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An ergodic sum related to the approximation by continued fractions

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Abstract. To each irrational number x is associated an infinite sequence of rational fractions $\frac{p_n}{q_n}$, known as the convergents of x. Consider the functions $q_n|q_n x - p_n| = \theta_n(x)$. We shall primarily be concerned with the computation, for almost all real x, of the ergodic sum

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log\theta_k(x) = -1 - \frac{1}{2}\log 2 \approx -1.34657.$$

Each irrational number x has a unique infinite, regular continued fraction expansion of the form

$$x = [a_1; a_2, a_3 \dots] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}$$

where the a_i are integers and $a_i > 0$ for $i > 1$. To x is associated an infinite sequence of rational fractions $\frac{p_n}{q_n} = [a_1; a_2, \dots, a_n]$, in lowest terms, known as the convergents of x. Define the functions $\theta_n(x)$ by the identity

$$\left| x - \frac{p_n}{q_n} \right| = \frac{\theta_n(x)}{q_n^2}.$$

Important metrical results on the $\theta_n(x)$ are proved in the papers [3],[5] and [7]. Since the convergents satisfy the following well-known inequality, usually attributed to Dirichlet,

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2},$$

we have $0 < \theta_n(x) < 1$.

It does not seem to have been observed that for almost all x

$$\lim_{n\to\infty} \frac{\log\theta_n(x)}{n} = 0. \tag{1}$$

To begin we supply a proof of this fact which was suggested by A. Rockett.

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From the sequence of inequalities (see [9]),

$$\frac{1}{q_n(q_n+q_{n+1})} \leq \left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}}$$

we get

$$\frac{1}{n}\log\left(\frac{q_n}{q_n+q_{n+1}}\right) \leq \frac{1}{n}\log\theta_n(x) \leq \frac{1}{n}\log\left(\frac{q_n}{q_{n+1}}\right).$$

The result then follows easily from the Khintchine–Lévy Theorem, which asserts that for almost all x

$$\lim_{n\to\infty}\frac{1}{n}\log q_n = \frac{\pi^2}{12\log 2}, \qquad \text{(see [2] or [9]).} \tag{2}$$

Now we look at the limiting average of the functions $\log\theta_n(x)$. While this average resembles those, such as $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\theta_k(x)$, computed in [3] and [5], its evaluation is complicated by the fact that $\log x$ is not continuous on the interval [0,1]. As a result, knowledge of the distribution function for $\theta_n(x)$ is not sufficient to prove the theorem. As in [3], we work with a form of the natural automorphic extension of the Gauss transform, derived from the extension originally given by Nakada [8].

Theorem 1. *For almost all x*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log\theta_k(x) = -1 - \frac{1}{2}\log 2 \approx -1.34657.$$

Let $\Lambda = ((0,1)\setminus\mathbb{Q})\times[0,1]$ and define the map $\widetilde{S}:\Lambda\to\Lambda$ by

$$\widetilde{S}(s,t) = \left(\frac{1}{s} - \left[\frac{1}{s}\right], \frac{1}{t+[\frac{1}{s}]}\right)$$

where $[x]$ is the greatest integer function. Let ν be the probability measure with density $m(s,t) = \frac{1}{\log 2}(1+st)^{-2}$. It was first observed by Nakada [8] that the dynamical system $(\Lambda, B, \nu, \widetilde{S})$ is ergodic. See also [1].

Consider the related self-mapping $\widetilde{T}$ of $\Omega = ((0,1)\setminus\mathbb{Q})\times[-\infty,-1]$ defined by

$$\widetilde{T}(x,y) = \left(\frac{1}{x} - \left[\frac{1}{x}\right], \frac{1}{y} - \left[\frac{1}{x}\right]\right).$$

Let $\varphi:\Lambda\to\Omega$ be the invertible function given by $\varphi(s,t) = (s, -\frac{1}{t})$. It is clear that φ maps Λ onto Ω and that $\widetilde{T} = \varphi\circ\widetilde{S}\circ\varphi^{-1}$.

