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Image partition regularity of matrices near 0 with real entries

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Abstract. We prove that many of the known results regarding image partition regularity and image partition regularity near zero for finite and infinite matrices with rational or integer entries have natural analogues for matrices with real entries over the reals, extending work by N. Hindman.

Contents

1. Introduction

One of the famous classical results of Ramsey Theory is van der Waerden's Theorem [12], which states that if $r, l \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $a, d \in \mathbb{N}$ such that $\{a, a+d, \ldots, a+ld\} \subseteq A_i$. In other words it says that the entries of $M\vec{x}$, where

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & l \end{pmatrix}$$

and

$$\vec{x} = \begin{pmatrix} a \\ d \end{pmatrix} \in \mathbb{N}^2,$$

are monochromatic.

This suggests the following definition:

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Definition 1.1. Let $u, v \in \mathbb{N}$ and let M be a $u \times v$ matrix with entries from $\mathbb{Q}$. The matrix M is image partition regular over $\mathbb{N}$ if and only if whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in \mathbb{N}^v$ such that $M\vec{x} \in C_i^u$.

Characterizations of those matrices with entries from $\mathbb{Q}$ that are *image partition regular* over $\mathbb{N}$ were obtained in [8]. The underlying fact behind the characterization of finite image partition regular matrices is that whenever $\mathbb{N}$ is finitely colored one color should contain a *central set* [5, Theorem 8.8] and hence satisfies the conclusion of the Central Sets Theorem [5, Proposition 8.21]. Algebraic definition of *central set* is also available in [10, Definition 4.42]. After the algebraic definition of central sets it becomes immediate that any set containing a central set is central. A natural extension of this notion to *central set near zero* was introduced in [7, Definition 4.1]. For any dense subsemigroup of $(\mathbb{R}, +)$ it was observed that there are some subsets of $\mathbb{R}$ living near zero which also satisfy some version of the Central Sets Theorem. Motivated by this observation, the notion of *Image Partition Regularity near zero* was introduced in [1].

Definition 1.2. Let S be a subsemigroup of $(\mathbb{R}, +)$ with $0 \in c\ell S$, let $u, v \in \mathbb{N}$, and let M be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then M is *image partition regular over S near zero* (abbreviated IPR/S_0) if and only if, whenever $S \setminus \{0\}$ is finitely colored and $\delta > 0$, there exists $\vec{x} \in S^v$ such that the entries of $M\vec{x}$ are monochromatic and lie in the interval $(-\delta, \delta)$.

The matrices which are *image partition regular near zero* are very interesting in themselves. *Image partition regularity near zero* over various subsemigroups of $(\mathbb{R}, +)$ containing 0 in the closure has been investigated extensively in [1]. One of the main objectives was that for finite matrices with rational entries *image partition regularity* and *image partition regularity near zero* are equivalent. But when we turn our attention to admissible infinite matrices these two notions are not equivalent. Here, an admissible infinite matrix is an $\omega \times \omega$ matrix, each row of which contains only finitely many nonzero elements, where ω is the first infinite ordinal number.

The characterization of finite matrices with real entries that are *image partition regular over* $\mathbb{R}^+$ were obtained in [6]. The definitions of *image partition regularity* and *image partition regularity near zero* for finite matrices with entries from $\mathbb{R}$ have natural generalizations for admissible infinite matrices.

In this paper we shall investigate matrices with real entries that are *image partition regular near zero* over $\mathbb{R}^+$. In Section 2 we shall characterize the finite matrices with real entries that are *image partition regular near zero* over $\mathbb{R}^+$ and prove that those matrices are actually the matrices with real entries that are image partition regular over $\mathbb{R}^+$. The following is a repetition of definition for finite matrices with entries from $\mathbb{R}$.

Definition 1.3. Let $u, v \in \mathbb{N}$ and M be a $u \times v$ matrix with entries from $\mathbb{R}$. Then M is *image partition regular over* $\mathbb{R}^+$ *near zero* (abbreviated IPR/$\mathbb{R}_0^+$) if and only if, whenever $\mathbb{R}^+$ is finitely colored and $\delta > 0$, there exists $\vec{x} \in (\mathbb{R}^+)^v$ such that the entries of $M\vec{x}$ are monochromatic and lie in the interval $(0, \delta)$.

In Section 3 we shall prove that the Milliken–Taylor matrices and insertion matrices (which we shall define in Definition 3.3 and Definition 3.7 respectively) with real entries are *image partition regular near zero* over $\mathbb{R}^+$.

We complete this section with some necessary facts regarding the compact right topological semigroup $(\beta\mathbb{R}_d^+, +)$, where $\mathbb{R}_d^+$ means $\mathbb{R}^+$ with the discrete topology. For details regarding the algebraic structure of βS_d for an arbitrary dense subsemigroup S of $\mathbb{R}^+$ one can see [7].

Definition 1.4. Define $0^+(\mathbb{R}^+) = \{p \in \beta\mathbb{R}_d^+ : (\forall\epsilon > 0)((0, \epsilon) \in p)\}$.

It is proved in [7, Lemma 2.5] that $0^+(\mathbb{R}^+)$ is a compact right topological subsemigroup of $(\beta\mathbb{R}_d^+, +)$ and therefore contains minimal idempotents of $0^+(\mathbb{R}^+)$.

Definition 1.5. A set $C \subseteq \mathbb{R}^+$ is *central near zero* if and only if there is a minimal idempotent p of $0^+(\mathbb{R}^+)$ containing C.

For our purpose we state the following version of Central Sets Theorem which follows directly from [1, Corollary 4.7].

