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The stable concordance genus

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Abstract. The concordance genus of a knot is the least genus of any knot in its concordance class. Although difficult to compute, it is a useful invariant that highlights the distinction between the three-genus and four-genus. In this paper we define and discuss the stable concordance genus of a knot. The stable concordance genus describes the behavior of the concordance genus under connected sum, and can be a valuable tool in calculating the concordance genus for certain families of knots. We will present several computations of the stable concordance genus and give a realization result.

Contents

1. Introduction

In studying knot concordance, a question immediately arises: For a given knot K, what is its simplest representative in the concordance group? From a geometric perspective, simplicity is most naturally described in terms of the genus of a knot. We therefore wish to find the least genus of a representative of K, the so-called concordance genus of K, $g_c(K)$.

Studying g_c was initially motivated by two observations. First, it clearly bounds the four-genus: $g_4 \leq g_c$. Secondly, Casson and Nakanishi independently observed that this inequality is not necessarily an equality. Casson observed that for the knot 6_2, $g_4 = 1$ and $g_c = 2$ (unpublished). Nakanishi constructed an infinite family of examples [11].

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Techniques for studying individual knots are well-developed. For instance, in [6, 7] Livingston determined g_c for all prime knots with 10 or fewer crossing, and in [5] the case of 11 crossings was, with a few exceptions, resolved.

The problem of determining g_c for naturally occurring families of knots is much more difficult. In this paper we will introduce a new tool to study the concordance genus, called the stable concordance genus, $\underrightarrow{g_c}$. It will be clear that $\underrightarrow{g_c} \leq g_c$ and that they are not always equal. However, we will identify a number of settings where they turn out to be equal. Section 3 will show calculations for several examples and determine the stable concordance genus for all prime knots of eight or fewer crossings except $7_7, 8_1, 8_{13}$ and 8_{21}. We will use this framework to determine the concordance genus of large families of knots.

In Section 4 we will determine g_c for arbitrary linear combinations of simple torus knots. The comparison to the four-genus of these knots produces Theorem 3, discussed in Section 5. This is a realization result for the four-genus relative to the concordance-genus. In the following, $\mathcal{C}$ denotes the knot concordance group, and $\mathcal{C}_{\mathbb{Q}} = \mathcal{C} \otimes \mathbb{Q}$.

Theorem 3. *For any j, $k \in \mathbb{Q}$, for which $1 \leq j \leq k$, there is some $K \in \mathcal{C}_{\mathbb{Q}}$ for which*

$$\underrightarrow{g_4}(K) = j, \quad \underrightarrow{g_c}(K) = k.$$

Furthermore, if $K \in \mathcal{C}$, given any $l \geq k$ then for some knot K' in the concordance class of K, $g_3(K') = l$.

Furthermore and as important, $\underrightarrow{g_c}$ provides a new perspective that opens up a host of interesting questions to investigate. In Section 5 we will discuss several questions for further research.

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2. Definition and properties

The stable four-genus, $\underrightarrow{g_4}$, is discussed by Livingston in [8] (notated as g_{st} in Livingston's work). It is defined as

$$\underrightarrow{g_4}(K) := \lim_{n \to \infty} \frac{g_4(nK)}{n},$$

where g_4 is the smooth four-genus of the knot. We define the stable concordance genus, $\underrightarrow{g_c}$, similarly in terms of g_c, the smooth concordance genus.

Definition 1. $\underrightarrow{g_c}(K) := \lim_{n \to \infty} \dfrac{g_c(nK)}{n}$.

That this is well-defined is an immediate consequence of Theorem 1 of Livingston's paper [8]. It is a direct consequence of basic properties of limits and the subadditivity of $g_c(K)$.

Since g_c is subadditive, we see that the stable concordance genus is multiplicative:

$$\underrightarrow{g_c}(mK) = \lim_{n\to\infty} \frac{g_c(nmK)}{n} = \lim_{nm\to\infty} \frac{g_c(nmK)}{\frac{nm}{m}} = m \lim_{nm\to\infty} \frac{g_c(nmK)}{nm} = m\underrightarrow{g_c}(K).$$

The stable concordance genus is first defined for knots, but any two concordant knots have the same concordance genus, and consequently the same stable concordance genus, so we can consider $\underrightarrow{g_c}$ to be a function on the concordance group, $\mathcal{C}$. We can extend $\underrightarrow{g_c}$, by multiplicativity, to be defined on $\mathcal{C}_{\mathbb{Q}} = \mathcal{C} \otimes \mathbb{Q}$.

Since the concordance genus is subadditive, the stable concordance genus is also subadditive. Although it is not strictly positive, $\underrightarrow{g_c}$ is at least nonnegative. Hence the stable concordance genus (like the stable four-genus [8]) is a seminorm. That is, it is a nonnegative function which is multiplicative and subadditive. Consequently, $\underrightarrow{g_c}$ satisfies a triangle inequality. Since $\underrightarrow{g_c}$ is a semi-norm, it can be much easier to compute than g_c, as will be seen in examples in Section 4.

We aim to understand $\underrightarrow{g_c}$ by looking at its unit ball,

$$B_{stc} = \{K \in \mathcal{C} | \underrightarrow{g_c}(K) \leq 1\}$$

(similarly B_{st4}, the unit ball for the stable four-genus) and particularly the restriction to two-dimensional subspaces. We will explore some basic examples of computation, with the goal of finding the unit ball of collections of knots of the form $xT_{2,n} + yT_{2,m}$.

To prepare to calculate $\underrightarrow{g_c}$ for basic examples, we first observe several properties of the invariant. Detailed definitions of the Alexander polynomial and classical knot signature can be found in many sources, such as [12], and as such are omitted here. Instead we simply observe several useful properties, which follow easily from properties mentioned in [7].

