

## Tensor products with bounded continuous functions

Dana P. Williams

ABSTRACT. We study the natural inclusions of  $C^b(X) \otimes A$  into  $C^b(X, A)$  and  $C^b(X, C^b(Y))$  into  $C^b(X \times Y)$ . In particular, excepting trivial cases, both these maps are isomorphisms only when  $X$  and  $Y$  are pseudocompact. This implies a result of Glicksberg showing that the Stone-Ćech compactification  $\beta(X \times Y)$  is naturally identified with  $\beta X \times \beta Y$  if and only if  $X$  and  $Y$  are pseudocompact.

### CONTENTS

1. Main results	70
2. Glicksberg’s theorem	72
3. Pseudocompact spaces	74
4. Proof of Theorem 3	75
References	77

Suppose that  $X$  is a locally compact Hausdorff space and that  $A$  is a  $C^*$ -algebra. The collection  $C^b(X, A)$  of bounded continuous  $A$ -valued functions on  $X$  is a  $C^*$ -algebra with respect to the supremum norm. (When  $A = \mathbf{C}$ , we write simply  $C^b(X)$ ). Elements in the algebraic tensor product  $C^b(X) \odot A$  will always be viewed as functions in  $C^b(X, A)$ , and the supremum norm on  $C^b(X, A)$  restricts to a  $C^*$ -norm on  $C^b(X) \odot A$ . Thus we obtain an injection  $\iota_1$  of the completion  $C^b(X) \otimes A$  into  $C^b(X, A)$ :

$$\iota_1 : C^b(X) \otimes A \hookrightarrow C^b(X, A),$$

and we can identify  $C^b(X) \otimes A$  with a subalgebra of  $C^b(X, A)$ . It is one of the fundamental examples in the theory that  $\iota_1$  is an isomorphism in the case that  $X$  is compact [8, Proposition B.16], and we want to investigate the general case here. Our main result identifies the range of  $\iota_1$  as those functions in  $C^b(X, A)$  whose range has compact closure. As a consequence we show — provided  $A$  is infinite dimensional — that  $\iota_1$  is an isomorphism if and only if  $X$  has the property that every

---

Received April 14, 2003.

*Mathematics Subject Classification.* Primary: 46L06; Secondary: 54D35, 54D20.

*Key words and phrases.* Tensor products,  $C^*$ -algebras, Stone-Ćech compactification, pseudo-compact.

continuous function on  $X$  is bounded. Such spaces are called *pseudocompact*. Since paracompact pseudocompact spaces are compact, it was tempting to make a blanket assumption of paracompactness. However, the arguments here use properties naturally associated to pseudocompactness rather than compactness, so it seemed worth the little bit of extra effort to include the general results. (Nevertheless, I have tried to organize the paper so that the niceties of noncompact pseudocompact spaces come at the end.) In fact, pseudocompactness arises again when we consider the case where  $A = C^b(Y)$  for a locally compact Hausdorff space  $Y$ . Then we are led to address the properties of the natural inclusion

$$\iota_2 : C^b(X, C^b(Y)) \hookrightarrow C^b(X \times Y).$$

In general, this map is an isomorphism when  $C^b(Y)$  is given the strict topology viewed as the multiplier algebra of  $C_0(Y)$  [1, Corollary 3.4]. Here, however, we are interested in the norm topology, and in Theorem 3 we show that  $\iota_2$  is an isomorphism if and — assuming that  $X$  is both infinite and pseudocompact — only if  $Y$  is pseudocompact. Combining these observations about  $\iota_1$  and  $\iota_2$  yields a well-known result about products of Stone-Ćech compactifications originally due to Glicksberg [5, Theorem 1] with simplified proofs given by Frolík [4] and Todd [10]. Recall that if  $X$  is a locally compact, then, following [3, §XI.8.2] for example, the Stone-Ćech compactification of  $X$  is a compact Hausdorff space  $\beta X$  together with a homeomorphism  $i^X$  of  $X$  onto a dense open subset of  $\beta X$  so that  $(\beta X, i^X)$  has the *extension property*: given any continuous map  $f$  of  $X$  into a compact Hausdorff space  $Y$ , there is a continuous map  $f^* : \beta X \rightarrow Y$  such that  $f = f^* \circ i^X$ . (The pair  $(\beta X, i^X)$  is unique up to the natural notion of equivalence.) In particular, if  $Y$  is locally compact Hausdorff, the extension property gives a surjection  $\varphi : \beta(X \times Y) \rightarrow \beta X \times \beta Y$ . Our results can be used to show that  $\varphi$  is a homeomorphism if and only if  $X$  and  $Y$  are pseudocompact (which is the special case of Glicksberg's result alluded to above).

