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Special Legendrian submanifolds in toric Sasaki–Einstein manifolds

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Abstract. We show every toric Sasaki–Einstein manifold S admits a special Legendrian submanifold L which arises as the link $\mathrm{fix}(\tau) \cap S$ of the fixed point set $\mathrm{fix}(\tau)$ of an anti-holomorphic involution τ on the cone $C(S)$. In particular, we obtain a special Legendrian torus $S^1 \times S^1$ in an irregular toric Sasaki–Einstein manifold which is diffeomorphic to $S^2 \times S^3$. Moreover, there exists a special Legendrian submanifold in $\sharp m(S^2 \times S^3)$ for each $m \geq 1$.

Contents

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1. Introduction

A Sasaki–Einstein manifold is a $(2n+1)$-dimensional Riemannian manifold (S, g) whose metric cone $(C(S), \overline{g}) = (\mathbb{R}_{>0} \times S, dr^2 + r^2 g)$ is a Ricci-flat Kähler manifold where r is the coordinate of $\mathbb{R}_{>0}$. Assuming that S is simply connected, then the cone $C(S)$ is a complex $(n+1)$-dimensional Calabi–Yau manifold which admits a holomorphic $(n+1)$-form Ω and a Kähler form ω on $C(S)$ satisfying the Monge–Ampère equation

$$\Omega \wedge \overline{\Omega} = c_{n+1} \omega^{n+1}$$

for a constant c_{n+1}. The real part Ω^{Re} of Ω is a calibration whose calibrated submanifolds are called *special Lagrangian submanifolds* [11]. An n-dimensional submanifold L in a Sasaki–Einstein manifold (S, g) is a *special Legendrian submanifold* if the cone $C(L)$ is a special Lagrangian submanifold in $C(S)$. We identify S with the hypersurface $\{r = 1\}$ in $C(S)$, and then L is regarded as the link $C(L) \cap S$ of $C(L)$.

Recently, toric Sasaki–Einstein manifolds have been constructed [3, 8, 9, 16]. The purpose of this paper is to construct a special Legendrian submanifold in every toric Sasaki–Einstein manifold. For a toric Sasaki manifold (S, g), the metric cone $(C(S), \overline{g})$ is a toric Kähler variety. Then there exists an anti-holomorphic involution τ on $C(S)$.

Theorem 1.1. *Let (S, g) be a compact simply connected toric Sasaki–Einstein manifold. Then the link $\mathrm{fix}(\tau) \cap S$ is a special Legendrian submanifold.*

The fixed point set of an isometric and anti-holomorphic involution is called the *real form*. It is well known that a real form of a Calabi–Yau manifold is a special Lagrangian submanifold. The point of Theorem 1.1 is to show that the real form $\mathrm{fix}(\tau)$ arises as the cone of the link $\mathrm{fix}(\tau) \cap S$. We have a generalization of Theorem 1.1 as follows:

Theorem 1.2. *Let (S, g) be a compact toric Sasaki manifold. Then the link $\mathrm{fix}(\tau) \cap S$ is a totally geodesic Legendrian submanifold.*

A typical example of Sasaki–Einstein manifolds is the odd-dimensional unit sphere S^{2n+1} with the standard metric, then the cone is the complex space $\mathbb{C}^{n+1} \backslash \{0\}$. Special Lagrangian cones in $\mathbb{C}^{n+1} \backslash \{0\}$ are regarded as special Lagrangian subvarieties in $\mathbb{C}^{n+1}$ with an isolated singularity at the origin. Joyce had provided the theory of special Lagrangian submanifolds in $\mathbb{C}^{n+1}$ with conical singularities [15]. Many examples of special Lagrangian submanifolds in $\mathbb{C}^{n+1}$ with the isolated singularity at the origin had been constructed [5, 12, 14, 19]. These special Lagrangian cones induce special Legendrian submanifolds in the sphere S^{2n+1}. Recently, Haskins and Kapouleas gave a construction of special Legendrian immersions into the sphere S^{2n+1} [13]. Special Legendrian submanifolds have also the aspect of minimal Legendrian submanifolds. On the sphere S^{2n+1}, the standard Sasaki–Einstein structure is regular and induced from the Hopf fibration

$S^{2n+1} \to \mathbb{C}P^n$. Some special Legendrian submanifolds in S^{2n+1} arise as lifts of minimal Lagrangian submanifolds in $\mathbb{C}P^n$ [5, 19].

There exist two interesting points of our theorems. One is that we can construct a special Legendrian submanifold in every toric Sasaki–Einstein manifold which is not necessarily the sphere S^{2n+1}. The other is that some of these special Legendrian submanifolds are totally geodesic Legendrian submanifolds in irregular Sasaki–Einstein manifolds. A Sasaki–Einstein manifold of dimension 3 is finitely covered by the standard 3-sphere S^3. Hence we will consider the case of Sasaki–Einstein manifolds whose dimension are greater than or equal to 5. Gauntlett, Martelli, Sparks and Wardram provided a family of explicit Sasaki–Einstein metrics $g_{p,q}$ on $S^2 \times S^3$ [9]. Let $Y_{p,q}$ denote the Sasaki–Einstein manifold $(S^2 \times S^3, g_{p,q})$.

