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A reflexivity criterion for Hilbert C^*-modules over commutative C^*-algebras

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Abstract. A C^*-algebra A is C^*-reflexive if any countably generated Hilbert C^*-module M over A is C^*-reflexive, i.e., the second dual module M'' coincides with M. We show that a commutative C^*-algebra A is C^*-reflexive if and only if for any sequence I_k of mutually orthogonal nonzero C^*-subalgebras, the canonical inclusion $\oplus_k I_k \subset A$ doesn't extend to an inclusion of $\prod_k I_k$.

Contents

1. Introduction

The aim of the present paper is to study the C^*-reflexivity property for Hilbert C^*-modules over C^*-algebras. The motivation comes from three sources. First, this property appears in our study of dynamical systems and group actions, where it was shown that some information about orbits can be detected from C^*-reflexivity of the corresponding Hilbert C^*-modules [3, 4]. Second, C^*-reflexive Hilbert C^*-modules are a natural setting for A-bilinear

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functions on them. Third, there was a series of papers providing various sufficient [7, 13, 4] and necessary [9] conditions for Hilbert C^*-modules over commutative C^*-algebras to be C^*-reflexive. The main result of this paper is a criterion for C^*-reflexivity in the commutative case.

Let us recall some basic facts about the dual and the second dual of a Hilbert C^*-module [9] (see also [6]). For a Hilbert C^*-module M over a C^*-algebra A, the *dual* Banach module M' is defined [9] as the set of all A-module bounded linear maps from M to A (such maps are called *functionals*). Iterating this procedure, one gets the second dual module M''.

There are isometric inclusions $M \subset M'' \subset M'$ for any Hilbert C^*-module M. The identifications are defined as follows. First of all we have the map $M \to M'$, $m \mapsto \widehat{m}$, $\widehat{m}(s) = \langle s, m\rangle$ for any $s \in M$. Then we can define the map $M \to M''$, $m \mapsto \dot{m}$, $\dot{m}(f) = f(m)$ for any $f \in M'$. Finally, $M'' \to M'$, $F \mapsto \widetilde{F}$ is defined by $\widetilde{F}(m) = F(\widehat{m})$. The A-valued inner product of M can be extended to M'' by the formula $\langle F, G\rangle = G(\widetilde{F})$ and thus M'' becomes a Hilbert C^*-module [9].

A Hilbert C^*-module M is *self-dual* if $M' = M$. There are very few C^*-algebras, for which all Hilbert C^*-modules are self-dual. Only finite-dimensional C^*-algebras have this property [2]. *C^*-reflexivity* (i.e., $M'' = M$) is a more common property. For example, all countably generated Hilbert C^*-modules over the C^*-algebra of compact operators with adjoined unit are C^*-reflexive [12].

Due to the Kasparov stabilization theorem [5], any countably generated Hilbert C^*-module over a C^*-algebra A is C^*-reflexive if the standard Hilbert A-module $H_A = l_2(A)$ is C^*-reflexive. We call a C^*-algebra A *C^*-reflexive* if H_A is C^*-reflexive.

It was shown by Paschke [10] that infinite-dimensional von Neumann algebras are not C^*-reflexive. On the positive, it is known that $C(X)$ is C^*-reflexive for nice spaces X.

Theorem 1.1. *Let X be a compact metric space. Then $C(X)$ is C^*-reflexive.*

The first version of a proof was given by Mishchenko [7]. Then Trofimov [13] realized that the formulation in [7] was too general and provided a proof for any compact X with a certain property L, which, in fact, is the same as the property of being a Baire space. Although the main part of the proof in [13] is correct, it was overlooked that implicitly X was assumed to be a *metric* space. Trofimov's proof was corrected in [4].

Many examples of C^*-reflexive modules arising from group actions were obtained in our previous papers [3, 4].

The main result of this paper is the criterion for C^*-reflexivity for commutative C^*-algebras, which is given in either topological or algebraic terms.

2. Topological preliminaries: the Baire property and the Stone–Čech compactification

Definition 2.1 ([8, p. 155])**.** A space X is said to be a *Baire space* if the following condition holds: Given any countable collection $\{A_n\}$ of closed subsets of X each of which has empty interior in X, their union $\cup A_n$ also has empty interior in X.

