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Classification of homotopy Dold manifolds

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Abstract. In his Math. Zeitschr. paper of 1956 A. Dold defined manifolds for the purpose of generating unoriented cobordism groups. In the present paper a complete piecewise linear and topological classification and partial smooth classification of manifolds homotopy equivalent to a Dold manifold have been done by determining: (1) the normal invariants of the Dold manifolds, (2) the surgery obstruction of a normal invariant and (3) the action of the Wall surgery obstruction groups on the diffeomorphism, piecewise linear and topological homeomorphism classes of homotopy Dold manifolds (to be made precise in the body of the paper).

Contents

1. Introduction

Let X be a compact, connected, smooth, piecewise linear (PL) or topological manifold with or without boundary ∂X. By a homotopy smoothing (respectively homotopy PL or TOP triangulation) of the manifold X we mean a pair (M, f), where M is a smooth, PL or topological manifold and $f : (M, \partial M) \to (X, \partial X)$ is a simple homotopy equivalence of pairs, for which $f\mid_{\partial M}\colon \partial M \to \partial X$ is a diffeomorphism (resp. a PL or TOP homeomorphism). Two homotopy smoothings (resp. homotopy PL or TOP triangulations) (M, f) and (M', f') are said to be equivalent if there is a diffeomorphism (resp. PL or TOP homeomorphism)

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$h : (M, \partial M) \to (M', \partial M')$ for which the maps $f' \circ h$ and f are homotopic relative to the boundary ∂M. The set of equivalence classes of homotopy smoothings (resp. PL or TOP triangulations) of the manifold X is denoted by $hS(X)$ (resp. $hT_{\mathrm{PL}}(X)$ or $hT_{\mathrm{TOP}}(X)$) and is a pointed set with base point (X, id_X). If we use the notation CAT = O, PL, or TOP, then later in the paper we shall call "homotopy smoothings, homotopy PL or TOP triangulations", simply as "homotopy CAT structures". Also $hS(X)$, $hT_{\mathrm{CAT}}(X)$ are sometimes referred simply as "structure sets".

The standard method of determining the sets $hS(X)$ (resp. $hT_{\mathrm{CAT}}(X)$, CAT = PL or TOP) for various concrete manifolds X is the analysis of the following Sullivan-Wall surgery exact sequences:

$$\to L_{n+1}(\pi_1(X), w(X)) \overset{\delta_{\mathrm{O}}}{\to} hS(X) \overset{\eta_{\mathrm{O}}}{\to} [X/\partial X, G/\mathrm{O}] \overset{\theta_{\mathrm{O}}}{\to} L_n(\pi_1(X), w(X)),$$

$$\to L_{n+1}(\pi_1(X), w(X)) \overset{\delta_{\mathrm{PL}}}{\to} hT_{\mathrm{PL}}(X) \overset{\eta_{\mathrm{PL}}}{\to} [X/\partial X, G/\mathrm{PL}] \overset{\theta_{\mathrm{PL}}}{\to} L_n(\pi_1(X), w(X)),$$

and

$$\to L_{n+1}(\pi_1(X), w(X)) \overset{\delta_{\mathrm{TOP}}}{\to} hT_{\mathrm{TOP}}(X) \overset{\eta_{\mathrm{TOP}}}{\to} [X/\partial X, G/\mathrm{TOP}] \overset{\theta_{\mathrm{TOP}}}{\to} L_n(\pi_1(X), w(X)),$$

where $n = \dim X \geq 5$, and where the first and the last terms are Wall's surgery obstruction groups, second terms are as defined above and the third terms are sets of normal invariants of X; the first maps δ are the realization maps (or actions), the second maps η are the forgetful type of maps (or Pontrjagin-Thom type maps) and the last maps θ are the surgery obstruction maps, for details one can refer to the book of Wall [17].

In order to determine $hS(X), hT_{\mathrm{PL}}(X)$, or $hT_{\mathrm{TOP}}(X)$ one must compute the groups $L_n(\pi_1(X), w(X))$, $[X/\partial X, G/\mathrm{CAT}]$, where CAT = O, PL, TOP, and also the maps θ and the actions δ in the above exact sequences.

The purpose of this paper is to determine completely the sets $hT_{\mathrm{PL}}(X)$, and $hT_{\mathrm{TOP}}(X)$, and partially the sets $hS(X)$, for $X = D^m \times P(r,s)$, where D^m is the disk of dimension $m \geq 0$ and $P(r,s)$, is the *Dold manifold* defined as the quotient $(S^r \times \mathbb{C}P^s)/\sim$, where $(x,y) \sim (x',y')$ if and only if $x' = -x$, and $y' = \bar{y}$.

The main results of this paper are Theorem (7.5), Theorem (7.6), Theorems (4.4) and (4.5), Propositions (5.1), (5.2), (5.3), (5.4) and (5.5), Propositions (6.2), (6.3) and (6.4), and Propositions (3.1) and (3.2). In addition to these, many results about Dold manifolds not very accessible in the literature have been derived.

The paper has been arranged in the following fashion: In Section 2 we give some equivalent definitions and basic (co)homological properties of Dold manifolds. In Section 3 we study a map, intimately related to the Browder-Livesay invariants associated with the Dold manifolds, and prove Propositions (3.1) and (3.2). In Section 4 we calculate the normal invariants both in the PL and topological cases for Dold manifolds and prove Theorems (4.4) and (4.5). In Section 5 we study the action map δ_{CAT}, (CAT = PL or TOP) of the groups $L_{n+1}((\mathbb{Z}/2)^{\pm})$ on the homotopy CAT structures $hT_{\mathrm{CAT}}(D^m \times P(r,s))$ and prove Propositions (5.1), (5.2), (5.3), (5.4) and (5.5). In Section 6 we calculate the image of the surgery obstruction maps θ_{CAT}, (CAT = PL or TOP) in various dimensions and orientabilities of Dold manifolds and prove Propositions (6.2), (6.3) and (6.4). In the last Section 7 we give some remarks about the homotopy smoothings of Dold manifolds which can be derived from the calculations of Sections 5, and 6, we summarize the calculations of $hT_{\mathrm{CAT}}(X)$, (CAT = PL or TOP), of $X = D^m \times P(r,s)$ in terms of exact sequences as Theorem (7.5), and finally we determine the structure of $hT_{\mathrm{CAT}}(P(r,s))$, (CAT = PL or TOP) in all cases as Theorem (7.6).

Techniques of the proofs are similar to the ones in Haršiladze [5], Haršiladze [7], and López de Medrano [10]. We have tried to make the paper as self contained as possible for the sake of readability.

2. Orientation and integral (co)homology of Dold manifolds

We start by giving an alternative description of Dold manifolds, defined in the introduction, which will be more useful in the later sections (see [3], [16]).

2.1. Description. A Dold manifold can be written as the total space of a fibre bundle over $\mathbb{R}P^r$ with fibre $\mathbb{C}P^s$:

$$(*)\qquad \mathbb{C}P^s \xrightarrow{\text{incl}} P(r,s) \xrightarrow{\text{proj}} \mathbb{R}P^r$$

We recall some properties of $P(r,s)$; we assume that $r,s > 1$:

From the homotopy exact sequence of the fibre bundle $(*)$ one gets that the fundamental group $\pi_1(P(r,s)) = \mathbb{Z}/2$. Also note that the total Stiefel-Whitney class of $P(r,s)$ is given by (see [3], page 30; see also [16])

$$W(P(r,s)) = (1+e_1)^r(1+e_1+e_2)^{s+1},$$

where $e_1 \in H^1(\mathbb{R}P^r;\mathbb{Z}/2)$, $e_2 \in H^2(\mathbb{C}P^s;\mathbb{Z}/2)$ are generators. So the first Stiefel-Whitney class of $P(r,s)$ is given by $w_1(P(r,s)) = (r+s+1)e_1$. Thus

$$w_1(P(r,s)) = \begin{cases} 0 & \text{if} \quad r+s+1 \quad \text{is} \quad \text{even} \\ \neq 0 & \text{if} \quad r+s+1 \quad \text{is} \quad \text{odd.} \end{cases}$$

Therefore,

$$P(r,s) \quad \text{is} \quad \begin{cases} \text{orientable} & \text{if} \quad r+s+1 \quad \text{is} \quad \text{even} \\ \text{non-orientable} & \text{if} \quad r+s+1 \quad \text{is} \quad \text{odd.} \end{cases}$$

Using cell structures of the Dold manifolds, or considering the double cover $\mathbb{Z}/2 \to S^r \times \mathbb{C}P^s \to P(r,s)$ one can calculate the integral (co)homology groups of Dold manifolds, these are as follows: ([3], [4]); I am also indebted to Prof. Stong for enlightening me on the various ways of looking at the Dold manifolds and their integral cohomology.

For r odd $H^*(P(r,s);\mathbb{Z})$ has one copy of $\mathbb{Z}$ in each dimension $0,4,8,\ldots,4[s/2], r+4, r+8,\ldots,r+4[s/2]$, and its torsion is given by

$$\sum_{q=2}^{(r-1)+4[s/2]} \sum_{\substack{i=1,j=0 \\ 2i+4j=q}}^{(r-1)/2,[s/2]} \mathbb{Z}/2 \oplus \sum_{q=3}^{r+4[s/2]} \sum_{\substack{i=1,j=0 \\ 2i+4j+1=q}}^{(r-1)/2,[s/2]} \mathbb{Z}/2.$$

For r even, s odd, $H^*(P(r,s);\mathbb{Z})$ has one copy of $\mathbb{Z}$ in each dimension $0,4,8,\dots,$ $2(s-1), r+2, r+6, \dots, r+2s$, and its torsion is given by

$$\sum_{q=2}^{r+2s-2} \sum_{\substack{i=1,j=0\\2i+4j=q}}^{r/2,(s-1)/2} \mathbb{Z}/2 \oplus \sum_{q=3}^{r+2s-1} \sum_{\substack{i=1,j=0\\2i+4j+1=q}}^{r/2,(s-1)/2} \mathbb{Z}/2.$$

For r even, s even, $H^*(P(r,s);\mathbb{Z})$ has one copy of $\mathbb{Z}$ in each dimension $0,4,8,\dots,$ $2s, r+2, r+6, \dots, r+2s-2$, and its torsion is given by

$$\sum_{q=2}^{r+2s} \sum_{\substack{i=1,j=0\\2i+4j=q}}^{r/2,s/2} \mathbb{Z}/2 \oplus \sum_{q=3}^{r+2s-3} \sum_{\substack{i=1,j=0\\2i+4j+1=q}}^{r/2,s/2} \mathbb{Z}/2.$$

One can write down the integral homology of the connected manifold $P(r,s)$ using universal coefficient theorem.