## 1. Main results

Recall that a subset of a topological space is called *precompact* if its closure is compact.

**Theorem 1.** *If  $X$  is a locally compact Hausdorff space and if  $A$  is a  $C^*$ -algebra, then  $f \in C^b(X, A)$  is in  $C^b(X) \otimes A$  if and only if the range of  $f$ ,  $R(f) := \{f(x) : x \in X\}$ , is precompact.*

**Proof.** Suppose that  $f \in C^b(X) \otimes A$ . To show that  $R(f)$  is precompact, it will suffice to see that  $R(f)$  is totally bounded. So we fix  $\epsilon > 0$  and try to show that  $R(f)$  can be covered by finitely many  $\epsilon$ -balls. Choose  $\sum_{i=1}^n f_i \otimes a_i \in C^b(X) \odot A$  so that

$$\left\| f - \sum_{i=1}^n f_i \otimes a_i \right\|_{\infty} < \frac{\epsilon}{2}.$$

Since

$$\left\| \left( \sum_{i=1}^n f_i \otimes a_i \right)(x) - \left( \sum_{i=1}^n f_i \otimes a_i \right)(y) \right\| \leq \sum_{i=1}^n |f_i(x) - f_i(y)| \|a_i\|,$$

and since the image of each  $f \in C^b(X)$  is bounded, we can find points  $x_1, \dots, x_r \in X$  such that the open sets

$$U_k := \left\{ x \in X : \left\| \left( \sum_{i=1}^n f_i \otimes a_i \right)(x) - \left( \sum_{i=1}^n f_i \otimes a_i \right)(x_k) \right\| < \frac{\epsilon}{2} \right\}$$

cover  $X$ . For convenience, let  $b_k := \left( \sum_{i=1}^n f_i \otimes a_i \right)(x_k)$ , and let

$$B_\epsilon(b_k) = \{a \in A : \|a - b_k\| < \epsilon\},$$

be the ball of radius  $\epsilon$  at  $b_k$ . Now if  $x \in U_k$ , then

$$\begin{aligned} \left\| f(x) - \left( \sum_{i=1}^n f_i \otimes a_i \right)(x_k) \right\| &\leq \left\| f(x) - \left( \sum_{i=1}^n f_i \otimes a_i \right)(x) \right\| + \left\| \left( \sum_{i=1}^n f_i \otimes a_i \right)(x) - b_k \right\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since the  $U_k$  cover  $X$ , it follows that

$$R(f) \subset \bigcup_{k=1}^r B_\epsilon(b_k).$$

Thus  $R(f)$  is totally bounded, and  $\overline{R(f)}$  must be compact.

Now assume that  $\overline{R(f)}$  is compact and therefore totally bounded. Thus given  $\epsilon > 0$ , there are elements  $\{b_k\}_{k=1}^r \subset A$  such that

$$\overline{R(f)} \subset \bigcup_{k=1}^r B_\epsilon(b_k).$$

Let  $U_k := f^{-1}(B_\epsilon(b_k))$ . Then  $\mathcal{U} = \{U_k\}_{k=1}^r$  is an open cover of  $X$ . First, we suppose that there is a partition of unity on  $X$  subordinate to  $\mathcal{U}$ . That is, we assume that there are functions  $f_k \in C^b(X)$  with  $0 \leq f_k \leq 1$ ,  $\text{supp } f_k \subset U_k$  and  $\sum_{k=1}^r f_k(x) = 1$  for all  $x \in X$ . Then  $\sum_{k=1}^r f_k \otimes b_k \in C^b(X) \odot A$ , and for all  $x \in X$ , we have

$$\begin{aligned} \left\| \left( \sum_{k=1}^r f_k \otimes b_k \right)(x) - f(x) \right\| &= \left\| \sum_{k=1}^r f_k(x) b_k - \sum_{k=1}^r f_k(x) f(x) \right\| \\ &\leq \sum_{k=1}^r f_k(x) \|b_k - f(x)\| \\ &\leq \epsilon \sum_{k=1}^r f_k(x) = \epsilon. \end{aligned}$$