Theorem 2.2 (Baire category theorem, see [8, Theorem 48.2])**.** *If X is a compact Hausdorff space or a complete metric space, then X is a Baire space.*

Theorem 2.3 ([8, Theorem 48.5])**.** *Let X be a space; let (Y, d) be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for all $x \in X$, where $f : X \to Y$. If X is a Baire space, the set of points at which f is continuous is dense in X.*

Proofs of the statements of the following theorem can be found, e.g., in [1, Sect. 3.6].

Theorem 2.4. *Suppose, X is a compact Hausdorff space with a dense subset Y. Then the following properties are equivalent:*

1. *X is the Stone–Čech compactification βY.*
2. *Any bounded continuous function on Y can be extended to a continuous function on X.*

3. Hilbert C^*-modules preliminaries

Recall that the standard Hilbert C^*-module $H_A = l_2(A)$ is the set of all sequences $(a_1, a_2, \ldots)$, $a_1, a_2, \ldots \in A$, such that the series $\sum_{i=1}^{\infty} a_i^* a_i$ is norm convergent in A.

For $H_A = l_2(C(X))$ the dual module can be described as follows (see, e.g., [6], Prop. 2.5.5):

$$(1) \qquad H'_A = \left\{ f = (f_1, f_2, \dots) : f_i \in C(X),\ \sup_N \left\| \sum_{i=1}^{N} f_i^* f_i \right\| < \infty \right\},$$

$\|f\|^2 = \sup_N \left\| \sum_{i=1}^{N} f_i^* f_i \right\|$, where $\|\cdot\|$ denotes the sup norm on X.

Unfortunately, there is no similar description of the second dual module H''_A for general A, but in the commutative case we have some results on elements of H''_A. The proof of the following statement is close to an argument of [13].

Lemma 3.1. *Suppose $(F_1, F_2, \dots) \in H'_A$ represents an element $F \in H''_A$. Let E be the continuity set of the point-wise limit $\Phi(x) := \sum_i F_i^*(x) F_i(x)$. Then, for any $x_0 \in E$ and any $f \in H'_A$, the limit of $\sum_i F_i^*(x_0) f_i(x_0)$ equals the value of the continuous function $F(f)$ at this point.*

Remark 3.2. By Theorem 2.3 the set E is dense in X.

Proof. Take $x_0 \in E$ and $\varepsilon > 0$. Choose a neighborhood $U_0 \ni x_0$ such that $|\Phi(x) - \Phi(x_0)| < \varepsilon^2$ for any $x \in U_0$. Choose N such that

$$\sum_{i=N+1}^{\infty} F_i^*(x_0)F_i(x_0) < \varepsilon^2.$$

Choose a neighborhood $U_1 \subset U_0$ of x_0 such that

$$\left|\sum_{i=1}^{N} F_i^*(x)F_i(x) - \sum_{i=1}^{N} F_i^*(x_0)F_i(x_0)\right| < \varepsilon^2 \qquad \forall\, x \in U_1$$

(this is possible because of continuity of this finite sum of continuous functions). Then, for any $x \in U_1$,

$$\begin{aligned}\left|\sum_{i=N+1}^{\infty} F_i^*(x)F_i(x)\right| &= \left|\Phi(x) - \sum_{i=1}^{N} F_i^*(x)F_i(x)\right| \\ &\le |\Phi(x) - \Phi(x_0)| + \left|\Phi(x_0) - \sum_{i=1}^{N} F_i^*(x_0)F_i(x_0)\right| \\ &\quad + \left|\sum_{i=1}^{N} F_i^*(x)F_i(x) - \sum_{i=1}^{N} F_i^*(x_0)F_i(x_0)\right| < 3\,\varepsilon^2.\end{aligned}$$

Because of the isometric embedding $H''_A \subset H'_A$ (cf. [9]) this means that, for any continuous function $\lambda : X \to [0,1]$ with $\operatorname{supp}\lambda \subset U_1$ and $\lambda(x_0) = 1$, we have the following estimate of the norm of an element of H''_A:

$$\left\|\lambda\, F - \sum_{i=1}^{N} \lambda F_i^* \widehat{e}_i\right\| < \sqrt{3}\varepsilon,$$

where $\widehat{e}_i$ are the images of the standard basis elements under the natural isometric inclusion $H_A \subset H''_A$. For any $f \in H'_A$

$$\begin{aligned}\sqrt{3}\varepsilon\,\|f\| &> \left|\left(\lambda\, F - \sum_{i=1}^{N} \lambda F_i^* \widehat{e}_i\right)(f)\Big|_{x_0}\right| \\ &= \left|F(f)(x_0) - \sum_{i=1}^{N} F_i^*(x_0) f_i(x_0)\right|.\end{aligned}$$

