

# Morita transforms of tensor algebras

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ABSTRACT. We show that if  $M$  and  $N$  are  $C^*$ -algebras and if  $E$  (resp.  $F$ ) is a  $C^*$ -correspondence over  $M$  (resp.  $N$ ), then a Morita equivalence between  $(E, M)$  and  $(F, N)$  implements an isometric functor between the categories of Hilbert modules over the tensor algebras of  $\mathcal{T}_+(E)$  and  $\mathcal{T}_+(F)$ . We show that this functor maps absolutely continuous Hilbert modules to absolutely continuous Hilbert modules and provides a new interpretation of Popescu’s reconstruction operator.

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## 1. Introduction

Suppose  $M$  is a  $C^*$ -algebra and  $E$  is a  $C^*$ -correspondence over  $M$  in the sense of [10]. This means, first of all, that  $E$  is a (right) Hilbert  $C^*$ -module, and secondly, that if  $\mathcal{L}(E)$  denotes the space of all bounded adjointable module maps on  $E$ , then  $E$  becomes a left  $M$ -module via a  $C^*$ -homomorphism  $\varphi_M$  from  $M$  into  $\mathcal{L}(E)$ . To emphasize the connection between  $E$  and  $M$ , we will call the pair,  $(E, M)$ , a  $C^*$ -correspondence pair. Form the Fock space built from  $(E, M)$ ,  $\mathcal{F}(E)$ . This is the direct sum  $\sum_{n \geq 0} E^{\otimes n}$ , where  $E^{\otimes n}$  is the internal tensor product of  $E$  with itself  $n$  times. (The tensor products are balanced over  $M$ .) The Fock space  $\mathcal{F}(E)$  is, itself, a  $C^*$ -correspondence

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over  $M$  and we write  $\varphi_{M\infty}$  for the left action of  $M$ . For  $\xi \in E$ ,  $T_\xi$  denotes the creation operator on  $\mathcal{F}(E)$  determined by  $\xi$ , i.e., for  $\eta \in \mathcal{F}(E)$ ,  $T_\xi\eta = \xi \otimes \eta$ . We let  $\mathcal{T}_+(E)$  denote the norm closed subalgebra of  $\mathcal{L}(\mathcal{F}(E))$  generated by  $\varphi_{M\infty}(M)$  and  $\{T_\xi \mid \xi \in E\}$ , and we call  $\mathcal{T}_+(E)$  *the tensor algebra of  $E$  or of  $(E, M)$* . In [8, Definition 2.1], we introduced the following definition.

**Definition 1.** We say that two  $C^*$ -correspondence pairs  $(E, M)$  and  $(F, N)$  are *Morita equivalent* in case there is a  $C^*$ -equivalence bimodule  $\mathcal{X}$  in the sense of [16, Definition 7.5] such that

$$\mathcal{X} \otimes_N F \simeq E \otimes_M \mathcal{X}$$

as  $C^*$ -correspondences. In this case, we say that  $\mathcal{X}$  *implements* a Morita equivalence between  $(E, M)$  and  $(F, N)$ .

Observe that the equation  $\mathcal{X} \otimes_N F \simeq E \otimes_M \mathcal{X}$  is equivalent to the equation  $\mathcal{X} \otimes_N F \otimes_N \tilde{\mathcal{X}} \simeq E$  and to the equation  $F \simeq \tilde{\mathcal{X}} \otimes_M E \otimes_M \mathcal{X}$ , where  $\tilde{\mathcal{X}}$  is the dual or opposite module of  $\mathcal{X}$ . We showed there that if  $(E, M)$  and  $(F, N)$  are Morita equivalent, then the tensor algebras  $\mathcal{T}_+(E)$  and  $\mathcal{T}_+(F)$  are Morita equivalent in the sense of [1]. It follows that  $\mathcal{T}_+(E)$  and  $\mathcal{T}_+(F)$  have isometrically isomorphic representation theories. However, when looking at the formulas involved in the isomorphism between the representation theories, certain details become obscure. Our objective in this note is to show that simply tensoring with  $\mathcal{X}$  implements an explicit isometric isomorphism between the representation theories of  $\mathcal{T}_+(E)$  and  $\mathcal{T}_+(F)$  in a fashion that preserves important properties that we shall introduce shortly. The first step is to have a clear picture of the representation theory of an operator tensor algebra.

## 2. The representations of $\mathcal{T}_+(E)$

We begin with a restatement of Theorem 3.10 in [10].

**Theorem 2.** *Let  $\rho$  be a completely contractive representation of  $\mathcal{T}_+(E)$  on a Hilbert space  $H$ . Define  $\sigma : M \rightarrow B(H)$  by the formula  $\sigma(a) = \rho \circ \varphi_{M\infty}(a)$  and define  $T : E \rightarrow B(H)$  by the formula  $T(\xi) = \rho(T_\xi)$ . Then  $\sigma$  is a  $C^*$ -representation of  $M$  on  $H$  and  $T$  is a completely contractive bimodule map in the sense that  $T(\varphi_M(a)\xi b) = \sigma(a)T(\xi)\sigma(b)$  for all  $a, b \in M$  and all  $\xi \in E$ . Conversely, given a  $C^*$ -representation  $\sigma : M \rightarrow B(H)$  and a completely contractive bimodule map  $T : E \rightarrow B(H)$ , there is a unique completely contractive representation  $\rho : \mathcal{T}_+(E) \rightarrow B(H)$  such that  $\sigma = \rho \circ \varphi_M$  and  $T(\xi) = \rho(T_\xi)$  for all  $\xi \in E$ .*

If  $T$  is a completely contractive bimodule map with respect to a  $C^*$ -representation  $\sigma$  of  $M$ , then we call  $(T, \sigma)$  a *completely contractive covariant pair*. We call the completely contractive representation  $\rho$  of  $\mathcal{T}_+(E)$  that  $(T, \sigma)$  determines the *integrated form* of  $(T, \sigma)$  and write  $\rho = T \times \sigma$ . Theorem 2 begs the question: How does one construct completely contractive

covariant pairs? For this purpose, we need to recall the definition of Rieffel's induced representation [17]. If  $\sigma : M \rightarrow B(H)$  is a Hilbert space representation of  $M$ , then we may build the Hilbert space  $E \otimes_\sigma H$ , which is the separated completion of the algebraic tensor product  $E \otimes H$  in the pre-inner product defined by the formula

$$\langle \xi \otimes h, \eta \otimes k \rangle := \langle h, \sigma(\langle \xi, \eta \rangle)k \rangle, \quad \xi \otimes h, \eta \otimes k \in E \otimes H.$$

The representation  $\sigma^E$  of  $\mathcal{L}(E)$  on  $E \otimes_\sigma H$  defined by the formula,  $\sigma^E(T) := T \otimes I$ ,  $T \in \mathcal{L}(E)$ , is called the representation of  $\mathcal{L}(E)$  induced by  $\sigma$ . The following theorem is essentially Lemma 2.5 of [7].