- $g_3 \geq g_c \geq g_4 \geq \frac{1}{2}|\sigma|$.
- Consequently, $\underrightarrow{g_c} \geq \underrightarrow{g_4} \geq \frac{1}{2}|\sigma|$ (recall that $\sigma(K\#J) = \sigma(K) + \sigma(J)$).
- The same inequalities hold for the Tristram–Levine signatures, so in fact $\underrightarrow{g_c} \geq \frac{1}{2}|\sigma_t(K)|$ for $t \in [0, 1]$.
- Further, $B_{stc} \subset B_{st4} \subset B_\sigma$ (where B_σ is the region in which the Tristram–Levine signatures all have values of two or less).
- $g_3(K) \geq \frac{1}{2}\deg(\Delta_K(t))$, the degree of the Alexander polynomial of K.

Note that the Tristram–Levine signature is defined to be

$$\sigma_t(K) = \lim_{\epsilon\to 0} \frac{1}{2}(\sigma'_{t-\epsilon}(K) + \sigma'_{t+\epsilon}(K))$$

where $\sigma'_t(K) = \text{signature}((1-e^{2\pi it})V + (1-e^{-2\pi it})V^T)$ and V is a Seifert matrix for K.

Theorem 1 (Fox–Milnor, [2]). *If K is slice, then $\Delta_K(t) = \pm t^k f(t)f(t^{-1})$ for some polynomial $f(t)$ and $k \in \mathbb{Z}$.*

As a consequence, if K is concordant to J, then

$$\Delta_K(t)\Delta_J(t) = \pm t^k f(t)f(t^{-1})$$

for some polynomial $f(t)$. So if we can write $\Delta_K(t) = \pm t^k h(t)h(t^{-1})g(t)$, for some polynomials $h(t)$ and $g(t)$, where $g(t)$ has no factors of the form $g_1(t)g_1(t^{-1})$, then we may conclude $g_c(K) \geq \frac{1}{2}\deg(g(t))$. In Section 3 we observe that in conjunction with jumps in the signature function, we can use this lower bound for the concordance genus to also bound $\underset{\rightarrow}{g_c}$ from below.

3. Preliminary examples

To begin exploration of $\underset{\rightarrow}{g_c}$ we calculate values for prime knots with eight crossings or fewer. In the following discussion, values of classical invariants including signature and Alexander polynomial are as given on KnotInfo [1].

The following are results of basic properties of limits:

Proposition 1. *The stable concordance genus is bounded above by the concordance genus.*

Proposition 2. *If $\frac{1}{2}|\sigma(K)| = g_c(K)$, then $\underset{\rightarrow}{g_c}(K) = g_c(K) = \frac{1}{2}|\sigma(K)|$.*

Proposition 2 applies to 15 prime knots of eight or fewer crossings:

$$3_1, 5_1, 5_2, 6_1, 7_2, 7_3, 7_4, 7_5, 8_8, 8_9, 8_{10}, 8_{11}, 8_{15}, 8_{19}, 8_{20}.$$

This includes the slice knots. In fact, as a special case of Proposition 2, all slice knots have stable genus zero.

Corollary 1. *If K is slice, K is stably slice (that is, $\underset{\rightarrow}{g_c} = \underset{\rightarrow}{g_4} = 0$).*

$\mathbf{4_1}$: The figure eight knot is negative amphichiral, so

$$g_4(2 * 4_1) = g_4(4_1\# - 4_1) = 0.$$

Therefore $g_4(2n4_1) = g_c(2n4_1) = 0$. Using the fact that the stable concordance genus is well-defined,

$$\underset{\rightarrow}{g_c}(4_1) = \lim_{n\to\infty} \frac{g_c(n4_1)}{n} = \lim_{k\to\infty} \frac{g_c(2k4_1)}{2k} = \lim_{k\to\infty} 0 = 0.$$

So $\underset{\rightarrow}{g_c}(4_1) = 0$. Since $g_c(4_1) = 1$, this is an example for which $\underset{\rightarrow}{g_c}(K) \neq g_c(K)$.

In fact, we can use the same technique to see that the stable concordance genus vanishes for all knots for which $g_c(nK) = 0$ for some n. This is exactly the knots of finite order in $\mathcal{C}$.

Proposition 3. *Any knot which has finite order in $\mathcal{C}$ is stably slice. In particular, negative amphichiral knots are stably slice.*

For prime knots of eight or fewer crossings, this applies to

$$6_3, 8_3, 8_{12}, 8_{17}, 8_{18}$$

as well as several of the previously mentioned knots including the slice knots.

At this point there remain thirteen prime knots of eight or fewer crossings which we have not yet calculated. We can compute nine of these by examining the relationship between the Alexander polynomial and signature function more closely.

Let $\mathrm{j}_\rho(K)$ denote the jump in the signature function of K at ρ. The following lemma is the key ingredient to proving Proposition 4, as we will see below. This lemma is a consequence of the work of Milnor [10] and Matumoto [9]. It was first formally stated by Garoufalidis [3], although a complete proof is not given in the literature.

Lemma 1. *If ρ is a root of the Alexander polynomial on S^1, then*

$$|\mathrm{j}_\rho(K)| = 2\, a_\rho,$$

where:

(a) *a_ρ is an integer.*
(b) *$a_\rho \leq \mathrm{mult}(\rho, \Delta_K(t))$, where $\mathrm{mult}(\rho, \Delta_K(t))$ is the multiplicity of ρ in $\Delta_K(t)$.*
(c) *$a_\rho \equiv \mathrm{mult}(\rho, \Delta_K(t)) \bmod 2$.*

Moreover, $\mathrm{j}_\rho(K) = -\mathrm{j}_{\bar\rho}(K)$, where $\bar\rho$ denotes the complex conjugate of ρ.