□

Lemma 3.3. *A sequence $(F_1, F_2, \dots) \in H'_A$ defines an element F of H''_A if and only if for each $f \in H'_A$, there exists a continuous function α_f such that the point-wise limit of the series $\sum_i F_i^* f_i$ coincides with α_f on the dense set E of continuity points of $\sum_i F_i^* F_i$. In this case $F(f) = \alpha_f$.*

Remark 3.4. The mentioned point-wise limit always exists because at a point all our sequences become $l_2(\mathbb{C})$-sequences.

Proof. The "only if" statement was proved in Lemma 3.1.

Conversely, let us define a functional F by the formula $F(f) = \alpha_f$. Evidently, it is an A-functional defined on H'_A. It is bounded by the Cauchy–Buniakovskiy inequality. It remains to show that, for the natural isometric embedding $H''_A \hookrightarrow H'_A$, $F \mapsto \widetilde{F}$, the element $\widetilde{F}$ corresponds to $(F_1, F_2, \dots)$, i.e., $\widetilde{F}(e_i) = F_i^*$. Indeed, $\widetilde{F}(e_i) = F(\widehat{e}_i) = \sum_k F_k^*(\widehat{e}_i)_k = F_i^*$, because in this case the point-wise limit is everywhere continuous. Here $(\widehat{e}_i)_k$ denotes the k-component of $\widehat{e}_i$. □

4. A sufficient property for $l_2(C(X))$ to be C^*-reflexive

We start with a proof of a stronger version of [13] (see [4] for a corrected version of [13]).

Theorem 4.1. *Suppose a compact Hausdorff space X does not contain a copy of the Stone–Čech compactification $\beta\mathbb{N}$ of natural numbers $\mathbb{N}$ as a closed subset. Then $l_2(C(X))$ is C^*-reflexive.*

Proof. Denote for brevity $A := C(X)$, $H_A := l_2(C(X))$. Since $H''_A \subset H'_A$, its elements are represented by series as in (1). Such an element is in H_A if and only if this series is norm-convergent.

Let $F = (F_1, F_2, \dots) \in H''_A$ and set

$$K_F = \inf_k \sup_{m>k} \sup_{x\in X} \sum_{i=k}^{m} |F_i(x)|^2 = \inf_k \sup_{m>k} \left\| \sum_{i=k}^{m} |F_i|^2 \right\|.$$

Obviously, $K_F \le \|F\|^2$, where $\|F\|^2$ is the least number C such that

$$\sup_{x\in X} \sum_{i=1}^{\infty} |F_i(x)|^2 \le C.$$

By the Cauchy criterion, $K_F = 0$ if and only if $(F_1, F_2, \dots) \in H_A$.

We will argue as follows: we will suppose that $H''_A \neq H_A$ and will prove that $\beta\mathbb{N} \subset X$. So we have an element $(F_1, F_2, \dots) \in H''_A$ such that $K_F > 0$.

There exists a number $m(1)$ such that the estimate

$$\sum_{i=1}^{m(1)-1} |F_i(x)|^2 > \|F\|^2 - K_F/3$$

holds for at least one $x \in X$.

Set

$$U_1 = \left\{ x \in X : \sum_{i=1}^{m(1)-1} |F_i(x)|^2 > \|F\|^2 - K_F/3 \right\} \subset X.$$

Set $F^{(1)} = F$, $F^{(2)} = (0, \dots, 0, F_{m(1)}, F_{m(1)+1}, \dots)$, where the first $m(1)-1$ terms are zeroes. Then $F^{(2)} \in H''_A \setminus H_A$ and $K_{F^{(2)}} = K_F \le \|F^{(2)}\|^2$.

There exists a number $m(2) > m(1)$ such that the estimate

$$\sum_{i=m(1)}^{m(2)-1} |F_i(x)|^2 > \|F^{(2)}\|^2 - K_F/3$$

holds for at least one $x \in X$.