**Theorem 3.** *Let  $\sigma : M \rightarrow B(H)$  be a  $C^*$ -representation. A completely contractive linear map  $T$  from  $E$  to  $B(H)$  is a bimodule map with respect to  $\sigma$  if and only if there is an operator  $\tilde{T} : E \otimes_\sigma H \rightarrow H$  with  $\|\tilde{T}\| \leq 1$  such that  $\tilde{T}\sigma^E \circ \varphi_M = \sigma\tilde{T}$  and  $T(\xi)h = \tilde{T}(\xi \otimes h)$ , for all  $\xi \otimes h \in E \otimes_\sigma H$ .*

Thus the completely contractive bimodule maps are in bijective correspondence with (contractive) intertwiners. The space of intertwiners of  $\sigma$  and  $\sigma^E \circ \varphi_M$  is a key player in our theory and to keep the notation manageable, when there is no risk of confusion in the context under discussion, we shall not distinguish notationally between bimodule maps  $T$  and the corresponding intertwiner  $\tilde{T}$ . Further, for reasons that will be explained in a minute, we frequently also denote bimodule maps by lower case fraktur letters from the end of the alphabet, as we do now.

**Definition 4.** Let  $\sigma : M \rightarrow B(H)$  be a  $C^*$ -representation. The  $\sigma$ -dual of  $E$ , denoted  $E^\sigma$ , is defined to be  $\{\mathfrak{z} \in B(H, E \otimes_\sigma H) \mid \mathfrak{z}\sigma = \sigma^E \circ \varphi_M \mathfrak{z}\}$ . We write  $E^{\sigma*}$  for the space  $\{\mathfrak{z}^* \mid \mathfrak{z} \in E^\sigma\}$  and we write  $\mathbb{D}(E^{\sigma*})$  for  $\{\mathfrak{z}^* \in E^{\sigma*} \mid \|\mathfrak{z}^*\| < 1\}$ , i.e.,  $\mathbb{D}(E^{\sigma*})$  is the open unit ball in  $E^{\sigma*}$ .

Thanks to Theorem 3,  $\overline{\mathbb{D}(E^{\sigma*})}$  labels all the completely contractive representations  $\rho$  of  $\mathcal{T}_+(E)$  with the property that  $\rho \circ \varphi_{M_\infty} = \sigma$ . The reason we introduced  $E^\sigma$ , instead of focusing exclusively on  $E^{\sigma*}$  is that  $E^\sigma$  is a  $W^*$ -correspondence over  $\sigma(M)'$ . (A  $W^*$ -correspondence is a  $C^*$ -correspondence with some additional structure that we discuss below.) For  $\mathfrak{z}_1, \mathfrak{z}_2 \in E^\sigma$ ,  $\langle \mathfrak{z}_1, \mathfrak{z}_2 \rangle := \mathfrak{z}_1^* \mathfrak{z}_2$ , and the  $\sigma(M)'$ -bimodule actions are given by the formula

$$a \cdot \mathfrak{z} \cdot b := (I_E \otimes a)\mathfrak{z}b, \quad a, b \in \sigma(M)', \mathfrak{z} \in E^\sigma,$$

where the products on the right are just composition of the maps involved. The reason for introducing the notation  $\overline{\mathbb{D}(E^{\sigma*})}$  and writing elements in this ball as lower case  $\mathfrak{z}$ 's,  $\mathfrak{w}$ 's, etc., is that we may view an element  $F \in \mathcal{T}_+(E)$  as a function  $\widehat{F}$  on  $\overline{\mathbb{D}(E^{\sigma*})}$  via the formula

$$\widehat{F}(\mathfrak{z}) := \mathfrak{z} \times \sigma(F), \quad \mathfrak{z} \in \overline{\mathbb{D}(E^{\sigma*})}.$$

Functions of the form  $\widehat{F}$  are bonafide  $B(H_\sigma)$ -valued analytic functions on  $\overline{\mathbb{D}(E^{\sigma*})}$  with additional very interesting properties, and they can be studied with function-theoretic techniques. (See [7, 6, 5].) For the purpose of

emphasizing the function-theoretic properties of the  $\widehat{F}$ 's, it seems preferable to write their arguments as  $\mathfrak{z}$ 's instead of  $T$ 's. But when representation-theoretic features need emphasis, the use of  $T$  and  $T \times \sigma$  is sometimes preferable.

### 3. The functor

Our objective in this section is to show that Morita equivalence of  $C^*$ -correspondence pairs  $(E, M)$  and  $(F, N)$  gives rise to a natural isometric isomorphism between representation theory of  $\mathcal{T}_+(E)$  and  $\mathcal{T}_+(F)$ .

**Theorem 5.** *Suppose  $(E, M)$  and  $(F, N)$  are Morita equivalent  $C^*$ -correspondence pairs via an  $M, N$ -equivalence bimodule  $\mathcal{X}$  and correspondence isomorphism  $W : E \otimes_M \mathcal{X} \rightarrow \mathcal{X} \otimes_N F$ . Suppose further that  $\sigma : N \rightarrow B(H)$  is a  $C^*$ -representation and let  $\sigma^\mathcal{X} : M \rightarrow B(\mathcal{X} \otimes_\sigma H)$  be the representation of  $M$  induced by  $\mathcal{X}$ . Then for each  $\mathfrak{z}^* \in \overline{\mathbb{D}(F^{\sigma^*})}$ ,  $\mathfrak{z}^{*\mathcal{X}} := (I_\mathcal{X} \otimes \mathfrak{z}^*)(W \otimes I_H)$  lies in  $\overline{\mathbb{D}(E^{\sigma^{\mathcal{X}^*})}$  and the map  $\mathfrak{z}^* \rightarrow \mathfrak{z}^{*\mathcal{X}}$  is an isometric surjection onto  $\overline{\mathbb{D}(E^{\sigma^{\mathcal{X}^*})}$ .*