We relegate the proof of Lemma 1 to Appendix A. We will give a direct proof, considering a diagonalization of the Hermitian form

$$B_t = (1 - t^{-1})V + (1 - t)V^t.$$

An alternate proof considers Milnor's definition of σ_θ signatures [10], which are equivalent to the jump function defined above, as shown by Matumoto [9].

Proposition 4. *If a knot, K, has Alexander polynomial $\Delta_K(t) = f(t)^x g(t)$ and $\mathrm{j}_\rho(K) = \pm 2x$ where $f(t)$ is the minimal polynomial for ρ in $\mathbb{Z}[t, t^{-1}]$, then for any J concordant to K, $f(t)^x$ is a factor of $\Delta_J(t)$.*

Proof. This is an immediate consequence of Lemma 1. In particular, for $f(t)$ the minimal polynomial of ρ with $\mathrm{j}_\rho(K) = \pm 2x = \mathrm{j}_\rho(J)$ (since the signature function is a concordance invariant), then $x \leq \mathrm{mult}(\rho, \Delta_J(t))$ and hence $f(t)^x$ is a factor of $\Delta_J(t)$. □

To expedite the discussions of the application of this proposition, we define two new polynomials.

Definition 2. The concordance polynomial of a knot, K, is the maximal degree polynomial which divides the Alexander polynomial of all knots concordant to K. We will denote it $\Delta^c_K(t)$.

This is well-defined up to multiplication by $\pm t^k$. Notice that $\Delta^c_K(t)$ divides $\Delta_J(t)$ for all $J \sim K$. Since $\Delta^c_K(t)$ divides $\Delta_K(t)$ in particular, we see that $\Delta^c_K(t)$ is simply a product of the factors of $\Delta_K(t)$ which also divide each $\Delta_J(t)$ for $J \sim K$.

Definition 3. The jump polynomial of a knot, K, is given by

$$\Delta^j_K(t) := \prod_{f_i(t)} f_i(t)^{\mathrm{j}_i(K)}$$

where f_i are the irreducible factors of $\Delta_K(t) = \prod f_i(t)^{x_i(t)}$, and

$$\mathrm{j}_i(K) := \max\left\{\left|\frac{1}{2}\mathrm{j}_\alpha(K)\right| : \alpha \text{ is a root of } f_i(t)\right\}.$$

The following are immediate consequences of these definitions and the previous results.

Proposition 5. *The jump polynomial of K divides the concordance polynomial of K, and both divide the Alexander polynomial of K. In particular,*

$$\deg(\Delta^j_K(t)) \leq \deg(\Delta^c_K(t)) \leq \deg(\Delta_K(t))$$

Proposition 6. *The concordance polynomial is a concordance invariant. Furthermore, $\frac{1}{2}\deg(\Delta^c_K(t)) \leq g_c(K)$.*

Proof. We observed above that $\Delta^c_K(t)$ divides $\Delta_J(t)$ for all $J \sim K$. Consequently $\deg(\Delta^c_K(T)) \leq \deg(\Delta_J(t))$ for all $J \sim K$, so in particular,

$$\deg(\Delta^c_K(T)) \leq \min\{\deg(\Delta_J(t)) : J \sim K\}.$$

Then since $\frac{1}{2}\deg(\Delta_J(t)) \leq g_3(J)$ for each $J \sim K$, we have

$$\begin{aligned}\frac{1}{2}\deg(\Delta^c_K(t)) &\leq \min\left\{\frac{1}{2}\deg(\Delta_J(t)) : J \sim K\right\} \\ &\leq \min\{g_3(J) : J \sim K\} = g_c(K).\end{aligned}$$ □

Proposition 7. *The degree of the jump polynomial is exactly the sum*

$$\deg \Delta^j_K(t) = \sum_i (\deg f_i(t)) * j_i(K)$$

where $f_i(t)$ and $j_i(K)$ are as given in the definition of the jump polynomial. Moreover, one half of this value is a lower bound for the concordance genus of K.

6_2: We begin by checking the signature and Alexander polynomial. The signature is $\sigma(6_2) = -2$, and $\Delta_{6_2}(t) = 1 - 3t + 3t^2 - 3t^3 + t^4$, which is irreducible in $\mathbb{Z}[t, t^{-1}]$. The concordance genus is $g_c(6_2) = 2$. So we have $1 = \frac{1}{2}|\sigma(K)| \leq \underrightarrow{g_c}(6_2) \leq g_c(6_2) = 2$. The Tristram–Levine signature jumps at the two complex roots of $\Delta_{6_2}(t)$, α and $\overline{\alpha}$ by two. While we cannot get a stronger bound directly from the Tristram–Levine signatures, we can use the jump function to show that half the degree of the Alexander polynomial of 6_2 (or in similar cases, a factor of the Alexander polynomial) does bound $\underrightarrow{g_c}$.

In this case, $\sigma_\omega(n6_2)$ jumps by $-2n$ at α (and $\overline{\alpha}$), i.e. $\mathrm{j}_\alpha(n6_2) = -2n$. The degree of the corresponding irreducible factor (which is in this case $\Delta_{6_2}(t)$) is 4. Hence by Proposition 7, for all $n \geq 1$,

$$2n = \frac{1}{2}\deg(1 - 3t + 3t^2 - 3t^3 + t^4)^n \leq g_c(n6_2).$$

Finally, we see that

$$2 = \frac{\frac{1}{2}\deg(\Delta_{6_2}(t)^n)}{n} \leq \underrightarrow{g_c}(6_2) \leq g_c(6_2) = 2.$$

We conclude that $\underrightarrow{g_c}(6_2) = 2$.