3. Browder-Livesay invariant associated to Dold manifolds

Let $X = P(r,s)$, and $Y = P(r-1,s)$, $r,s > 1$. Then the inclusion $Y \subset X$ induces isomorphism of fundamental groups $\pi_1(Y) \cong \pi_1(X) = \mathbb{Z}/2$. It easily follows from the alternative description of Dold manifolds given above that the pair (X,Y) is a Browder-Livesay pair according to the definition of Haršiladze [7]. Let $n = \dim X = r+2s$, and let t denote the generator of the group $(\mathbb{Z}/2) \cong \pi_1(X) \cong \pi_1(Y)$. Let $\omega^X :$ $\pi_1(X) \to \mathbb{Z}/2 = \{+1,-1\}$ denote the orientation homomorphism (or orientation character) of X and ω^Y the same for Y. Further, let $\epsilon = \pm 1$ denote the number $\omega^X(t),\ \ 0 \neq t \in \mathbb{Z}/2$. It then follows from the definition of a Browder-Livesay pair that $\omega^Y(t) = -\epsilon$. Let $\mathrm{BL}(X,Y)$ denote the group $L_{n+\epsilon}(0)$. The geometric meaning of this group can be seen as follows:

Suppose we have a simple homotopy equivalence $f : (M,\partial M) \to (X,\partial X)$, where M is some manifold for which $f\mid_{f^{-1}(\partial Y)}: f^{-1}(\partial Y) \to \partial Y$ is also a simple homotopy equivalence. For every such simple homotopy equivalence there is defined a *Browder-Livesay invariant* η with values $\eta(f)$ in the group $\mathrm{BL}(X,Y) = L_{n+\epsilon}(0)$, such that $\eta(f) = 0$ if and only if the map f is homotopic *rel* ∂M to a map f_1 for which the map $f_1\mid_{f_1^{-1}(Y)}: f_1^{-1}(Y) \to Y$ is a simple homotopy equivalence. We will say in short that f_1 *is a splitting along* Y, *or* f *splits along* Y.

Let $L(X) \overset{\text{def}}{=} L_n(\pi_1(X),\omega^X)$. Further, let $V = X\backslash U$, where U is a tubular neighbourhood of Y in X. In this case $\partial V = \partial X_{\partial Y} \cup \hat{Y}$, $\hat{Y}$ is a double covering of Y and $\partial X_{\partial Y}$ is the complement of a tubular neighbourhood of ∂Y in ∂X. One then has the following diagram of chain complexes (see [7], diagram (8), and also [6], Theorem 1 and its proof):

$$\text{(D1)}\quad \begin{tikzcd} \to L(Y\times I^2) \arrow[r] \arrow[d, phantom, "\wr"] & L(V\times I) \arrow[r] \arrow[d, phantom, "\wr"] & L(X\times I) \arrow[r, "\partial"] \arrow[d, phantom, "\wr"] & \mathrm{BL}(X,Y) \arrow[r] \arrow[d, phantom, "\wr"] & L(Y) \arrow[d, phantom, "\wr"] \\ \to L(X\times I^2) \arrow[r] & \mathrm{BL}(X\times I, Y\times I) \arrow[r] & L(Y\times I) \arrow[r] & L(V) \arrow[r] & L(X) \end{tikzcd}$$

which can be extended indefinitely on the left and where the vertical $\wr$'s denote isomorphism of the homology groups of the chain complexes.

In algebraic notation the diagram looks like the following. For brevity, we have denoted here and afterwards $L_n(\pi, \omega)$ by $L_n(\pi^\omega)$.

(D2)
$$\begin{tikzcd} {} \arrow[r, "r"] & L_{n+1}(0) \arrow[r, "c"] \arrow[d, phantom, "\wr"] & L_{n+1}((\mathbb{Z}/2)^{\omega^X}) \arrow[r, "\partial = r\circ t"] \arrow[d, phantom, "\wr"] & L_{n+\epsilon}(0) \arrow[r, "t\circ c"] \arrow[d, phantom, "\wr"] & {} \\ {} \arrow[r, "\partial = r\circ t"] & L_{n+\epsilon+1}(0) \arrow[r, "t\circ c"] & L_n((\mathbb{Z}/2)^{\omega^Y}) \arrow[r, "r"] & L_n(0) \arrow[r, "c"] & {} \end{tikzcd}$$

All the horizontal maps in the diagram are expressible in terms of algebraically defined maps
$$c : L_n(0) \to L_n((\mathbb{Z}/2)^{\omega^X}),$$
defined by functoriality,
$$r : L_n((\mathbb{Z}/2)^{\omega^Y}) \to L_n(0),$$
the transfer, and
$$t : L_n((\mathbb{Z}/2)^{\omega^X}) \to L_{n-1+\epsilon}((\mathbb{Z}/2)^{\omega^X}),$$
multiplication of a quadratic form by the generator $t \in \mathbb{Z}/2$. The map
$$\partial : L(X \times I) \to \mathrm{BL}(X, Y)$$
which factors through the Browder-Livesay invariant η as follows:
$$\partial = \eta \circ \delta : L(X \times I) \xrightarrow{\delta} hT_{\mathrm{CAT}}(X) \xrightarrow{\eta} \mathrm{BL}(X, Y),$$
(CAT = PL or TOP), coincides with
$$r \circ t : L_{n+1}((\mathbb{Z}/2)^{\omega^X}) \to L_{n+\epsilon}(0).$$

We need to compute ∂ in various cases. We have to consider two cases: when $\omega^X = +$, that is ω^X is the constant map taking every element of the fundamental group to $+1$, and when $\omega^X = -$, that is when ω^X maps the fundamental group onto $\{+1, -1\}$.

Case I: $\omega^X = +$. In this case $\epsilon = +1$, implying $\omega^Y = -$ and the diagram (D2) looks like:

(D3)
$$\begin{tikzcd} {} \arrow[r, "r"] & L_{n+1}(0) \arrow[r, "c"] \arrow[d, phantom, "\wr"] & L_{n+1}((\mathbb{Z}/2)^+) \arrow[r, "\partial = r\circ t"] \arrow[d, phantom, "\wr"] & L_{n+1}(0) \arrow[r, "t\circ c"] \arrow[d, phantom, "\wr"] & {} \\ {} \arrow[r, "\partial = r\circ t"] & L_{n+2}(0) \arrow[r, "t\circ c"] & L_n((\mathbb{Z}/2)^-) \arrow[r, "r"] & L_n(0) \arrow[r, "c"] & {} \end{tikzcd}$$

which gives:
$$\frac{\mathrm{Ker}\ [L_{n+1}((\mathbb{Z}/2)^+) \xrightarrow{\partial = r\circ t} L_{n+1}(0)]}{\mathrm{Im}\ [L_{n+1}(0) \xrightarrow{c} L_{n+1}((\mathbb{Z}/2)^+)]} \cong \frac{\mathrm{Ker}\ [L_n((\mathbb{Z}/2)^-) \xrightarrow{r} L_n(0)]}{\mathrm{Im}\ [L_{n+2}(0) \xrightarrow{t\circ c} L_n((\mathbb{Z}/2)^-)]}.$$

From the computations of Wall's surgery groups in [17] of the groups 0, and $\mathbb{Z}/2$, that is
$$L_4(0) = \mathbb{Z}, \quad L_4((\mathbb{Z}/2)^+) = \mathbb{Z} \oplus \mathbb{Z}; \quad L_4((\mathbb{Z}/2)^-) = L_2((\mathbb{Z}/2)^\pm) = L_2(0) = \mathbb{Z}/2;$$
$$L_3((\mathbb{Z}/2)^+) = \mathbb{Z}/2; L_3((\mathbb{Z}/2)^-) = L_1((\mathbb{Z}/2)^\pm) = L_{\mathrm{odd}}(0) = 0.$$
one derives readily the following:

3.1. Proposition. *If $\omega^X = +$ then $\epsilon = +1$, implying $\omega^Y = -$ and we have*

$$\partial : L_{n+1}((\mathbb{Z}/2)^+) \xrightarrow{r\circ t} L_{n+1}(0) = \begin{cases} \textit{zero map} & \textit{if } n \equiv 0,1,2 \bmod 4 \\ \textit{epi. } (\mathbb{Z})^2 \to \mathbb{Z} & \textit{if } n \equiv 3 \bmod 4. \end{cases}$$

Proof. In cases $n \equiv 0, 2, 3 \pmod 4$ the stated maps are the only choice for ∂. For $n \equiv 1 \pmod 4$ the choice of ∂ is a consequence of the fact that the map $c : L_2(0) \to L_2((\mathbb{Z}/2)^+)$ preserves the Arf invariants. □

Case II: $\omega^X = -$. In this case $\epsilon = -1$, implying $\omega^Y = +$ and the diagram (D2) looks like:

(D4)
$$\begin{array}{ccccccc} \xrightarrow{r} & L_{n+1}(0) & \xrightarrow{c} & L_{n+1}((\mathbb{Z}/2)^-) & \xrightarrow{\partial = r\circ t} & L_{n+\epsilon}(0) & \xrightarrow{t\circ c} \\ & \wr & & \wr & & \wr & \\ \xrightarrow{\partial = r\circ t} & L_{n+\epsilon+1}(0) & \xrightarrow{t\circ c} & L_n((\mathbb{Z}/2)^+) & \xrightarrow{r} & L_n(0) & \xrightarrow{c} \end{array}$$

In this case

$$\frac{\text{Ker } [L_{n+1}((\mathbb{Z}/2)^-) \xrightarrow{\partial = r\circ t} L_{n-1}(0)]}{\text{Im } [L_{n+1}(0) \xrightarrow{c} L_{n+1}((\mathbb{Z}/2)^-)]} \cong \frac{\text{Ker } [L_n((\mathbb{Z}/2)^+) \xrightarrow{r} L_n(0)]}{\text{Im } [L_n(0) \xrightarrow{t\circ c} L_n((\mathbb{Z}/2)^+)]}.$$

One readily derives the following:

3.2. Proposition. *If $\omega^X = -$, $\epsilon = -1$, $\omega^Y = +$ we have*

$$\partial = r \circ t : L_{n+1}((\mathbb{Z}/2)^-) \to L_{n-1}(0)$$

is the zero map for all n.

Proof. Apart from cases $n \equiv 0, 1, 2 \pmod 4$, where the choice of ∂ is unique the other case $n \equiv 3 \pmod 4$ follows from the fact that $c : \mathbb{Z} = L_0(0) \to L_0((\mathbb{Z}/2)^-) = \mathbb{Z}/2$ is the zero map, because the Arf invariant of even symmetric forms are zero. □

4. Normal invariant of Dold manifolds

Let us first recall the following well-known theorem due to Sullivan and Kirby-Siebenmann (see [9]):

Let $K(A, n)$ denote an Eilenberg-Mac Lane space, Y denote the space with two nontrivial homotopy groups $\pi_2(Y) = \mathbb{Z}/2$; $\pi_4(Y) = \mathbb{Z}$, and k-invariant $\delta Sq^2 \in H^5(K(\mathbb{Z}/2, 2); \mathbb{Z})$, where Sq^2 is the Steenrod square and δ is the Bockstein homomorphism in cohomology, corresponding to the exact sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$. For a topological space X, let $X_{(2)}$ denotes its localization at 2, $X_{(0)}$ denotes its rationalization and $X[1/2]$ denotes its localization away from 2 (see [14]), then:

4.1. Theorem (Sullivan, Kirby-Siebenmann)**.** *We have the following homotopy equivalences:*

$$G/\mathrm{TOP}_{(2)} \cong \prod_{i>1} K(\mathbb{Z}/2, 4i-2) \times \prod_{i>1} K(\mathbb{Z}_{(2)}, 4i),$$

$$G/\mathrm{PL}_{(2)} \cong Y_{(2)} \times \prod_{i>1} K(\mathbb{Z}/2, 4i-2) \times \prod_{i>1} K(\mathbb{Z}_{(2)}, 4i),$$

and the following homotopy equivalences:

$$G/\mathrm{TOP}[1/2] \overset{h_{\mathrm{TOP}}}{\cong} BO^{\otimes}[1/2] \overset{h_{\mathrm{PL}}}{\cong} G/\mathrm{PL}[1/2].$$