**Proof.** For  $\mathfrak{z}^* \in \overline{\mathbb{D}(F^{\sigma^*})}$  set  $\mathfrak{z}_1^* := \begin{bmatrix} 0 & \mathfrak{z}^* \\ 0 & 0 \end{bmatrix}$  acting on  $H \oplus (F \otimes_\sigma H)$ . Then  $\mathfrak{z}_1^*$  commutes with  $\begin{bmatrix} \sigma & 0 \\ 0 & \sigma^F \circ \varphi_N \end{bmatrix}$ . Consequently,  $I_\mathcal{X} \otimes \mathfrak{z}_1^* = \begin{bmatrix} 0 & I_\mathcal{X} \otimes \mathfrak{z}^* \\ 0 & 0 \end{bmatrix}$  acting on  $\mathcal{X} \otimes_\sigma H \oplus \mathcal{X} \otimes_{\sigma^F \circ \varphi_N} (F \otimes_\sigma H)$  commutes with  $\begin{bmatrix} \sigma^\mathcal{X} & 0 \\ 0 & (\sigma^F \circ \varphi_N)^\mathcal{X} \end{bmatrix}$ . Since  $W \otimes I_H : E \otimes \mathcal{X} \otimes_\sigma H \rightarrow \mathcal{X} \otimes F \otimes_\sigma H$  intertwines  $(\sigma^\mathcal{X})^E \circ \varphi_M$  and  $(\sigma^F \circ \varphi_N)^\mathcal{X}$  by hypothesis, we see that  $\begin{bmatrix} 0 & (I_\mathcal{X} \otimes \mathfrak{z}^*)(W \otimes I_H) \\ 0 & 0 \end{bmatrix}$  commutes with  $\begin{bmatrix} \sigma^\mathcal{X} & 0 \\ 0 & (\sigma^\mathcal{X})^E \circ \varphi_M \end{bmatrix}$ . Since  $\|(I_\mathcal{X} \otimes \mathfrak{z}^*)(W \otimes I_H)\| = \|\mathfrak{z}^*\|$ , it follows that  $\mathfrak{z}^{*\mathcal{X}} := (I_\mathcal{X} \otimes \mathfrak{z}^*)(W \otimes I_H)$  lies in  $\overline{\mathbb{D}(E^{\sigma^{\mathcal{X}^*})}$  and that the map  $\mathfrak{z}^* \rightarrow \mathfrak{z}^{*\mathcal{X}}$  is isometric. Finally, to see that the map is surjective, we appeal to [17, Theorem 6.23]: Let  $\mathfrak{w}^* \in \overline{\mathbb{D}(E^{\sigma^{\mathcal{X}^*})}$ . Then  $\mathfrak{w}^*$  intertwines  $\sigma^\mathcal{X}$  and  $(\sigma^\mathcal{X})^E \circ \varphi_M$  by hypothesis. Consequently,  $\mathfrak{w}^*(W \otimes I_H)^{-1}$  intertwines  $\sigma^\mathcal{X}$  and  $(\sigma^F \circ \varphi_N)^\mathcal{X}$ . That is  $\begin{bmatrix} 0 & \mathfrak{w}^*(W \otimes I_H)^{-1} \\ 0 & 0 \end{bmatrix}$  lies in the commutant of  $\begin{bmatrix} \sigma^\mathcal{X}(\mathcal{L}(\mathcal{X})) & 0 \\ 0 & (\sigma^F \circ \varphi_N)^\mathcal{X}(\mathcal{L}(\mathcal{X})) \end{bmatrix} = \begin{bmatrix} \sigma & 0 \\ 0 & (\sigma^F \circ \varphi_N) \end{bmatrix}^\mathcal{X}(\mathcal{L}(\mathcal{X}))$  and so, by [17, Theorem 6.23],  $\begin{bmatrix} 0 & \mathfrak{w}^*(W \otimes I_H)^{-1} \\ 0 & 0 \end{bmatrix}$  must have the form  $I_\mathcal{X} \otimes \begin{bmatrix} \mathfrak{z}_{11} & \mathfrak{z}_{12} \\ \mathfrak{z}_{21} & \mathfrak{z}_{22} \end{bmatrix}$ , where  $\begin{bmatrix} \mathfrak{z}_{11} & \mathfrak{z}_{12} \\ \mathfrak{z}_{21} & \mathfrak{z}_{22} \end{bmatrix}$  lies in the commutant of  $\begin{bmatrix} \sigma & 0 \\ 0 & (\sigma^F \circ \varphi_N) \end{bmatrix}$ . Since  $I_\mathcal{X} \otimes \begin{bmatrix} \mathfrak{z}_{11} & \mathfrak{z}_{12} \\ \mathfrak{z}_{21} & \mathfrak{z}_{22} \end{bmatrix}$  maps  $\mathcal{X} \otimes_N (F \otimes_\sigma H)$  to  $\mathcal{X} \otimes_\sigma H$  and is zero on

$\mathcal{X} \otimes_{\sigma} H$ , it follows that  $\begin{bmatrix} \mathfrak{z}_{11} & \mathfrak{z}_{12} \\ \mathfrak{z}_{21} & \mathfrak{z}_{22} \end{bmatrix} = \begin{bmatrix} 0 & \mathfrak{z}_{12} \\ 0 & 0 \end{bmatrix}$  for  $\mathfrak{z}_{12} \in \overline{\mathbb{D}(F^{\sigma^*})}$ , proving that the map  $\mathfrak{z}^* \rightarrow \mathfrak{z}^{*\mathcal{X}}$  is surjective.  $\square$

**Definition 6.** If  $(E, M)$  and  $(F, N)$  are Morita equivalent  $C^*$ -correspondence pairs via an equivalence  $M, N$ -bimodule  $\mathcal{X}$ , then the map  $(T, \sigma) \rightarrow (T^{\mathcal{X}}, \sigma^{\mathcal{X}})$  from the representation theory of  $\mathcal{T}_+(E)$  to the representation theory of  $\mathcal{T}_+(F)$  defined by  $\mathcal{X}$  will be called the *Morita transform* determined by  $\mathcal{X}$ .

We like to think of the Morita transform as a generalized conformal map.

#### 4. Morita equivalence and absolute continuity

Our focus in this section will be on Morita equivalence in the context of  $W^*$ -algebras and  $W^*$ -correspondences. As we noted above, a  $W^*$ -correspondence is a  $C^*$ -correspondence with additional structure. We begin by highlighting what the additional structure is and how to deal with it. So, throughout this section  $M$  and  $N$  will be  $W^*$ -algebras and  $E$  (resp.  $F$ ) will be a  $W^*$ -correspondence over  $M$  (resp.  $N$ ). This means, in particular, that  $E$  and  $F$  are *self-dual* Hilbert  $C^*$ -modules over  $M$  and  $N$ , respectively, in the sense of Paschke [11, Section 3, p. 449], and that the left actions of  $M$  and  $N$  are given by *normal* representations,  $\varphi_M$  and  $\varphi_N$  of  $M$  and  $N$  into  $\mathcal{L}(E)$  and  $\mathcal{L}(F)$ , respectively. (Recall that Paschke showed that in the setting of self-dual Hilbert modules over  $W^*$ -algebras, every continuous module map is adjointable and  $\mathcal{L}(E)$  is a  $W^*$ -algebra by [11, Corollary 3.5 and Proposition 3.10].) To avoid technical distractions, we assume that  $\varphi_M$  and  $\varphi_N$  are faithful and unital.