8_5: We can also apply Proposition 7 to Alexander polynomials which are products of several irreducible factors. In this case, to get a sharp bound we require that the signature function jump at roots of each factor of $\Delta_{8_5}(t)$. The Alexander polynomial of 8_5 is

$$\begin{aligned}\Delta_{8_5}(t) &= 1 - 3t + 4t^2 - 5t^3 + 4t^4 - 3t^5 + t^6\\ &= (1 - t + t^2)(1 - 2t + t^2 - 2t^3 + t^4).\end{aligned}$$

The signature functions jumps by 2 at α (and $\bar{\alpha}$), the root of $1 - t + t^2$, and also by 2 at β (and $\bar{\beta}$), where β is one of the roots of $1 - 2t + t^2 - 2t^3 + t^4$. Hence by a similar argument to above, applied to both factors, and we may conclude that

$$3n = \frac{1}{2}[\deg(1 - t + t^2)^n + \deg(1 - 3t + 3t^2 - 3t^3 + t^4)^n] \leq g_c(n8_5).$$

So we have $3 = \frac{1}{2}\deg(\Delta_{8_5}(t)) \leq \underrightarrow{g_c}(8_5) \leq g_c(8_5) = 3$.

We can similarly calculate the stable concordance genus of

$$7_6, 8_2, 8_4, 8_6, 8_7, 8_{14}, 8_{16}.$$

In each of these cases the stable concordance genus is equal to the concordance genus. There are four prime knots of eight or fewer crossings for which the stable concordance genus is as of yet undetermined: $7_7, 8_1, 8_{13}, 8_{21}$.

4. Torus knots

The stable concordance genus is particularly interesting when we use it to examine larger collections of knots under connect sum. Here we will look for the stable concordance genus unit ball restricted to sums of the form $xK + yJ$, with K, J torus knots. Richard Litherland has dealt with the 4-genus of such sums of torus knots $xT_{2,n} + yT_{2,m}$. Here we will present a calculation of the concordance genus, stable and unstable, for these knots, and contrast to the 4-genus calculation. The 4-genus calculation mirrors Livingston's calculation of $xT_{2,7} + yT_{2,11}$ in [8]. This is summarized in Theorem 2 below

Having calculated the stable concordance genus of $T_{2,3} = 3_1$ and $T_{2,5} = 5_1$ in the previous section, we begin with sums of these two knots.

$\mathbf{xT_{2,3} + yT_{2,5}}$: The signature function of $xT_{2,3}+yT_{2,5}$ jumps at $1/10$, $1/6$, and $3/10$ in $[0, 1/2]$, taking on the values: $0 \in [0, 1/10)$, $2y \in [1/10, 1/6)$, $2x + 2y \in [1/6, 3/10)$, and $2x + 4y \in [3/10, 1/2]$. We will first look at the stable four-genus for this family of knots. The signature function gives us the bounds

$$\underset{\rightarrow}{g_4} \geq |y|$$
$$\underset{\rightarrow}{g_4} \geq |x + y|$$
$$\underset{\rightarrow}{g_4} \geq |x + 2y|.$$

Considering each of these inequalities for $\underset{\rightarrow}{g_4} \leq 1$, we bound a region in the plane (this is the signature ball B_σ defined in Section 2). We then check the corner points of this region, and see that since $\underset{\rightarrow}{g_4}(T_{2,3}) = 1$, $\underset{\rightarrow}{g_4}(-T_{2,3} + T_{2,5}) = 1$ and $\underset{\rightarrow}{g_4}(-2T_{2,3} + T_{2,5}) = 1$. Since $\underset{\rightarrow}{g_4}(xK + yJ) = \underset{\rightarrow}{g_4}(-xK - yJ)$ this is enough to determine that this region is in fact the unit ball for the stable 4-genus (Figure 1). Although it does not represent a corner point, $\underset{\rightarrow}{g_4}(T_{2,5}) = 2$ as we saw earlier, which is consistent with this calculation.

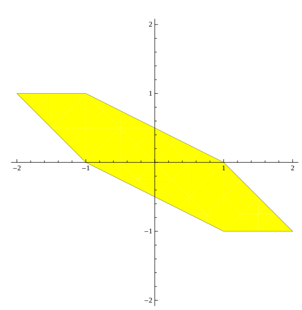
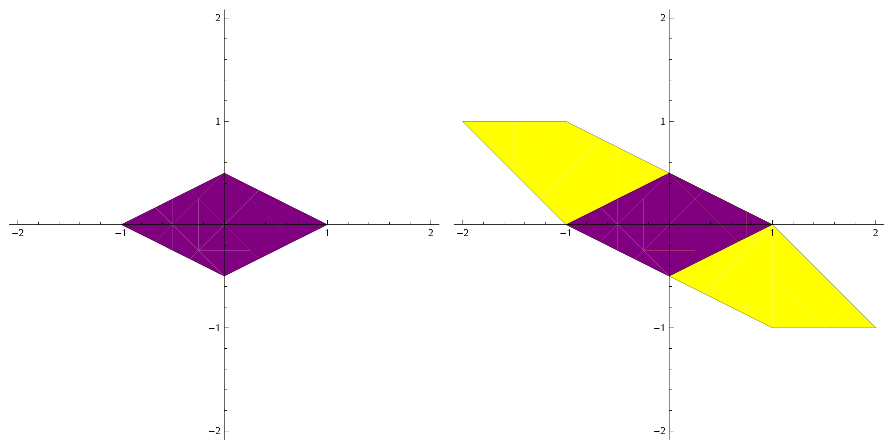


FIGURE 1. Stable four-genus unit ball for $xT_{2,3} + yT_{2,5}$.