Theorem 1.6. *Let $\mathcal{T}$ be the set of all those sequences $\langle y_n\rangle_{n=1}^\infty$ in $\mathbb{R}$ such that $\lim_{n\to\infty} y_n = 0$ and let C be central near zero in $\mathbb{R}^+$. Let $F \in \mathcal{P}_f(\mathcal{T})$. There exist a sequence $\langle a_n\rangle_{n=1}^\infty$ in $\mathbb{R}^+$ such that $\sum_{n=1}^\infty a_n$ converges and a sequence $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and for each $L \in \mathcal{P}_f(\mathbb{N})$ and each $f \in F$, $\sum_{n\in L}(a_n + \sum_{t\in H_n} f(t)) \in C$.*

The Central Sets Theorem was first discovered by H. Furstenberg [5, Proposition 8.21] for the semigroup $\mathbb{N}$ and considering sequences in $\mathbb{Z}$. However the most general version of Central Sets Theorem is available in [2, Theorem 2.2].

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2. Finite matrices

We show in this section that for finite matrices with entries from $\mathbb{R}$ *image partition regularity near zero* depends only on the fact that they have images in any central sets near zero. We start with the following well-known definition.

Definition 2.1. Let M be a $u \times v$ matrix with entries from $\mathbb{R}$. Then M is a *first entries matrix* if:

(1) No row of M is row is $\vec{0}$.
(2) The first nonzero entry of each row is positive.
(3) If the first nonzero entries of any two rows occur in the same column, then those entries are equal.

If M is a first entries matrix and c is the first nonzero entry of some row M, then c is called a *first entry* of M.

Now we prove a lemma which is important for determination of finite matrices with real entries that are *image partition regularity near zero* over $\mathbb{R}^+$.

Lemma 2.2. *Let M be a $u \times v$ first entry matrix with entries from $\mathbb{R}$ and let C be a central set near zero in $\mathbb{R}^+$. Then there exist sequences $\langle x_{1,n}\rangle_{n=1}^{\infty}$, $\langle x_{2,n}\rangle_{n=1}^{\infty}, \ldots, \langle x_{v,n}\rangle_{n=1}^{\infty}$ in $\mathbb{R}^+$ such that for any $i \in \{1, 2, \ldots, v\}$, $\sum_{t=1}^{\infty} x_{i,t}$ converges and for each $F \in \mathcal{P}_f(\mathbb{N})$, $M\vec{x}_F \in C^u$, where*

$$\vec{x}_F = \begin{pmatrix} \sum_{t\in F} x_{1,t} \\ \vdots \\ \sum_{t\in F} x_{v,t} \end{pmatrix}.$$

Proof. Let C be a central set near zero in $\mathbb{R}^+$. We proceed by induction on v. Assume first that $v = 1$. We can assume M has no repeated rows, so in this case we have $M = (c)$ for some $c \in \mathbb{R}^+$. Pick a sequence $\langle y_n\rangle_{n=1}^{\infty}$ [7, Theorem 3.1] with $FS(\langle y_n\rangle_{n=1}^{\infty}) \subseteq C$, where $\sum_{n=1}^{\infty} y_n$ converges and for each $n \in \mathbb{N}$ let $x_{1,n} = \frac{y_n}{c}$. The sequence $\langle x_{1,n}\rangle_{n=1}^{\infty}$ is as required.

Now let $v \in \mathbb{N}$ and assume the theorem is true for v. Let M be a $u\times(v+1)$ first entries matrix with entries from $\mathbb{R}$. By rearranging the rows of M and adding additional rows to M if need be, we may assume that we have some $r \in \{1, 2, \ldots, u-1\}$ and some $d \in \mathbb{R}^+$ such that

$$a_{i,1} = \begin{cases} 0 & \text{if } i \in \{1, 2, \ldots, r\}, \\ d & \text{if } i \in \{r+1, r+2, \ldots, u\}. \end{cases}$$

Let B be the $r\times v$ matrix with entries $b_{i,j} = a_{i,j+1}$. Pick sequences $\langle z_{1,n}\rangle_{n=1}^{\infty}$, $\langle z_{2,n}\rangle_{n=1}^{\infty}$, $\ldots$, $\langle z_{v,n}\rangle_{n=1}^{\infty}$ in $\mathbb{R}^+$ as guaranteed by the induction hypothesis for the matrix B. For each $i \in \{r+1, r+2, \ldots, u\}$ and each $n \in \mathbb{N}$, let

$$y_{i,n} = \sum_{j=2}^{v+1} a_{i,j} z_{j-1,n}.$$

Now for each $i \in \{r+1, r+2, \ldots, u\}$, $\sum_{t=1}^{\infty} y_{i,t}$ converges. We take $y_{r,n} = 0$ for all $n \in \mathbb{N}$.

Now C being central set near zero in $\mathbb{R}^+$, by Theorem 1.6 we can pick sequence $\langle k_n\rangle_{n=1}^{\infty}$ in $\mathbb{R}^+$ and a sequence $\langle H_n\rangle_{n=1}^{\infty}$ of finite nonempty subsets of $\mathbb{N}$ such that $\max H_n < \min H_{n+1}$ for each n and for each $i \in \{r, r+1, \ldots, u\}$, $FS(\langle k_n + \sum_{t\in H_n} y_{i,t}\rangle_{n=1}^{\infty}) \subseteq C$, where $\sum_{n=1}^{\infty} k_n$ converges.

For each $n \in \mathbb{N}$, let $x_{1,n} = \frac{k_n}{d}$ and note that $k_n = k_n + \sum_{t\in H_n} y_{r,t} \in C \subseteq \mathbb{R}^+$. For $j \in \{2, 3, \ldots, v+1\}$, let $x_{j,n} = \sum_{t\in H_n} z_{j-1,t}$. Certainly for each $i \in \{1, 2, \ldots, v+1\}$, $\sum_{t=1}^{\infty} x_{i,t}$ converges. We claim that the sequences $\langle x_{j,n}\rangle_{n=1}^{\infty}$ are as required. To see this, let F be a finite nonempty subset of $\mathbb{N}$. We need to show that for each $i \in \{1, 2, \ldots, u\}$, $\sum_{j=1}^{v+1} a_{i,j} \sum_{n\in F} x_{j,n} \in C$. So let $i \in \{1, 2, \ldots, u\}$ be given.