A key role in this theory is played by Paschke's Theorem 3.2 in [11], which says among other things that any Hilbert  $C^*$ -module  $E$  over a  $W^*$ -algebra has a canonical embedding into a self-dual Hilbert module over the algebra, which should be viewed as a canonical completion of  $E$ . This allows us to perform  $C^*$ -algebraic constructions and pass immediately to the completions to obtain  $W^*$ -algebraic objects. For instance, if  $E$  is a Hilbert  $W^*$ -module over  $M$ , then we may form the  $C^*$ -tensor square,  $E^{\otimes 2} = E \otimes_M E$ , which is not, in general, a  $W^*$ -correspondence over  $M$ . However, its self-dual completion is. More generally, we can form the  $C^*$ -Fock space built from  $(E, M)$ ,  $\mathcal{F}_c(E)$ , as we did at the outset of this note. Then we let  $\mathcal{F}(E)$  be the self-dual completion of  $\mathcal{F}_c(E)$  in the sense of [11, Theorem 3.2], and call  $\mathcal{F}(E)$  the *Fock space of the  $W^*$ -correspondence  $E$* . Similarly, we form  $\mathcal{F}_c(F)$  and  $\mathcal{F}(F)$ . We write  $\varphi_{M\infty}$  for the left action of  $M$  on both  $\mathcal{F}_c(E)$  and on  $\mathcal{F}(E)$ . This should cause no confusion, since every element of  $\mathcal{L}(\mathcal{F}_c(E))$  has a unique extension to an element of  $\mathcal{L}(\mathcal{F}(E))$ , by [11, Corollary 3.7], and the process of mapping each element in  $\mathcal{L}(\mathcal{F}_c(E))$  to its extension in  $\mathcal{L}(\mathcal{F}(E))$  gives an isometric embedding of  $\mathcal{L}(\mathcal{F}_c(E))$  in  $\mathcal{L}(\mathcal{F}(E))$ . Likewise,  $\varphi_{N\infty}$  denotes the left action of  $N$  on both  $\mathcal{F}_c(F)$  and  $\mathcal{F}(F)$ . The creation

operator  $T_\xi$  on  $\mathcal{F}_c(E)$  determined by  $\xi \in E$ , therefore has a unique extension to  $\mathcal{F}(E)$  and we do not distinguish notationally between the original and the extension. But in the  $W^*$ -setting we let  $\mathcal{T}_+(E)$  denote the norm closed subalgebra of  $\mathcal{L}(\mathcal{F}(E))$  generated by  $\varphi_{M^\infty}(M)$  and  $\{T_\xi \mid \xi \in E\}$ , and we call  $\mathcal{T}_+(E)$  *the tensor algebra of  $E$*  or of  $(E, M)$ . That is, we focus on the tensor algebra as living on the  $W^*$ -Fock space  $\mathcal{F}(E)$ . We view  $\mathcal{T}_+(F)$  similarly. Finally, we let  $H^\infty(E)$  denote the ultra-weak closure of  $\mathcal{T}_+(E)$  in  $\mathcal{L}(\mathcal{F}(E))$ , and we let  $H^\infty(F)$  denote the ultra-weak closure of  $\mathcal{T}_+(F)$  in  $\mathcal{L}(\mathcal{F}(F))$ . The algebras  $H^\infty(E)$  and  $H^\infty(F)$  are called the *Hardy algebras* of  $E$  and  $F$ , respectively.

In the special case when  $M = \mathbb{C} = E$ , we see that  $\mathcal{F}_c(E) = \mathcal{F}(E) = \ell^2(\mathbb{N})$ ,  $\mathcal{T}_+(E)$  is the disc algebra  $A(\mathbb{D})$  and  $H^\infty(E) = H^\infty(\mathbb{T})$ . More generally, when  $M = \mathbb{C}$  and  $E = \mathbb{C}^d$ ,  $\mathcal{T}_+(E)$  is Popescu's noncommutative disc algebra and  $H^\infty(E)$  is his noncommutative Hardy algebra [13]. Somewhat later, Davidson and Pitts studied  $H^\infty(\mathbb{C}^d)$  under the name *noncommutative analytic Toeplitz algebra* [2].

**Definition 7.** If  $M$  and  $N$  are  $W^*$ -algebras and if  $E$  and  $F$  are  $W^*$ -correspondences over  $M$  and  $N$ , respectively, we say that  $(E, M)$  and  $(F, N)$  are *Morita equivalent* in case there is a self-dual  $M-N$  equivalence bimodule  $\mathcal{X}$  in the sense of [16, Definition 7.5] such that

$$\mathcal{X} \otimes_N F \simeq E \otimes_M \mathcal{X}$$

as  $W^*$ -correspondences. In this case, we say that  $\mathcal{X}$  *implements* a Morita equivalence between  $(E, M)$  and  $(F, N)$ .

We emphasize that the modules  $\mathcal{X} \otimes_N F$  and  $E \otimes_M \mathcal{X}$  are self-dual completions of the balanced  $C^*$ -tensor products. A completely contractive representation of a  $W^*$ -correspondence pair  $(E, M)$  on a Hilbert space  $H$  is a pair  $(T, \sigma)$  where  $\sigma$  is a normal representation of  $M$  on  $H$  and where  $T$  is an ultra-weakly continuous, completely contractive bimodule map from  $E$  to  $B(H)$ . However, as we noted in [7, Remark 2.6], the ultra-weak continuity of  $T$  follows automatically from the bimodule property of  $T$  and the normality of  $\sigma$ .

Our goal is to show that Morita equivalence in the sense of Definition 7 preserves absolute continuity in the sense of the following definition, which was inspired by the important paper of Davidson, Li and Pitts [3].

**Definition 8.** Let  $(T, \sigma)$  be a completely contractive covariant representation of  $(E, M)$  on  $H$  and assume that  $\sigma$  is a normal representation of  $M$ . Then a vector  $x \in H$  is called *absolutely continuous* if and only if the functional  $a \rightarrow \langle (T \times \sigma)(a)x, x \rangle$ ,  $a \in \mathcal{T}_+(E)$ , extends to an ultra-weakly continuous linear functional on  $H^\infty(E)$ . The collection of all absolutely continuous vectors in  $H$  is denoted  $\mathcal{V}_{\text{ac}}(T, \sigma)$ , and we say  $(T, \sigma)$  and  $T \times \sigma$  are absolutely continuous in case  $\mathcal{V}_{\text{ac}}(T, \sigma) = H$ .