To calculate the stable concordance genus unit ball, we generalize the calculation given in the previous section for the knot 6_2. Notice that

$$\Delta_{xT_{2,3}+yT_{2,5}}(t) = (1-t+t^2)^{|x|}(1-t+t^2-t^3+t^4)^{|y|}$$

and since $\sigma_\omega(t)$ jumps at the roots of each factor by $2x$ and $2y$, the jump polynomial is $\Delta^j_{xT_{2,3}+yT_{2,5}} = \Delta_{xT_{2,3}+yT_{2,5}}$. So, by Proposition 7, we have $\underrightarrow{g_c}(xT_{2,3}+yT_{2,5}) \geq |x| + 2|y|$. Thus the unit ball for $\underrightarrow{g_c}(xT_{2,3}+yT_{2,5})$ is contained in the ball defined by these equations, but furthermore, $\underrightarrow{g_c}(T_{2,3}) = 1$, and $\underrightarrow{g_c}(T_{2,5}) = 2$, so by linearity this is the unit ball (Figure 2).



(A) Stable concordance genus unit ball.

(B) Stable concordance genus unit ball overlaid stable four-genus unit ball.

FIGURE 2. The stable concordance genus unit ball for $xT_{2,3}+yT_{2,5}$ is different from the stable four-genus unit ball.

We observe in particular that the unit ball for the stable four-genus is different from the unit ball for the stable concordance genus. This is the primary observation that motivates Theorem 3.

$\mathbf{T_{2,n}}$: For any torus knot of the form $T_{2,n}$, $\underrightarrow{g_c}(T_{2,n}) = \frac{|n|-1}{2}$. For any such knot, there is a surface of genus $\frac{|n|-1}{2}$ whose boundary is the knot, so $\underrightarrow{g_c}(T_{2,n}) \leq g_3(T_{2,n}) \leq \frac{|n|-1}{2}$. On the other hand, $\sigma(T_{2,n}) = \frac{|n|-1}{2}$, so $\underrightarrow{g_c}(T_{2,n}) \geq \underrightarrow{g_4}(T_{2,n}) \geq \frac{|n|-1}{2}$. This will assist us in a general calculation of knots of the form $xT_{2,n}+yT_{2,m}$. We conclude that $\underrightarrow{g_c}(T_{2,n}) = \frac{|n|-1}{2}$.

$\mathbf{xT_{2,n}+yT_{2,m}}$: As long as the Alexander polynomials of $T_{2,n}$ and $T_{2,m}$ have distinct factors, then the argument for $xT_{2,3}+yT_{2,5}$ follows through with any family of knots of the form $xT_{2,n}+yT_{2,m}$. We claim the following:

Theorem 2. *The stable concordance genus of a knot of the form $xT_{2,n}+yT_{2,m}$ is*

$$\frac{|n|-1}{2}|x|+\frac{|m|-1}{2}|y|$$

for any $n,m\in\mathbb{Z}$ with $n<m$, $kn\neq m$ for any $k\in\mathbb{Z}$ and any $x,y\in\mathbb{Q}$.

For ease of exposition, we'll assume $n,m>0$ in the following proof, and continuing throughout the paper. The results for negative n and m follow similarly with appropriate absolute value signs peppered in.

Proof. The Alexander polynomial for these knots is $\Delta_{T_{2,n}}(t)=\frac{(t^{2n}-1)(t-1)}{(t^2-1)(t^n-1)}$ (recall further that $\Delta_{K\#J}(t)=\Delta_K\Delta_J$). Notice, $\Delta_{T_{2,n}}$ is a product of the cyclotomic polynomials $\Phi_i(t)$ for i a factor of $2n$ other than 2 or n. Since n is odd, $\Phi_{2n}(t)$ is a factor of $\Delta_{T_{2,n}}(t)$. And, if $n\neq mk$ then $\Phi_{2n}(t)$ is not a factor of $\Delta_{T_{2,m}}(t)$. Recall that $\Phi_{2n}(t)$ is the minimal polynomial of the $2n^{th}$ primitive roots of unity, ξ_{2n}^k (for $1\leq k<n$ and $\gcd(k,n)=1$).

For each of the primitive $2n^{th}$ roots of unity, the signature function jumps (specifically, $\mathrm{j}_{\xi_{2n}}(xT_{2,n}+yT_{2,m})=\pm 2x$ and $\mathrm{j}_{\xi_{2m}}(xT_{2,n}+yT_{2,m})=\pm 2y$). Thus, so as long as n and m are have factors distinct from the other (that is, $n\neq mk$ and $m\neq nk$), then the signature function for K jumps at a root of $\Delta_{T_{2,n}}(t)$ which is not a root of $\Delta_{T_{2,m}}(t)$ and vice versa. Applying Proposition 7, we have $\underrightarrow{g_c}(xT_{2,n}+yT_{2,m})\geq\frac{n-1}{2}|x|+\frac{m-1}{2}|y|$. As computed above, $\underrightarrow{g_c}(T_{2,n})=\frac{n-1}{2}$. We conclude then that the unit ball for such knots is defined by the inequality $1\geq\underrightarrow{g_c}(xT_{2,n}+yT_{2,m})\geq\frac{n-1}{2}|x|+\frac{m-1}{2}|y|$. Hence, more generally, $\underrightarrow{g_c}(xT_{2,n}+yT_{2,m})=\frac{n-1}{2}|x|+\frac{m-1}{2}|y|$. □

In fact, we see that for $x,y\in\mathbb{Z}$, this is the same as $g_3(xT_{2,n}+yT_{2,m})$, so we have proven the following.