Case 1. $i \leq r$. Then

$$\begin{aligned}\sum_{j=1}^{v+1} a_{i,j} \cdot \sum_{n\in F} x_{j,n} &= \sum_{j=2}^{v+1} a_{i,j} \cdot \sum_{n\in F} \sum_{t\in H_n} z_{j-1,t} \\ &= \sum_{j=1}^{v} b_{i,j} \cdot \sum_{t\in G} z_{j,t} \in C.\end{aligned}$$

where $G = \bigcup_{n\in F} H_n$.

Case 2. $i > r$. Then

$$\begin{aligned}\sum_{j=1}^{v+1} a_{i,j} \cdot \sum_{n\in F} x_{j,n} &= d \cdot \sum_{n\in F} x_{1,n} + \sum_{j=2}^{v+1} a_{i,j} \cdot \sum_{n\in F} x_{j,n} \\ &= \sum_{n\in F} d x_{1,n} + \sum_{n\in F} \sum_{t\in H_n} \sum_{j=2}^{v+1} a_{i,j} z_{j-1,t} \\ &= \sum_{n\in F} \left(k_n + \sum_{t\in H_n} y_{i,t} \right) \in C. \qquad \square\end{aligned}$$

Theorem 2.3. *Let M be a $u \times v$ matrix with entries from $\mathbb{R}$. Then the following are equivalent.*

(1) *M is IPR/$\mathbb{R}^+$.*
(2) *M is IPR/$\mathbb{R}_0^+$.*
(3) *For every central set C in $\mathbb{R}^+$ there exists $\vec{x} \in (\mathbb{R}^+)^v$ such that $M\vec{x} \in C^u$.*
(4) *For every central set near zero C in $\mathbb{R}^+$ there exists $\vec{x} \in (\mathbb{R}^+)^v$ such that $M\vec{x} \in C^u$.*

Proof. (1) ⇒ (4) Let C be a central set near zero in $\mathbb{R}^+$ and M be IPR/$\mathbb{R}^+$. So by [6, Theorem 4.1], there exist $m \in \{1, 2, \ldots u\}$ and a $u \times m$ first entries matrix B such that for all $\vec{y} \in (\mathbb{R}^+)^m$ there exists $\vec{x} \in (\mathbb{R}^+)^v$ such that $M\vec{x} = B\vec{y}$. Then by the above Lemma 2.2 there exist some $\vec{y} \in (\mathbb{R}^+)^m$ such that $B\vec{y} \in C^u$.

(4) ⇒ (2) Let $\bigcup_{i=1}^{r} C_i$ be a finite partition of $\mathbb{R}^+$ and $\epsilon > 0$. Then some C_i is central near zero so that $C_i \cap (0, \epsilon)$ is also central near zero. Hence by (4) there exists $\vec{x} \in (\mathbb{R}^+)^v$ such that $M\vec{x} \in C_i \cap (0, \epsilon)$.

(2) ⇒ (1) is obvious.

(1) ⇔ (3) follows from [6, Theorem 4.1]. □

3. Infinite matrices

We now prove that certain infinite matrices with real entries are also image partition regular near zero over $\mathbb{R}^+$. K. Milliken and A. Taylor independently proved a theorem from which it can be derived that certain infinite matrices, called Milliken–Taylor Matrices, are image partition regular over $\mathbb{N}$. Some generalizations of this celebrated theorem are also available in [9, Corollary 3.6], [1, Theorem 5.7], [3, Theorem 2.6].

Definition 3.1. Let $m \in \omega$, let $\vec{a} = \langle a_i \rangle_{i=0}^m$ be a sequence in $\mathbb{R} \setminus \{0\}$, and let $\vec{x} = \langle x_n \rangle_{n=0}^\infty$ be a sequence in $\mathbb{R}$. The *Milliken–Taylor system* determined by $\vec{a}$ and $\vec{x}$, $MT(\vec{a}, \vec{x}) = \{\sum_{i=0}^m a_i \cdot \sum_{t \in F_i} x_t :$ each $F_i \in \mathcal{P}_f(\omega)$ and if $i < m$, then $\max F_i < \min F_{i+1}\}$

If $\vec{c}$ is obtained from $\vec{a}$ by deleting repetitions, then for any infinite sequence $\vec{x}$, one has $MT(\vec{a}, \vec{x}) \subseteq MT(\vec{c}, \vec{x})$, so it suffices to consider sequences $\vec{c}$ without adjacent repeated entries.

Definition 3.2. Let $\vec{a}$ be a finite or infinite sequence in $\mathbb{R}$ with only finitely many nonzero entries. Then $c(\vec{a})$ is the sequence obtained from $\vec{a}$ by deleting all zeroes and then deleting all adjacent repeated entries. The sequence $c(\vec{a})$ is the *compressed form* of $\vec{a}$. If $\vec{a} = c(\vec{a})$, then $\vec{a}$ is a *compressed sequence*.

Definition 3.3. Let $\vec{a}$ be a compressed sequence in $\mathbb{R} \setminus \{0\}$. A *Milliken–Taylor matrix determined by $\vec{a}$* is an $\omega \times \omega$ matrix M such that the rows of M are all possible rows with finitely many nonzero entries and compressed form equal to $\vec{a}$.