Thus, if such a partition of unity exists, then  $f$  belongs to the closure of  $C^b(X) \odot A$  and  $f \in C^b(X) \otimes A$  as required. If  $X$  were paracompact, then given any finite cover  $\mathcal{U}$ , there is a partition of unity  $\{f_k\}_{k=1}^r$  on  $X$  subordinate to  $\mathcal{U}$  [8, Proposition 4.34]. Since, out of stubbornness,  $X$  is not assumed to be paracompact, we will have to take advantage of the special nature of the covers  $\mathcal{U}$  involved and the extension property of the Stone-Ćech compactification  $\beta X$  of  $X$ . Since  $\overline{R(f)}$  is compact and  $f : X \rightarrow \overline{R(f)}$  is continuous, there is a continuous function  $F : \beta X \rightarrow \overline{R(f)}$  such that  $F \circ i^X = f$ . Let  $V_k := F^{-1}(B_\epsilon(b_k))$ . Then  $\mathcal{V} = \{V_k\}$  is a finite open cover of  $\beta X$ , and there is a partition of unity  $\{\varphi_k\}$  on  $\beta X$  subordinate to  $\mathcal{V}$ . Since

$U_k = (i^X)^{-1}(V_k)$ , it follows that if  $f_k := \varphi_k \circ i^X$ , then  $\{f_k\}$  is a partition of unity on  $X$  subordinate to  $\mathcal{U}$ . This completes the proof.  $\square$

**Corollary 2.** *Suppose that  $X$  is a locally compact Hausdorff space and that  $A$  is a  $C^*$ -algebra. If  $X$  is pseudocompact, then  $R(f)$  is compact for each  $f \in C^b(X, A)$ , and*

$$(1) \quad C^b(X) \otimes A = C^b(X, A)$$

(that is,  $\iota_1$  is an isomorphism). Conversely, if  $A$  is not finite dimensional, then (1) holds only if  $X$  is pseudocompact. If  $A$  is finite dimensional, then (1) always holds.

**Proof.** Suppose that  $X$  is pseudocompact. It is not hard to see that the continuous image of a paracompact space in a metric space is compact (Corollary 8). Thus  $R(f)$  is compact for each  $f \in C^b(X, A)$  and (1) follows from Theorem 1.

Now assume that  $A$  is infinite dimensional and that  $X$  is not pseudocompact. It follows from [9, Theorem 1.23], that the unit ball of  $A$  is not compact. Thus there is a sequence  $\{a_k\}_{k=1}^{\infty}$  of elements of  $A$  of norm at most one such that  $\{a_k\}_{k=1}^{\infty}$  is not totally bounded. Since  $X$  is not pseudocompact, there are precompact open sets  $U_n$  in  $X$  whose closures are locally finite and pairwise disjoint (Lemma 6). Fix  $x_n \in U_n$ . Then by Urysohn's Lemma (cf. [7, Proposition 1.7.5]), there are  $f_n \in C_c(X)$  such that  $0 \leq f_n \leq 1$ ,  $f_n(x_n) = 1$  and  $\text{supp } f_n \subset U_n$ . Since  $\{\overline{U_n}\}$  is locally finite,

$$f(x) := \sum_{n=1}^{\infty} f_n(x)a_n$$

defines a function  $f \in C^b(X, A)$  whose range contains  $\{a_n\}_{n=1}^{\infty}$ . Thus  $f$  is not in the range of  $\iota_1$ , and (1) does not hold.

Since every finite dimensional  $C^*$ -algebra is a direct sum of matrix algebras [6, Theorem 6.3.8] and since it is easy to see that  $C^b(X, M_n) \cong M_n(C^b(X))$ , the last assertion is an easy consequence of another standard example in the theory of tensor products:  $M_n(C^b(X)) \cong C^b(X) \otimes M_n$  [8, Proposition B.18].  $\square$

Now we want to consider the case where  $A$  is of the form  $C^b(Y)$  for some locally compact Hausdorff space  $Y$ , and investigate the natural inclusion  $\iota_2$ .