**Remark 9.** The definition of an absolutely continuous vector just given is not quite the one given in [4, Definition 3.1]. However, by [4, Remark 3.2], it is equivalent to the one given there. Also, by virtue of [4, Theorem 4.11],  $T \times \sigma$  extends to an ultra-weakly continuous completely contractive representation of  $H^\infty(E)$  if and only if  $T \times \sigma$  is absolutely continuous.

**Theorem 10.** *Suppose that  $(E, M)$  and  $(F, N)$  are  $W^*$ -correspondence pairs that are Morita equivalent via an equivalence bimodule  $\mathcal{X}$ . If  $(\mathfrak{z}^*, \sigma)$  is a completely contractive covariant representation of  $(F, N)$ , where  $\sigma$  is normal, then*

$$(1) \quad \mathcal{X} \otimes_\sigma \mathcal{V}_{\text{ac}}(\mathfrak{z}^*, \sigma) = \mathcal{V}_{\text{ac}}(\mathfrak{z}^{\mathcal{X}*}, \sigma^{\mathcal{X}}).$$

*In particular,  $(\mathfrak{z}^*, \sigma)$  is absolutely continuous if and only if  $(\mathfrak{z}^{\mathcal{X}*}, \sigma^{\mathcal{X}})$  is absolutely continuous.*

The proof of this theorem rests on a calculation of independent interest. Recall that each  $\mathfrak{z} \in \mathbb{D}(F^\sigma)$  determines a completely positive map  $\Phi_{\mathfrak{z}}$  on  $\sigma(N)'$  via the formula

$$\Phi_{\mathfrak{z}}(a) := \mathfrak{z}^*(I_E \otimes a)\mathfrak{z}, \quad a \in \sigma(N)'.$$

Recall, also, that the commutant of  $\sigma^{\mathcal{X}}(M)$  is  $I_{\mathcal{X}} \otimes \sigma(N)'$ , by [17, Theorem 6.23].

**Lemma 11.** *With the notation as in Theorem 10,*

$$\Phi_{\mathfrak{z}^{\mathcal{X}}} = I_{\mathcal{X}} \otimes \Phi_{\mathfrak{z}},$$

*i.e., for all  $a \in \sigma(N)'$ ,  $\Phi_{\mathfrak{z}^{\mathcal{X}}}(I_{\mathcal{X}} \otimes a) = I_{\mathcal{X}} \otimes \Phi_{\mathfrak{z}}(a)$ .*

**Proof.** By Theorem 5,  $\mathfrak{z}^{\mathcal{X}} = (W \otimes I_H)^*(I_{\mathcal{X}} \otimes \mathfrak{z})$ . Consequently, for all  $a \in \sigma(N)'$ ,

$$\begin{aligned} \Phi_{\mathfrak{z}^{\mathcal{X}}}(I_{\mathcal{X}} \otimes a) &= \mathfrak{z}^{\mathcal{X}*}(I_E \otimes (I_{\mathcal{X}} \otimes a))\mathfrak{z}^{\mathcal{X}} \\ &= (I_{\mathcal{X}} \otimes \mathfrak{z}^*)(W \otimes I_H)(I_{E \otimes \mathcal{X}} \otimes a)(W \otimes I_H)^*(I_{\mathcal{X}} \otimes \mathfrak{z}) \\ &= (I_{\mathcal{X}} \otimes \mathfrak{z}^*)(I_{\mathcal{X} \otimes F} \otimes a)(I_{\mathcal{X}} \otimes \mathfrak{z}) \\ &= (I_{\mathcal{X}} \otimes \mathfrak{z}^*)(I_{\mathcal{X}} \otimes (I_F \otimes a))(I_{\mathcal{X}} \otimes \mathfrak{z}) \\ &= I_{\mathcal{X}} \otimes \Phi_{\mathfrak{z}}(a). \quad \square \end{aligned}$$

For the proof of Theorem 10, we need one more ingredient:

**Definition 12.** Let  $\Phi$  be a completely positive map on a  $W^*$ -algebra  $M$ . A positive element  $a \in M$  is called *superharmonic* with respect to  $\Phi$  in case  $\Phi(a) \leq a$ . A superharmonic element  $a \in M$  is called a *pure superharmonic* element in case  $\Phi^n(a) \rightarrow 0$  ultra-strongly as  $n \rightarrow \infty$ .

**Proof of Theorem 10.** In [4, Theorem 4.7], we proved that the absolutely continuous subspace for  $(\mathfrak{z}^*, \sigma)$  is the closed linear span of the ranges of all the pure superharmonic operators for  $\Phi_{\mathfrak{z}}$ , i.e., the projection onto  $\mathcal{V}_{\text{ac}}(\mathfrak{z}^*, \sigma)$  is the supremum taken over all the projections  $P$ , where  $P$  is the projection

onto the range of a pure superharmonic operator for  $\Phi_3$ . From Lemma 11 we see that  $a \in N$  is a pure superharmonic operator for  $\Phi_3$  if and only if  $I_{\mathcal{X}} \otimes a$  is a pure superharmonic operator for  $\Phi_{3,\mathcal{X}}$ . Since the range projection of  $I_{\mathcal{X}} \otimes a$  is  $I_{\mathcal{X}} \otimes P$ , if  $P$  is the range projection of  $a$ , the equation (1) is immediate.  $\square$

## 5. Stabilization and reconstruction

We return to the  $C^*$ -setting, although everything we will say has an analogue in the  $W^*$ -setting. So let  $N$  be a  $C^*$ -algebra and let  $F$  be a  $C^*$ -correspondence over  $N$ . We are out to identify a special pair  $(E, M)$  that is Morita equivalent to  $(F, N)$  and is a kind of stabilization of  $(F, N)$ . As we will see,  $(E, M)$  will have a representation theory that is closely connected to Popescu's reconstruction operator.