It follows from [7, Lemma 2.5] that

$$\begin{aligned} 0^+(\mathbb{R}) &= \big\{p \in \beta\mathbb{R}_d : (\forall \epsilon > 0)\big((0, \epsilon) \in p\big)\big\} \quad \text{and} \\ 0^-(\mathbb{R}) &= \big\{p \in \beta\mathbb{R}_d : (\forall \epsilon > 0)\big((-\epsilon, 0) \in p\big)\big\} \end{aligned}$$

are both right ideals of $0^+(\mathbb{R}) \cup 0^-(\mathbb{R})$. Further given $c \in \mathbb{R} \setminus \{0\}$ and $p \in \beta\mathbb{R}_d \setminus \{0\}$, the product $c \cdot p$ is defined in $(\beta\mathbb{R}_d, \cdot)$. One has $A \subseteq \mathbb{R}$ is a member of $c \cdot p$ if and only if $c^{-1}A = \{x \in \mathbb{R} : c \cdot x \in A\}$ is a member of p.

Lemma 3.4. *Let $p \in 0^+(\mathbb{R})$, and let $c \in \mathbb{R}^+$. Then $c \cdot p \in 0^+(\mathbb{R})$ and $(-c) \cdot p \in 0^-(\mathbb{R})$.*

Proof. Let $\epsilon > 0$. We need to show that $(0, \epsilon) \in c \cdot p$. Now $(0, \epsilon/c) \in p$. But $(0, \epsilon/c) \subseteq c^{-1}(0, \epsilon)$. Hence $c^{-1}(0, \epsilon) \in p$ so that $(0, \epsilon) \in c \cdot p$.

The proof of the other is similar. □

Definition 3.5. Let $\langle w_n \rangle_{n=0}^\infty$ be a sequence in $\mathbb{R}$. A *sum subsystem* of $\langle w_n \rangle_{n=0}^\infty$ is a sequence $\langle x_n \rangle_{n=0}^\infty$ such that there exists a sequence $\langle H_n \rangle_{n=0}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \omega$, $\max H_n < \min H_{n+1}$ and $x_n = \sum_{t \in H_n} w_t$.

Recall $FS(\langle x_n \rangle_{n=k}^\infty) = \{\sum_{n \in F} x_n : F \in \mathcal{P}_f(\omega)$ and $\min F \geq k\}$, for a given a sequence $\langle x_n \rangle_{n=0}^\infty$ and $k \in \omega$. The proof of the following theorem is taken nearly verbatim from [1, Theorem 5.7].

Theorem 3.6. *Let $m \in \omega$ and $\vec{a} = \langle a_i \rangle_{i=0}^m$ be a compressed sequence in $\mathbb{R} \setminus \{0\}$ with $a_0 > 0$, and let M be a Milliken–Taylor matrix determined by $\vec{a}$, Then M is IPR/$\mathbb{R}_0^+$. In fact, given any sequence $\langle w_n \rangle_{n=0}^\infty$ in $\mathbb{R}^+$ such that $\sum_{n=0}^\infty w_n$ converges, whenever $r \in \mathbb{N}$, $\mathbb{R}^+ = \bigcup_{i=1}^r C_i$, and $\delta > 0$, there exist $i \in \{1, 2, \ldots, r\}$ and a sum subsystem $\langle x_n \rangle_{n=0}^\infty$ of $\langle w_n \rangle_{n=0}^\infty$ such that $MT(\vec{a}, \vec{x}) \subseteq C_i \cap (0, \delta)$.*

Proof. By [10, Lemma 5.11] we can choose an idempotent

$$p \in \bigcap_{k=0}^{\infty} c\ell_{\beta\mathbb{R}_d} FS(\langle w_n \rangle_{n=k}^\infty).$$

Since $\sum_{n=0}^\infty w_n$ converges, $p \in 0^+(\mathbb{R})$. Let $q = a_0 \cdot p + a_1 \cdot p + \cdots + a_m \cdot p$. Then by Lemma 3.4 and the previously mentioned fact that $0^+(\mathbb{R})$ and $0^-(\mathbb{R})$ are both right ideals of $0^+(\mathbb{R}) \cup 0^-(\mathbb{R})$, we have that $q \in 0^+(\mathbb{R})$. So it suffices to show that whenever $Q \in q$, there is a sum subsystem $\langle x_n \rangle_{n=0}^\infty$ of $\langle w_n \rangle_{n=0}^\infty$ such that $MT(\vec{a}, \vec{x}) \subseteq Q$. Let $Q \in q$ be given. Assume first that $m = 0$. Then $(a_0)^{-1} Q \in p$ so by [10, Theorem 5.14] there is a sum subsystem $\langle x_n \rangle_{n=0}^\infty$ of $\langle w_n \rangle_{n=0}^\infty$ such that $FS(\langle x_n \rangle_{n=0}^\infty) \subseteq (a_0)^{-1} Q$. Then $MT(\vec{a}, \vec{x}) \subseteq Q$. Now assume that $m > 0$. Define

$$P(\emptyset) = \{x \in \mathbb{R} : -(a_0 \cdot x) + Q \in a_1 \cdot p + a_2 \cdot p + \cdots + a_m \cdot p\}.$$

Since $Q \in q$ we have

$$a_0^{-1} \cdot \{x \in \mathbb{R} : -x + Q \in a_1 \cdot p + a_2 \cdot p + \cdots + a_m \cdot p\} \in p$$

which shows that

$$P(\emptyset) = \{x \in \mathbb{R} : -(a_0 \cdot x) + Q \in a_1 \cdot p + a_2 \cdot p + \cdots + a_m \cdot p\} \in p.$$

Given x_0 define

$$P(x_0) = \{y \in \mathbb{R} : -(a_0 \cdot x_0 + a_1 \cdot y) + Q \in a_2 \cdot p + a_3 \cdot p + \cdots + a_m \cdot p\}.$$

If $x_0 \in P(\emptyset)$, then $-(a_0 \cdot x_0) + Q \in a_1 \cdot p + a_2 \cdot p + \cdots + a_m \cdot p$ and so

$$\{y \in \mathbb{R} : -(a_1 \cdot y) + (-(a_0 \cdot x_0) + Q) \in a_2 \cdot p + a_3 \cdot p + \cdots + a_m \cdot p\} \in p$$

and thus $P(x_0) \in p$.