**Theorem 3.** *Suppose that  $X$  and  $Y$  are locally compact Hausdorff spaces. If  $Y$  is pseudocompact, then the natural inclusion  $\iota_2 : C^b(X, C^b(Y)) \hookrightarrow C^b(X \times Y)$  is an isomorphism. If  $X$  is pseudocompact and infinite, then  $\iota_2$  is an isomorphism only if  $Y$  is pseudocompact.*

We'll postpone the proof until §4 so that we can see how Glicksberg's result follows from Theorems 1 and 3.

## 2. Glicksberg's theorem

If  $Z$  is compact Hausdorff and if  $h : X \rightarrow Z$  is a homeomorphism of  $X$  onto a dense subset of  $Z$ , then  $h(X)$  is open in  $Z$  [3, §XI.8.3], and the extension property

of  $\beta X$  implies that there is a unique continuous surjection  $\varphi : \beta X \rightarrow Z$  such that  $\varphi \circ i^X = h$ . Equivalently, we get a commutative diagram of algebra homomorphisms

$$(2) \quad \begin{array}{ccc} & & C(\beta X) \\ & \nearrow i_*^X & \uparrow \varphi^* \\ C_0(X) & \xrightarrow{h_*} & C(Z), \end{array}$$

where, for example,

$$h_*(f)(z) = \begin{cases} 0 & \text{if } z \notin h(X), \text{ and} \\ f(x) & \text{if } z = h(x) \text{ for some } x \in X, \end{cases}$$

and  $\varphi^*(f)(z) = f(\varphi(z))$ . Note that  $\varphi^*$  is the unique homomorphism making (2) commute. In particular, if  $X$  and  $Y$  are locally compact spaces, then we have a commutative diagram

$$\begin{array}{ccc} & & \beta(X \times Y) \\ & \nearrow i^{X \times Y} & \downarrow \varphi \\ X \times Y & \xrightarrow{i^X \times i^Y} & \beta X \times \beta Y, \end{array}$$

and it is natural to ask when  $\varphi$  is a homeomorphism so that we can identify  $(\beta X \times \beta Y, i^X \times i^Y)$  with  $(\beta(X \times Y), i^{X \times Y})$ . Since  $\varphi$ , and hence  $\varphi^*$ , is unique and since the extension property of  $\beta X$  easily implies  $C^b(X) \cong C(\beta X)$ , we can find  $\varphi$  by combining the following natural maps:

$$\begin{array}{ccccc} & & C(\beta(X \times Y)) & \xrightarrow{\cong} & C^b(X \times Y) \\ & \nearrow i_*^{X \times Y} & \uparrow \varphi^* & & \uparrow \iota_2 \\ C_0(X \times Y) & & & & C^b(X, C^b(Y)) \\ & \searrow (i^X \times i^Y)_* & & & \uparrow \iota_1 \\ & & C(\beta X \times \beta Y) & \xrightarrow{\cong} C(\beta X) \otimes C(\beta Y) & \xrightarrow{\cong} C^b(X) \otimes C^b(Y). \end{array}$$

Thus  $\varphi^*$  is an isomorphism exactly when both  $\iota_1$  and  $\iota_2$  are surjective. Thus if both  $X$  and  $Y$  are infinite, so that, for example,  $A = C^b(Y)$  is infinite dimensional, then  $\iota_1$  is an isomorphism if and only if  $X$  is pseudocompact (Corollary 2), and if  $X$  is pseudocompact, then  $\iota_2$  is an isomorphism if and only if  $Y$  is pseudocompact (Theorem 3). Since  $\varphi^*$  is an isomorphism exactly when  $\varphi$  is a homeomorphism, we have proved a special case of Glicksberg's result [5, Theorem 1].

**Theorem 4** (Glicksberg). *Suppose that  $X$  and  $Y$  are infinite locally compact spaces. Then the natural map  $\varphi : \beta(X \times Y) \rightarrow \beta X \times \beta Y$  is a homeomorphism if and only if both  $X$  and  $Y$  are pseudocompact.*

Of course, Glicksberg considered arbitrary products and only assumed that  $X$  and  $Y$  are completely regular. We have dispensed with completely regular spaces out of prejudice, and the extension to arbitrary products is not difficult and is discussed in [10].