As a consequence of this the normal invariants for a manifold X can be calculated using the following fibre squares, where CAT = PL or TOP:

$$\begin{array}{ccc} G/\mathrm{CAT} & \overset{P^{G/\mathrm{CAT}}_{(2)}}{\longrightarrow} & G/\mathrm{CAT}_{(2)} \\ P^{G/\mathrm{CAT}}[1/2]\downarrow & & \downarrow u \\ BO^{\otimes}[1/2] \cong G/\mathrm{CAT}[1/2] & \overset{\mathrm{ph}\circ h_{\mathrm{CAT}}}{\longrightarrow} & G/\mathrm{CAT}_{(0)} = \prod_{i>0} K(\mathbb{Q},4i) \end{array}$$

where ph stands for the Pontrjagin character, and u_* in homotopy coincides with the inclusion $\phi : \mathbb{Z}_{(2)} \subset \mathbb{Q}$ for $k > 1$, and with 2ϕ for $k = 1$. These give by definition, the following exact sequences for any CW-complex X:

$$0 \to [X, G/\mathrm{TOP}] \overset{\Phi^{G/\mathrm{TOP}}}{\longrightarrow} KO^0(X)[1/2] \oplus \sum_{i>1} H^{4i-2}(X;\mathbb{Z}/2) \oplus \sum_{i>1} H^{4i}(X;\mathbb{Z}_{(2)}) \overset{\Psi^{G/\mathrm{TOP}}}{\longrightarrow} \sum_{i>0} H^{4i}(X;\mathbf{Q}) \to 0$$

and

$$0 \to [X, G/\mathrm{PL}] \overset{\Phi^{G/\mathrm{PL}}}{\longrightarrow} KO^0(X)[1/2] \oplus [X, Y_{(2)}] \oplus \sum_{i>1} H^{4i-2}(X;\mathbb{Z}/2) \oplus \sum_{i>1} H^{4i}(X;\mathbb{Z}_{(2)}) \overset{\Psi^{G/\mathrm{PL}}}{\longrightarrow} \sum_{i>0} H^{4i}(X;\mathbf{Q}) \to 0.$$

Here $\Phi^{G/\mathrm{CAT}} = \left(\left(P^{G/\mathrm{CAT}}[1/2]\right)_* \oplus \left(P^{G/\mathrm{CAT}}_{(2)}\right)_*\right)\Delta$, $\Psi^{G/\mathrm{CAT}} = \nabla(-\mathrm{ph}\circ h_{\mathrm{cat}} \oplus u)$, $\Delta(x) = (x,x)$ and $\nabla(x,y) = x + y$.

Let

$$\Pi = Y \times \prod_{i>1} K(\mathbb{Z}/2, 4i-2) \times \prod_{i>1} K(\mathbb{Z}, 4i),$$
$$\Pi_{(2)} = Y_{(2)} \times \prod_{i>1} K(\mathbb{Z}/2, 4i-2) \times \prod_{i>1} K(\mathbb{Z}_{(2)}, 4i),$$
$$\Pi[1/2] = Y[1/2] \times \prod_{i>1} K(\mathbb{Z}[1/2], 4i).$$

Then from the fibre square:

$$\begin{array}{ccc} \Pi & \xrightarrow{P^{\Pi}_{(2)}} & \Pi_{(2)} \\ {\scriptstyle P^{\Pi}[1/2]}\downarrow & & \downarrow{\scriptstyle u} \\ \Pi[1/2] & \xrightarrow{j} & \Pi_{(0)} = \prod_{i>0} K(\mathbb{Q},4i) \end{array}$$

we also get an exact sequence:

$$0 \to [X, \Pi] \xrightarrow{\Phi^{\Pi}} [X, Y[1/2]] \oplus \sum_{i>1} H^{4i}(X; \mathbb{Z}[1/2]) \oplus [X, Y_{(2)}] \oplus \sum_{i>1} H^{4i-2}(X; \mathbb{Z}/2) \oplus \sum_{i>1} H^{4i}(X; \mathbb{Z}_{(2)}) \xrightarrow{\Psi^{\Pi}} \sum_{i>0} H^{4i}(X; \mathbf{Q}) \to 0,$$

Φ^{Π} and Ψ^{Π} have similar definitions as above. Now, we have the following result of Rudyak ([11], Theorems 1):

4.2. Theorem (Rudyak)**.** *Let X be a finite CW-complex with no odd torsion in homology, then $(P^{\Pi}_{(2)})_* : [X, \Pi] \to [X, \Pi_{(2)}]$ and $(P^{G/\mathrm{CAT}}_{(2)})_* : [X, G/\mathrm{CAT}] \to [X, (G/\mathrm{CAT})_{(2)}]$, are monomorphisms.*

We shall prove:

4.3. Theorem. *For $X = P(r, s)$, $r, s > 1$, if we identify $G/\mathrm{PL}_{(2)}$ and $\Pi_{(2)}$ under the homotopy equivalence given in Theorem (4.1), then the groups $\mathrm{Im}\ (P^{\Pi}_{(2)})_*$ and $\mathrm{Im}\ (P^{G/\mathrm{PL}}_{(2)})_*$ are isomorphic.*

Proof. Since $X = P(r, s)$, $r, s > 1$ is a finite complex with no odd torsion in homology, by Theorem (4.2) it follows that $[X, G/\mathrm{PL}]$, and $[X, \Pi]$ are finitely generated abelian groups which do not have any odd torsions and whose $\mathbb{Z}$-ranks and 2-torsions are same as the $\mathbb{Z}_{(2)}$-ranks and 2-torsions of

$$[X, G/\mathrm{PL}] \otimes \mathbb{Z}_{(2)} \cong [X, G/\mathrm{PL}_{(2)}] \cong [X, \Pi_{(2)}] \cong [X, \Pi] \otimes \mathbb{Z}_{(2)},$$

so, if we identify $G/\mathrm{PL}_{(2)}$ and $\Pi_{(2)}$ under the homotopy equivalence given in Theorem (4.1) $\mathrm{Im}\ (P^{\Pi}_{(2)})_*$ and $\mathrm{Im}\ (P^{G/\mathrm{PL}}_{(2)})_*$ are isomorphic. This proves the theorem. $\square$

The above discussion yields that:

4.4. Theorem. *For $X = P(r, s)$, $r, s > 1$,*

$$[X, G/\mathrm{TOP}] \cong \sum_{i>1} H^{4i-2}(X; \mathbb{Z}/2) \oplus \sum_{i>1} H^{4i}(X; \mathbb{Z}),$$
$$[X, G/\mathrm{PL}] \cong [X, \Pi] \cong [X, Y] \oplus \sum_{i>1} H^{4i-2}(X; \mathbb{Z}/2) \oplus \sum_{i>1} H^{4i}(X; \mathbb{Z}).$$

Hence the normal invariant of $X = P(r, s)$, $r, s > 1$ in the topological case is completely determined and the normal invariant in the PL case is determined once we determine $[X, Y]$. We recall for ready reference the following calculations in ([10], [11]): $[\mathbb{C}P^n, Y] = \mathbb{Z}$ for $n \geq 2$,

$$[\mathbb{R}P^n, Y] = \begin{cases} \mathbb{Z}/2 & \text{if } n = 2, 3, \\ \mathbb{Z}/4 & \text{if } n \geq 4. \end{cases}$$

Towards determining $[X, Y]$ we recall the alternative description of X

$$\mathbb{C}P^s \xrightarrow{\text{incl}} X = P(r, s) \xrightarrow{\text{proj}} \mathbb{R}P^r,$$

and recall from the calculations of Section 2

$$H^4(P(r,s);\mathbb{Z}) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z}/2 & \text{if } r\geq 4,\ s\geq 2 \\ \mathbb{Z} & \text{if } r=3,\ s\geq 2 \\ \mathbb{Z}\oplus\mathbb{Z} & \text{if } r=2,\ s\geq 2 \end{cases}$$

Case I: $r\geq 4,\ s\geq 2$. Let $a\in H^4(\mathbb{R}P^r;\mathbb{Z})=\mathbb{Z}/2$, and α,β are generators of $H^4(P(r,s);\mathbb{Z})=\mathbb{Z}\oplus\mathbb{Z}/2$.

Now from the definition of Y we have a fibration $K(\mathbb{Z},4)\xrightarrow{j} Y\xrightarrow{p} K(\mathbb{Z}/2,2)$ for which ΩY has zero k-invariant $k\in H^4(K(\mathbb{Z}/2,1);\mathbb{Z})$, and also for $P(r,s)$ the operation $\delta Sq^2: H^2(P(r,s);\mathbb{Z}/2)\to H^5(P(r,s);\mathbb{Z})$ is zero. So we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & H^4(P(r,s);\mathbb{Z}) \arrow[r, "j_*"] & {[P(r,s),Y]} \arrow[r, "p_*"] & H^2(P(r,s);\mathbb{Z}/2) \arrow[r] & 0 \\ 0 \arrow[r] & H^4(\mathbb{R}P^r;\mathbb{Z}) \arrow[r, "\chi"] \arrow[u, "\mathrm{proj}^*"] & {[RP^r,Y]} \arrow[r, "p_*"] \arrow[u, "\mathrm{proj}^Y"] & H^2(\mathbb{R}P^r;\mathbb{Z}/2) \arrow[r] \arrow[u, "\mathrm{proj}^*"] & 0. \end{tikzcd}$$

Clearly the vertical maps are nonzero. Now, using the calculations mentioned above for $[\mathbb{R}P^n,Y]$, we get

$$j_*(\beta)=j_*\circ\mathrm{proj}^*(a)=\mathrm{proj}^Y\circ\chi(a)=\mathrm{proj}^Y(2a')=2(\mathrm{proj}^Y(a')).$$

Consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^4(P(r,s);\mathbb{Z}) \arrow[r, "j_*"] \arrow[d, "\mathrm{incl}^*"] & {[P(r,s),Y]} \arrow[r, "p_*"] \arrow[d, "\mathrm{incl}^Y"] & H^2(P(r,s);\mathbb{Z}/2) \arrow[r] \arrow[d, "\mathrm{incl}^*"] & 0 \\ 0 \arrow[r] & H^4(\mathbb{C}P^s;\mathbb{Z}) \arrow[r, "\chi"] & {[\mathbb{C}P^s,Y]} \arrow[r, "p_*"] & H^2(\mathbb{C}P^s;\mathbb{Z}/2) \arrow[r] & 0, \end{tikzcd}$$

which reduces to

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}\oplus\mathbb{Z}/2 \arrow[r, "j_*"] \arrow[d, "\mathrm{incl}^*"] & {[P(r,s),Y]} \arrow[r, "p_*"] \arrow[d, "\mathrm{incl}^Y"] & \mathbb{Z}/2\oplus\mathbb{Z}/2 \arrow[r] \arrow[d, "\mathrm{incl}^*"] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r, "p_*"] & \mathbb{Z}/2 \arrow[r] & 0. \end{tikzcd}$$

Now the extreme left vertical arrow is onto, the extreme right vertical arrow is a projection onto the factor $H^0(\mathbb{R}P^r;\mathbb{Z}/2)\otimes H^2(\mathbb{C}P^s;\mathbb{Z}/2)$. So the middle vertical arrow is also onto. It follows from this that $[P(r,s),Y]$ contains a $\mathbb{Z}$ summand with $\mathrm{incl}^Y(1)=1$, $j_*=\times 2$, and p_* maps the generator of $\mathbb{Z}$ as $p_*(1)=(\bar{0},\bar{1})\in\mathbb{Z}/2\oplus\mathbb{Z}/2$ mapping onto the factor $H^0(\mathbb{R}P^r;\mathbb{Z}/2)\otimes H^2(\mathbb{C}P^s;\mathbb{Z}/2)$.