Given $n \in \{1, 2, \ldots, m-1\}$ and $x_0, x_1, \ldots, x_{n-1}$, let $P(x_0, x_1, \ldots, x_{n-1}) = \{y \in \mathbb{R} : -(a_0 \cdot x_0 + \cdots + a_{n-1} \cdot x_{n-1} + a_n \cdot y) + Q \in a_{n+1} \cdot p + \cdots + a_m \cdot p\}$. If $x_0 \in P(\emptyset)$ and for each $i \in \{1, 2, \ldots, n-1\}$, $x_i \in P(x_0, x_1, \ldots, x_{i-1})$, then $P(x_0, x_1, \ldots, x_{n-1}) \in p$.

Now given $x_0, x_1, \ldots, x_{m-1}$, let

$$\begin{aligned} &P(x_0, x_1, \ldots, x_{m-1}) \\ &\qquad = \{y \in \mathbb{R} : a_0 \cdot x_0 + a_1 \cdot x_1 + \cdots + a_{m-1} \cdot x_{m-1} + a_m \cdot y \in Q\}. \end{aligned}$$

If $x_0 \in P(\emptyset)$ and for each $i \in \{1, 2, \ldots, m-1\}$, $x_i \in P(x_0, x_1, \ldots, x_{i-1})$, then $P(x_0, x_1, \ldots, x_{m-1}) \in p$. Given any $B \in p$, let $B^\star = \{x \in B : -x + B \in p\}$. Then $B^\star \in p$ and by [10, Lemma 4.14], for each $x \in B^\star$, $-x + B^\star \in p$.

Choose $x_0 \in P(\emptyset)^\star \cap FS(\langle w_n\rangle_{n=0}^\infty)$ and choose $H_0 \in \mathcal{P}_f(\mathbb{N})$ such that $x_0 = \sum_{t\in H_0} w_t$. Let $n \in \omega$ and assume that we have chosen $x_0, x_1, \ldots, x_n$ and $H_0, H_1, \ldots, H_n$ such that:

(1) If $k \in \{0, 1, \ldots, n\}$, then $H_k \in \mathcal{P}_f(\omega)$ and $x_k = \sum_{t\in H_k} w_t$.
(2) If $k \in \{0, 1, \ldots, n-1\}$, then $\max H_k < \min H_{k+1}$.
(3) If $\emptyset \neq F \subseteq \{0, 1, \ldots, n\}$, then $\sum_{t\in F} x_t \in P(\emptyset)^\star$.
(4) If $k \in \{1, 2, \ldots, \min\{m, n\}\}$, $F_0, F_1, \ldots, F_k \in \mathcal{P}_f(\{0, 1, \ldots, n\})$, and for each $j \in \{0, 1, \ldots, k-1\}$, $\max F_j < \min F_{j+1}$, then

$$\sum_{t\in F_k} x_t \in P\left(\sum_{t\in F_0} x_t, \sum_{t\in F_1} x_t \ldots, \sum_{t\in F_{k-1}} x_t\right)^\star.$$

All hypotheses hold at $n = 0$, (2) and (4) vacuously.

Let $v = \max H_n$. For $r \in \{0, 1, \ldots, n\}$, let

$$E_r = \left\{\textstyle\sum_{t\in F} x_t : \emptyset \neq F \subseteq \{r, r+1, \ldots, n\}\right\}.$$

For $k \in \{0, 1, \ldots, m-1\}$ and $r \in \{0, 1, \ldots, n\}$, let

$$W_{k,r} = \left\{\left(\sum_{t\in F_0} x_t, \ldots, \sum_{t\in F_k} x_t\right) : F_0, F_1, \ldots, F_k \in \mathcal{P}_f(\{0, 1, \ldots, r\}) \right.$$
$$\left. \text{and for each } i \in \{0, 1, \ldots, k-1\}, \max F_i < \min F_{i+1}\right\}$$

Note that $W_{k,r} \neq \emptyset$ if and only if $k \leq r$.

If $y \in E_0$, then $y \in P(\emptyset)^\star$, so $-y + P(\emptyset)^\star \in p$ and $P(y) \in p$. If $k \in \{1, 2, \ldots, m-1\}$ and $(y_0, y_1, \ldots, y_k) \in W_{k,m}$, then $y_k \in P(y_0, y_1, \ldots, y_{k-1})$ so $P(y_0, y_1, \ldots, y_k) \in p$ and thus $P(y_0, y_1, \ldots, y_k)^\star \in p$. If $r \in \{0, 1, \ldots, n-1\}$, $k \in \{0, 1, \ldots, \min\{m-1, r\}\}$, $(y_0, y_1, \ldots, y_k) \in W_{k,r}$, and $z \in E_{r+1}$, then $z \in P(y_0, y_1, \ldots, y_k)^\star$ so $-z + P(y_0, y_1, \ldots, y_k)^\star \in p$.

If $n = 0$, let $x_1 \in FS(\langle w_t\rangle_{t=v+1}^\infty) \cap P(\emptyset)^\star \cap \{-x_0 + P(\emptyset)^\star\} \cap P(x_0)^\star$ and pick $H_1 \in \mathcal{P}_f(\mathbb{N})$ such that $\min H_1 > v$ and $x_1 = \sum_{t\in H_1} w_t$. The hypotheses are satisfied. Now assume that $n \geq 1$ and pick

$$\begin{aligned} x_{n+1} \in\ & FS(\langle w_t\rangle_{t=v+1}^\infty) \cap P(\emptyset)^\star \cap \bigcap_{y\in E_0} \{-y + P(\emptyset)^\star\} \\ & \cap \bigcap_{k=0}^{\min\{m-1,n\}} \bigcap_{(y_0,y_1,\ldots,y_k)\in W_{k,m}} P(y_0, y_1, \ldots, y_k)^\star \\ & \cap \bigcap_{r=0}^{n-1} \bigcap_{k=0}^{\min\{m-1,r\}} \bigcap_{(y_0,y_1,\ldots,y_k)\in W_{k,r}} \bigcap_{z\in E_{r+1}} \left(-z + P(y_0, y_1, \ldots, y_k)^\star\right). \end{aligned}$$

Pick $H_{n+1} \in \mathcal{P}_f(\mathbb{N})$ such that $\min H_{n+1} > v$ and $x_{n+1} = \sum_{t\in H_{n+1}} w_t$.