Set

$$U_2 = \left\{ x \in X : \sum_{i=m(1)}^{m(2)-1} |F_i(x)|^2 > \|F^{(2)}\|^2 - K_F/3 \right\} \subset X.$$

Proceeding as above, we get an increasing sequence of numbers $m(k)$ and a sequence of nonempty open sets $U_k \subset X$ such that

$$U_k = \left\{ x \in X : \sum_{i=m(k-1)}^{m(k)-1} |F_i(x)|^2 > \|F^{(k)}\|^2 - K_F/3 \right\}.$$

Suppose that $\overline{U}_j \cap \overline{U}_l \neq \emptyset$ for some $j, l,\ j < l$. Take $x_0 \in \overline{U}_j \cap \overline{U}_l$. Then

$$\sum_{i=m(j-1)}^{m(j)-1} |F_i(x_0)|^2 \geq \|F^{(j)}\|^2 - K_F/3; \tag{2}$$

$$\sum_{i=m(l-1)}^{m(l)-1} |F_i(x_0)|^2 \geq \|F^{(l)}\|^2 - K_F/3 \geq K_F - K_F/3 = 2K_F/3. \tag{3}$$

Summing up (2) and (3), we get

$$\|F^{(j)}\|^2 \geq \sum_{i=m(j-1)}^{m(l)-1} |F_i(x_0)|^2 \geq \|F^{(j)}\|^2 - K_F/3 + 2K_F/3$$
$$= \|F^{(j)}\|^2 + K_F/3.$$

The obtained contradiction proves that the open sets U_k, $k \in \mathbb{N}$, (and their closures) do not intersect. Choose a sequence of points $x_k \in U_k$.

If $E \subset X$ is the (dense) set of continuity points of $\sum_i F_i^* F_i$ then one can assume also that $x_k \in E$ for each k.

Let $\mathbb{N} := \{x_1, x_2, \dots\}$. We wish to show that the closure $\overline{\mathbb{N}}$ of $\mathbb{N}$ in X is homeomorphic to $\beta\mathbb{N}$. This is equivalent (see Theorem 2.4) to the following property: any bounded function on $\mathbb{N}$ can be extended to a continuous function on $\overline{\mathbb{N}}$.

Our functional F should be able to be evaluated on elements of H'_A. In particular, take any bounded sequence $\{\lambda_k\}$, $\lambda_k \in \mathbb{C}$. Choose functions $g_k : X \to [0,1]$, $\operatorname{supp} g_k \subset U_k$, $g_k(x_k) = 1$. Then the sequence

$$(f_1, f_2, \dots) = (\lambda_1 |F_1(x_1)| g_1, \dots, \lambda_1 |F_{m(1)}(x_1)| g_1,$$
$$\lambda_2 |F_{m(1)+1}(x_2)| g_2, \dots, \lambda_2 |F_{m(2)}(x_2)| g_2, \dots)$$

belongs to H'_A. By Lemma 3.1, the series $\sum_i F_i^* f_i$ converges over $\mathbb{N}$ pointwise to a continuous function $F(f) \in C(X)$. In our case,

$$F(f)(x_k) = \lambda_k \cdot \sum_{i=m(k)}^{m(k+1)-1} F_i^*(x_k) F_i(x_k).$$

Thus, varying the sequence $\{\lambda_k\}$, we can obtain any bounded sequence of complex numbers, as the sequence of values $F(f)(x_k)$. Therefore, any bounded function on $\mathbb{N}$, which is automatically continuous on $\mathbb{N}$, should be extendable to a contuinuous function on $\overline{\mathbb{N}} \subset X$ and on entire X. □

5. A criterion for C^*-reflexivity

A more careful analysis of the argument in the previous theorem implies that instead of embedding $\mathbb{N}$ (and then $\beta\mathbb{N}$), we should embed something coarsely equivalent to $\mathbb{N}$, but more compatible with the topology on X.

Recall that if $\{A_k\}$ is a sequence of Banach spaces then one can form their direct product $\prod_k A_k$ (resp. direct sum $\oplus_k A_k$), which is the Banach space of all bounded sequences $(a_1, a_2, \ldots)$, $a_k \in A_k$, (resp. of all sequences with $\lim_{k\to\infty} \|a_k\| = 0$) with the norm $\|(a_1, a_2, \ldots)\| = \sup_k \|a_k\|$. If all A_k are C^*-algebras then both $\oplus_k A_k$ and $\prod_k A_k$ are C^*-algebras.