Form the Fock space over  $F$ ,  $\mathcal{F}(F)$ , and let  $M = \mathcal{K}(\mathcal{F}(F))$ . Also, let  $P_0$  be the projection onto the sum  $F \oplus F^{\otimes 2} \oplus F^{\otimes 3} \oplus \dots$  in  $\mathcal{F}(F)$ . Then  $P_0$  lies in  $\mathcal{L}(\mathcal{F}(F))$ , which is the multiplier algebra of  $M = \mathcal{K}(\mathcal{F}(F))$ . We set  $E := P_0\mathcal{K}(\mathcal{F}(F))$  and endow  $E$  with its obvious structure as a right Hilbert  $C^*$ -module over  $\mathcal{K}(\mathcal{F}(F))$ . Note that  $\mathcal{L}(E) = P_0\mathcal{L}(\mathcal{F}(F))P_0$ . Define  $R : \mathcal{F}(F) \otimes F \rightarrow \mathcal{F}(F)$  by the formula  $R(\xi \otimes f) = \xi \otimes f$ , where the first  $\xi \otimes f$ , the argument of  $R$ , is viewed as an element in  $\mathcal{F}(F) \otimes_N F$ , while the second  $\xi \otimes f$ , the image of  $R(\xi \otimes f)$ , is viewed as an element of  $\mathcal{F}(F)$ . It appears that  $R$  is the identity map. However, this is only because we have suppressed the isomorphisms between  $F^{\otimes n} \otimes F$  and  $F^{\otimes(n+1)}$ . The map  $R$  is adjointable, and its adjoint is given by the formulae  $R^*(a) = 0$ , if  $a \in N$ , viewed as the zero<sup>th</sup> component of  $\mathcal{F}(F)$ , while  $R^*(\xi_1 \otimes \xi_2 \otimes \xi_3 \otimes \dots \otimes \xi_n) = (\xi_1 \otimes \xi_2 \otimes \dots \otimes \xi_{n-1}) \otimes \xi_n$ , if  $n \geq 1$  and  $\xi_1 \otimes \xi_2 \otimes \xi_3 \otimes \dots \otimes \xi_n$  is a decomposable element of  $F^{\otimes n} \subseteq \mathcal{F}(F)$ . In particular,  $RR^* = P_0$ . We define  $\varphi_M : M \rightarrow \mathcal{L}(E)$  by the formula

$$\varphi_M(a) := R(a \otimes I_F)R^*, \quad a \in M.$$

Observe that  $\varphi_M$  extends naturally to the multiplier algebra of  $M$ , which is  $\mathcal{L}(\mathcal{F}(F))$  and  $\varphi_M(I) = P_0$ . Consequently,  $E$  is an essential left module over  $M$ .

**Proposition 13.** *If  $\mathcal{X} = \mathcal{F}(F)$ , then  $\mathcal{X}$  is an equivalence bimodule between  $M = \mathcal{K}(\mathcal{F}(F))$  and  $N$  and the map  $W$  from  $E \otimes_M \mathcal{X}$  to  $\mathcal{X} \otimes_N F$  defined by the formula*

$$W(P_0a \otimes \xi) = R^*P_0a\xi, \quad P_0a \otimes \xi \in E \otimes_M \mathcal{X},$$

*is an isomorphism of  $M, N$ -correspondences. Consequently,  $(E, M)$  and  $(F, N)$  are Morita equivalent.*

**Proof.** By definition,  $\mathcal{X}$  is an equivalence bimodule implementing a Morita equivalence between  $M$  and  $N$ . Also, it is clear that  $W$  is a right  $N$ -module map. To see that  $W$  is a left  $M$ -module map, it may be helpful to emphasize that the tensor product  $E \otimes_M \mathcal{X}$  is balanced over  $M$ . So, if  $P_0$  and  $I$  were in  $\mathcal{K}(\mathcal{F}(F))$  (which they aren't; they're only multipliers of  $\mathcal{K}(\mathcal{F}(F))$ ), then



$P_0a \otimes \xi$  could be replaced by  $P_0 \otimes \xi$ , which in turn could be replaced by  $I \otimes P_0\xi$ . Further, sending  $I \otimes P_0\xi$  to  $P_0\xi$  effects an isomorphism between  $E \otimes_M \mathcal{X}$  and  $P_0\mathcal{F}(F)$ . It results that  $W$  is effectively  $R^*$ . The following equation, then, gives the desired result.

$$\begin{aligned} W\varphi_M(b)(P_0a \otimes \xi) &= W(R(b \otimes I_F)R^*)P_0a \otimes \xi \\ &= (b \otimes I_F)R^*a\xi \\ &= (b \otimes I_F)W(P_0a \otimes \xi). \end{aligned}$$

The fact that  $W$  is isometric is another easy computation: For all  $a, b \in M$ , and  $\xi, \eta \in F$ ,

$$\begin{aligned} \langle P_0a \otimes \xi, P_0b \otimes \eta \rangle &= \langle \xi, a^*P_0b\eta \rangle \\ &= \langle P_0a\xi, P_0b\eta \rangle \\ &= \langle R^*a\xi, R^*b\eta \rangle \\ &= \langle W(P_0a \otimes \xi), W(P_0b \otimes \eta) \rangle. \end{aligned}$$

(Note that we have used the fact that  $P_0 = RR^*$  when passing from the second line to the third.) Since  $\mathcal{K}(\mathcal{F}(F))\mathcal{F}(F) = \mathcal{F}(F)$ ,  $P_0\mathcal{K}(\mathcal{F}(F))\mathcal{F}(F) = P_0\mathcal{F}(F)$ , and so  $R^*P_0\mathcal{K}(\mathcal{F}(F))\mathcal{F}(F) = R^*P_0\mathcal{F}(F) = \mathcal{F}(F) \otimes F$ . This shows that  $W$  is surjective.  $\square$

**Definition 14.** Given a  $C^*$ -correspondence pair  $(F, N)$ , we call the  $C^*$ -correspondence pair  $(E, M) = (P_0\mathcal{K}(\mathcal{F}(F)), \mathcal{K}(\mathcal{F}(F)))$  constructed in Proposition 13 the *canonical stabilization* of  $(F, N)$ , and we call  $(\mathcal{F}(F), W)$  the *canonical  $(E, M), (F, N)$ -equivalence*.

