{C} \to \mathcal{C}^{\text{str}}$ is defined on objects by $L([V]) = V$ for all $V \in \mathcal{C}^{\text{str}}$, and $LX = GX$ if X is not in the image of R (this is well-defined since R is injective on objects). We define $\gamma_X\colon RLX \to X$ by

$$\gamma_X = ([GX] \xrightarrow{\chi_{GX}} FGX \xrightarrow{\epsilon_X} X)$$

if X is not in the image of R, and $\gamma_X = \text{id}$ if $X = [V]$ for some $V \in \mathcal{C}^{\text{str}}$. Since R is fully faithful, we can define L on morphisms by requiring $\gamma\colon RL \to Id$ to be a functorial isomorphism. By definition $\beta = \text{id}\colon Id \to LR$ is a natural isomorphism, so that L is a quasi-inverse equivalence for R. We shall show in addition that $L\gamma$ is the identity; since γR is the identity by definition, this actually says that β and γ are the adjunction morphisms of an adjoint equivalence. Since R is faithful, it suffices to check that $RL\gamma_X$ is the identity for all X. Now the application of L

to morphisms is defined through naturality of γ, so that we have a commutative diagram

$$\begin{tikzcd} RLRLX \arrow[r, "RLf"] \arrow[d, "\gamma_{RLX}"] & RLX \arrow[d, "\gamma_X"] \\ RLX \arrow[r, "f"] & X \end{tikzcd}$$

for any $f\colon RLX \to X$. Specializing $f = \gamma_X$ yields $RL\gamma_X = \mathrm{id}$ since $\gamma_{RLX} = \mathrm{id}$.

Now we use the obvious fact that any category equivalent to a monoidal category is itself a monoidal category, to define a new monoidal category structure $(\diamondsuit, I^\diamondsuit, \alpha^\diamondsuit, \lambda^\diamondsuit, \rho^\diamondsuit)$ on $\mathcal{C}$. More precisely, we define $X\diamondsuit Y := R(LX * LY)$, and let $I_\diamondsuit := R(\emptyset)$. We calculate $L(X\diamondsuit Y) = LR(LX * LY) = LX * LY$ and $L(I^\diamondsuit) = LR(\emptyset) = \emptyset$. We define $\alpha^\diamondsuit = \mathrm{id}$, which makes sense since

$$(X\diamondsuit Y)\diamondsuit Z = R(L(X\diamondsuit Y) * LZ) = R(LX * LY * LZ) = \\ R(LX * L(Y\diamondsuit Z)) = X\diamondsuit(Y\diamondsuit Z).$$

It goes without saying that Mac Lane's pentagon commutes. Further we define the left and right unit constraints to be $\lambda^\diamondsuit_X = \rho^\diamondsuit_X = \gamma_X$, which makes sense since

$$X\diamondsuit I^\diamondsuit = R(LX\diamondsuit LR(\emptyset)) = R(LX * \emptyset) = RLX$$

and similarly $I^\diamondsuit\diamondsuit X = RLX$. Although $I^\diamondsuit$ is not a strict unit object, we do have

$$X\diamondsuit I^\diamondsuit\diamondsuit Y = R(LX * L(I^\diamondsuit) * LY) = R(LX * \emptyset * LY) = R(LX * LY) = X\diamondsuit Y$$

for all $X, Y \in \mathcal{C}$, and

$$\rho^\diamondsuit_X\diamondsuit Y = R(L(\rho^\diamondsuit_X) * LY) = R(L(\gamma_X) * LY) = R(\mathrm{id} * LY) = \mathrm{id},$$

and similarly $X\diamondsuit\lambda^\diamondsuit_Y = \mathrm{id}$, so that the coherence triangle for the unit constraints commutes.

We claim that $L\colon (\mathcal{C}, \diamondsuit, \alpha^\diamondsuit, I^\diamondsuit, \lambda^\diamondsuit, \rho^\diamondsuit) \to \mathcal{C}^{\mathrm{str}}$ is a strict monoidal functor. In fact we already know that L is strictly compatible with tensor products and neutral object, and since both associativity constraints are identities, the hexagon for a monoidal functor commutes trivially. But we also know that $L(\lambda^\diamondsuit_X) = L(\rho^\diamondsuit_X) = L(\gamma_X) = \mathrm{id}$, so the two coherence triangles also commute trivially.

Combining with the monoidal category equivalence $F\colon \mathcal{C} \to \mathcal{C}^{\mathrm{str}}$, we have a monoidal category equivalence

$$FL\colon (\mathcal{C}, \diamondsuit, \alpha^\diamondsuit, I^\diamondsuit, \lambda^\diamondsuit, \rho^\diamondsuit) \to (\mathcal{C}, \otimes, \alpha, I, \lambda, \rho)$$

(we do not care to specify its monoidal category structure). But the functor FL is naturally isomorphic to the identity, by $FLX \xrightarrow{\chi^{-1}_{LX}} RLX \xrightarrow{\gamma_X} X$, hence the identity is also a monoidal functor between the same two categories.

We have managed to exhibit a monoidal category structure on $\mathcal{C}$ equivalent to the original one, but in which the associativity constraint is the identity. We complete the proof by applying Theorem 3.2 to make the old unit object I, which is isomorphic to $I^\diamondsuit$, the new unit object of a strict monoidal category structure equivalent to $\diamondsuit$. To be able to do so, we have to make sure that I is not in the image of $\diamondsuit$, for which it suffices to have I not in the image of R. If I *is* in the image of R, then $\mathcal{U}I = \{V\} \times \mathcal{U}FV$ for some $V \in \mathcal{C}^{\mathrm{str}}$. Should this be the case, however, we can simply change the "underlying" functor $\mathcal{U}$, modifying it as in Remark 4.2

by an automorphism T of <u>*Sets*</u> such that TUI is not a cartesian product of two sets at all. $\square$

For our first application recall that *a* tensor product of a right and a left A-module is (any) solution of the familiar universal problem for A-bilinear maps to abelian groups. Any tensor product of A-bimodules becomes an A-bimodule in a natural way. Now our result above says that among the possible tensor products we can make a smart choice (The Tensor Product, as it were) such that tensor products are strictly associative.