Thus we obtain that

$$[P(r,s),Y]=\mathbb{Z}\oplus\mathbb{Z}/4,\quad \text{for } r\geq 4,\ s\geq 2.$$

Case II: $r=3,\ s\geq 2$. In this case the exact sequence

$$0\to H^4(P(3,s);\mathbb{Z})\xrightarrow{j_*}[P(3,s),Y]\xrightarrow{p_*}H^2(P(3,s);\mathbb{Z}/2)\to 0$$

reduces to

$$0\to\mathbb{Z}\xrightarrow{j_*}[P(3,s),Y]\xrightarrow{p_*}\mathbb{Z}/2\oplus\mathbb{Z}/2\to 0.$$

Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^4(P(3,s);\mathbb{Z}) \arrow[r, "j_*"] \arrow[d, "\mathrm{incl}^*"'] & {[P(3,s),Y]} \arrow[r, "p_*"] \arrow[d, "\mathrm{incl}^Y"'] & H^2(P(3,s);\mathbb{Z}/2) \arrow[r] \arrow[d, "\mathrm{incl}^*"'] & 0 \\
0 \arrow[r] & H^4(\mathbb{C}P^s;\mathbb{Z}) \arrow[r, "\chi"] & {[\mathbb{C}P^s,Y]} \arrow[r, "p_*"] & H^2(\mathbb{C}P^s;\mathbb{Z}/2) \arrow[r] & 0,
\end{tikzcd}$$

which reduces to

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "j_*"] \arrow[d, "\mathrm{incl}^*"'] & {[P(3,s),Y]} \arrow[r, "p_*"] \arrow[d, "\mathrm{incl}^Y"'] & \mathbb{Z}/2\oplus\mathbb{Z}/2 \arrow[r] \arrow[d, "\mathrm{incl}^*"'] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r, "p_*"] & \mathbb{Z}/2 \arrow[r] & 0.
\end{tikzcd}$$

Now the extreme right vertical arrow is a projection onto the factor $H^0(\mathbb{R}P^3;\mathbb{Z}/2)\otimes H^2(\mathbb{C}P^s;\mathbb{Z}/2)$. The extreme left vertical arrow is an isomorphism. So the middle vertical arrow is also onto. It follows from this that $[P(3,s),Y]$ contains a $\mathbb{Z}$ summand with $\mathrm{incl}^Y(1) = 1$, $j_* = \times 2$, and p_* maps the generator of $\mathbb{Z}$ as $p_*(1) = (\bar{0},\bar{1}) \in \mathbb{Z}/2\oplus\mathbb{Z}/2$ mapping onto the factor $H^0(\mathbb{R}P^3;\mathbb{Z}/2)\otimes H^2(\mathbb{C}P^s;\mathbb{Z}/2)$.

Considering again the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^4(P(3,s);\mathbb{Z}) \arrow[r, "j_*"] & {[P(3,s),Y]} \arrow[r, "p_*"] & H^2(P(3,s);\mathbb{Z}/2) \arrow[r] & 0 \\
0 \arrow[r] & H^4(\mathbb{R}P^3;\mathbb{Z}) \arrow[r, "\chi"] \arrow[u, "\mathrm{proj}^*"] & {[\mathbb{R}P^3,Y]} \arrow[r, "p_*"] \arrow[u, "\mathrm{proj}^Y"] & H^2(\mathbb{R}P^3;\mathbb{Z}/2) \arrow[r] \arrow[u, "\mathrm{proj}^*"] & 0,
\end{tikzcd}$$

which reduces to

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "j_*"] & {[P(3,s),Y]} \arrow[r, "p_*"] & \mathbb{Z}/2\oplus\mathbb{Z}/2 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & \mathbb{Z}/2 \arrow[r, "\cong"] \arrow[u, "\mathrm{proj}^Y"] & \mathbb{Z}/2 \arrow[r] \arrow[u, "\mathrm{proj}^*"] & 0.
\end{tikzcd}$$

Now the extreme right vertical arrow is an injection onto the factor $H^2(\mathbb{R}P^3;\mathbb{Z}/2)\otimes H^0(\mathbb{C}P^s;\mathbb{Z}/2)$. So the middle vertical arrow is also an injection onto a $\mathbb{Z}/2$-summand of $[P(3,s),Y]$. It follows from this that p_* maps the generator of this $\mathbb{Z}/2$ as $p_*(1) = (\bar{1},\bar{0}) \in \mathbb{Z}/2\oplus\mathbb{Z}/2$ mapping onto the factor $H^2(\mathbb{R}P^3;\mathbb{Z}/2)\otimes H^0(\mathbb{C}P^s;\mathbb{Z}/2)$.

Thus we conclude that

$$[P(r,s),Y] = \mathbb{Z}\oplus\mathbb{Z}/2, \text{ for } r = 3,\ s \geq 2.$$

Case III: $r = 2,\ s \geq 2$. In this case the exact sequence

$$0 \to H^4(P(2,s);\mathbb{Z}) \xrightarrow{j_*} [P(2,s),Y] \xrightarrow{p_*} H^2(P(2,s);\mathbb{Z}/2) \to 0$$

reduces to

$$0 \to \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{j_*} [P(2,s),Y] \xrightarrow{p_*} \mathbb{Z}/2\oplus\mathbb{Z}/2 \to 0.$$

Consider the following commutative diagram corresponding to the double covering $\mathrm{covproj} : S^2 \times \mathbb{C}P^s \to P(2,s)$:

$$\begin{tikzcd}
0 \arrow[r] & H^4(P(2,s);\mathbb{Z}) \arrow[r, "j_*"] \arrow[d, "\mathrm{covproj}^*"'] & {[P(2,s),Y]} \arrow[r, "p_*"] \arrow[d, "\mathrm{covproj}^Y"'] & H^2(P(2,s);\mathbb{Z}/2) \arrow[r] \arrow[d, "\mathrm{covproj}^*"'] & 0 \\
0 \arrow[r] & H^4(S^2 \times \mathbb{C}P^s;\mathbb{Z}) \arrow[r, "\chi"] & {[S^2 \times \mathbb{C}P^s, Y]} \arrow[r, "p_*"] & H^2(S^2 \times \mathbb{C}P^s;\mathbb{Z}/2) \arrow[r] & 0,
\end{tikzcd}$$

which reduces to

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "j_*"] \arrow[d, "\mathrm{covproj}^*"'] & {[P(2,s),Y]} \arrow[r, "p_*"] \arrow[d, "\mathrm{covproj}^Y"'] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r] \arrow[d, "\mathrm{covproj}^*"'] & 0 \\
0 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\times 2"] & {[S^2 \times \mathbb{C}P^s, Y]} \arrow[r, "p_*"] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r] & 0.
\end{tikzcd}$$

Now the extreme right vertical arrow and the extreme left vertical arrows are isomorphisms. So the middle vertical arrow is also an isomorphism. It suffices therefore to calculate $[S^2 \times \mathbb{C}P^s, Y]$.

For this consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^4(S^2 \times \mathbb{C}P^s;\mathbb{Z}) \arrow[r, "j_*"] & {[S^2 \times \mathbb{C}P^s, Y]} \arrow[r, "p_*"] & H^2(S^2 \times \mathbb{C}P^s;\mathbb{Z}/2) \arrow[r] & 0 \\
0 \arrow[r] & H^4(\mathbb{C}P^s;\mathbb{Z}) \arrow[r, "\chi"] \arrow[u, "\mathrm{proj}_2^*"'] & {[\mathbb{C}P^s, Y]} \arrow[r, "p_*"] \arrow[u, "\mathrm{proj}_2^Y"'] & H^2(\mathbb{C}P^s;\mathbb{Z}/2) \arrow[r] \arrow[u, "\mathrm{proj}_2^*"'] & 0,
\end{tikzcd}$$

which reduces to

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "j_*"] & {[S^2 \times \mathbb{C}P^s, Y]} \arrow[r, "p_*"] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[u, "\mathrm{proj}_2^*"'] & \mathbb{Z} \arrow[r, "p_*"] \arrow[u, "\mathrm{proj}_2^Y"'] & \mathbb{Z}/2 \arrow[r] \arrow[u, "\mathrm{proj}_2^*"'] & 0.
\end{tikzcd}$$

From similar considerations as in Case I it follows that $[S^2 \times \mathbb{C}P^s, Y]$ contains a $\mathbb{Z}$-summand and $j_* = \times 2$ on the factor $H^0(S^2;\mathbb{Z}) \otimes H^4(\mathbb{C}P^s;\mathbb{Z})$, and p_* maps the generator as $p_*(1) = (\bar{0}, \bar{1})$ onto the summand $H^0(S^2;\mathbb{Z}/2) \otimes H^2(\mathbb{C}P^s;\mathbb{Z}/2)$. Consider next the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^4(S^2 \times \mathbb{C}P^s;\mathbb{Z}) \arrow[r, "j_*"] & {[S^2 \times \mathbb{C}P^s, Y]} \arrow[r, "p_*"] & H^2(S^2 \times \mathbb{C}P^s;\mathbb{Z}/2) \arrow[r] & 0 \\
0 \arrow[r] & H^4(S^2;\mathbb{Z}) \arrow[r, "\chi"] \arrow[u, "\mathrm{proj}_1^*"'] & {[S^2, Y]} \arrow[r, "p_*"] \arrow[u, "\mathrm{proj}_1^Y"'] & H^2(S^2;\mathbb{Z}/2) \arrow[r] \arrow[u, "\mathrm{proj}_1^*"'] & 0,
\end{tikzcd}$$

which reduces to

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "j_*"] & {[S^2 \times \mathbb{C}P^s, Y]} \arrow[r, "p_*"] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & \mathbb{Z}/2 \arrow[r, "p_*"] \arrow[u, "\mathrm{proj}_1^Y"'] & \mathbb{Z}/2 \arrow[r] \arrow[u, "\mathrm{proj}_1^*"'] & 0.
\end{tikzcd}$$

The extreme right vertical arrow is an injection onto the factor $H^2(S^2;\mathbb{Z}/2)\otimes H^0(\mathbb{C}P^s;\mathbb{Z}/2)$. So the middle vertical arrow is also an injection onto a $\mathbb{Z}/2$-summand of $[S^2\times\mathbb{C}P^s,Y]$. It follows from this that p_* maps the generator of this $\mathbb{Z}/2$ as $p_*(1)=(\bar{1},\bar{0})\in\mathbb{Z}/2\oplus\mathbb{Z}/2$ mapping onto the factor $H^2(S^2;\mathbb{Z}/2)\otimes H^0(\mathbb{C}P^s;\mathbb{Z}/2)$.