Corollary 2. *For any knot of the form $xT_{2,n}+yT_{2,m}$ with $n,m\in\mathbb{Z}$ with $n<m$, $kn\neq m$ for any $k\in\mathbb{Z}$ and any $x,y\in\mathbb{Z}$,*

$$g_c(xT_{2,n}+yT_{2,m})=\underrightarrow{g_c}(xT_{2,n}+yT_{2,m})=\frac{|n|-1}{2}|x|+\frac{|m|-1}{2}|y|.$$

5. A realization result

We observed in the case of $xT_{2,3}+yT_{2,5}$, the unit ball for the stable four-genus is not the same as that for the stable concordance genus. A similar calculation in the more general case of $xT_{2,n}+yT_{2,m}$ allows us to construct examples that show the following theorem.

Theorem 3. *For any j, $k\in\mathbb{Q}$, for which $1\leq j\leq k$, there is some $K\in\mathcal{C}_{\mathbb{Q}}$ for which $\underrightarrow{g_4}(K)=j$, $\underrightarrow{g_c}(K)=k$.*

Proof. We will use the fact, from Theorem 2, that the stable concordance genus of the sum of torus knots $xT_{2,2n+1} + yT_{2,2m+1}$ is $n|x| + m|y|$ (for $n, m \in \mathbb{Z}$ with $n < m$, $k(2n+1) \neq (2m+1)$). We also will use that the stable four genus calculation from the previous section generalizes to certain connect sums of knots $T_{2,n}$.

According to a calculation by Rick Litherland, for these knots the stable four-genus is determined by the signature function [8]. The signature function gives a set of bounds of the form:

$$\underset{\rightarrow}{g_4}(K) \geq |y|$$
$$\vdots$$
$$\underset{\rightarrow}{g_4}(K) \geq |n'x + m'y|$$
$$\vdots$$
$$\underset{\rightarrow}{g_4}(K) \geq |nx + (m-1)y|$$
$$\underset{\rightarrow}{g_4}(K) \geq |nx + my|$$

where $0 \leq n' \leq n$ and $0 \leq m' \leq m$, and where the sequence above has increasing values of n' and m'. While the particular sequence of inequalities depends on n and m and the relative orders of their corresponding roots of unity, we can determine the order of a few based on the fact that $n < m$. Since $n < m$, the m^{th} root of unity occurs before the n^{th} root of unity. Since these roots give the jumps in the signature function, we know that the signature jumps first at $\frac{1}{m}$ by $2y$, and that the last jump before $t = \frac{1}{2}$ occurs at $t = \frac{m-1}{2m}$ and is also a jump by $2y$. In particular, we know the signature in the interval $t = \frac{m-1}{2m}$ to $t = \frac{1}{2}$ gives the bound $\underset{\rightarrow}{g_4}(K) \geq |nx + my|$. We claim that this is the highest bound given by the signature function (and hence by [8] is equal to the stable four genus) for small negative values of x.

Proof of Claim. Amongst all of the lines $n'x + m'y = 1$ with $0 \leq n' \leq n$ and $0 \leq m' \leq m$, the lowest y-intercept occurs for lines with $m' = m$. As discussed above, the only such line given by the signature function bounds will be the line $nx + my = 1$. So in fact the line $nx + my = 1$ gives the upper boundary of the unit ball on the y-axis. Furthermore, all other lines have an y-intercept of at least $\frac{1}{m-1}$. Hence, in the second quadrant the line $nx + my = 1$ must continue to be the upper boundary until it crosses one of these lines. Each of these lines has negative slope, so they are all above the line $y = \frac{1}{m-1}$ in the second quadrant. Thus, for $d(n,m) := \frac{1}{n(1-m)}$ (the x-coordinate of intersection of the line $nx+my = 1$ with the line $y = \frac{1}{m-1}$), we have that for any x with $d(n,m) \leq x \leq 0$, the line $nx+my = 1$ gives the upper boundary of the stable four-genus unit ball, restricted to the plane $xT_{2,2n+1}+yT_{2,2m+1}$. Thus we have proven the claim that $\underset{\rightarrow}{g_4}(K) \geq |nx+my|$ is a tight bound for small negative values of x. $\square$

By multiplicativity, if we choose a pair (x, y) on this line, which satisfies $\frac{k}{j} = n(-x)+my = n|x|+m|y|$, then for $K = jxT_{2,2n+1}+jyT_{2,2m+1}$, we have that $\underrightarrow{g_c}(K) = k$ and $\underrightarrow{g_4}(K) = j$. A simple linear algebra computation shows us that (x, y) should be $(\frac{1}{2n}(1 - \frac{k}{j}), \frac{1}{2m}(1 + \frac{k}{j}))$. Then by choosing n and m sufficiently large, so that $d(n, m) \leq 0$ (note that x is already negative, and by simplifying the left inequality, we see that such an n and m can always be chosen), we guarantee that such an (x, y) produces the desired values of $\underrightarrow{g_c}$ and $\underrightarrow{g_4}$. □

Corollary 3. *In the case of Theorem* 3 *where $K \in \mathcal{C}$ (so in particular, K is a $\mathbb{Z}$ linear combination of torus knots), then given any $l \geq k$, there is a knot K' in the concordance class of K, such that $g_3(K') = l$, $\underrightarrow{g_4}(K') = j$, and $\underrightarrow{g_c}(K') = k$.*