Hypotheses (1) and (2) hold directly. For hypothesis (3) assume that $\emptyset \neq F \subseteq \{0, 1, \ldots, n+1\}$ and $n+1 \in F$. If $F = \{n+1\}$ we have directly that $x_{n+1} \in P(\emptyset)^\star$, so assume that $\{n+1\} \subsetneq F$ and let $G = F \setminus \{n+1\}$. Let $y = \sum_{t\in G} x_t$. Then $y \in E_0$ so $x_{n+1} \in -y + P(\emptyset)^\star$ and so $\sum_{t\in F} x_t \in P(\emptyset)^\star$.

To verify hypothesis (4), let $k \in \{1, 2, \ldots, \min\{m, n+1\}\}$ and assume that $F_0, F_1, \ldots, F_k \in \mathcal{P}_f(\{0, 1, \ldots, n+1\})$ and for each $j \in \{0, 1, \ldots, k-1\}$, $\max F_j < \min F_{j+1}$. We can assume that $n+1 \in F_k$. For $l \in \{0, 1, \ldots, k-1\}$ let $y_l = \sum_{t\in F_l} x_t$. Then $k-1 \le \min\{m-1, n\}$ and $(y_0, y_1, \ldots, y_{k-1}) \in W_{k-1,m}$. If $F_k = \{n+1\}$, then $\sum_{t\in F_k} x_t = x_{n+1} \in P(y_0, y_1, \ldots, y_{k-1})^\star$. So assume that $\{n+1\} \subsetneq F_k$ and let $F'_k = F_k \setminus \{n+1\}$. Let $r = \max F_{k-1}$. Then $r < \min F'_k$ so $r \le n-1$, $k-1 \le \min\{m-1, r\}$, and $(y_0, y_1, \ldots, y_{k-1}) \in W_{k-1,r}$. Let $z = \sum_{t\in F'_k} x_t$. Then $z \in E_{r+1}$ so $x_{n+1} \in -z + P(y_0, y_1, \ldots, y_{k-1})^\star$. Hence we have

$$\sum_{t\in F_k} x_t \in P\left(\sum_{t\in F_0} x_t, \sum_{t\in F_1} x_t \ldots, \sum_{t\in F_{k-1}} x_t\right)^\star. \qquad \square$$

Next we define another class of infinite matrices with real entries, called *insertion* matrices, which are also image partition regular near zero over $\mathbb{R}^+$. The notion of insertion matrix was first introduced in [11, Definition 4.8].

Definition 3.7. Let γ, $\delta \in \omega \bigcup \{\omega\}$ and let C be a $\gamma \times \delta$ matrix with finitely many nonzero entries in each row. For each $t < \delta$, let B_t be a finite matrix of dimension $u_t \times v_t$. Let $R = \{(i, j) : i < \gamma \text{ and } j \in \times_{t<\delta}\{0, 1, \ldots, u_t - 1\}\}$. Given $t < \delta$ and $k \in \{0, 1, \ldots, u_t - 1\}$, denote by $\vec{b}_k^{(t)}$ the k^{th} row of B_t. Then D is an insertion matrix of $\langle B_t\rangle_{t<\delta}$ into C if and only if the rows of D are all rows of the form

$$c_{i,0} \cdot \vec{b}^{(0)}_{j(0)} \frown c_{i,1} \cdot \vec{b}^{(1)}_{j(1)} \frown \cdots$$

where $(i, j) \in R$.

For example, if

$$C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 1 & 1 \\ 5 & 7 \end{pmatrix}$$

and

$$B_1 = \begin{pmatrix} 0 & 1 \\ 3 & 3 \end{pmatrix},$$

then the insertion matrix is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 5 & 7 & 0 & 0 \\ 5 & 7 & 0 & 0 \\ 2 & 2 & 0 & 1 \\ 2 & 2 & 3 & 3 \\ 10 & 14 & 0 & 1 \\ 10 & 14 & 3 & 3 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 5 & 7 & 0 & 0 \\ 2 & 2 & 0 & 1 \\ 2 & 2 & 3 & 3 \\ 10 & 14 & 0 & 1 \\ 10 & 14 & 3 & 3 \end{pmatrix}$$

is an insertion matrix of $\langle B_t \rangle_{t<2}$ into C.

We now show that if $\vec{a} = \langle a_i \rangle_{i=0}^{l}$ is a compressed sequence in $\mathbb{R} \setminus 0$ with $l > 0$ and $a_0 > 0$ then the insertion matrix is $IPR/\mathbb{R}_0^+$. The proof of the following theorem is similar to that of [11, Theorem 4.12].

Theorem 3.8. *Let $\vec{a} = \langle a_0, a_1, \ldots, a_l \rangle$ be a compressed sequence in $\mathbb{R} \setminus 0$ with $l > 0$ and $a_0 > 0$, let C be a Milliken–Taylor matrix for $\vec{a}$, and for each $t < \omega$, let B_t be a $u_t \times v_t$ finite matrx with entries from $\mathbb{R}$ which is image partition regular matrix over $\mathbb{R}^+$. Then any insertion matrix of $\langle B_t \rangle_{t<\omega}$ into C is image partition regular near zero, i.e., $IPR/\mathbb{R}_0^+$.*

Proof. Pick by [7, Lemma 2.5] and [10, Corollary 2.6] some minimal idempotent p of $0^+(\mathbb{R})$. Let $q = a_0 \cdot p + a_1 \cdot p + \cdots + a_l \cdot p$. Then by Lemma 3.4 and the previously mentioned fact that $0^+(\mathbb{R})$ and $0^-(\mathbb{R})$ are both right ideals of $0^+(\mathbb{R}) \cup 0^-(\mathbb{R})$, we have that $q \in 0^+(\mathbb{R})$. Let $\mathcal{G}$ be a finite partition of $\mathbb{R}^+$ and pick $A \in \mathcal{G}$ such that $A \in q$.