Finally we consider the following commutative diagram corresponding to the inclusion $i:S^2\times S^2\hookrightarrow S^2\times\mathbb{C}P^s$

$$\begin{tikzcd}
0 \arrow[r] & H^4(S^2\times\mathbb{C}P^s;\mathbb{Z}) \arrow[r, "j_*"] \arrow[d, "i^*"] & {[S^2\times\mathbb{C}P^s,Y]} \arrow[r, "p_*"] \arrow[d, "i^Y"] & H^2(S^2\times\mathbb{C}P^s;\mathbb{Z}/2) \arrow[r] \arrow[d, "i^*"] & 0 \\
0 \arrow[r] & H^4(S^2\times S^2;\mathbb{Z}) \arrow[r, "\chi"] & {[S^2\times S^2,Y]} \arrow[r, "p_*"] & H^2(S^2\times S^2;\mathbb{Z}/2) \arrow[r] & 0,
\end{tikzcd}$$

which reduces to

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "j_*"] \arrow[d, "i^*"] & {[S^2\times\mathbb{C}P^s,Y]} \arrow[r, "p_*"] \arrow[d, "i^Y"] & \mathbb{Z}/2\oplus\mathbb{Z}/2 \arrow[r] \arrow[d, "i^*"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & {[S^2\times S^2,Y]} \arrow[r, "p_*"] & \mathbb{Z}/2\oplus\mathbb{Z}/2 \arrow[r] & 0.
\end{tikzcd}$$

The extreme right vertical arrow is an isomorphism, and the extreme left vertical arrow is onto, hence the middle vertical arrow is also onto.

Now it is an analogous but easy exercise, which can be left to the reader, to show that $[S^2\times S^2,Y]\cong\mathbb{Z}\otimes\mathbb{Z}/2\otimes\mathbb{Z}/2$. From the above three considerations we can conclude that

$$[S^2\times\mathbb{C}P^s,Y]\cong[P(2,s),Y]\cong\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2.$$

We can summarize the above calculations in the form of the following

4.5. Theorem. *For Dold manifolds $P(r,s)$, $r,\ s\geq 2$,*

$$[P(r,s),Y]=\begin{cases}\mathbb{Z}\oplus\mathbb{Z}/4 & \text{if } r\geq 4,\ s\geq 2\\ \mathbb{Z}\oplus\mathbb{Z}/2 & \text{if } r=3,\ s\geq 2\\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2 & \text{if } r=2,\ s\geq 2.\end{cases}$$

5. The action δ_{CAT} of $L_{n+1}((\mathbb{Z}/2)^{\pm})$ on the homotopy CAT structures of $P(r,s)$

Let $X=D^m\times P(r,s)$, $r,s>1$, where D^m is the standard m-dimensional disk.

5.1. Proposition. *Let $n=\dim X\equiv 3\pmod 4$, $r+s+1$ even. So X is orientable. Then the action δ_{CAT} of the group $L_4((\mathbb{Z}/2)^+)=\mathbb{Z}\oplus\mathbb{Z}$ on $hT_{\mathrm{CAT}}(X)$ is trivial when restricted to the subgroup $\mathbb{Z}=\mathrm{Im}[L_4(0)\to L_4((\mathbb{Z}/2)^+)]$, and is nontrivial on the remaining summand.*

Proof. The action of the subgroup (direct summand) $\mathbb{Z}=\mathrm{Im}[L_4(0)\to L_4((\mathbb{Z}/2)^+)]$ on the set $hT_{\mathrm{CAT}}(X)$, CAT = PL or TOP., is obtained by taking connected sum with a homotopy sphere, which in the CAT case is trivial.

We examine the action of the remaining summand $\mathbb{Z}$ on $hT_{\mathrm{CAT}}(X)$. Let $Y = D^m \times P(r-1,s)$. Then $\mathrm{BL}(X,Y) = L_{r+2s+m+1}(0) = L_4(0) = \mathbb{Z}$. In addition, the composite

$$\mathbb{Z} \oplus \mathbb{Z} = L_4((\mathbb{Z}/2)^+) \overset{\delta_{\mathrm{CAT}}}{\to} hT_{\mathrm{CAT}}(X) \overset{\eta}{\to} L_4(0) = \mathrm{BL}(X,Y) = \mathbb{Z}$$

is such that following sequence is exact by the considerations of Section 3:

$$\mathbb{Z} = L_4(0) \overset{i=c}{\to} L_4((\mathbb{Z}/2)^+) \overset{\eta\circ\delta_{\mathrm{CAT}}=rot}{\to} L_4(0) = \mathbb{Z}.$$

Now, the map η is an invariant of homotopy CAT structures of the manifold X, and for the proof of the proposition we shall prove that the kernel of the map δ_{CAT} is given by $\operatorname{Ker}\delta_{\mathrm{CAT}} = \operatorname{Im}[L_4(0) \to L_4((\mathbb{Z}/2)^+)]$. But by the last exact sequence we have

$$\operatorname{Ker}\delta_{\mathrm{CAT}} \subseteq \operatorname{Ker}\eta\circ\delta_{\mathrm{CAT}} = \operatorname{Im}[L_4(0) \to L_4((\mathbb{Z}/2)^+)] = \mathbb{Z}.$$

So this together with the observation in the first line of the proof gives the result. □

5.2. Proposition. *Let* $n = \dim X \equiv 1 \pmod 4$. *Then the action* δ_{CAT} *of the group* $L_2((\mathbb{Z}/2)^{\pm})$ *on* $hT_{\mathrm{CAT}}(X)$ *is trivial.*

Proof. Since $L_2((\mathbb{Z}/2)^{\pm}) = L_2(0)$, this group acts on the homotopy CAT structures by taking connected sum with a homotopy sphere, which in the CAT case is trivial. □

5.3. Proposition. *Let* $D^m = *$, *and* $n = \dim X \equiv 0 \pmod 2$ *and* X *is not orientable. Then the groups* $L_1((\mathbb{Z}/2)^-) = 0$ *and* $L_3((\mathbb{Z}/2)^-) = 0$, *so the actions* δ_{CAT} *of these groups on* $hT_{\mathrm{CAT}}(P(r,s))$ *are trivial.*

5.4. Proposition. *Let* $n = \dim X \equiv 2 \pmod 4$, $r+s+1$, *even. So that* X *is orientable. Then the action* δ_{CAT} *of the group* $L_3((\mathbb{Z}/2)^+) = \mathbb{Z}/2$ *on* $hT_{\mathrm{CAT}}(X)$ *is trivial.*

Proof. If $Y = D^m \times P(r-1,s)$, then $\mathrm{BL}(X,Y) = L_{r+2s+m+1}(0) = L_3(0) = 0$. So the composite

$$L_3((\mathbb{Z}/2)^+) \overset{\delta_{\mathrm{CAT}}}{\to} hT_{\mathrm{CAT}}(X) \overset{\eta}{\to} \mathrm{BL}(X,Y) = 0$$

is obviously the zero map. Hence

$$\operatorname{Ker}\delta_{\mathrm{CAT}} \subseteq \operatorname{Ker}\eta\circ\delta_{\mathrm{CAT}} = L_3((\mathbb{Z}/2)^+) = \mathbb{Z}/2.$$

Now, realize the nonzero element of the group $L_3((\mathbb{Z}/2)^+)$ by a normal map $f : M \to D^m \times P(r,s) \times I$ such that

$$f\,|_{\partial_- M}\colon \partial_- M \to D^m \times P(r,s) \times 0 \cup \partial(D^m \times P(r,s)) \times I$$

is a CAT homeomorphism (CAT = PL or TOP). The map $\partial : L_3((\mathbb{Z}/2)^+) \to L_3(0) = 0$ is obviously zero. Therefore, by the relation between ∂ and the Browder-Livesay invariant mentioned in Section 3, the homotopy CAT structure $f\,|_{\partial_+ M}\colon \partial_+ M \to D^m \times P(r,s) \times 1$ is split along $D^m \times P(r-1,s) \times 1$. We denote the map $f\,|_{\partial_+ M}$ by f_1.

Thus the map

$$f_1\,|_{f_1^{-1}(D^m\times P(r-1,s))}\colon f_1^{-1}(D^m \times P(r-1,s)) \to D^m \times P(r-1,s)$$

is a simple homotopy equivalence, and is a CAT-homeomorphism on the boundary, that is, it is a homotopy CAT structure. We show that it is trivial. Now the map

$$f\mid_{f^{-1}(D^m\times P(r-1,s)\times I)}\colon f^{-1}(D^m\times P(r-1,s)\times I)\to D^m\times P(r-1,s)\times I$$

is normal and realizes some element of the group $L_2((\mathbb{Z}/2)^-)$. Therefore, according to Proposition (5.2) the homotopy CAT structure $f_1\mid_{f_1^{-1}(D^m\times P(r-1,s))}$ is trivial. Now if $\widehat{U}$ is the tubular neighbourhood of $D^m\times P(r-1,s)$ in $D^m\times P(r,s)$, then $D^m\times P(r,s)\backslash\widehat{U}=D^m\times D^r\times\mathbb{C}P^s$, so $D^m\times P(r,s)=\widehat{U}\cup D^m\times D^r\times\mathbb{C}P^s$. Similarly $\partial_+M=f_1^{-1}(\widehat{U})\cup f_1^{-1}(D^m\times D^r\times\mathbb{C}P^s)$, and f_1 gives simple homotopy equivalences on each piece. From the above observation $f_1\mid_{f_1^{-1}(\widehat{U})}\colon f_1^{-1}(\widehat{U})\to\widehat{U}$ is trivial. Also f_1 is a CAT homeomorphism along the common boundary of $f_1^{-1}(D^m\times D^r\times\mathbb{C}P^s)$ and $f_1^{-1}(\widehat{U})$. Hence the homotopy CAT structure f_1 is trivial if and only if the map

$$\hat{f}_1\overset{\mathrm{def}}{=}f_1\mid_{f_1^{-1}(D^m\times D^r\times\mathbb{C}P^s)}\colon f_1^{-1}(D^m\times D^r\times\mathbb{C}P^s)\to D^m\times D^r\times\mathbb{C}P^s$$

is trivial (i.e., a CAT homeomorphism).