Corollary 4.4. *Let A be any ring. Then one can choose, for all M, N in the category ${}_A\mathcal{M}_A$ of A-A-bimodules, a concrete realization $M \otimes_A N$ of the tensor product over A, in such a way that for any three $M, N, P \in {}_A\mathcal{M}_A$ we have $(M \otimes_A N) \otimes_A P = M \otimes_A (N \otimes_A P) =: M \otimes_A N \otimes_A P$, $M \otimes_A A = A \otimes_A M = A$, and for any $m \in M$, $n \in N$, $p \in P$, and $a \in A$ we have $(m \otimes n) \otimes p = m \otimes (n \otimes p)$ as well as $a \otimes m = am$ and $m \otimes a = ma$.*

A quasibialgebra H is a k-algebra for which the category ${}_H\mathcal{M}$ carries a monoidal category structure for which the underlying functor to the category of k-modules strictly preserves tensor products, but does not satisfy the coherence conditions for a monoidal functor. In particular, H has a comultiplication, which is used to endow the tensor product (over k) of two H-modules with an H-module structure. We refer the reader to [1] for details.

Corollary 4.5. *Let H be a quasibialgebra. Then one can choose, for all $M, N \in {}_H\mathcal{M}$, a left H-module $M \odot N$ isomorphic to the usual H-module $M \otimes N$ (with the diagonal module structure) in such a way that $(M \odot N) \odot P = M \odot (N \odot P) =: M \odot N \odot P$ for all $M, N, P \in {}_H\mathcal{M}$, and $M \otimes k = M = k \otimes M$.*

Observe that the underlying k-module of $M \odot N$ is of course a tensor product over k of the underlying k-modules of M and of N, simply since this was true for $M \otimes N$. In particular, we have elements $m \odot n \in M \odot N$ for $m \in M$ and $n \in N$ so that $\odot\colon M \times N \to M \odot N$ is a universal k-bilinear map. However, we do not have $(m \odot n) \odot p = m \odot (n \odot p)$ in $M \odot N \odot P$.

5. Likely extensions

It appears likely that our results can be extended to "strictify" certain additional structures on monoidal categories along with the tensor product. Some such (and some other) extensions of our main theorem are projected below, some of which seem easy to achieve, while others may pose more serious problems; I have not checked any details.

1. If $\mathcal{C}$ is a category of structured sets, and we are given the structure of a left rigid monoidal category on $\mathcal{C}$, then we should be able to find an equivalent strict monoidal category structure on $\mathcal{C}$ and a duality functor $(-)^*$ on $\mathcal{C}$ such that $(X \otimes Y)^* = Y^* \otimes X^*$, functorially in $X, Y \in \mathcal{C}$, and $I^* = I$.
2. If $\mathcal{C}$ is a category of structured sets, and we are given the structure of a left and right rigid monoidal category on $\mathcal{C}$, then we should be able to find an equivalent strict monoidal category structure on $\mathcal{C}$, a left duality functor $(-)^*$ and a right duality functor ${}^*(-)$ on $\mathcal{C}$, such that $(X \otimes Y)^* = Y^* \otimes X^*$ and ${}^*(X^*) = X$ functorially in $X, Y \in \mathcal{C}$, and $I^* = I$.

3. If we are given the structure of a ribbon category on a category $\mathcal{C}$ of structured sets, we should be able to find an equivalent strict monoidal category structure on $\mathcal{C}$ and a duality functor $(-)^*$ such that $(X\otimes Y)^* = Y^*\otimes X^*$ and $X^{**}=X$ functorially in $X,Y\in\mathcal{C}$, and $I^*=I$.
4. We should be able to choose, for any three rings R,S,T and any two bimodules $M\in {}_R\mathcal{M}_S$ and $N\in {}_S\mathcal{M}_T$, a realization $M\otimes_S N\in {}_R\mathcal{M}_T$ of the tensor product over S, in such a way that triple tensor products are always strictly associative, and tensoring a bimodule with one of the rings does not change it. (Clearly this project falls somewhat outside of the formalism of monoidal categories).
5. Can one extend the choices in the preceding project to make them compatible with all of the functors ${}_R\mathcal{M}_S\to {}_{R'}\mathcal{M}_{S'}$ induced by ring maps $R'\to R$ and $S'\to S$, or at least those functors induced by inclusions?
6. Given monoidal categories of structured sets $\mathcal{C}$ and $\mathcal{C}'$ and a monoidal functor $F\colon\mathcal{C}\to\mathcal{C}'$, can one find equivalent strict monoidal category structures on $\mathcal{C}$ and on $\mathcal{C}'$, and a strict monoidal functor $F'\colon\mathcal{C}\to\mathcal{C}'$ isomorphic to F?
7. If F as above is not monoidal, but only tensor product preserving, perhaps incoherently, can one at least find equivalent strict monoidal category structures on $\mathcal{C}$ and on $\mathcal{C}'$, and a functor F' isomorphic to F such that the new tensor products fulfill $F'(X\otimes Y)=F'(X)\otimes F'(Y)$ for all objects $X,Y\in\mathcal{C}$ (though certainly not functorially)?

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