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# Compatible ideals and radicals of Ore extensions

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Abstract. For a ring endomorphism $\alpha$ and an $\alpha$-derivation $\delta$, we introduce $\alpha$-compatible ideals which are a generalization of $\alpha$-rigid ideals and study the connections of the prime radical and the upper nil radical of $R$ with the prime radical and the upper nil radical of the Ore extension $R[x;\alpha,\delta]$ and the skew power series $R[[x;\alpha]]$. As a consequence we obtain a generalization of Hong, Kwak and Rizvi, 2005.

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## 1. Introduction

Throughout the paper $R$ always denotes an associative ring with unity. $R[x;\alpha,\delta]$ will stand for the Ore extension of $R$, where $\alpha$ is an endomorphism and $\delta$ an $\alpha$-derivation of $R$, that is, $\delta$ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$. We use $P(R)$, $N_r(R)$ and $N(R)$ to denote the prime radical, the upper nil radical and the set of all nilpotent elements of $R$, respectively.

According to Krempa [15], an endomorphism $\alpha$ of a ring $R$ is called *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. $R$ is called an $\alpha$-rigid ring [9] if there exists a rigid endomorphism $\alpha$ of $R$. Note that any rigid endomorphism of a ring is a monomorphism and $\alpha$-rigid rings are reduced by Hong et al. [9]. Properties of $\alpha$-rigid rings have been studied in Krempa [15], Hirano [7] and Hong et al. [11] and [9].

On the other hand, a ring $R$ is called 2-*primal* if $P(R) = N(R)$ (see [2]). Every reduced ring is obviously a 2-primal ring. Moreover, 2-primal rings have been extended to the class of rings which satisfy $N_r(R) = N(R)$, but the converse does not hold ([3], Example 3.3). Observe that $R$ is a 2-primal ring if and only if $P(R) = N_r(R) = N(R)$, if and only if $P(R)$ is a completely semiprime ideal (i.e., $a^2 \in P(R)$ implies $a \in P(R)$ for $a \in R$) of $R$. We refer to [2, 3, 7, 10, 14, 18, 19]

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for more detail on 2-primal rings. Recall that a ring $R$ is called *strongly prime* if $R$ is prime with no nonzero nil ideals. An ideal $P$ of $R$ is *strongly prime* if $R/P$ is a strongly prime ring. All (strongly) prime ideals are taken to be proper. We say an ideal $P$ of a ring $R$ is *minimal* (*strongly*) *prime* if $P$ is minimal among (strongly) prime ideals of $R$. Note that (see [17])

$$N_r(R) = \cap\{P \mid P \text{ is a minimal strongly prime ideal of } R\}.$$

Recall that an ideal $P$ of $R$ is *completely prime* if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$. Every completely prime ideal of $R$ is strongly prime and every strongly prime ideal is prime.

According to Hong et al. [11], for an endomorphism $\alpha$ of a ring $R$, an $\alpha$-ideal $I$ is called an *$\alpha$-rigid ideal* if $a\alpha(a) \in I$ implies $a \in I$ for $a \in R$. Hong et al. [11] studied connections between the $\alpha$-rigid ideals of $R$ and the related ideals of some ring extensions. They also studied the relationship of $P(R)$ and $N_r(R)$ of $R$ with the prime radical and the upper nil radical of the Ore extension $R[x; \alpha, \delta]$ of $R$ in the cases when either $P(R)$ or $N_r(R)$ is an $\alpha$-rigid ideal of $R$, obtaining the following result: Let $P(R)$ (resp. $N_r(R)$) be an $\alpha$-rigid $\delta$-ideal of $R$. Then

$$P(R[x; \alpha, \delta]) \subseteq P(R)[x; \alpha, \delta]$$

(resp. $N_r(R[x; \alpha, \delta]) \subseteq N_r(R)[x; \alpha, \delta]$). Hong et al. [11] provided an example to show that the condition "$P(R)$ is $\alpha$-rigid" is not superfluous.