Now $\hat{f}_1$ is a homotopy CAT structure, and

$$f\mid_{f^{-1}(D^m\times D^r\times\mathbb{C}P^{s-1}\times I)}\colon f^{-1}(D^m\times D^r\times\mathbb{C}P^{s-1}\times I)\to D^m\times D^r\times\mathbb{C}P^{s-1}\times I$$

is normal and realizes some element of the group $L_1(0)=0$. Thus the homotopy CAT structure

$$f\mid_{f^{-1}(D^m\times D^r\times\mathbb{C}P^{s-1})}=\hat{f}_1\mid_{\hat{f}_1{}^{-1}(D^m\times D^r\times\mathbb{C}P^{s-1})}$$

is trivial. Now let U_1 be a tubular neighbourhood of $D^m\times D^r\times\mathbb{C}P^{s-1}$ in $D^m\times D^r\times\mathbb{C}P^s$. Then $D^m\times D^r\times\mathbb{C}P^s\backslash U_1=D^m\times D^r\times D^{2s}$. Thus $D^m\times D^r\times\mathbb{C}P^s=U_1\cup D^m\times D^r\times D^{2s}$. Similarly one has $f_1^{-1}(D^m\times D^r\times\mathbb{C}P^s)=f_1^{-1}U_1\cup f_1^{-1}(D^m\times D^r\times D^{2s})$. $\hat{f}_1$ gives simple homotopy equivalence on each piece, and is a CAT-homeomorphism on the boundary, hence is a homotopy CAT structure. Now the set $hT_{\mathrm{CAT}}(D^m\times D^r\times D^{2s})$ consists of the identity element only, therefore, the homotopy CAT structure $\hat{f}_1$ on each piece of the above decomposition is trivial, and hence $\hat{f}_1$ is trivial. Hence the homotopy CAT structure f_1 is trivial., and we get that $\mathbb{Z}/2=L_3((\mathbb{Z}/2)^+)\subseteq\operatorname{Ker}\delta_{\mathrm{CAT}}$. Therefore $\operatorname{Ker}\delta_{\mathrm{CAT}}=\mathbb{Z}/2$. □

5.5. Proposition. *Let $n=\dim X\equiv 3\pmod 4$, $r+s+1$ is odd (so X is non-orientable). Then the action δ_{CAT} of the group $L_4((\mathbb{Z}/2)^-)=\mathbb{Z}/2$ on $hT_{\mathrm{CAT}}(X)$ is trivial.*

Proof. Realize the nontrivial element of the group $L_4((\mathbb{Z}/2)^-)=\mathbb{Z}/2$ by a normal map $f:M\to D^m\times P(r,s)\times I$ such that

$$f\mid_{\partial_-M}\colon\partial_-M\to D^m\times P(r,s)\times 0\cup\partial(D^m\times P(r,s))\times I$$

is a CAT-homeomorphism (CAT = PL, or TOP). The map $\partial=r\circ t:L_4((\mathbb{Z}/2)^-)\to L_2(0)$ is zero by (3.2), so by the relation between ∂ and the Browder-Livesay invariant mentioned in Section 3 we get that the homotopy CAT structure $f\mid_{\partial_+M}\colon\partial_+M\to D^m\times P(r,s)\times 1$ splits along the submanifold $D^m\times P(r-1,s)\times 1$, and gives a homotopy CAT structure of the manifold $D^m\times P(r-1,s)\times 1$. Denote the map $f\mid_{\partial_+M}$ by f_1.

The map

$$f\mid_{f^{-1}(D^m\times P(r-1,s)\times I)}\colon f^{-1}(D^m\times P(r-1,s)\times I)\to D^m\times P(r-1,s)\times I$$

is normal and defines an element of the group $L_3((\mathbb{Z}/2)^+)$, (the image of the map $L_4((\mathbb{Z}/2)^-) \to L_3((\mathbb{Z}/2)^+)$, referring to the second vertical map of the diagram (D4)). This element is nontrivial. Moreover,

$$\begin{aligned} &f\mid_{f^{-1}(D^m\times P(r-1,s)\times 0\cup\partial(D^m\times P(r-1,s))\times I)}\colon \\ &\qquad f^{-1}(D^m \times P(r-1,s)\times 0 \cup \partial(D^m \times P(r-1,s))\times I) \\ &\qquad\qquad \to D^m \times P(r-1,s)\times 0 \cup \partial(D^m \times P(r-1,s))\times I \end{aligned}$$

is a CAT-homeomorphism (CAT = PL, or TOP). So, by Theorem (5.4) we get that

$$f\mid_{f^{-1}(D^m\times P(r-1,s)\times 1)}\colon f^{-1}(D^m \times P(r-1,s)\times 1) \to D^m \times P(r-1,s)\times 1$$

is trivial (that is a CAT-homeomorphism).

Now, if U is a tubular neighbourhood of $D^m \times P(r-1,s)$ in $D^m \times P(r,s)$, then $D^m \times P(r,s)\backslash U$ is homeomorphic to $D^m \times D^r \times \mathbb{C}P^s$, or $D^m \times P(r,s) = U \cup D^m \times D^r \times \mathbb{C}P^s$. Similarly $\partial_+ M = f_1^{-1}(U) \cup f_1^{-1}(D^m \times D^r \times \mathbb{C}P^s)$, and f_1 gives simple homotopy equivalences on each piece. From the above observation $f_1\mid_{f_1^{-1}(\widehat{U})}\colon f_1^{-1}(\widehat{U}) \to \widehat{U}$ is trivial. Also f_1 is a CAT homeomorphism along the common boundary of $f_1^{-1}(D^m \times D^r \times \mathbb{C}P^s)$ and $f_1^{-1}(U)$. Hence the homotopy CAT structure f_1 is trivial if and only if the map

$$\hat{f}_1 \overset{\text{def}}{=} f_1\mid_{f_1^{-1}(D^m\times D^r\times\mathbb{C}P^s)}\colon f_1^{-1}(D^m \times D^r \times \mathbb{C}P^s) \to D^m \times D^r \times \mathbb{C}P^s$$

is trivial (i.e., a CAT homeomorphism).

Now $\hat{f}_1$ is a homotopy CAT structure, and

$$f\mid_{f^{-1}(D^m\times D^r\times\mathbb{C}P^{s-1}\times I)}\colon f^{-1}(D^m \times D^r \times \mathbb{C}P^{s-1}\times I) \to D^m \times D^r \times \mathbb{C}P^{s-1}\times I$$

is normal and realizes some element of the group $L_2(0)$. Thus, by Theorem (5.2), the homotopy CAT structure

$$f\mid_{f^{-1}(D^m\times D^r\times\mathbb{C}P^{s-1})} = \hat{f}_1\mid_{\hat{f}_1^{-1}(D^m\times D^r\times\mathbb{C}P^{s-1})}$$

is trivial. Now let U_1 be a tubular neighbourhood of $D^m \times D^r \times \mathbb{C}P^{s-1}$ in $D^m \times D^r \times \mathbb{C}P^s$. Then $D^m \times D^r \times \mathbb{C}P^s \backslash U_1 = D^m \times D^r \times D^{2s}$. Thus $D^m \times D^r \times \mathbb{C}P^s = U_1 \cup D^m \times D^r \times D^{2s}$. Similarly one has $f_1^{-1}(D^m \times D^r \times \mathbb{C}P^s) = f_1^{-1}U_1 \cup f_1^{-1}(D^m \times D^r \times D^{2s})$. $\hat{f}_1$ gives simple homotopy equivalence on each piece, and is a CAT-homeomorphism on the boundary, hence is a homotopy CAT structure. Now the set $hT_{\mathrm{CAT}}(D^m \times D^r \times D^{2s})$ consists of the identity element only, therefore, the homotopy CAT structure $\hat{f}_1$ on each piece of the above decomposition is trivial, and hence $\hat{f}_1$ is trivial. Hence the homotopy CAT structure f_1 is trivial., and we get that $\mathbb{Z}/2 = L_4((\mathbb{Z}/2)^-) \subseteq \operatorname{Ker}\delta_{\mathrm{CAT}}$. Therefore $\operatorname{Ker}\delta_{\mathrm{CAT}} = \mathbb{Z}/2$. That is δ_{CAT} is trivial. □

Thus in Propositions (5.1), (5.2), (5.3), (5.4), (5.5) the kernel of the action map δ_{CAT} in the Sullivan-Wall exact sequence for manifolds of the form $D^m \times P(r,s)$ has been calculated in all possible cases.

6. The surgery obstruction map $\theta_{\mathrm{CAT}} : [X/\partial X, G/\mathrm{CAT}] \to L_n((\mathbb{Z}/2)^{\pm})$

We first recall that if $X = D^m \times P(r,s)$, the map $\partial : L_{n+1}((\mathbb{Z}/2)^{\omega^X}) \to L_{n+\epsilon}(0)$ is defined as the composite:

$$L_{n+1}((\mathbb{Z}/2)^{\omega^X}) \xrightarrow{\delta_{\mathrm{CAT}}} hT_{\mathrm{CAT}}(D^m \times P(r,s)) \xrightarrow{\eta} \mathrm{BL}(D^m \times P(r,s), D^m \times P(r-1,s)),$$

6.1. Lemma. *Let $X = D^m \times P(r,s)$, $r, s > 1$, $\dim X = n+1$. If $x \in L_{n+1}((\mathbb{Z}/2)^{\omega^X})$ is realized by a normal map $F : M \to X$ which is a* CAT *homeomorphism on the boundary* (CAT = PL *or* TOP)*, then $\partial(x) = 0$.*

Proof. Let $X_1 = D^m \times P(r-1,s)$, then $\pi_1(X_1) = \pi_1(X)$, and $\omega^{X_1} = -\omega^X$.

Realize the element $-x$ by a normal map $f : N \to X_1 \times I$, such that $f |_{\partial_- N}: \partial_- N \to X_1 \times 0 \cup \partial X_1 \times I$ is a CAT homeomorphism (CAT = PL or TOP). By definition of ∂, the obstruction to splitting the homotopy CAT structure $f |_{\partial_+ N}: \partial_+ N \to X_1 \times 1$ along the submanifold $Y_1 = D^m \times P(r-2,s)$ is equal to $-\partial x$. Consider the connected sum of the manifolds M and N, and also the sum of X and $X_1 \times I$. The normal maps F and f define a normal map $F_1 : M\#N \to X\#X_1 \times I$. According to the construction of surgery obstructions the map F_1 is a simple homotopy equivalence, and considered as an element of the group $L_{n+1}((\mathbb{Z}/2)^{\omega^X})$, is equal to zero; but $\pi_1(X\#X_1 \times I) \neq \mathbb{Z}/2$. However, by Wall ([17]; Th. 9.4), one can change F_1 using simultaneous surgeries along 1-cycles in the manifolds $M\#N$ and $X\#X_1 \times I$, without changing the boundaries, which make the fundamental groups equal to $\mathbb{Z}/2$. We obtain as a result of theses surgeries a normal map $F_2 : M_2 \to X_2$. Since on one component of the boundary the map F_2 splits, it follows from ([10]; Lemma 1, Section 1.2.2) that F_2 splits on the other component of the boundary too. Therefore $\partial(x) = 0$. □

6.2. Proposition. *Let $X = D^m \times P(r,s)$, then in the group $L_4((\mathbb{Z}/2)^+)$ the normal maps $F : M \to X$, which are* CAT *homeomorphism* (CAT = PL *or* TOP) *on the boundary, realize the elements of the subgroup* $\mathrm{Im}[L_4(0) \to L_4((\mathbb{Z}/2)^+)]$ *and only these.*

Proof. The elements of the image of the group $L_4(0)$ is realized by maps $f : M^0\#X \to X$, where M^0 is a Milnor Plumbed manifold and f is the map which is identity on X and maps M^0 to a point.