The measure $\mu = \varphi^*\nu$ is defined by

$$\mu(D) = \frac{1}{\log}\int_{\varphi^{-1}(D)} \frac{1}{(1+st)^2} dsdt,$$

where D is a borel subset of Ω. It follows by an application of the chain rule that μ has the density $p(x,y) = \frac{1}{\log 2}(x-y)^{-2}$. As constructed, μ is invariant under the action of $\widetilde{T}$ and φ defines an isomorphism between the dynamical systems $(\Lambda, B, \nu, \widetilde{S})$ and $(\Omega, B, \mu, \widetilde{T})$. It follows that $(\Omega, B, \mu, \widetilde{T})$ is an ergodic dynamical system. The ergodicity of $\widetilde{T}$ is central to the proof of Theorem 1 that follows.

Let $T(x) = \frac{1}{x} - [\frac{1}{x}]$ be the classical Gauss map and let $\pi(x_1, x_2) = x_1$ be projection on the first factor. Then independent of y, $\pi \circ \widetilde{T}(x,y) = T(x)$ and $\pi^*(\mu)$ is the classical Gauss measure, which is an invariant measure for T, with density $g(x) = \frac{1}{\log 2}\frac{1}{1+x}$.

For irrational $x = [0; a_1, a_2 \dots]$, the Gauss map T acts as a shift on continued fraction expansions with $T(x) = [0; a_2, a_3 \dots]$. Even when x is rational, T acts as a shift on the finite continued fraction expansion and the iterates are defined until $T^n(x) = 0$. We assume henceforth that $x = [0; a_1, a_2 \dots]$ is irrational. The iterates $\widetilde{T}^n(x)$ are then defined for all positive integers n. If $y = -[a_{-1}; a_{-2}, \dots] \in (-\infty, -1]$, with a possibly finite continued fraction expansion and x is as above then $\widetilde{T}^n(x,y) = (\hat{x}, \hat{y})$ where $\hat{x} = [0; a_{n+1}, a_{n+2} \dots]$ and $\hat{y} = -[a_n; a_{n-1}, a_{n-2}, \dots, a_1, a_{-1} \dots]$.

Define the function

$$F(x,y) = \log\left(\frac{1}{x} - \frac{1}{y}\right) = \log\left(\frac{y-x}{xy}\right).$$

We show that F is integrable on $\Omega = (0,1) \times (-\infty, -1]$ with respect to the density p. It shall soon be clear that F is very useful for computing the quantity $\log \theta_n(x)$.

$$\begin{aligned}
&\int_\Omega \log\left(\frac{y-x}{xy}\right) p(x,y) \quad dxdy \\
&= \frac{1}{\log 2}\int_{-\infty}^{-1}\int_0^1 \frac{\log\left(\frac{y-x}{xy}\right)}{(x-y)^2} \quad dxdy \\
&= \frac{1}{\log 2}\int_{-\infty}^{-1} \frac{1 + \log(1-y) - \log(-y)}{y(y-1)} \quad dy \\
&= \frac{1}{\log 2}\lim_{h\to\infty}\Bigg[\log(-y) + \log(1-y) + \frac{1}{2}(\log(-y))^2 \\
&\quad + \frac{1}{2}(\log(1-y))^2 - \log(-y)\log(1-y)\Bigg] \quad \Bigg]\Bigg|_{-h}^{-1} \\
&= \frac{1}{\log 2}\Bigg[\left(\log 2 + \frac{1}{2}(\log 2)^2\right) \\
&\quad - \lim_{h\to\infty}\left(\frac{\frac{1}{2}\log(-y) - \frac{1}{2}\log(1-y) + 1}{\frac{1}{\log(-y)}} + \frac{\frac{1}{2}\log(1-y) - \frac{1}{2}\log(-y) + 1}{\frac{1}{\log(1-y)}}\right)\Bigg]
\end{aligned}$$

where the last limit is zero by L'Hospital's rule.

Since F is μ-integrable and $\widetilde{T}^n(x,y)$ is defined on a set of full measure for all $n \geq 0$, it is a direct consequence of the Birkhoff Ergodic Theorem (see [4] or [2]) that for almost all $(x,y) \in \Omega$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\widetilde{T}^i(x,y)) = \int_\Omega F(x,y)p(x,y) \quad dxdy. \tag{3}$$

This value was just computed to be $1 + \frac{1}{2}\log 2$.

It is known, see, e.g., [2], that if two numbers α, β have continued fractions expansions which agree in their first n digits, then $|\alpha - \beta| < 2^{-n+1}$. Thus for $y, y' \in [-\infty, -1]$, $\quad |\pi_2 \circ \widetilde{T}^n(x,y) - \pi_2 \circ \widetilde{T}^n(x,y')| < 2^{-n+1}$.