**Remark 5.** Note that even if one of  $X$  and  $Y$  fails to be pseudocompact, Theorem 4 does not preclude the possibility that  $\beta(X \times Y)$  and  $\beta X \times \beta Y$  are homeomorphic. It only asserts that the natural, and only useful, way to identify them fails. See [5, §6] for further thoughts on this.

### 3. Pseudocompact spaces

Before turning to the proof of Theorem 3, some preliminaries on pseudocompact spaces are in order. Of course, everything sketched here is well-known, but distilling the specifics from the literature could be tedious.

**Lemma 6.** *If  $X$  is locally compact and not pseudocompact, then there is a sequence  $\{V_n\}$  of nonempty open sets in  $X$  such that each point in  $X$  meets at most one  $\overline{V_n}$ .*

**Proof.** Since  $X$  is not pseudocompact, there is a unbounded continuous nonnegative real-valued function  $f$  on  $X$ . Thus there is a sequence  $\{x_n\} \subset X$  such that  $f(x_{n+1}) > f(x_n) + 1$  for all  $n$ . By composing with a continuous piecewise linear function on  $\mathbf{R}$ , we may as well assume that  $f(x_n) = n$ . Then we can let  $V_n := f^{-1}((n - \frac{1}{4}, n + \frac{1}{4}))$ .  $\square$

Recall that a family  $\{U_i\}$  of subsets of  $X$  is called *locally finite* if every point in  $X$  has a neighborhood meeting at most finitely many  $U_i$ .

**Proposition 7.** *A locally compact space  $X$  is pseudocompact if and only if every countable locally finite collection of nonempty open sets in  $X$  is finite.*

**Proof.** Let  $\{U_n\}$  be locally finite sequence of nonempty open sets in  $X$ . Fix  $x_n \in U_n$ . By Urysohn's Lemma, there is a continuous function  $f_n$  on  $X$  such that  $0 \leq f_n \leq 1$ ,  $f_n(x_n) = 1$  and  $\text{supp } f_n \subset U_n$ . Since  $\{U_n\}$  is locally finite,  $f = \sum f_n$  is continuous (and unbounded) on  $X$ . Thus  $X$  is not pseudocompact. This, together with Lemma 6, establishes the result.  $\square$

Since, by definition, every open cover of a paracompact space has a locally finite subcover, it follows immediately from Proposition 7 that a pseudocompact paracompact space is compact. Since metric spaces are paracompact and since the inverse image of a locally finite cover is locally finite, we obtain the following easy corollary of Proposition 7.

**Corollary 8.** *If  $X$  is pseudocompact and if  $f : X \rightarrow Y$  is a continuous function from  $X$  into a metric space  $Y$ , then  $f(X)$  is compact.*

Nevertheless, there certainly are locally compact noncompact pseudocompact spaces. For example, if  $\omega_1$  is the first uncountable ordinal, then  $[0, \omega_1)$  is a countably compact locally compact space in the order topology, and countably compact spaces are easily seen to be pseudocompact. Moreover, if  $\omega$  is the first countable ordinal, then  $[0, \omega_1] \times [0, \omega] \setminus \{(\omega_1, \omega)\}$  is pseudocompact and not even countably compact [3, §XI.3 Ex. 2]. More generally,  $X$  is pseudocompact if and only if the corona set  $\beta X \setminus X$  contains no nonempty closed  $G_\delta$  sets [5, p. 370]. This observation — together with an old result of Čech showing that closed  $G_\delta$  sets in the corona of a noncompact locally compact space always have cardinality at least that of the continuum [2, p. 835] — has a curious consequence. If  $x \in \beta X \setminus X$ , then  $\beta X \setminus \{x\}$  is always pseudocompact [5, pp. 380–1].

As it happens, the product of two locally compact pseudocompact spaces is pseudocompact [5, Theorem 4(a)]. We give a short proof of the special case of this we need in §4. Some care is called for as, in general, the product of (not necessarily locally compact) pseudocompact spaces need not be pseudocompact [3, p. 245].