Proof. Suppose K as calculated above is a knot (in particular x and y are integers). It may be that $g_3(K) \neq l$. If necessary, we may lower the three-genus to k, by definition of $\underrightarrow{g_c}(K)$ (without changing $\underrightarrow{g_c}$ or $\underrightarrow{g_4}$). Let $K' \sim K$ be such that $g_3(K') = \underrightarrow{g_c}(K) = k$. Let J be a slice knot with $g_3(J) = 1$ (for instance 6_1). The three-genus is additive, so $g_3(xJ) = x$. Then if $K'' = K'\#(l-k)J$, we have $g_3(K'') = g_3(K')+g_3((l-k)J) = k+(l-k) = l$. Since J was slice, we still have $\underrightarrow{g_c}(K'') = k$ and $\underrightarrow{g_4}(K'') = j$. □

Note. In the proof of Theorem 3, we have not required (and in fact usually may not assume) that x and y be integer values. Hence, we have only completed the proof in $\mathcal{C}_{\mathbb{Q}}$ and only claim it to be true for the stable invariants, not the concordance genus and four genus (since these are not defined for $\mathcal{C}_{\mathbb{Q}}$). A more detailed examination of the stable four genus may yield different examples in which we can demand integer values of our coefficients. In this case, we can refine the result to give an actual knot.

Conjecture 1. *For any j, k, and $l \in \mathbb{Z}$, for which $1 \leq j \leq k \leq l$, there is some knot K for which $g_4(K) = j$, $g_c(K) = k$, and $g_3(K) = l$.*

It has been previously observed in work by Casson and also in work of Nakanishi [11] that the gap between $g_4(K)$ and $g_c(K)$ can be made arbitrarily large. A proof of this conjecture would confirm that we can additionally construct K to have a given value for $g_c(K)$. On the other hand, it is evident that for any $K \in \mathcal{C}_{\mathbb{Q}}$, some integer multiple of K is a knot. Since $\underrightarrow{g_c}$ is multiplicative, as a consequence of Theorem 3, we have found the following result, similar to that of Casson and Nakanishi:

Corollary 4. *The difference of the invariants $\underrightarrow{g_c}(K) - \underrightarrow{g_4}(K)$ for $K \in \mathcal{C}$ can be made arbitrarily large.*

There are many other open questions raised by this invariant. We'll conclude with listing several of them.

- In the examples in the Section 4 we observed that $\underset{\rightarrow}{g_4}(K) = g_4(K)$ and $\underset{\rightarrow}{g_c}(K) = g_c(K)$. We saw in Section 3 that this is not always the case. It is unknown whether this gap can be made arbitrarily large.
- Livingston gives an example in [8] of a knot with rational (non-integer) stable four genus. On the other hand, there are no known knots with rational (non-integer) stable concordance genus.
- In all of the examples calculated in this paper, if $\underset{\rightarrow}{g_c}(K) = k$, then for some integer multiple of K, $\frac{g_c(nK)}{n} = k$. Of course, it is not necessarily true that a limit of a sequence must appear in that sequence. It is an open question whether there is a knot K for which $\underset{\rightarrow}{g_c}(K) = k$, but there is no multiple n of K such that $\frac{g_c(nK)}{n} = k$.
- A special case of the previous question: Does there exist a knot K which is not finite order in the concordance group but $\underset{\rightarrow}{g_c}(K) = 0$? It is clearly true that if K is torsion in the concordance group, then $\underset{\rightarrow}{g_c}(K) = 0$. If the converse is true, it could prove to be a very useful tool to identify torsion in the concordance group. It is known that there is two-torsion in the concordance group, but it is still unknown whether there is any other torsion in the concordance group.
- We observed that if $g_4(K) = 0$ then $g_c(K) = 0$. Does the same hold for $\underset{\rightarrow}{g_4}$ and $\underset{\rightarrow}{g_c}$? This is in fact related to the previous question. If it is true that $\underset{\rightarrow}{g_c}(K) = 0$ only when K is torsion in the concordance group, and similarly that $\underset{\rightarrow}{g_4}(K) = 0$ only when K is torsion in the concordance group, then it must also be true that whenever $\underset{\rightarrow}{g_4}(K) = 0$ then $\underset{\rightarrow}{g_c}(K) = 0$ as well. Otherwise, there may be a distinction between the stable invariants which cannot arise for the classical invariants.

Appendix A. Proof of Lemma 1

Recall, Lemma 1, as stated in Section 3:

Lemma 1. *If ρ is a root of the Alexander polynomial on S^1, then*

$$|\mathrm{j}_\rho(K)| = 2\, a_\rho,$$

where:

(a) *a_ρ is an integer.*
(b) *$a_\rho \leq \mathrm{mult}(\rho, \Delta_K(t))$, where $\mathrm{mult}(\rho, \Delta_K(t))$ is the multiplicity of ρ in $\Delta_K(t)$.*
(c) *$a_\rho \equiv \mathrm{mult}(\rho, \Delta_K(t)) \bmod 2$.*

Moreover, $\mathrm{j}_\rho(K) = -\mathrm{j}_{\bar\rho}(K)$, where $\bar\rho$ denotes the complex conjugate of ρ.

Proof. We begin by proving part (b). Consider the $\mathbb{Q}(t)$ Hermitian form given by $B_t = (1-t^{-1})V + (1-t)V^t$. Notice that $(1-t^{-1})^n \Delta_K(t) = \det(B_t)$.

The matrix B_t can be diagonalized. In particular, there is a matrix A with $\det(A) = 1$ and AB_tA^* is diagonal, where A^* is the conjugate transpose. Choosing A carefully, one can insist that the diagonal matrix AB_tA^* has rational functions on the diagonal, and avoid having factors $p(t)$ of the Alexander polynomial as denominators. Then we can see a direct relationship between jumps in the signature function and factors of the Alexander polynomial. The following argument shows that the matrix can in fact be diagonalized in such a way. This argument is a special case of a result proved by Gilmer and Livingston [4].