Our next task is to prove that if the equality (3) holds for a given (x,y) then it holds for (x,y') for all $y' \in [-\infty, -1]$. In essence, the equality is true for almost all x independent of y. Fix x and suppose that the equality (3) holds for $(x,y) \in \Omega$. Let $y' \in [-\infty, -1]$. To simplify the computation write $\widetilde{T^i}(x,y) = (x_i, y_i)$ and $\widetilde{T^i}(x,y') = (x_i, y_i')$. Keep in mind that y_i and y_i' are negative numbers. Then

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\left|F(\widetilde{T^i}(x,y)) - F(\widetilde{T^i}(x,y'))\right| &= \frac{1}{n}\sum_{i=1}^{n}\left|\log\left(\frac{y_i - x_i}{x_i y_i}\right) - \log\left(\frac{y_i' - x_i}{x_i y_i'}\right)\right| \\
&= \frac{1}{n}\sum_{i=1}^{n}\Bigg|\Big(\log\big(x_i - y_i\big) - \log(x_i) - \log(-y_i)\Big) \\
&\quad - \Big(\log\big(x_i - y_i'\big) - \log(x_i) - \log(-y_i')\Big)\Bigg| \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\left|\log\frac{x_i - y_i}{x_i - y_i'}\right| + \frac{1}{n}\sum_{i=1}^{n}\left|\log\frac{y_i'}{y_i}\right|. \qquad (4)
\end{aligned}$$

There is no loss of generality in supposing that $x_i - y_i \geq x_i - y_i'$, since the absolute value of the log of the quotient in the first sum of (4) is the same either way the inequality goes. Then by an earlier observation

$$0 < (x_i - y_i) - (x_i - y_i') = y_i' - y_i < 2^{-i+1}.$$

Since $x_i - y_i' > 1$,

$$\frac{x_i - y_i}{x_i - y_i'} < 1 + 2^{-i+1}(x_i - y_i')^{-1} < 1 + 2^{-i+1}.$$

Now take logs and apply the standard estimate that comes from the alternating series for $\log x$ to get

$$\log\frac{x_i - y_i}{x_i - y_i'} < \log(1 + 2^{-i+1}) < 2^{-i+1}.$$

It follows that the first sum in (4) converges to zero as n goes to ∞. By a similar argument the same conclusion can be reached for the second sum in (4). This shows that if the equality (3) holds for some $(x,y) \in \Omega$ then it holds for any $(x,y') \in \Omega$.

We are now close to completing the the proof of Theorem 1. Two identities from the classical theory will link the above to our main theorem. If $x = [0; a_1, a_2 \ldots]$ then

$$[a_n; a_{n-1}, \ldots, a_1] = \frac{q_n}{q_{n-1}} \quad \text{(see [9])} \tag{5}$$

and

$$\theta_n(x) = \left(\frac{1}{T^n(x)} + \frac{q_{n-1}}{q_n}\right)^{-1} \quad \text{(see [6, p. 29, (11)]).} \tag{6}$$

Given $x \in (0,1)$, let $(x_0, y_0) = \widetilde{T}(x, \infty) = (T(x), -[1/x]) \in \Omega$. As above define $\widetilde{T^i}(x_0, y_0) = (x_i, y_i)$. If $x = [0; a_1, a_2 \dots]$ then for $i > 0$

$$(x_{i-1}, y_{i-1}) = ([0; a_{i+1}, a_{i+2} \dots], -[a_i; a_{i-1}, a_{i-2}, \dots, a_1]) = \left(T^i(x), -\frac{q_i}{q_{i-1}}\right)$$

where we have used (5) above. From (6) and the definition of F,

$$\begin{aligned} -F(\widetilde{T^i}(x_0, y_0)) &= -\log\left(\frac{1}{x_i} - \frac{1}{y_i}\right) \\ &= \log\left(\frac{1}{T^{i+1}(x)} + \frac{q_i}{q_{i+1}}\right)^{-1} \\ &= \log \theta_{i+1}(x). \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log \theta_{i+1}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} -F(\widetilde{T}^i(x_0, y_0))$$

converges to $-1 - \frac{1}{2}\log 2$ for almost all $x_0 \in (0,1)$, independent of y_0, and consequently for almost all $x \in (0,1)$. The proof of Theorem 1 is complete.

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