Now $\{x \in \mathbb{R} : -x + A \in a_1 \cdot p + \cdots + a_l \cdot p\} \in a_0 \cdot p$ so that

$$D_0 = \{x \in \mathbb{R} : -a_0 \cdot x + A \in a_1 \cdot p + \cdots + a_l \cdot p\} \in p.$$

Then $D_0^\star \in p$ (as used in Theorem 3.6).

Let $\alpha_0 = 0$ and inductively let $\alpha_{n+1} = \alpha_n + v_n$.

So pick by Theorem 2.3, $x_0, x_1, \ldots, x_{\alpha_1 - 1} \in \mathbb{R}^+$ such that

$$B_0 \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{\alpha_1 - 1} \end{pmatrix} \in (D_0^\star)^{u_0}.$$

Let H_0 be the set of entries of

$$B_0 \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{\alpha_1 - 1} \end{pmatrix}.$$

Inductively, let $n \in \mathbb{N}$ and assume that we have chosen $\langle x_t \rangle_{t=0}^{\alpha_n - 1}$ in $\mathbb{R}^+$, $\langle D_k \rangle_{k=0}^{n-1}$ in p, and $\langle H_k \rangle_{k=0}^{n-1}$ in the set $\mathcal{P}_f(\mathbb{N})$ of finite nonempty subsets of $\mathbb{N}$ such that for $r \in \{0, 1, \ldots, n-1\}$:

(I) H_r is the set of entries of

$$B_r \begin{pmatrix} x_{\alpha_r} \\ x_{\alpha_r + 1} \\ \vdots \\ x_{\alpha_{r+1} - 1} \end{pmatrix}.$$

(II) If $\emptyset \neq F \subseteq \{0, 1, \ldots, r\}$, $k = \min F$, and for each $t \in F$, $y_t \in H_t$, then $\sum_{t \in F} y_t \in D_k^\star$.

(III) If $r < n-1$, then $D_{r+1} \subseteq D_r$.

(IV) If $m \in \{0, 1, \ldots, l-1\}$ and $F_0, F_1, \ldots, F_m$ are all nonempty subsets of $\{0, 1, \ldots, r\}$, for each $i \in \{0, 1, \ldots, m-1\}$, $\max F_i < \min F_{i+1}$, and for each $t \in \bigcup_{i=0}^m F_i$, $y_t \in H_t$, then

$$-\sum_{t=0}^{m} a_i \cdot \sum_{t \in F_i} y_t + A \in a_{m+1} \cdot p + a_{m+2} \cdot p + \cdots + a_l \cdot p.$$

(V) If $r < n-1$, $F_0, F_1, \ldots, F_{l-1}$ are nonempty subsets of $\{0, 1, \ldots, r\}$, for each $i \in \{0, 1, \ldots, m-1\}$, $\max F_i < \min F_{i+1}$, and for each $t \in \bigcup_{i=0}^m F_i$, $y_t \in H_t$, then $D_{r+1} \subseteq a_l^{-1}(-\sum_{t=0}^{l-1} a_i \cdot \sum_{t \in F_i} y_t + A)$.

(VI) if $r < n-1$, $m \in \{0, 1, \ldots, l-2\}$, $F_0, F_1, \ldots, F_m$ are nonempty subsets of $\{0, 1, \ldots, r\}$, for each $i \in \{0, 1, \ldots, m-1\}$, $\max F_i < \min F_{i+1}$, and for each $t \in \bigcup_{i=0}^m F_i$, $y_t \in H_t$, then $D_{r+1} \subseteq \{x \in \mathbb{R} : -a_{m+1} \cdot x + (-\sum_{t=0}^m a_i \cdot \sum_{t \in F_i} y_t + A) \in a_{m+2} \cdot p + a_{m+3} \cdot p + \cdots + a_l \cdot p\}$.

At $n = 1$, hypotheses (I), (II),and (IV) hold directly while (III), (V), and (VI) are vacuous.

For $m \in \{0, 1, \ldots, l-1\}$, let $G_m = \{\sum_{t=0}^m a_i \cdot \sum_{t \in F_i} y_t : F_0, F_1, \ldots, F_m$ are nonempty subsets of $\{0, 1, \ldots, n-1\}$, for each $i \in \{0, 1, \ldots, m-1\}$, $\max F_i < \min F_{i+1}$, and for each $t \in \bigcup_{i=0}^m F_i$, $y_t \in H_t\}$.