The other elements of the group $L_4((\mathbb{Z}/2)^+)$ are not realized because the map $\partial = \eta \circ \delta_{\mathrm{CAT}}$ is different from zero (see 5.1). □

6.3. Proposition. *The groups $L_2((\mathbb{Z}/2)^{\pm})$ and $L_4((\mathbb{Z}/2)^-)$ are completely realized by normal maps of closed manifolds into the manifolds $X = P(r,s)$.*

Proof. In both cases the nontrivial element of the group $L_n((\mathbb{Z}/2)^{\omega^X})$ belongs to the image of the natural map $L_n(0) \to L_n((\mathbb{Z}/2)^{\omega^X})$, refer to the proofs of Propositions (5.2) and (5.5). Let x be the nontrivial element of the group $L_n((\mathbb{Z}/2)^{\omega^X})$ which can be considered an element of $L_n(0)$, and let the latter be realized by a normal map $F_1 : M_1 \to X_1 = D^r \times \mathbb{C}P^s$, (which is possible since $D^r \times \mathbb{C}P^s = I \times D^{r-1} \times \mathbb{C}P^s$) and such that

$$F_1 |_{\partial_- M_1}: \partial_- M_1 \to 0 \times D^{r-1} \times \mathbb{C}P^s \cup I \times S^{r-2} \times \mathbb{C}P^s$$

is a CAT homeomorphism (CAT = PL or TOP). We show that the map $F_1|_{\partial M_1}\colon \partial M_1 \to S^{r-1} \times \mathbb{C}P^s$ can be assumed to be a CAT homeomorphism. Indeed, if $\dim X_1 \equiv 2 \pmod 4$, then for M_1 we can take the manifold $X_1 \# K$, where K is a Kervaire manifold, and for F_1 we can take the map which is identity on X_1 and takes K to a point. In the case $\dim X_1 \equiv 0 \pmod 4$, this is essentially the proof of the first part of Proposition (5.5). This makes it possible to 'glue' X_1 to $X = P(r,s)$ and in exactly the same way to 'glue' M_1 to some manifold M, giving us a normal map $F : M \to X$ realizing the element x. (Here the words 'glue' have the following meaning: Let $P(r-1,s) \subset P(r,s) = X$ have the tubular neighbourhood U. Then $X \backslash U = X_1$. So $X = U \cup X_1$ with $U \cap X_1 = S^{r-1} \times \mathbb{C}P^s$ and $F_1|_{\partial M_1}\colon \partial M_1 \to S^{r-1} \times \mathbb{C}P^s$ is a CAT homeomorphism. Thus $F_1 : M_1 \to X_1$ extends to $F : M = M_1 \cup_{F_1|_{\partial M_1}} U \to X_1 \cup U = X$, where $F|_U\colon U \to U$ is identity.) □

6.4. Proposition. *The group $L_3((\mathbb{Z}/2)^+) \cong \mathbb{Z}/2$ is realized by normal maps $F : M \to D^m \times P(r,s)$ which are* CAT *homeomorphisms* (CAT = PL *or* TOP) *on the boundary, $r, s > 1$.*

Proof. This follows directly from Proposition (5.4) and Lemma (6.1). □

Thus in Propositions (6.2), (6.3), and (6.4) we have found the image of the surgery obstruction map θ_{CAT} in the CAT version of Sullivan-Wall surgery exact sequence (CAT = PL or TOP) for the manifolds $D^m \times P(r,s)$.

7. PL and TOP classification theorems and remarks on homotopy smoothings

Owing to the existence of natural maps of the smooth version of Sullivan Wall surgery exact sequence to the CAT versions of Sullivan-Wall surgery exact sequences (CAT = PL or TOP):

(D6)

$$\begin{tikzcd}
L_{n+1}(\pi_1(X), w(X)) \arrow[r, "\delta_{\mathrm{O}}"] \arrow[d, equal] & hS(X) \arrow[r, "\eta_{\mathrm{O}}"] \arrow[d] & {[X/\partial X, G/\mathrm{O}]} \arrow[r, "\theta_{\mathrm{O}}"] \arrow[d] & L_n(\pi_1(X), w(X)) \arrow[d, equal] \\
L_{n+1}(\pi_1(X), w(X)) \arrow[r, "\delta_{\mathrm{CAT}}"] & hT_{\mathrm{CAT}}(X) \arrow[r, "\eta_{\mathrm{CAT}}"] & {[X/\partial X, G/\mathrm{CAT}]} \arrow[r, "\theta_{\mathrm{CAT}}"] & L_n(\pi_1(X), w(X)),
\end{tikzcd}$$

one can draw many conclusions from the results of Section 5 (Propositions (5.1), (5.2), (5.3), (5.4), and (5.5)) and Section 6 (Propositions (6.2), (6.3), and (6.4)).

7.1. Proposition. *Let $X = D^m \times P(r,s)$, $n = \dim X \equiv 3 \pmod 4$, $r, s > 1$, $r+s+1$ even. So X is orientable. Then the action δ_O of the group $L_4((\mathbb{Z}/2)^+) = \mathbb{Z} \oplus \mathbb{Z}$ on $hS(X)$ is trivial when restricted to elements of a subgroup of $\mathbb{Z} = \mathrm{Im}[L_4(0) \to L_4((\mathbb{Z}/2)^+)]$ and is nontrivial otherwise.*

7.2. Proposition. *Let $D^m = *$, and $n = \dim X \equiv 0 \pmod 2$ and X is not orientable. Then $L_1((\mathbb{Z}/2)^-) = 0$ and $L_3((\mathbb{Z}/2)^-) = 0$, so the actions δ_O of these groups on $hS(P(r,s))$ are trivial.*

7.3. Proposition. *Let $X = D^m \times P(r,s)$. then in the group $L_4((\mathbb{Z}/2)^+)$ the normal maps $F : M \to X$, which are diffeomorphism on the boundary, realize the elements of a subgroup of* $\mathrm{Im}[L_4(0) \to L_4((\mathbb{Z}/2)^+)]$.

7.4. Proposition. *The groups $L_2((\mathbb{Z}/2)^{\pm})$ and $L_4((\mathbb{Z}/2)^{-})$ are completely realized by normal maps of closed smooth manifolds into the manifolds $X = P(r,s)$.*

Finally we summarize the calculations made in the previous sections in the form of the following:

7.5. Theorem (Classification Theorem 1)**.** *Let $X = D^m \times P(r,s)$, $r,s > 1$, where D^m is an m-dimensional disk. Then there are following exact sequences* (CAT = PL *or* TOP):

(1) *If* $\dim X \equiv 3 \pmod 4$, $r+s+1$ *even, that is X is orientable, then*

$$\to L_4((\mathbb{Z}/2)^+) \overset{\delta_{\mathrm{CAT}}}{\to} hT_{\mathrm{CAT}}(X) \overset{\eta_{\mathrm{CAT}}}{\to} [X/\partial X, G/\mathrm{CAT}] \overset{\theta_{\mathrm{CAT}}}{\to} L_3((\mathbb{Z}/2)^+),$$

reduces to

$$0 \to \mathbb{Z} \overset{\delta_{\mathrm{CAT}}}{\to} hT_{\mathrm{CAT}}(X) \overset{\eta_{\mathrm{CAT}}}{\to} [X/\partial X, G/\mathrm{CAT}] \overset{\theta_{\mathrm{CAT}}}{\to} \mathbb{Z}/2 \to 0.$$

(2) *If* $\dim X \equiv 2 \pmod 4$, $r+s+1$ *even, that is X is orientable, then*

$$\to L_3((\mathbb{Z}/2)^+) \overset{\delta_{\mathrm{CAT}}}{\to} hT_{\mathrm{CAT}}(X) \overset{\eta_{\mathrm{CAT}}}{\to} [X/\partial X, G/\mathrm{CAT}] \overset{\theta_{\mathrm{CAT}}}{\to} L_2((\mathbb{Z}/2)^+),$$

reduces to

$$0 \overset{\delta_{\mathrm{CAT}}}{\to} hT_{\mathrm{CAT}}(X) \overset{\eta_{\mathrm{CAT}}}{\to} [X/\partial X, G/\mathrm{CAT}] \overset{\theta_{\mathrm{CAT}}}{\to} \mathbb{Z}/2 \to 0.$$

(3) *If* $\dim X \equiv 0 \pmod 4$, $r+s+1$ *even, that is X is orientable, then*

$$\to L_1((\mathbb{Z}/2)^+) \overset{\delta_{\mathrm{CAT}}}{\to} hT_{\mathrm{CAT}}(X) \overset{\eta_{\mathrm{CAT}}}{\to} [X/\partial X, G/\mathrm{CAT}] \overset{\theta_{\mathrm{CAT}}}{\to} L_0((\mathbb{Z}/2)^+),$$

reduces to

$$0 \overset{\delta_{\mathrm{CAT}}}{\to} hT_{\mathrm{CAT}}(X) \overset{\eta_{\mathrm{CAT}}}{\to} [X/\partial X, G/\mathrm{CAT}] \overset{\theta_{\mathrm{CAT}}}{\to} \mathbb{Z} \to 0.$$

(4) *If* $\dim X \equiv 0 \pmod 2$, $r+s+1$ *odd, that is X is non orientable, then*

$$\to L_{\mathrm{odd}}((\mathbb{Z}/2)^-) \overset{\delta_{\mathrm{CAT}}}{\to} hT_{\mathrm{CAT}}(X) \overset{\eta_{\mathrm{CAT}}}{\to} [X/\partial X, G/\mathrm{CAT}] \overset{\theta_{\mathrm{CAT}}}{\to} L_{even}((\mathbb{Z}/2)^-),$$

reduces to

$$0 \overset{\delta_{\mathrm{CAT}}}{\to} hT_{\mathrm{CAT}}(X) \overset{\eta_{\mathrm{CAT}}}{\to} [X/\partial X, G/\mathrm{CAT}] \overset{\theta_{\mathrm{CAT}}}{\to} \mathbb{Z}/2 \to 0.$$

(5) *If* $\dim X \equiv 1 \pmod 4$, *X is orientable or non orientable, then*

$$\to L_2((\mathbb{Z}/2)^{\pm}) \overset{\delta_{\mathrm{CAT}}}{\to} hT_{\mathrm{CAT}}(X) \overset{\eta_{\mathrm{CAT}}}{\to} [X/\partial X, G/\mathrm{CAT}] \overset{\theta_{\mathrm{CAT}}}{\to} L_1((\mathbb{Z}/2)^{\pm}),$$

reduces to

$$0 \overset{\delta_{\mathrm{CAT}}}{\to} hT_{\mathrm{CAT}}(X) \overset{\eta_{\mathrm{CAT}}}{\to} [X/\partial X, G/\mathrm{CAT}] \overset{\theta_{\mathrm{CAT}}}{\to} 0.$$

(6) *If* $\dim X \equiv 3 \pmod 4$, $r+s+1$ *odd, that is X is non orientable, then*

$$\to L_4((\mathbb{Z}/2)^-) \overset{\delta_{\mathrm{CAT}}}{\to} hT_{\mathrm{CAT}}(X) \overset{\eta_{\mathrm{CAT}}}{\to} [X/\partial X, G/\mathrm{CAT}] \overset{\theta_{\mathrm{CAT}}}{\to} L_3((\mathbb{Z}/2)^-),$$

reduces to

$$0 \overset{\delta_{\mathrm{CAT}}}{\to} hT_{\mathrm{CAT}}(X) \overset{\eta_{\mathrm{CAT}}}{\to} [X/\partial X, G/\mathrm{CAT}] \overset{\theta_{\mathrm{CAT}}}{\to} 0.$$

Proof. Combining Propositions (5.1), (5.2), (5.3, (5.4), (5.5), and Propositions (6.2), (6.3), (6.4) we get the result. □

This theorem and Theorems (4.4) and (4.5) together determine $hT_{\mathrm{CAT}}(P(r,s))$ completely, where CAT = PL or TOP, once we analyze the maps θ_{CAT}, η_{CAT}, and δ_{CAT} bit more closely:

7.6. Theorem (Classification Theorem 2)**.** *Consider Dold manifolds $P(r,s)$ with $r,s>1$, $r+2s=4k+j$, $j=1,\dots,4$. Then for $k\geq 1$:* (*Coefficients of integral cohomologies are dropped*)

$$\text{(1(i))}\qquad hT_{\mathrm{TOP}}(P(r,s)^{4k+1})\cong\sum_{i=2}^{k}H^{4i-2}(P(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(P(r,s))$$

(1(ii)) $hT_{\mathrm{PL}}(P(r,s)^{4k+1})\cong$

$$\begin{cases}(\mathbb{Z}\oplus\mathbb{Z}/4)\oplus\sum_{i=2}^{k}H^{4i-2}(P(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(P(r,s)) & \text{if } r\geq 4,\ s\geq 2,\\ (\mathbb{Z}\oplus\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(P(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(P(r,s)) & \text{if } r=3,\ s\geq 2,\\ (\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(P(r,s);\mathbb{Z}/2) & \\ \qquad\oplus\sum_{i=2}^{k}H^{4i}(P(r,s)) & \text{if } r=2,\ s\geq 2,\end{cases}$$

$$\text{(2(i))}\qquad hT_{\mathrm{TOP}}(P(r,s)^{4k+2})\cong\sum_{i=2}^{k}H^{4i-2}(P(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(P(r,s))$$

(2(ii)) $hT_{\mathrm{PL}}(P(r,s)^{4k+2})\cong$

$$\begin{cases}(\mathbb{Z}\oplus\mathbb{Z}/4)\oplus\sum_{i=2}^{k}H^{4i-2}(P(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(P(r,s)) & \text{if } r\geq 4,\ s\geq 2,\\ (\mathbb{Z}\oplus\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(P(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i}(P(r,s)) & \text{if } r=3,\ s\geq 2,\\ (\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2)\oplus\sum_{i=2}^{k}H^{4i-2}(P(r,s);\mathbb{Z}/2) & \\ \qquad\oplus\sum_{i=2}^{k}H^{4i}(P(r,s)) & \text{if } r=2,\ s\geq 2,\end{cases}$$

$$\text{(3(i))}\quad hT_{\mathrm{TOP}}(P(r,s)_{+}^{4k+3})\cong\mathbb{Z}\oplus\sum_{i=2}^{k}H^{4i-2}(P(r,s);\mathbb{Z}/2)\oplus\mathbb{Z}/2\oplus\sum_{i=2}^{k}H^{4i}(P(r,s))$$

$$(3(\mathrm{ii}))\quad hT_{\mathrm{PL}}(P(r,s)^{4k+3}_{+}) \cong \begin{cases} (\mathbb{Z}\oplus\mathbb{Z}/4)\oplus\mathbb{Z}\oplus\sum\limits_{i=2}^{k} H^{4i-2}(P(r,s);\mathbb{Z}/2)\oplus\mathbb{Z}/2 \\ \qquad \oplus\sum\limits_{i=2}^{k} H^{4i}(P(r,s)) & \text{if } r\geq 4,\ s\geq 2, \\ (\mathbb{Z}\oplus\mathbb{Z}/2)\oplus\mathbb{Z}\oplus\sum\limits_{i=2}^{k} H^{4i-2}(P(r,s);\mathbb{Z}/2)\oplus\mathbb{Z}/2 \\ \qquad \oplus\sum\limits_{i=2}^{k} H^{4i}(P(r,s)) & \text{if } r=3,\ s\geq 2, \\ (\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2)\oplus\mathbb{Z}\oplus\sum\limits_{i=2}^{k} H^{4i-2}(P(r,s);\mathbb{Z}/2) \\ \qquad \oplus\mathbb{Z}/2\oplus\sum\limits_{i=2}^{k} H^{4i}(P(r,s)) & \text{if } r=2,\ s\geq 2, \end{cases}$$

$$(4(\mathrm{i}))\qquad hT_{\mathrm{TOP}}(P(r,s)^{4k+3}_{-}) \cong \sum_{i=2}^{k+1} H^{4i-2}(P(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k} H^{4i}(P(r,s))$$

$$(4(\mathrm{ii}))\quad hT_{\mathrm{PL}}(P(r,s)^{4k+3}_{-}) \cong \begin{cases} (\mathbb{Z}\oplus\mathbb{Z}/4)\oplus\sum\limits_{i=2}^{k+1} H^{4i-2}(P(r,s);\mathbb{Z}/2) \\ \qquad \oplus\sum\limits_{i=2}^{k} H^{4i}(P(r,s)) & \text{if } r\geq 4,\ s\geq 2, \\ (\mathbb{Z}\oplus\mathbb{Z}/2)\oplus\sum\limits_{i=2}^{k+1} H^{4i-2}(P(r,s);\mathbb{Z}/2) \\ \qquad \oplus\sum\limits_{i=2}^{k} H^{4i}(P(r,s)) & \text{if } r=3,\ s\geq 2, \\ (\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2)\oplus\sum\limits_{i=2}^{k+1} H^{4i-2}(P(r,s);\mathbb{Z}/2) \\ \qquad \oplus\sum\limits_{i=2}^{k} H^{4i}(P(r,s)) & \text{if } r=2,\ s\geq 2, \end{cases}$$

$$(5(\mathrm{i}))\qquad hT_{\mathrm{TOP}}(P(r,s)^{4k+4}_{+}) \cong \sum_{i=2}^{k+1} H^{4i-2}(P(r,s);\mathbb{Z}/2)\oplus\sum_{i=2}^{k} H^{4i}(P(r,s))$$

(5(ii)) $hT_{\mathrm{PL}}(P(r,s)_+^{4k+4}) \cong$

$$\begin{cases} (\mathbb{Z}\oplus\mathbb{Z}/4)\oplus\sum_{i=2}^{k+1} H^{4i-2}(P(r,s);\mathbb{Z}/2) \\ \qquad \oplus\sum_{i=2}^{k} H^{4i}(P(r,s)) & \text{if } r\geq 4,\ s\geq 2, \\ (\mathbb{Z}\oplus\mathbb{Z}/2)\oplus\sum_{i=2}^{k+1} H^{4i-2}(P(r,s);\mathbb{Z}/2) \\ \qquad \oplus\sum_{i=2}^{k} H^{4i}(P(r,s)) & \text{if } r=3,\ s\geq 2, \\ (\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2)\oplus\sum_{i=2}^{k+1} H^{4i-2}(P(r,s);\mathbb{Z}/2) \\ \qquad \oplus\sum_{i=2}^{k} H^{4i}(P(r,s)) & \text{if } r=2,\ s\geq 2, \end{cases}$$

(6(i)) $$hT_{\mathrm{TOP}}(P(r,s)_-^{4k+4}) \cong \sum_{i=2}^{k} H^{4i-2}(P(r,s);\mathbb{Z}/2)\oplus(\mathbb{Z}/2)^2\oplus\sum_{i=2}^{k+1} H^{4i}(P(r,s))$$

(6(ii)) $hT_{\mathrm{PL}}(P(r,s)_-^{4k+4}) \cong$

$$\begin{cases} (\mathbb{Z}\oplus\mathbb{Z}/4)\oplus\sum_{i=2}^{k} H^{4i-2}(P(r,s);\mathbb{Z}/2)\oplus(\mathbb{Z}/2)^2 \\ \qquad \oplus\sum_{i=2}^{k+1} H^{4i}(P(r,s)) & \text{if } r\geq 4,\ s\geq 2, \\ (\mathbb{Z}\oplus\mathbb{Z}/2)\oplus\sum_{i=2}^{k} H^{4i-2}(P(r,s);\mathbb{Z}/2)\oplus(\mathbb{Z}/2)^2 \\ \qquad \oplus\sum_{i=2}^{k+1} H^{4i}(P(r,s)) & \text{if } r=3,\ s\geq 2, \\ (\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2)\oplus\sum_{i=2}^{k} H^{4i-2}(P(r,s);\mathbb{Z}/2) \\ \qquad \oplus(\mathbb{Z}/2)^2\oplus\sum_{i=2}^{k+1} H^{4i}(P(r,s)) & \text{if } r=2,\ s\geq 2, \end{cases}$$

Proof. Case (1(i),(ii)). This case is trivial from Theorem (7.5).

Case (2(i),(ii)). For $\dim P(r,s)\equiv 2 \pmod 4$, ≥ 6, $P(r,s)$ orientable or not. For example $\dim P(r,s)=4k+2,\ k\geq 1$.

In this case $\theta_{\mathrm{CAT}} : [(P(r,s))^{4k+2}, G/\mathrm{CAT}] \to \mathbb{Z}/2$ coincides with the projection map $\phi_{4k+2} : [P(r,s), G/\mathrm{CAT}] \to H^{4k+2}(P(r,s);\mathbb{Z}/2)=\mathbb{Z}/2$.

Case (3(i),(ii)). For $\dim P(r,s)\equiv 3 \pmod 4$, ≥ 6, $P(r,s)$ orientable, so r has to be necessarily odd, and s necessarily even. For example $\dim P(r,s)=4k+3,\ k\geq 1$.

In this case $i : P(r-1,s)\hookrightarrow P(r,s)$ induces

$$[(P(r,s))_+^{4k+3}, G/\mathrm{CAT}] \xrightarrow{\mathrm{i}^*} [(P(r-1,s))^{4k+2}, G/\mathrm{CAT}],$$

and $\theta_{\mathrm{CAT}} : [(P(r,s))^{4k+3}, G/\mathrm{CAT}] \to \mathbb{Z}/2$ coincides with i^* composed with the projection

$$\phi_{4k+2} : [(P(r-1,s))^{4k+2}, G/\mathrm{CAT}] \to H^{4k+2}((P(r-1,s))^{4k+2};\mathbb{Z}/2)=\mathbb{Z}/2.$$

Case (4(i),(ii)). This case is trivial from Theorem (7.5).

Case (5(i),(ii)). For $\dim P(r,s) \equiv 0 \pmod 4$, ≥ 6, $P(r,s)$ orientable (r has to be necessarily even, and s necessarily odd), for example $\dim P(r,s) = 4k+4$, $k \geq 1$.

In this case $\theta_{\mathrm{CAT}} : [(P(r,s))^{4k+4}, G/\mathrm{CAT}] \to \mathbb{Z}$ coincides with the projection $\phi_{4k+4} : [(P(r,s))^{4k+4}, G/\mathrm{CAT}] \to H^{4k+4}(P(r,s);\mathbb{Z}) = \mathbb{Z}$.

Case (6(i),(ii)). For $\dim P(r,s) \equiv 0 \pmod 4$, and $\geq 6, P(r,s)$ non orientable (r has to be necessarily even, and s necessarily also even), for example $\dim P(r,s) = 4k+4$, $k \geq 1$.

In this case $j : P(r-2,s) \hookrightarrow P(r,s)$ induces

$$[(P(r,s))^{4k+4}_{+}, G/\mathrm{CAT}] \xrightarrow{\mathrm{j}^*} [(P(r-2,s))^{4k+2}, G/\mathrm{CAT}],$$

and $\theta_{\mathrm{CAT}} : [(P(r,s))^{4k+4}, G/\mathrm{CAT}] \to \mathbb{Z}/2$ coincides with j^* composed with the projection

$$\phi_{4k+2} : [(P(r-2,s))^{4k+2}, G/\mathrm{CAT}] \to H^{4k+2}((P(r-2,s))^{4k+2}; \mathbb{Z}/2) = \mathbb{Z}/2.$$

□

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