In [6], the authors defined $\alpha$-compatible rings, which are a generalization of $\alpha$-rigid rings. A ring $R$ is called *$\alpha$-compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. Moreover, $R$ is said to be *$\delta$-compatible* if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. If $R$ is both $\alpha$-compatible and $\delta$-compatible, we say that $R$ is *$(\alpha, \delta)$-compatible.* In this case, clearly the endomorphism $\alpha$ is injective. In ([6], Lemma 2.2), the authors showed that $R$ is $\alpha$-rigid if and only if $R$ is $\alpha$-compatible and reduced. Thus the $\alpha$-compatible ring is a generalization of an $\alpha$-rigid ring to the more general case where $R$ is not assumed to be reduced.

Motivated by the above facts, for an endomorphism $\alpha$ of a ring $R$, we define $\alpha$-compatible ideals in $R$ which are a generalization of $\alpha$-rigid ideals. For an ideal $I$, we say that $I$ is an *$\alpha$-compatible ideal* of $R$ if for each $a, b \in R$, $ab \in I \Leftrightarrow a\alpha(b) \in I$. Moreover, $I$ is said to be a *$\delta$-compatible ideal* if for each $a, b \in R$, $ab \in I \Rightarrow a\delta(b) \in I$. If $I$ is both $\alpha$-compatible and $\delta$-compatible, we say that $I$ is an *$(\alpha, \delta)$-compatible ideal.* The definition is quite natural, in the light of its similarity with the notion of $\alpha$-rigid ideals. Here, in Proposition 2.4, we will show that $I$ is an $\alpha$-rigid ideal if and only if $I$ is an $\alpha$-compatible ideal and completely semiprime.

In this paper we first give some examples of $(\alpha, \delta)$-compatible ideals which are not $\alpha$-rigid. Then, we study connections between $(\alpha, \delta)$-compatible ideals of $R$ and related ideals of some ring extensions. Also we investigate the relationship of $P(R)$ and $N_r(R)$ of $R$ with the prime radical and the upper nil radical of the Ore extension $R[x; \alpha, \delta]$ and the skew power series $R[[x; \alpha]]$.

Recall that an ideal $I$ of $R$ is called an *$\alpha$-ideal* if $\alpha(I) \subseteq I$; $I$ is called *$\alpha$-invariant* if $\alpha^{-1}(I) = I$; $I$ is called a *$\delta$-ideal* if $\delta(I) \subseteq I$; $I$ is called an *$(\alpha, \delta)$-ideal* if it is both an $\alpha$- and a $\delta$-ideal. If $I$ is an $(\alpha, \delta)$-ideal, then $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ is an endomorphism and $\bar{\delta} : R/I \to R/I$ defined by $\bar{\delta}(a + I) = \delta(a) + I$ is an $\bar{\alpha}$-derivation.

## 2. The prime and upper nil radicals of Ore extensions

In this section, our focus of study will be on the prime and the upper nil radicals of a ring $R$ and those of the Ore extension $R[x;\alpha,\delta]$ and the skew power series ring $R[[x;\alpha]]$.

**Proposition 2.1.** *Let $I$ be an ideal of a ring $R$. Then the following statements are equivalent:*

(1) *$I$ is an $(\alpha,\delta)$-compatible ideal.*
(2) *$R/I$ is $(\overline{\alpha},\bar{\delta})$-compatible.*

**Proof.** (1)⇒(2) Clearly $I$ is an $(\alpha,\delta)$-ideal. Let $(a+I)(b+I)=0$ in $R/I$. Then $ab\in I$. Hence $a\alpha(b)\in I$. Thus $(a+I)\overline{\alpha}(b+I)=0$. Similarly, $(a+I)\overline{\alpha}(b+I)=0$ implies $(a+I)(b+I)=0$. If $(a+I)(b+I)=0$, then $ab\in I$, so that $a\delta(b)\in I$. Hence $(a+I)\bar{\delta}(b+I)=0$.

(2)⇒(1) This is similar to the proof of (1)⇒(2). □

**Lemma 2.2.** *Let $I$ be an $\alpha$-compatible ideal of a ring $R$. Then $I$ is $\alpha$-invariant.*

**Proof.** Let $\alpha(a)\in I$. Then $1\alpha(a)\in I$ and so $a\in I$, since $I$ is $\alpha$-compatible. Thus $I$ is an $\alpha$-invariant ideal. □

Recall from [16] that a one-sided ideal $I$ of a ring $R$ has the *insertion of factors property* (or simply, IFP) if $ab\in I$ implies $aRb\subseteq I$ for $a,b\in R$ (H.E. Bell in 1970 introduced this notion for $I=0$).