Let $U \subset X$ be an open subset. Then there is a canonical inclusion of $C_0(U) = \mathrm{Ker}\big(C(X) \to C(X \setminus U)\big)$ into $C(X)$. For a sequence $\{U_k\}$, $U_k \subset X$, $k \in \mathbb{N}$, of open disjoint sets, there is always a canonical inclusion $\oplus_k C_0(U_k) \subset C(X)$. Sometimes this canonical inclusion can be extended to an inclusion of $\prod_k C_0(U_k)$ into $C(X)$. In this case we call such inclusion canonical as well.

Existence of such inclusion of ideals can be expressed in topological terms: the canonical inclusion of $\sqcup_k U_k$ in X extends to the canonical inclusion of the Gelfand spectrum Y of $\prod_k C_0(U_k)$ into X.

Example 5.1. Suppose, $X = [0, 1]$ and $U_k = (\frac{1}{2^{k+1}}, \frac{1}{2^k})$. Then the inclusion $\oplus_k C_0(U_k) \subset C(X)$ doesn't extend to an inclusion of $\prod_k C_0(U_k)$. Indeed, if we take $f_k \in C_0(U_k)$ with $\|f_k\| = 1$ then the function on X that coincides with f_k on each U_k is not continuous on X.

Example 5.2. Let $X = \beta\mathbb{N}$, $U_k = \{k\} \subset \mathbb{N}$. Then $C_0(U_k) = \mathbb{C}$, and there is a canonical inclusion of $\prod_k C_0(U_k)$ into $l^\infty = C(\beta\mathbb{N})$ (in fact, they coincide).

Lemma 5.3. *Let $C(X)$ be not C^*-reflexive. Then there exists a sequence $\{U_k\}$ of disjoint open subsets of X such that $\prod_k C_0(U_k)$ is canonically included into $C(X)$.*

Proof. Let $(F_1, F_2, \ldots) \in H''_A \setminus H_A$. As in the proof of Theorem 4.1, one can construct

- a number $K > 0$ (one can take $K = 2K_F/3$);
- an increasing sequence $\{m(k)\}_{k\in\mathbb{N}}$ of integers;

- a sequence $U_1, U_2, \dots$ of open subsets of X

such that:

(1) $\overline{U}_i \cap \overline{U}_j = \emptyset$ if $i \neq j$;
(2) $K < \sum_{i-m(k)}^{m(k+1)-1} F_i^*(x) F_i(x) \leq \|F\|^2$ for any $x \in U_k$.

Let $\lambda_k \in C_0(U_k)$. Note that $\lambda_k \cdot F \in C_0(U_k) \subset C(X)$ for any $F \in C(X)$. Note that the function $g_k(x) = \sum_{i-m(k)}^{m(k+1)-1} F_i^*(x) F_i(x)$ is invertible on U_k. For a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$, set

$$\begin{aligned} f^\lambda &= (f_1, f_2, \dots) \\ &= (\lambda_1 F_1 g_1^{-1}, \dots, \lambda_1 F_{m(1)} g_1^{-1}, \lambda_2 F_{m(1)+1} g_2^{-1}, \dots, \lambda_2 F_{m(2)} g_2^{-1}, \dots). \end{aligned}$$

If the sequence λ is bounded (i.e., lies in $\prod_k C_0(U_k)$) then $f^\lambda \in H'_A$.

Let $Y = \overline{\sqcup_k U_k} \setminus \sqcup_k U_k$. Then $X \setminus Y$ is dense in X and, for any $x \in X \setminus Y$, the series $\sum_i F_i^*(x) f_i(x)$ converges either to 0, if $x \in X \setminus \sqcup_k U_k$, or to $\lambda_k(x)$, if $x \in U_k$.

Define a map $\prod_k C_0(U_k) \to C(X)$ by $\lambda \mapsto F(f^\lambda)$. It is well-defined due to continuity of $F(f)$ for each $f \in H'_A$. And it is obviously injective and coincides with the canonical inclusion of each $C_0(U_k)$ into $C(X)$. $\square$

Lemma 5.4. *Let there exist a sequence $\{I_k\}$ of nontrivial left ideals in a C^*-algebra A such that:*

(1) $I_k^* I_l = 0$ whenever $k \neq l$.
(2) $\prod_k I_k$ canonically embeds into A.

Then A is not C^*-reflexive.