For ease of exposition, we will work over the real numbers. Then we can guarantee that the irreducible factor corresponding to the jump at $\rho = e^{i\theta}$ is $p(t) = t^2 - 2\cos(\theta)t + 1$. We hope to see that any Hermitian matrix with entries in $\mathbb{R}(t)$ can be diagonalized (using simultaneous row and conjugate column operations) over the field of fractions so that no diagonal entry has a power of $p(t)$ in its denominator. We will call this relation congruence.

Let R be the ring formed from $\mathbb{R}(t)$ by inverting all nonzero elements that are not multiples of $p = p(t)$. In R, every element can be written as $\frac{r}{s}p^m$ for some r, s, relatively prime to p, and some $m \geq 0$. Note that B_t has entries in R. So our goal is to see we can diagonalize B_t (or more generally, a matrix B with entries in R) over R.

Consider all congruent matrices to B. Choose one with a diagonal entry of the form ap^k with k minimal. If necessary, transpose rows and columns so that this minimum occurs at the (1, 1) entry. If $k = 0$, we can use this diagonal entry to clear out a row and column and proceed by induction on a smaller matrix. So suppose $k > 0$.

We consider the top left 2×2 corner. Following the Euclidean algorithm, we perform row operations on all rows but the first (and corresponding column operations), so that the upper left corner is of the form

$$\begin{pmatrix} ap^k & bp^m \\ \bar{b}p^m & cp^n \end{pmatrix}$$

(where a, b, and c are units in R and $\bar{b}$ denotes the complex conjugate of b), and all other entries in the first row and column are 0.

Suppose $k > m$. We also have $k < n$ by minimality of k, so $m < k < n$. Multiply the second row by $1/\bar{b}$ and add it to the first row and perform the corresponding column operation. We now have

$$\begin{pmatrix} ap^k + 2p^m + \alpha p^n & * \\ * & * \end{pmatrix}$$

where α is a unit in R. Since $m < k < n$, we can write the (1, 1) entry as $p^m(ap^{k-m} + 2 + \alpha p^{n-m})$. Since $ap^{k-m} + 2 + \alpha p^{n-m}$ is clearly prime to p (we can evaluate at the roots of p and get 2), we now have a new diagonal entry with p exponent less than k, contradicting our assumption that k was minimal.

It follows that we must have $k \leq m$. In this case we can clear the (1, 2) and (2, 1) entries, and complete our argument by induction. This concludes the proof of part (b).

Part (a) is an immediate consequence of the fact that knot signatures are always even.

Part (c) follows directly from considering the diagonalized matrix from the proof of part (b).

□

References

[1] Cha, J. C., and Livingston, C. KnotInfo: table of knot invariants. http://www.indiana.edu/~knotinfo, October 2013.

[2] Fox, Ralph H.; Milnor, John W. Singularities of 2-spheres in 4-space and cobordism of knots. *Osaka J. Math.* **3** (1966), 257–267. MR0211392 (35 #2273), Zbl 0146.45501.

[3] Garoufalidis, Stavros. Does the Jones polynomial determine the signature of a knot? Preprint, 2003. arXiv:math/0310203.

[4] Gilmer, P. M., and Livingston, C. Signature jumps and the Alexander polynomial for links. In preparation.

[5] Kearney, M. Kate. The concordance genus of 11–crossing knots. *J. Knot Theory Ramifications* **22** (2013), no. 13, 1350077, 17 pp. MR3152219, Zbl 06254735, arXiv:1208.5059, doi: 10.1142/S0218216513500776.

[6] Livingston, Charles. The concordance genus of knots. *Algebr. Geom. Topol.* **4** (2004), 1–22. MR2031909 (2005e:57023), Zbl 1055.57007, doi: 10.2140/agt.2004.4.1.

[7] Livingston, Charles. The concordance genus of a knot. II. *Algebr. Geom. Topol.* **9** (2009), no. 1, 167–185. MR2482072 (2009m:57012), Zbl 1171.57004, arXiv:0807.0765, doi: 10.2140/agt.2009.9.167.

[8] Livingston, Charles. The stable 4-genus of knots. *Algebr. Geom. Topol.* **10** (2010), no. 4, 2191–2202. MR2745668 (2012c:57017), Zbl 1213.57015, arXiv:0904.3054, doi: 10.2140/agt.2010.10.2191.

[9] Matumoto, Takao. On the signature invariants of a nonsingular complex sesquilinear form. *J. Math. Soc. Japan* **29** (1977), no. 1, 67–71. MR0437456 (55 #10386), Zbl 0376.15013.

[10] Milnor, John W. Infinite cyclic coverings. *Conference on the Topology of Manifolds (Michigan State Univ., E. Lansing, Mich., 1967)* 115–133. *Prindle, Weber & Schmidt, Boston, Mass.*, 1968. MR0242163 (39 #3497), Zbl 0179.52302.

[11] Nakanishi, Yasutaka. A note on unknotting number. *Math. Sem. Notes Kobe Univ.* **9** (1981), no. 1, 99–108. MR0634000 (83d:57005), Zbl 0481.57002.

[12] Rolfsen, Dale. Knots and Links. Corrected reprint of the 1976 original. Mathematics Lecture Series, 7. *Publish or Perish Inc., Houston, TX*, 1990. xiv+439 pp. ISBN: 0-914098-16-0. MR1277811 (95c:57018), Zbl 0854.57002.

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