The following proposition extends ([6], Lemma 2.1).

**Proposition 2.3.** *Let $I$ be an $(\alpha,\delta)$-compatible ideal of $R$ and $a,b\in R$.*

(1) *If $ab\in I$, then $a\alpha^n(b)\in I$ and $\alpha^n(a)b\in I$ for every positive integer $n$. Conversely, if $a\alpha^k(b)$ or $\alpha^k(a)b\in I$ for some positive integer $k$, then $ab\in I$.*
(2) *If $ab\in I$, then $\alpha^m(a)\delta^n(b),\delta^n(a)\alpha^m(b)\in I$ for any nonnegative integers $m,n$.*

**Proof.** (1) If $ab\in I$, then $\alpha^n(a)\alpha^n(b)\in I$, since $I$ is an $\alpha$-ideal. Hence $\alpha^n(a)b\in I$, since $I$ is $\alpha$-compatible. If $\alpha^k(a)b\in I$, then $\alpha^k(a)\alpha^k(b)\in I$, since $I$ is $\alpha$-compatible. Hence $\alpha^k(ab)\in I$ and $ab\in I$, since $I$ is $\alpha$-invariant, by Lemma 2.2.

(2) It is enough to show that $\delta(a)\alpha(b)\in I$. If $ab\in I$, then by (1) and $\delta$-compatibility of $I$, $\alpha(a)\delta(b)\in I$. Hence $\delta(a)b=\delta(ab)-\alpha(a)\delta(b)\in I$. Thus $\delta(a)b\in I$ and $\delta(a)\alpha(b)\in I$, since $I$ is $\alpha$-compatible. □

**Proposition 2.4.** *Let $R$ be a ring, $I$ be an ideal of $R$ and $\alpha: R\to R$ be an endomorphism of $R$. Then the following conditions are equivalent:*

(1) *$I$ is an $\alpha$-rigid ideal of $R$.*
(2) *$I$ is $\alpha$-compatible, semiprime and satisfies the IFP property.*
(3) *$I$ is $\alpha$-compatible and completely semiprime.*

*If $\delta$ is an $\alpha$-derivation of $R$, then the following are equivalent:*

(4) *$I$ is an $\alpha$-rigid $\delta$-ideal of $R$.*
(5) *$I$ is $(\alpha,\delta)$-compatible, semiprime and satisfies the IFP property.*
(6) *$I$ is $(\alpha,\delta)$-compatible and completely semiprime.*

**Proof.** (1)⇒(2) This follows from ([11], Propositions 2.2 and 2.4).

(2)⇒(1) Let $a\alpha(a) \in I$. Then $a^2 \in I$, since $I$ is $\alpha$-compatible. Hence $aRa \subseteq I$, since $I$ satisfies the IFP property. Thus $a \in I$, since $I$ is semiprime.

(4)⇒(6) By (1)⇒(3), $I$ is $\alpha$-compatible and completely semiprime. We show that $a\delta(b) \in I$, when $ab \in I$. If $ab \in I$, then $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b \in \delta(I) \subseteq I$. Thus $(\alpha(a)\delta(b))^2 = \delta(ab)\alpha(a)\delta(b) - \delta(a)b\alpha(a)\delta(b) \in I$, because $\delta(ab), b\alpha(a) \in I$. Since $I$ is completely semiprime, we have $\alpha(a)\delta(b) \in I$ and so $a\delta(b) \in I$, by Proposition 2.3. □

In [6], the authors give some examples of $(\alpha, \delta)$-compatible rings which are not $\alpha$-rigid. Note that there exists a ring $R$ for which every nonzero proper ideal is $\alpha$-compatible but $R$ is not $\alpha$-compatible. For example, consider the ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where $F$ is a field, and the endomorphism $\alpha$ of $R$ is defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ for $a, b, c \in F$.

The following examples show that there exist $\alpha$-compatible ideals that are not $\alpha$-rigid.