For $k \in \{0, 1, \ldots, n-1\}$, let $E_k = \{\sum_{t \in F} y_t :$ if $\phi \neq F \subseteq \{0, 1, \ldots, r\}$, $k = \min F$, and for each $t \in F$, $y_t \in H_t\}$. Given $b \in E_k$, we have that $b \in D_k^\star$ by hypothesis (II) and so $-b + D_k^\star \in p$. If $d \in G_{l-1}$, then by (IV), $-d + A \in a_l \cdot p$ so that $a_l^{-1}(-d + A) \in p$. If $m \in \{0, 1, \ldots, l-2\}$ and $d \in G_m$, then by (IV), $-d + A \in a_{m+1} \cdot p + a_{m+2} \cdot p + \cdots + a_l \cdot p$ so that $\{x \in \mathbb{R} : -a_{m+1} \cdot x + (-d + A) \in a_{m+2} \cdot p + a_{m+3} \cdot p + \cdots + a_l \cdot p\} \in p$. Thus we have that

$D_n \in p$, where $D_n = D_{n-1} \cap \bigcap_{k=0}^{n-1} \bigcap_{b\in E_k} (-b + D_k^\star) \cap \bigcap_{d\in G_{l-1}} a_l^{-1}(-d + A)$ $\cap \bigcap_{m=0}^{l-2} \bigcap_{d\in G_m} \{x \in \mathbb{R} : -a_{m+1} \cdot x + (-d + A) \in a_{m+2} \cdot p + a_{m+3} \cdot p + \cdots + a_l \cdot p\}$. (Here, if say $l = 1$ or $n < l$, we are using the convention that $\bigcap \emptyset = \mathbb{R}$.)

Pick, again by Theorem 2.3, $x_{\alpha_n}, x_{\alpha_n+1}, \ldots, x_{\alpha_{n+1}-1} \in \mathbb{R}^+$ such that

$$B_n \begin{pmatrix} x_{\alpha_n} \\ x_{\alpha_n+1} \\ \vdots \\ x_{\alpha_{n+1}-1} \end{pmatrix} \in (D_n^\star)^{u_n}.$$

Let H_n be the set of entries of

$$B_n \begin{pmatrix} x_{\alpha_n} \\ x_{\alpha_n+1} \\ \vdots \\ x_{\alpha_{n+1}-1} \end{pmatrix}.$$

Then hypotheses (I), (III),(V), and (VI) hold directly.

To verify hypothesis (II), let $\emptyset \neq F \subseteq \{0, 1, \ldots, r\}$, let $k = \min F$, and for each $t \in F$, let $y_t \in H_t$. If n does not belong to F, then $\sum_{t\in F} y_t \in D_k^\star$ by hypothesis (II) at $n-1$, so assume that $n \in F$. If $F = \{n\}$, then we have that $y_n \in D_n^\star$ directly so assume that $F \neq \{n\}$. Let $b = \sum_{t\in F\setminus\{n\}} y_t$. Then $b \in E_k$ and so $y_n \in -b + D_k^\star$ and thus $b + y_n \in D_k^\star$ as required.

To verify hypothesis (IV), let $m \in \{0, 1, \ldots, l-1\}$ and $F_0, F_1, \ldots, F_m$ are nonempty subsets of $\{0, 1, \ldots, n\}$, such that for each $i \in \{0, 1, \ldots, m-1\}$, $\max F_i < \min F_{i+1}$, and for each $t \in \bigcup_{i=0}^m F_i$, $y_t \in H_t$. If $m = 0$, then $\sum_{t\in F_0} y_t \in D_0^\star$ by (II) and (III) so that $-a_0 \cdot \sum_{t\in F_0} y_t + A \in a_1 \cdot p + a_2 \cdot p + \cdots + a_l \cdot p$ as required.

So assume that $m > 0$. Let $k = \min F_m$ and $j = \max F_{m-1}$. Then $\sum_{t\in F_m} y_t \in D_k^\star$ by (II) $\subseteq D_{j+1}$ by (III) $\subseteq \{x \in \mathbb{R} : -a_m \cdot x + (-\sum_{t=0}^{m-1} a_i \cdot \sum_{t\in F_i} y_t + A) \in a_{m+1} \cdot p + a_{m+2} \cdot p + \cdots + a_l \cdot p\}$ by (VI) as required.

The induction being complete, we claim that whenever $F_0, F_1, \ldots, F_l$ are nonempty subsets of ω such that for each $i \in \{0, 1, \ldots, l-1\}$, $\max F_i < \min F_{i+1}$, and for each $t \in \bigcup_{i=0}^m F_i$, $y_t \in H_t$, then $-\sum_{i=0}^l a_i \cdot \sum_{t\in F_i} y_t \in A$. To see this, let $k = \min F_l$ and let $j = \max F_{l-1}$. Then $\sum_{t\in F_l} y_t \in D_k^\star \subseteq D_{j+1} \subseteq a_l^{-1}(-\sum_{t=0}^{l-1} a_i \cdot \sum_{t\in F_i} y_t + A)$ by hypothesis (V), and so $\sum_{i=0}^l a_i \cdot \sum_{t\in F_i} y_t \in A$ as claimed.

Let Q be an insertion matrix of $\langle B_t \rangle_{t<\omega}$ into C. We claim that all entries of $Q\vec{x}$ are in A. To see this, let $\gamma < \omega$ be given and let $j \in \times_{t<\omega}\{0, 1, \ldots, u_t - 1\}$, so that $c_{\gamma,0} \cdot \vec{b}_{j(0)}^{(0)} \frown c_{\gamma,1} \cdot \vec{b}_{j(1)}^{(1)} \frown \ldots$ is a row of Q, say row δ. For each $t \in \{0, 1, \ldots, m\}$, let $y_t = \sum_{k=0}^{v_t-1} b_{j(t),k}^{(t)} \cdot x_{\alpha_t+k}$ (so that $y_t \in H_t$). Therefore we have $\sum_{q=0}^\infty q_{\delta,s} \cdot x_s = \sum_{t=0}^m c_{\gamma,t} \cdot y_t$. Choose nonempty subsets $F_0, F_1, \ldots, F_l$ of $\{0, 1, \ldots, m\}$ such that for each $i \in \{0, 1, \ldots, l-1\}$, $\max F_i < \min F_{i+1}$, and for each $t \in F_i$, $c_{\gamma,t} = a_i$. (One can do this because C is a Milliken–Taylor matrix for $\vec{a}$.) Then $\sum_{t=0}^{m} c_{\gamma,t} \cdot y_t = \sum_{i=0}^{l} a_i \cdot \sum_{t \in F_i} y_t \in A$. $\square$

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