**Lemma 9.** *Suppose that  $X$  and  $Y$  are locally compact Hausdorff spaces with  $X$  compact and  $Y$  pseudocompact. Then  $X \times Y$  is pseudocompact.*

**Proof.** It will suffice to prove that if  $\{U_n\}$  and  $\{V_n\}$  are infinite sequences of nonempty open sets in  $X$  and  $Y$ , respectively, then  $\{U_n \times V_n\}$  is not locally finite. Since  $Y$  is pseudocompact,  $\{V_n\}$  can't be locally finite and there is a  $y \in Y$  such that every neighborhood of  $y$  meets infinitely many  $V_n$ . Let

$$\Omega := \{(n, W) : n \in \mathbf{N}, W \text{ is a neighborhood of } y \text{ and } W \cap V_n \neq \emptyset\}.$$

It is easy to see that  $\Omega$  is directed [3, Definition X.1.2]. Thus if we choose  $y_{(n,W)} \in W \cap V_n$ , then  $\{y_\omega\}_{\omega \in \Omega}$  is a net in  $Y$  converging to  $y$ . On the other hand, if for each  $(n, W) \in \Omega$  we choose  $x_{(n,W)} \in U_n$ , then, since  $X$  is compact,  $\{x_\omega\}_{\omega \in \Omega}$  has a subnet  $\{x_{\omega_i}\}_{i \in I}$  which converges to some  $x \in X$ . Thus if  $U$  and  $V$  are neighborhoods of  $x$  and  $y$ , respectively, then  $\{(x_{\omega_i}, y_{\omega_i})\}$  is eventually in  $U \times V$ . Thus if  $\omega_i := (n_i, W_i)$ , then we eventually have  $(U \times V) \cap (U_{n_i} \times V_{n_i}) \neq \emptyset$ . It follows that every neighborhood of  $(x, y)$  eventually meets infinitely many  $U_n \times V_n$ . Thus  $X \times Y$  is pseudocompact.  $\square$

**Remark 10.** Using the observation that a space  $X$  is pseudocompact if and only if every bounded continuous function on  $X$  attains its maximum on  $X$ , it is not hard to see that Theorem 4 implies that the product of locally compact pseudocompact spaces is pseudocompact [5, Theorem 4(a)]. Let  $F \in C^b(X \times Y)$ . Then Theorem 4 implies that  $F$  has a unique continuous extension  $F^*$  to  $C(\beta X \times \beta Y)$ . Let  $f^*(x) := F(x, \cdot)$ . Since  $X$  is pseudocompact, there is a  $\bar{x} \in X$  such that  $\|f^*(\bar{x})\|_\infty = \sup_{x \in X} \|f^*(x)\|_\infty$ . But since  $Y$  is pseudocompact, there is a  $\bar{y} \in Y$  such that  $\|f(\bar{x})\|_\infty = f(\bar{x})(\bar{y})$ . Thus,  $F$  assumes its maximum at  $(\bar{x}, \bar{y})$ , and  $X \times Y$  is pseudocompact.

### 4. Proof of Theorem 3

If  $Y$  is compact, then the surjectivity of  $\iota_2$  is fairly standard. However, if  $Y$  is only pseudocompact, then the proof of surjectivity given here depends on a result of Frolík's [4, Lemma 1.3].<sup>1</sup> Since this result is the heart of the proof, and since the published version has some annoying typos, we include the proof here for completeness.

**Lemma 11** (Frolík). *Suppose that  $X$  and  $Y$  are locally compact Hausdorff spaces with  $Y$  pseudocompact. If  $f$  is a bounded real-valued function on  $X \times Y$ , then*

$$g(x) := \sup_{y \in Y} f(x, y)$$

*defines a continuous function on  $X$ .*

<sup>1</sup>This result is also essential to both Frolík's [4] and Todd's [10] proofs of Glicksberg's result.

**Proof.** Since continuity is a local property, we may as well assume that  $X$  is compact. Since  $g$  is the supremum of continuous functions on  $X$ , it is easy to see that  $g$  is lower semicontinuous; that is,  $\{x \in X : g(x) > b\}$  is open for all  $b \in \mathbf{R}$ . Thus it will suffice to see that  $g$  is also upper semicontinuous so that  $\{x \in X : g(x) < b\}$  is open for all  $b \in \mathbf{R}$ . If  $g$  is not upper semicontinuous, then there must be a  $x_0 \in X$  and a  $\epsilon > 0$  such that every neighborhood  $U$  of  $x_0$  contains a  $x$  such that  $g(x) > g(x_0) + 3\epsilon$ . In particular, there is a  $(x, y) \in U \times Y$  such that

$$(3) \quad f(x, y) > g(x_0) + 3\epsilon.$$

We claim that we can choose points  $(x_n, y_n)$ , and neighborhoods  $V_n$  of  $y_n$ ,  $U_n$  of  $x_n$  and  $U'_n$  of  $x_0$  such that:

- (a)  $\sup\{|f(x, y) - f(x', y')| : (x, y), (x', y') \in U_n \times V_n\} < \epsilon$ ,
- (b)  $\sup\{|f(x, y) - f(x', y')| : (x, y), (x', y') \in U'_n \times V_n\} < \epsilon$ ,
- (c)  $U'_{n+1} \subset U'_n$ ,
- (d)  $U_n \subset U'_{n-1}$  provided  $n \geq 2$ , and
- (e)  $f(x_n, y_n) > g(x_0) + 3\epsilon$ .

For convenience, let  $U'_0 = X$ . Assume we have chosen  $(x_i, y_i)$ ,  $V_i$ ,  $U_i$  and  $U'_i$  for  $i < n$ . Then by assumption we can choose  $(x_n, y_n) \in U'_{n-1} \times Y$  such that (e) holds. The existence of  $V_n$ ,  $U_n$  and  $U'_n$  such that (a)–(d) hold follows easily from the continuity of  $f$ .

Since  $X \times Y$  is pseudocompact (Lemma 9), there is a  $(\bar{x}, \bar{y}) \in X \times Y$  such that every neighborhood of  $(\bar{x}, \bar{y})$  meets infinitely many  $U_n \times V_n$ . Let  $U$  and  $V$  be neighborhoods of  $\bar{x}$  and  $\bar{y}$ , respectively, so that  $(x, y) \in U \times V$  implies

$$(4) \quad |f(x, y) - f(\bar{x}, \bar{y})| < \epsilon.$$

By assumption, there is a  $n$  such that there exists  $(x', y') \in (U_n \times V_n) \cap (U \times V)$ . Using (4) together with (a) and (e), we have

$$(5) \quad f(\bar{x}, \bar{y}) > g(x_0) + 2\epsilon.$$

On the other hand, there is a  $m > n$  such that there exists  $(x'', y'') \in (U_m \times V_m) \cap U \times V$ . But (c) and (d) imply that

$$U_m \subset U'_{m-1} \subset U'_n.$$

Thus  $(x'', y'') \in (U'_n \times V_n) \cap (U \times V)$ . Now (4) implies  $|f(x'', y'') - f(\bar{x}, \bar{y})| < \epsilon$ , while (b) implies  $|f(x'', y'') - f(x_0, y')| < \epsilon$ . Thus

$$f(\bar{x}, \bar{y}) < f(x_0, y') + \epsilon \leq g(x_0) + \epsilon.$$

This contradicts (3), and completes the proof.  $\square$

**Proof of Theorem 3.** Let  $F \in C^b(X \times Y)$  and define  $f : X \rightarrow C^b(Y)$  by  $f(x)(y) := F(x, y)$ . Notice that  $F$  is in the image of  $\iota_2$  if and only if  $f$  is continuous. If  $Y$  is pseudocompact and  $x_0 \in X$ , then

$$G(x, y) := |f(x, y) - f(x_0, y)|$$

is continuous on  $X \times Y$ . Frolík's Lemma 11 implies that

$$g(x) := \sup_{y \in Y} G(x, y) = \|f(x) - f(x_0)\|_\infty$$

is continuous on  $X$ . Since  $g(x_0) = 0$ ,  $f$  is continuous at  $x_0$ . Since  $x_0$  is arbitrary,  $\iota_2$  is surjective whenever  $Y$  is pseudocompact.

Now suppose that  $X$  is infinite and pseudocompact and that  $Y$  is not pseudocompact. Let  $\{x_n\}$  be an infinite sequence in  $X$ . A simple argument (cf. [3, §VII.2.4]) shows that there are precompact neighborhoods  $U_n$  of  $x_n$  such that  $\overline{U_n} \cap \overline{U_m} = \emptyset$  if  $n \neq m$ . Since  $Y$  is not pseudocompact, there is a sequence of nonempty precompact open sets  $\{V_n\}$  with pairwise disjoint closures such that each point in  $y$  has a neighborhood meeting at most one  $\overline{V_n}$  (Lemma 6). It follows that