**Example 2.5.** Let $F$ be a field. Let $R = \left\{\begin{pmatrix} f & f_1 \\ 0 & f \end{pmatrix} \;\middle|\; f, f_1 \in F[x]\right\}$, where $F[x]$ is the ring of polynomials over $F$. Then $R$ is a subring of the $2 \times 2$ matrix ring over the ring $F[x]$. Let $\alpha : R \to R$ be an automorphism defined by $\alpha\left(\begin{pmatrix} f & f_1 \\ 0 & f \end{pmatrix}\right) = \begin{pmatrix} f & uf_1 \\ 0 & f \end{pmatrix}$, where $u$ is a fixed nonzero element of $F$. Let $p(x)$ be an irreducible polynomial in $F[x]$. Let $I = \left\{\begin{pmatrix} 0 & f_1 \\ 0 & 0 \end{pmatrix} \;\middle|\; f_1 \in \langle p(x)\rangle\right\}$, where $\langle p(x)\rangle$ is the principal ideal of $F[x]$ generated by $p(x)$. Then $I$ is an $\alpha$-compatible ideal of $R$ but it is not $\alpha$-rigid. Indeed, since $\begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix} \alpha\left(\begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in I$, but $\begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix} \notin I$ for $g(x) \notin \langle p(x)\rangle$. Thus $I$ is not $\alpha$-rigid.

**Example 2.6** ([12], Example 2)**.** Let $\mathbb{Z}_2$ be the field of integers modulo 2 and $A = \mathbb{Z}_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$ be the free algebra of polynomials with zero constant term in noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Note that $A$ is a ring without identity. Consider an ideal of $\mathbb{Z}_2 + A$, say $I$, generated by $a_0b_0$, $a_1b_2 + a_2b_1$, $a_0b_1 + a_1b_0$, $a_0b_2 + a_1b_1 + a_2b_0$, $a_2b_2$, $a_0rb_0$, $a_2rb_2$, $(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2)$ with $r \in A$ and $r_1r_2r_3r_4$ with $r_1, r_2, r_3, r_4 \in A$. Then $I$ satisfies the IFP property. Let $\alpha : R \to R$ be an inner automorphism (i.e., there exists an invertible element $u \in R$ such that $\alpha(r) = u^{-1}ru$ for each $r \in R$). Then $I$ is $\alpha$-compatible, since $I$ satisfies the IFP property. But $I$ is not $\alpha$-rigid, since $I$ is not completely semiprime.

**Definition 2.7.** Given $\alpha$ and $\delta$ as above and integers $j \geq i \geq 0$, let us write $f_i^j$ for the sum of all "words" in $\alpha$ and $\delta$ in which there are $i$ factors of $\alpha$ and $j - i$ factors of $\delta$. For instance, $f_j^j = \alpha^j$, $f_0^j = \delta^j$ and $f_{j-1}^j = \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \cdots + \delta\alpha^{j-1}$.

Note that if $I$ is an $(\alpha, \delta)$-ideal of $R$, then $I[x; \alpha, \delta]$ is an ideal of the Ore extension $R[x; \alpha, \delta]$.

**Theorem 2.8.** *Let $I$ be an $(\alpha,\delta)$-compatible semiprime ideal of $R$. Assume $f(x)=\sum_{i=0}^{n} a_i x^i$ and $g(x)=\sum_{j=0}^{m} b_j x^j \in R[x;\alpha,\delta]$. Then the following statements are equivalent:*

(1) $f(x)R[x;\alpha,\delta]g(x)\subseteq I[x;\alpha,\delta]$.
(2) $a_iRb_j\subseteq I$ *for each* $i,j$.

**Proof.** (1)⇒(2) Assume $f(x)R[x;\alpha,\delta]g(x)\subseteq I[x;\alpha,\delta]$. Then

$$(\dagger)\qquad (a_0+\cdots+a_nx^n)c(b_0+\cdots+b_mx^m)\in I[x;\alpha,\delta]\text{ for each } c\in R.$$