We want to illustrate the calculations of Proposition 13 in a concrete setting first considered by Popescu. For this purpose, we require two observations. First, recall that  $E$  has the form  $PM$ . In general, if  $M$  is a  $C^*$ -algebra and if  $E$  has the form  $PM$ , where  $P$  is a projection in the multiplier algebra of  $M$ , then we called  $(E, M)$  *strictly cyclic* in [10, Page 419]. In this case, if  $(T, \sigma)$  is a completely contractive covariant representation of  $(E, M)$  on a Hilbert space  $H$ , then  $E \otimes_\sigma H$  is really  $\sigma(P)H$ , where we have extended  $\sigma$  to the multiplier algebra of  $M$ , if  $M$  is not unital. Consequently, the intertwiner  $\tilde{T}$  really maps the *subspace*  $\sigma(P)H$  into  $H$  but the adjoint of  $\tilde{T}$  may be viewed as an operator *on*  $H$ , i.e., from  $H$  to  $H$ , with range contained in  $\sigma(P)H$ , of course. Second, observe that in general, if  $(T, \sigma)$  is a covariant representation of  $(F, N)$  on a Hilbert space  $H$ , then the representation induced from the canonical equivalence is  $(T^{\mathcal{F}(F)}, \sigma^{\mathcal{F}(F)})$ . We know  $\sigma^{\mathcal{F}(F)}$  represents  $\mathcal{K}(\mathcal{F}(F))$  on  $\mathcal{F}(F) \otimes_\sigma H$  via the ordinary action of  $\mathcal{K}(\mathcal{F}(F))$  on  $\mathcal{F}(F)$ , tensored with the identity operator on  $H$ , i.e.,  $\sigma^{\mathcal{F}(F)}(a) = a \otimes I_H$ . On the other hand, from Theorem 5,  $\widetilde{T^{\mathcal{F}(F)}} = (I_{\mathcal{F}(F)} \otimes \tilde{T})(W \otimes I_H)$ . But as we noted in the proof of Proposition 13,  $W$  is effectively  $R^*$ , and taking into account all the balancing that is taking place, we may write  $\widetilde{T^{\mathcal{F}(F)}} = (I_{\mathcal{F}(F)} \otimes \tilde{T})(R^* \otimes I_H)$ . Since, as we just remarked,

$\widetilde{T^{\mathcal{F}(F)}}$  maps from  $E \otimes_M \mathcal{F}(F) \otimes_\sigma H = P_0 \mathcal{K}(\mathcal{F}(F)) \otimes_{\mathcal{K}(\mathcal{F}(F))} \mathcal{F}(F) \otimes_\sigma H$ , which can be identified with the subspace  $P_0 \mathcal{F}(F) \otimes_\sigma H$  of  $\mathcal{F}(F) \otimes_\sigma H$ , it will be more convenient in the example below to work with the adjoint of  $\widetilde{T^{\mathcal{F}(F)}}$ ,

$$(2) \quad \left( \widetilde{T^{\mathcal{F}(F)}} \right)^* = (R \otimes I_H)(I_{\mathcal{F}(F)} \otimes \widetilde{T}^*),$$

and view  $\left( \widetilde{T^{\mathcal{F}(F)}} \right)^*$  as an operator in  $B(\mathcal{F}(F) \otimes_\sigma H)$ .

**Example 15.** In this example, we let  $N = \mathbb{C}$  and we let  $F = \mathbb{C}^d$ . We interpret  $\mathbb{C}^d$  as  $\ell^2(\mathbb{N})$ , if  $d = \infty$ . If  $(T, \sigma)$  is a completely contractive covariant representation of  $(\mathbb{C}^d, \mathbb{C})$  on a Hilbert space  $H$ , then  $\sigma$  is just the  $n$ -fold multiple of the identity representation of  $\mathbb{C}$ , where  $n$  is the dimension of  $H$ . Also,  $\widetilde{T}$  may be viewed in terms of a  $1 \times d$  matrix of operators on  $H$ ,  $[T_1, T_2, \dots, T_d]$ , such that  $\sum_{i=1}^d T_i T_i^* \leq I_H$ , i.e.  $[T_1, T_2, \dots, T_d]$  is a row contraction. When  $\mathbb{C}^d \otimes H$  is identified with the column direct sum of  $d$

copies of  $H$ , the formula for  $\widetilde{T} : \mathbb{C}^d \otimes H \rightarrow H$  is  $\widetilde{T} \left( \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_d \end{pmatrix} \right) = \sum_{i=1}^d T_i h_i$ .

Consequently,  $\widetilde{T}^* : H \rightarrow \mathbb{C}^d \otimes H$  is given by the formula

$$\widetilde{T}^* h = \begin{pmatrix} T_1^* h \\ T_2^* h \\ \vdots \\ T_d^* h \end{pmatrix}.$$

On the other hand,  $\mathcal{F}(\mathbb{C}^d) \otimes \mathbb{C}^d$  may be viewed as the column direct sum of  $d$  copies of  $\mathcal{F}(\mathbb{C}^d)$  and when this is done,  $R$  has a matricial representation as  $[R_1, R_2, \dots, R_d]$ , where  $R_i$  is the right creation operator on  $\mathcal{F}(\mathbb{C}^d)$  determined by the  $i^{\text{th}}$  canonical basis vector  $e_i = (0, 0, \dots, 0, 1, 0, \dots, 0)^\top$  for  $\mathbb{C}^d$ , i.e.,  $R_i \xi = \xi \otimes e_i$ . Notice that  $[R_1, R_2, \dots, R_d]$  is a *row isometry*, meaning that  $R_i$ 's are all isometries and that their range projections  $R_i R_i^*$  are mutually orthogonal. Thus, the formula for  $R : \mathcal{F}(\mathbb{C}^d) \otimes \mathbb{C}^d \rightarrow \mathcal{F}(\mathbb{C}^d)$

is  $R \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_d \end{pmatrix} = \sum_{i=1}^d R_i \xi_i$ . Consequently, in the context of this example,

equation (2) becomes

$$\begin{aligned}
\left(\widetilde{T^{\mathcal{F}(\mathbb{C}^d)}}\right)^* (\xi \otimes h) &= (R \otimes I_H)(I_{\mathcal{F}(\mathbb{C}^d)} \otimes \widetilde{T}^*)(\xi \otimes h) \\
&= (R \otimes I_H) \begin{pmatrix} \xi \otimes T_1^* h \\ \xi \otimes T_2^* h \\ \vdots \\ \xi \otimes T_d^* h \end{pmatrix} \\
&= \sum_{i=1}^d R_i \xi \otimes T_i^* h \\
&= \left( \sum_{i=1}^d R_i \otimes T_i^* \right) (\xi \otimes h),
\end{aligned}$$

i.e.,  $\left(\widetilde{T^{\mathcal{F}(\mathbb{C}^d)}}\right)^*$  is Popescu's *reconstruction operator*  $\sum_{i=1}^d R_i \otimes T_i^*$ .

The reconstruction operator first appeared implicitly in [12], where Popescu developed a characteristic operator function for noncommuting  $d$ -tuples of contractions. (In this connection it was used explicitly in [14].) The first place the term “reconstruction operator” appeared in the literature is [15, Page 50], which began circulating as a preprint in 2004. Since that time, the reconstruction operator has played an increasingly prominent role in Popescu's work. In addition, the reconstruction operator has popped up elsewhere in the literature, but without the name attached to it. One notable example is Orr Shalit's paper [18, Page 69]. There he attached a whole semigroup of them to representations of certain product systems of correspondences. Because of Example 15 we feel justified in introducing the following terminology.