$$\bigcup \overline{U_n} \times \overline{V_n}$$

is  $\sigma$ -compact and closed in  $X \times Y$ . Fix  $y_n \in V_n$ . Then  $\{(x_n, y_n)\}$  is also closed in  $X \times Y$ . Since  $\sigma$ -compact locally compact Hausdorff spaces are paracompact, Urysohn's Lemma implies there is a continuous function  $F$  on  $\bigcup \overline{U_n} \times \overline{V_n}$  such that  $0 \leq F \leq 1$ ,  $F(x_n, y_n) = 1$  for all  $n$  and  $\text{supp } F \subset \bigcup U_n \times V_n$ . We can extend  $F$  to a bounded continuous function on  $X \times Y$  by letting it be identically zero on the complement of  $\bigcup U_n \times V_n$ .

As above, let  $f(x) = F(x, \cdot)$ . If  $F$  were in the image of  $\iota_2$ , then  $x \mapsto f(x)$  would be continuous. Since  $X$  is pseudocompact,  $\{f(x)\}_{x \in X}$  is compact in  $C^b(Y)$  (Corollary 8). In particular,  $\{f(x_n)\}$  has a convergent subsequence  $\{f(x_{n_i})\}$ . Then there is a  $k$  such that  $i \geq k$  implies

$$(6) \quad \|f(x_{n_i}) - f(x_{n_k})\|_\infty < \frac{1}{2}.$$

Of course we can choose  $i \geq k$  such that  $n_i > n_k$ . Then  $(x_{n_i}, y_{n_k}) \notin \bigcup U_n \times V_n$  and

$$\begin{aligned} f(x_{n_i})(y_{n_k}) - f(x_{n_k})(y_{n_k}) &= F(x_{n_i}, y_{n_k}) - F(x_{n_k}, y_{n_k}) \\ &= 0 - 1 = -1, \end{aligned}$$

and this contradicts (6). This completes the proof. □

## References

- [1] Charles A. Akemann, Gert K. Pedersen, and Jun Tomiyama, *Multipliers of  $C^*$ -algebras*, J. Functional Analysis **13** (1973), 277–301, [MR 57 #10431](#), [Zbl 0258.46052](#).
- [2] Eduard Čech, *On bicomact spaces*, Ann. of Math. **38** (1937), 823–844, [Zbl 0017.42803](#).
- [3] James Dugundji, *Topology*, Allyn and Bacon Inc., Boston, Mass., 1966, [MR 33 #1824](#), [Zbl 0144.21501](#).
- [4] Zdeněk Frolík, *The topological product of two pseudocompact spaces*, Czechoslovak Math. J **10(85)** (1960), 339–349, [MR 22 #7099](#), [Zbl 0099.38503](#).
- [5] Irving Glicksberg, *Stone-Čech compactifications of products*, Trans. Amer. Math. Soc. **90** (1959), 369–382, [MR 21 #4405](#), [Zbl 0089.38702](#).
- [6] Gerard J. Murphy,  *$C^*$ -algebras and operator theory*, Academic Press Inc., Boston, MA, 1990, [MR 91m:46084](#), [Zbl 0714.46041](#).
- [7] Gert K. Pedersen, *Analysis now*, Graduate Texts in Mathematics, vol. 118, Springer-Verlag, New York, 1989, [MR 90f:46001](#), [Zbl 0668.46002](#).
- [8] Iain Raeburn and Dana P. Williams, *Morita equivalence and continuous-trace  $C^*$ -algebras*, Mathematical Surveys and Monographs, vol. 60, American Mathematical Society, Providence, RI, 1998, [MR 2000c:46108](#), [Zbl 0922.46050](#).
- [9] Walter Rudin, *Functional analysis*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York, 1973, [MR 51 #1315](#), [Zbl 0253.46001](#).
- [10] Christopher Todd, *On the compactification of products*, Canad. Math. Bull. **14** (1971), 591–592, [MR 46 #9935](#), [Zbl 0224.54027](#).

DEPARTMENT OF MATHEMATICS, BRADLEY HALL, HANOVER, NH 03755-3551, USA  
[dana.williams@dartmouth.edu](mailto:dana.williams@dartmouth.edu) <http://www.math.dartmouth.edu/~dana>

This paper is available via <http://nyjm.albany.edu:8000/j/2003/9-5.html>.