Hence $a_n\alpha^n(cb_m)\in I$. Thus $a_ncb_m\in I$, since $I$ is $\alpha$-compatible. Therefore $a_nf_i^j(cb_m)\in I$, by Proposition 2.3. Next, replace $c$ by $cb_{m-1}da_ne$, where $c,d,e\in R$. Then $(a_0+\cdots+a_nx^n)cb_{m-1}da_ne(b_0+\cdots+b_{m-1}x^{m-1})\in I[x;\alpha,\delta]$. Hence $a_n\alpha^n(cb_{m-1}da_neb_{m-1})\in I$ and $a_ncb_{m-1}da_neb_{m-1}\in I$, since $I$ is $\alpha$-compatible. Thus $(Ra_nRb_{m-1})^2\subseteq I$. Hence $Ra_nRb_{m-1}\subseteq I$, since $I$ is semiprime. Continuing this process, we obtain $a_nRb_k\subseteq I$, for $k=0,1,\cdots,m$. Hence by $(\alpha,\delta)$-compatibility of $I$, we get $(a_0+\cdots+a_nx^n)R[x;\alpha,\delta](b_0+\cdots+b_{m-1}x^{m-1})\subseteq I[x;\alpha,\delta]$. Using induction on $n+m$, we obtain $a_iRb_j\subseteq I$ for each $i,j$.

(2)⇒(1) This follows from Proposition 2.3. □

**Corollary 2.9.** *If $I$ is a (semi)prime $(\alpha,\delta)$-compatible ideal of $R$, then $I[x;\alpha,\delta]$ is a (semi)prime ideal of $R[x;\alpha,\delta]$.*

**Proof.** Assume that $I$ is a prime $(\alpha,\delta)$-compatible ideal of $R$. Let $f(x)=\sum_{i=0}^{n} a_ix^i$ and $g(x)=\sum_{j=0}^{m} b_jx^j\in R[x;\alpha,\delta]$ such that $f(x)R[x;\alpha,\delta]g(x)\subseteq I[x;\alpha,\delta]$. Then $a_iRb_j\subseteq I$ for each $i,j$, by Theorem 2.8. Assume $g(x)\notin I[x;\alpha,\delta]$. Hence $b_j\notin I$ for some $j$. Thus $a_i\in I$ for each $i=0,1,\cdots,n$, since $I$ is prime. Therefore $f(x)\in I[x;\alpha,\delta]$. Consequently, $I[x;\alpha,\delta]$ is a prime ideal of $R[x;\alpha,\delta]$. □

**Theorem 2.10.** *If each minimal prime ideal of $R$ is $(\alpha,\delta)$-compatible, then $P(R[x;\alpha,\delta])\subseteq P(R)[x;\alpha,\delta]$.*

**Proof.** This follows from Corollary 2.9. □

The following example shows that there exists a ring $R$ such that all minimal prime ideals are $\alpha$-compatible, but are not $\alpha$-rigid.

**Example 2.11.** Let $R=Mat_2(\mathbb{Z}_4)$ be the $2\times 2$ matrix ring over the ring $\mathbb{Z}_4$. Then $P(R)=\left\{\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\ \middle|\ a_{ij}\in\overline{2}\mathbb{Z}\right\}$ is the only prime ideal of $R$. Let $\alpha:R\to R$ be the endomorphism defined by $\alpha\left(\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\right)=\begin{pmatrix} a_{11} & -a_{12}\\ -a_{21} & a_{22}\end{pmatrix}$. Then $\alpha$ is an automorphism of $R$ and $P(R)$ is $\alpha$-compatible. However, $P(R)$ is not $\alpha$-rigid, since $\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\alpha\left(\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\right)\in P(R)$, but $\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\notin P(R)$.

**Theorem 2.12.** *If $P$ is a completely (semi)prime $(\alpha,\delta)$-compatible ideal of $R$, then $P[x;\alpha,\delta]$ is a completely (semi)prime ideal of $R[x;\alpha,\delta]$.*

**Proof.** Let $P$ be a completely prime ideal of $R$. $R/P$ is a domain, hence it is a reduced ring. $R/P$ is a $(\bar\alpha,\bar\delta)$-compatible ring, hence $R/P$ is $\bar\alpha$-rigid, by ([6], Lemma 2.2). Let $\overline{f(x)},\overline{g(x)}\in R/P[x;\bar\alpha,\bar\delta]$ such that $\overline{f(x)}\ \overline{g(x)}=0$. Then $\overline{f(x)}=0$ or $\overline{g(x)}=0$, by ([9], Proposition 6). Thus $R[x;\alpha,\delta]/P[x;\alpha,\delta]\cong R/P[x;\bar\alpha,\bar\delta]$ is a domain and $P[x;\alpha,\delta]$ is a completely prime ideal of $R[x;\alpha,\delta]$. □