Proof. Take $a_k \in I_k$ such that $\|a_k\| = 1$. Let $F = (a_1, a_2 \dots)$. As the series $\sum_k a_k^* a_k$ doesn't converge in norm, $F \notin H_A = l_2(A)$. Let us show that $F \in H''_A$. Take some $f = (f_1, f_2, \dots) \in H'_A$. Then we can define $F(f)$ as $F(f) = \sum_k a_k^* f_k := (a_1^* f_1, a_2^* f_2, \dots)$. As I_k is a left ideal, so $a_k^* f_k \in I_k$. As $f \in H'_A$, so the sequence $(a_1^* f, a_2^* f_2, \dots)$ is bounded, hence lies in $\prod_k I_k$, hence, by assumption, in A. Thus $F(f) \in A$ is well-defined. $\square$

So, we have proved the following theorem.

Theorem 5.5. *The module $l_2(C(X))$ is not C^*-reflexive if and only if there exists a sequence $\{U_k\}$ of open pairwise nonintersecting nonempty sets in X such that*

$$\prod_k C_0(U_k) \subset C(X).$$

Proof. This follows from the two preceding lemmas. If $A = C(X)$ then $C_0(U_k)$ are the (left) ideals required in the second Lemma. $\square$

Now, keeping in mind the Kasparov stabilization theorem and some evident topological argument, we can reformulate this theorem in the following way.

Theorem 5.6. *Any countable generated Hilbert C^*-module over $C(X)$ is C^*-reflexive if and only if there does not exist any sequence of orthogonal ideals $I_k \in C(X)$ such that $\prod_k I_k \subset C(X)$.*

We conjecture that the same condition gives a criterion for C^*-reflexivity for general (noncommutative) C^*-algebras.

6. An example

Let A be the C^*-subalgebra of l^∞ that consists of all sequences $\{a_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$. This C^*-algebra is the algebra of all continuous functions on the *Higson compactification* $\nu\mathbb{N}$ of $\mathbb{N}$ [11].

Theorem 6.1. *The C^*-algebra $A = C(\nu\mathbb{N})$ is C^*-reflexive.*

Proof. Assume the contrary. Then there exist disjoint open subsets U_k, $k \in \mathbb{N}$, of $\nu\mathbb{N}$ such that $\prod_k C_0(U_k) \subset A$. Being an open set of $\nu\mathbb{N}$, each U_k contains at least one point of $\mathbb{N}$. Let $n_k \in U_k$ be such a point. Each point of $\mathbb{N}$ is also an open set of $\nu\mathbb{N}$, and $\mathbb{C} \cong C_0(\{n_k\}) \subset C_0(U_k)$. Therefore, $\prod_k C_0(\{n_k\}) \subset A$. Take an arbitrary sequence

$$\{a_n\}_{n\in\mathbb{N}} \in \prod_k C_0(\{n_k\}).$$

Set $\mathbb{M} = \mathbb{N} \setminus \cup_k\{n_k\} = \{m_1, m_2, \ldots\}$. As $\{a_n\}_{n\in\mathbb{N}} \in \prod_k C_0(\{n_k\})$, so $a_n = 0$ for any $n \in \mathbb{M}$.

If $\mathbb{M}$ is finite then the sequence $\{a_n\}_{n\in\mathbb{N}} \in \prod_k C_0(\{n_k\})$ is (modulo several first terms) an arbitrary bounded sequence, which contradicts that this sequence lies in A. If $\mathbb{M}$ is infinite then for each n there is a number $m > n$ such that one can find integers k_1 and k_2 such that $n_{k_1} > m$, $m_{k_2} > m$ and $|n_{k_1} - m_{k_2}| = 1$. As $a_{m_{k_2}} = 0$ and $a_{n_{k_1}}$ may take an arbitrary value, so we get a contradiction with the condition $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$. Getting contradictions in both cases, we conclude that our assumption was false. □

Remark 6.2. Note that there exists a (noncanonical) inclusion $\beta\mathbb{N} \subset \nu\mathbb{N}$. Indeed, let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of integers such that

$$\lim_{k\to\infty} n_{k+1} - n_k = \infty.$$

Set $b_k = a_{n_k}$. The map $\{a_n\}_{n\in\mathbb{N}} \mapsto \{b_k\}_{k\in\mathbb{N}}$ gives a $*$-homomorphism from A to l^∞, and an easy check shows surjectivity of this map. Therefore, the map $k \mapsto n_k$ extends to a continuous injective map $\beta\mathbb{N} \to \nu\mathbb{N}$.

This shows that our sufficient condition from Section 4 is not a necessary condition.

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