**Definition 16.** If  $(T, \sigma)$  is a completely contractive covariant representation of a  $C^*$ -correspondence pair  $(F, N)$  on a Hilbert space  $H$ , then the adjoint of the intertwiner of the Morita transform of the canonical stabilization of  $(F, N)$  is called the *reconstruction operator* of  $(T, \sigma)$ ; i.e., the reconstruction operator of  $(T, \sigma)$  is defined to be  $\left(\widetilde{T^{\mathcal{F}(F)}}\right)^*$  viewed as an operator in  $B(\mathcal{F}(F) \otimes_{\sigma} H)$ .

Our analysis begs the questions: How unique is the canonical stabilization of a  $C^*$ -correspondence pair? Are there noncanonical stabilizations? In general there are many stabilizations that “compete” with the canonical stabilization. Organizing them seems to be a complicated matter. To see a little of what is possible, we will briefly outline what happens in the setting of Example 15. So fix  $(\mathbb{C}^d, \mathbb{C})$ . We shall assume  $d$  is finite to keep matters simple. We can stabilize  $\mathbb{C}$  as a  $C^*$ -algebra getting the compact operators on  $\ell^2(\mathbb{N})$ . It is important to do this explicitly, however. So let  $\mathcal{X}$  be column

Hilbert space  $\mathbf{C}_\infty$ . This is  $\ell^2(\mathbb{N})$  with the operator space structure it inherits as the set of all operators from  $\mathbb{C}$  to  $\ell^2(\mathbb{N})$ . Equivalently, it is the set of all infinite matrices  $T = (t_{ij})$  that represent a compact operator on  $\ell^2(\mathbb{N})$  and have the property that  $t_{ij} = 0$ , when  $j > 1$ . (See [1].) We then have  $\widetilde{\mathbf{C}_\infty} = \mathbf{R}_\infty$ , the row Hilbert space. Also, if  $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ , then  $\mathbf{C}_\infty$  is a  $\mathcal{K}, \mathbb{C}$ -equivalence bimodule. So, if  $E$  is any correspondence over  $\mathcal{K}$  that is equivalent to  $\mathbf{C}^d$ , then  $E$  must be isomorphic to

$$\mathbf{C}_\infty \otimes_{\mathbb{C}} \mathbf{C}^d \otimes_{\mathbb{C}} \mathbf{R}_\infty \simeq \mathbf{C}_d(\mathcal{K})$$

with its usual left and right actions of  $\mathcal{K}$ . Because  $\mathcal{K}$  is stable, there is an endomorphism  $\alpha$  of  $\mathcal{K}$  such that  $\mathbf{C}_d(\mathcal{K})$  is isomorphic to  ${}_\alpha\mathcal{K}$ . That is,  ${}_\alpha\mathcal{K}$  is  $\mathcal{K}$  as a right  $\mathcal{K}$ -module (the module product is just the product in  $\mathcal{K}$  and the  $\mathcal{K}$ -valued inner product is  $\langle \xi, \eta \rangle := \xi^* \eta$ .) The left action of  $\mathcal{K}$  is that which is implemented by  $\alpha$ , i.e.,  $a \cdot \xi := \alpha(a)\xi$ . General theory tells us this is the case, but we can see it explicitly as follows. Choose a Cuntz family of  $d$  isometries on  $\ell^2(\mathbb{N})$ ,  $\{S_i\}_{i=1}^d$ . (This means that  $S_i^* S_j = \delta_{ij} I$  and  $\sum_{i=1}^d S_i S_i^* = I$ .) Then, as is well known,  $\{S_i\}_{i=1}^d$  defines an endomorphism of  $\mathcal{K}$  via the formula  $\alpha(a) = \sum_{i=1}^d S_i a S_i^*$ . Note, too, that  $\alpha$  extends to be a *unital* endomorphism of  $B(\ell^2(\mathbb{N}))$  since  $\sum_{i=1}^d S_i S_i^* = I$ . On the other hand, define  $V : \mathbf{C}_d(\mathcal{K}) \rightarrow \mathcal{K}$  via the formula

$$V \left( \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \right) = \sum_{i=1}^d S_i a_i, \quad \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \in \mathbf{C}_d(\mathcal{K}).$$

Then it is a straightforward calculation to see that  $V$  is a correspondence isomorphism from  $\mathbf{C}_d(\mathcal{K})$  onto  ${}_\alpha\mathcal{K}$ . Thus  $\mathcal{X} = \mathbf{C}_\infty$  is an equivalence bimodule between  $({}_\alpha\mathcal{K}, \mathcal{K})$  and  $(\mathbf{C}^d, \mathbb{C})$  and  $({}_\alpha\mathcal{K}, \mathcal{K})$  is a bona fide contender for a stabilization of  $(\mathbf{C}^d, \mathbb{C})$ . Note that this time  ${}_\alpha\mathcal{K}$  is strictly cyclic, but the projection  $P$  is the identity.

Suppose, now, that  $(T, \sigma)$  is a completely contractive covariant representation of  $(\mathbf{C}^d, \mathbb{C})$  on a Hilbert space  $H$ . Then as before  $\sigma$  is an  $n$ -fold multiple of the identity representation of  $\mathbb{C}$  on  $\mathbb{C}$ , where  $n$  is the dimension of  $H$  and  $\widetilde{T} : \mathbf{C}^d \otimes H \rightarrow H$  may be viewed as a row contraction  $[T_1, T_2, \dots, T_d]$  of operators on  $H$ . The induced representation of  $\mathcal{K}$ ,  $\sigma^{\mathbf{C}_\infty}$  is the  $n$ -fold multiple of the identity representation of  $\mathcal{K}$  (same  $n$ ) and a calculation along the lines of that was carried out in Example 15 shows that  $(\widetilde{T^{\mathbf{C}_\infty}})^* = S_1 \otimes T_1^* + S_2 \otimes T_2^* + \dots + S_d \otimes T_d^*$  acting on  $\ell^2(\mathbb{N}) \otimes H$ . Thus,  $(\widetilde{T^{\mathbf{C}_\infty}})^*$  is an alternative for Popescu's reconstruction operator. How different from his reconstruction operator  $(\widetilde{T^{\mathbf{C}_\infty}})^*$  is remains to be seen. We

believe the difference could be very interesting. We believe that the dependence of  $(\widetilde{TC_\infty})^*$  on the Cuntz family  $\{S_i\}_{i=1}^d$  could be very interesting, also.

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