**Corollary 2.13** ([11], Proposition 3.8)**.** *If $P(R)$ is an $\alpha$-rigid $\delta$-ideal of $R$, then $P(R[x;\alpha,\delta]) \subseteq P(R)[x;\alpha,\delta]$.*

**Proof.** $P(R)$ is an $\alpha$-rigid $\delta$-ideal, hence $P(R)$ is a completely semiprime $(\alpha,\delta)$-compatible ideal of $R$, by Proposition 2.4. Therefore $P(R[x;\alpha,\delta]) \subseteq P(R)[x;\alpha,\delta]$, by Theorem 2.12. □

**Theorem 2.14.** *If $P$ is a strongly (semi)prime $(\alpha,\delta)$-compatible ideal of $R$, then $P[x;\alpha,\delta]$ is a strongly (semi)prime ideal of $R[x;\alpha,\delta]$.*

**Proof.** By Corollary 2.9, $P[x;\alpha,\delta]$ is a prime ideal of $R[x;\alpha,\delta]$. Hence

$$R[x;\alpha,\delta]/P[x;\alpha,\delta] \cong R/P[x;\overline{\alpha},\bar{\delta}]$$

is a prime ring. We claim that zero is the only nil ideal of $R/P[x;\overline{\alpha},\bar{\delta}]$. Let $J$ be a nil ideal of $R/P[x;\overline{\alpha},\bar{\delta}]$. Assume $I$ be the set of all leading coefficients of elements of $J$. First we show that $I$ is an ideal of $R/P$. Clearly $I$ is a left ideal of $R/P$. Let $\bar{a} \in I$ and $\bar{r} \in R/P$. Then there exists $\overline{f(x)} = \bar{a}_0 + \cdots + \bar{a}_{n-1}x^{n-1} + \bar{a}x^n \in J$. Hence $(\overline{f(x)}\bar{r})^m = 0$ for some nonnegative integer $m$. Thus $\bar{a}\ \overline{\alpha}^n(\bar{r}\bar{a})\cdots\overline{\alpha}^{(m-1)n}(\overline{ra})\overline{\alpha}^{mn}(\bar{r}) = 0$, since it is the leading coefficient of $(\overline{f(x)}\bar{r})^m$. Therefore $(\overline{ar})^m = 0$, since $R/P$ is $\overline{\alpha}$-compatible. Consequently, $I$ is an ideal of $R/P$. Clearly $I$ is a nil ideal of $R/P$. Hence $I = 0$ and so $J = 0$. Therefore $P[x;\alpha,\delta]$ is a strongly prime ideal of $R[x;\alpha,\delta]$. □

**Theorem 2.15.** *If each minimal strongly prime ideal of $R$ is $(\alpha,\delta)$-compatible, then $N_r(R[x;\alpha,\delta]) \subseteq N_r(R)[x;\alpha,\delta]$.*

**Corollary 2.16** ([11], Proposition 3.8)**.** *If $N_r(R)$ is an $\alpha$-rigid $\delta$-ideal of $R$, then $N_r(R[x;\alpha,\delta]) \subseteq N_r(R)[x;\alpha,\delta]$.*

**Proof.** $N_r(R)$ is an $\alpha$-rigid $\delta$-ideal, hence $N_r(R)$ is a completely semiprime $(\alpha,\delta)$-compatible ideal of $R$, by Proposition 2.4, and $N_r(R)$ is a strongly semiprime ideal of $R$. Therefore $N_r(R[x;\alpha,\delta]) \subseteq N_r(R)[x;\alpha,\delta]$, by Theorem 2.15. □

As a parallel result to Theorems 2.8, 2.10, 2.12, 2.14 and 2.15, we have the following results for the skew power series ring $R[[x;\alpha]]$.

**Proposition 2.17.** *Let $I$ be an $\alpha$-compatible semiprime ideal of $R$. Assume $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x;\alpha]]$. Then the following statements are equivalent:*

(1) *$f(x)R[[x;\alpha]]g(x) \subseteq I[[x;\alpha]]$.*
(2) *$a_iRb_j \subseteq I$ for each $i,j$.*

**Proof.** (1)⇒(2) Assume $f(x)R[[x;\alpha]]g(x) \subseteq I[[x;\alpha]]$. Let $c$ be an arbitrary element of $R$. Then we have the following:

$$(\S)\qquad \sum_{k=0}^{\infty}\left(\sum_{i+j=k} a_i x^i c b_j x^j\right) = \sum_{k=0}^{\infty}\left(\sum_{i+j=k} a_i \alpha^i(cb_j)\right)x^k \in I[[x;\alpha]].$$

By induction on $i+j$, we show that $a_iRb_j \subseteq I$ for all $i,j$. From Equation (§), we obtain $a_0Rb_0 \subseteq I$. This prove for $i+j=0$. Now suppose that $a_iRb_j \subseteq I$ for

$i + j \leq n - 1$. From Equation (§), we have

$$(\S\S) \qquad \sum_{i+j=k} a_i \alpha^i (cb_j) \in I \text{ for all } \ k \geq 0.$$

Let $c, d, e$ be arbitrary elements of $R$. For $k = n$ replace $c$ by $cb_n da_0 e$. Then we obtain $a_0 cb_n da_0 eb_n \in I$, since $a_0 Rb_j \subseteq I$ for each $j \leq n - 1$. Hence $(Ra_0 Rb_n)^2 \subseteq I$ and so $Ra_0 Rb_n \subseteq I$, since $I$ is semiprime. Continuing this process (replacing $c$ by $cb_j da_{n-j} e$ in Equation (§§) for $j = 1, 2, \cdots, n-1$ and using $\alpha$-compatibility of $I$), we obtain $a_i Rb_j \subseteq I$ for $i + j = n$.

(2)⇒(1) This follows from Proposition 2.3. □

**Corollary 2.18.** *If $I$ is a (semi)prime $\alpha$-compatible ideal of $R$, then $I[[x;\alpha]]$ is a (semi)prime ideal of $R[[x;\alpha]]$.*

**Theorem 2.19.** *If each minimal prime ideal of $R$ is $\alpha$-compatible, then $P(R[[x;\alpha]]) \subseteq P(R)[[x;\alpha]]$.*

**Proof.** This follows from Corollary 2.18. □

**Theorem 2.20.** *If $P$ is a completely (semi)prime $\alpha$-compatible ideal of $R$, then $P[[x;\alpha]]$ is a completely (semi)prime ideal of $R[[x;\alpha]]$.*

**Proof.** Let $P$ be a completely prime ideal of $R$. $R/P$ is a domain, hence it is a reduced ring. $R/P$ is an $\overline{\alpha}$-compatible ring, hence $R/P$ is $\overline{\alpha}$-rigid, by ([6], Lemma 2.2). Let $\overline{f(x)}, \overline{g(x)} \in R/P[[x;\overline{\alpha}]]$ such that $\overline{f(x)}\ \overline{g(x)} = 0$. Then $\overline{f(x)} = 0$ or $\overline{g(x)} = 0$, by ([9], Proposition 17). Thus $R[[x;\alpha]]/P[[x;\alpha]] \cong R/P[[x;\overline{\alpha}]]$ is a domain and $P[[x;\alpha]]$ is a completely prime ideal of $R[[x;\alpha]]$. □

**Corollary 2.21** ([11], Proposition 3.12)**.** *If $P(R)$ is an $\alpha$-rigid ideal of $R$, then $P(R[[x;\alpha]]) \subseteq P(R)[[x;\alpha]]$.*

**Theorem 2.22.** *If $P$ is a strongly prime $\alpha$-compatible ideal of $R$, then $P[[x;\alpha]]$ is a strongly prime ideal of $R[[x;\alpha]]$.*

**Theorem 2.23.** *If each minimal strongly prime ideal of $R$ is $\alpha$-compatible, then $N_r(R[[x;\alpha]]) \subseteq N_r(R)[[x;\alpha]]$.*

**Corollary 2.24** ([11], Proposition 3.12)**.** *If $N_r(R)$ is an $\alpha$-rigid ideal of $R$, then $N_r(R[[x;\alpha]]) \subseteq N_r(R)[[x;\alpha]]$.*

## References


[1] Annin, Scott. Associated primes over skew polynomials rings. *Comm. Algebra* **30** (2002), no. 5, 2511–2528. MR1904650 (2003c:16039). Erratum. *Comm. Algebra* **30** (2002), no. 10, 5109–5110. MR1940490 (2003k:16037).

[2] Birkenmeier, Gary F.; Heatherly, Henry E.; Lee, Enoch K. Completely prime ideals and associated radicals. *Ring theory* (Granville, OH, 1992), 102–129, *World Sci. Publ., River Edge, NJ*, 1993. MR1344225 (96e:16025), Zbl 0853.16022.

[3] Birkenmeier, Gary F.; Kim, Jin Yong; Park, Jae Keol. Regularity conditions and the simplicity of prime factor rings. *J. Pure Appl. Algebra* **115** (1997), no. 3, 213–230. MR1431833 (97k:16006), Zbl 0870.16006. Corrigendum. *J. Pure Appl. Algebra* **149** (2000), no. 1, 105. MR1760908 (2001d:16019).

[4] Ferrero, Miguel. Radicals of skew polynomial rings and skew Laurent polynomial rings. *Math. J. Okayama Univ.* **29** (1987), 119–126 (1988). MR0936735 (89d:16001), Zbl 0647.16004.

[5] Hashemi, E. Radicals of skew Laurent polynomial rings. To appear.

[6] Hashemi, E.; Moussavi, A. Polynomial extensions of quasi-Baer rings. *Acta Math. Hungar.* **107** (2005), no. 3, 207–224. MR2148584 (2006c:16048), Zbl 1081.16032.

[7] Hirano, Yasuyuki. Some studies on strongly $\pi$-regular rings. *Math. J. Okayama Univ.* **20** (1978), no. 2, 141–149. MR0519561 (80i:16020), Zbl 0394.16011.

[8] Hirano, Yasuyuki. On the uniqueness of rings of coefficients in skew polynomial rings. *Publ. Math. Debrecen* **54** (1999), no. 3-4, 489–495. MR1694531 (2000b:16055), Zbl 0930.16018.

[9] Hong, Chan Yong; Kim, Nam Kyun; Kwak, Tai Keun. Ore extensions of Baer and p.p.-rings. *J. Pure Appl. Algebra* **151** (2000), no. 3, 215–226. MR1776431 (2001f:16054), Zbl 0982.16021.

[10] Hong, Chan Yong; Kwak, Tai Keun. On minimal strongly prime ideals. *Comm. Algebra* **28** (2000), no. 10, 4867–4878. MR1779879 (2001g:16005), Zbl 0981.16001.

[11] Hong, Chan Yong; Kwak, Tai Keun; Rizvi, S. Tariq. Rigid ideals and radicals of Ore extensions. *Algebra Colloq.* **12** (2005), no. 3, 399–412. MR2144995 (2006g:16063).

[12] Huh, Chan; Lee, Yang; Smoktunowicz, Agata. Armendariz rings and semicommutative rings. *Comm. Algebra* **30** (2002), no. 2, 751–761. MR1883022 (2002j:16037), Zbl 1023.16005.

[13] Jordan, D. A. Bijective extensions of injective ring endomorphisms. *J. London Math. Soc.* **35** (1982), no. 3, 435–448. MR0657500 (83j:16002), Zbl 0486.16002.

[14] Kim, Nam Kyun; Kwak, Tai Keun. Minimal prime ideals in 2-primal rings. *Math. Japonica* **50** (1999), no. 3, 415–420. MR1727666 (2000h:16002), Zbl 0943.16002.

[15] Krempa, Jan. Some examples of reduced rings. *Algebra Colloq.* **3** (1996), no. 4, 289–300. MR1422968 (98e:16027), Zbl 0859.16019.

[16] Mason, Gordon. Reflexive ideals. *Comm. Algebra* **9** (1981), no. 17, 1709–1724. MR0631884 (83b:16017), Zbl 0468.16024.

[17] Rowen, Louis H. Ring Theory. Vol. I. Pure and Applied Math., 127. *Academic Press, Boston*, 1988. MR0940245 (89h:16001), Zbl 0651.16001.

[18] Shin, Gooyong. Prime ideals and sheaf representation of a pseudo symmetric ring. *Trans. Amer. Math. Soc.* **184** (1973), 43–60 (1974). MR0338058 (49 #2825), Zbl 0283.16021.

[19] Sun, Shu Hao. Noncommutative rings in which every prime ideal is contained in a unique maximal ideal. *J. Pure Appl. Algebra* **76** (1991), no. 2, 179–192. MR1145866 (93b:16058), Zbl 0747.16001.

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