Theorem 1.3. *There exists a special Legendrian torus $S^1 \times S^1$ in the toric Sasaki–Einstein manifold $Y_{p,q}$.*

Any simply connected toric Sasaki–Einstein 5-manifold is diffeomorphic to the m-fold connected sum $\sharp m(S^2 \times S^3)$ of $S^2 \times S^3$ for an integer $m \geq 0$ where $\sharp m(S^2 \times S^3)$ for $m = 0$ means the 5-sphere S^5. Boyer, Galicki, Nakamaye and Kollár showed that there exist many Sasaki–Einstein metrics on $\sharp m(S^2 \times S^3)$ for each $m \geq 1$ [3, 16]. Van Covering provided a toric Sasaki–Einstein metric on $\sharp m(S^2 \times S^3)$ for each odd $m > 1$ [24]. For any $m \geq 1$, Cho, Futaki and Ono showed that there exists an infinite inequivalent family of toric Sasaki–Einstein metrics on $\sharp m(S^2 \times S^3)$ [6]. We fix a toric Sasaki–Einstein metric on $\sharp m(S^2 \times S^3)$ and denote S by the toric Sasaki–Einstein manifold $\sharp m(S^2 \times S^3)$ with the metric. Let τ be the anti-holomorphic involution on the toric Kähler cone $C(S)$ constructed in §3.3. Then Theorem 1.1 implies the following corollary:

Corollary 1.4. *For any $m \geq 1$, the link $\mathrm{fix}(\tau) \cap S$ is a special Legendrian submanifold in $\sharp m(S^2 \times S^3)$.*

The paper is organized as follows. In Section 2, we recall basic facts about Sasakian geometry. We introduce weighted Calabi–Yau structures on the Kähler cones of Sasaki manifolds which characterize Sasaki–Einstein structures on Sasaki manifolds. In Section 3, we define special Legendrian submanifolds in Sasaki–Einstein manifolds and provide a method to find special Legendrian submanifolds by considering the fixed point set of an anti-holomorphic involution. We apply the method to toric Sasaki–Einstein manifolds, and prove Theorem 1.1. We also provide Theorem 1.2 as a generalization of Theorem 1.1. We show Theorem 1.3 and give examples of special Legendrian submanifolds.

2. Sasakian geometry

In this section, we will give a brief review of some elementary results in Sasakian geometry. For much of this material, we refer to [2] and [21]. We assume that S is a smooth manifold of dimension $(2n + 1)$.

2.1. Sasaki structures.

Definition 2.1. A Riemannian manifold (S, g) is a *Sasaki manifold* if and only if the metric cone $(C(S), \overline{g}) = (\mathbb{R}_{>0} \times S, dr^2 + r^2 g)$ is Kähler for a complex structure.

We identify the manifold S with the hypersurface $\{r = 1\}$ of $C(S)$. Let J and ω denote the complex structure and the Kähler form on the Kähler manifold $(C(S), \overline{g})$, respectively. The vector field $r\frac{\partial}{\partial r}$ is called the *Euler vector field* on $C(S)$. We define a vector field ξ and a 1-form η on $C(S)$ by

$$\xi = J\left(r\frac{\partial}{\partial r}\right), \quad \eta(X) = \frac{1}{r^2}\overline{g}(\xi, X),$$

for any vector field X on $C(S)$. The vector field ξ is a Killing vector field, i.e., $L_\xi \overline{g} = 0$, and $\xi + \sqrt{-1}\,J\xi = \xi - \sqrt{-1}\,r\frac{\partial}{\partial r}$ is a holomorphic vector field on $C(S)$. It follows from $L_\xi \eta = JL_{r\frac{\partial}{\partial r}}\eta = 0$ that

$$\eta(\xi) = 1, \quad i_\xi d\eta = 0, \tag{1}$$

where i_ξ means the interior product. The form η is expressed as

$$\eta = d^c \log r = \sqrt{-1}\,(\overline{\partial} - \partial)\log r$$

where d^c is the composition $-J \circ d$ of the exterior derivative d and the action of the complex structure $-J$ on differential forms. We define an action λ of $\mathbb{R}_{>0}$ on $C(S)$ by

$$\lambda_a(r, x) = (ar, x)$$

for $a \in \mathbb{R}_{>0}$ and $(r, x) \in \mathbb{R}_{>0} \times S = C(S)$. If we put $a = e^t$ for $t \in \mathbb{R}$, then it follows from $L_{r\frac{\partial}{\partial r}} = \frac{d}{dt}\lambda^*_{e^t}|_{t=0}$ that $\{\lambda_{e^t}\}_{t\in\mathbb{R}}$ is one parameter group of transformations such that $r\frac{\partial}{\partial r}$ is the infinitesimal transformation. The Kähler form ω satisfies $\lambda^*_a \omega = a^2\omega$ for $a \in \mathbb{R}_{>0}$ and

$$L_{r\frac{\partial}{\partial r}}\omega = 2\omega.$$

It implies that

$$\omega = \frac{1}{2}d(r^2\eta) = \frac{\sqrt{-1}}{2}\partial\overline{\partial}r^2.$$

Hence $\frac{1}{2}r^2$ is a Kähler potential on $C(S)$.

The 1-form η induces the restriction $\eta|_S$ on $S \subset C(S)$. Since $L_{r\frac{\partial}{\partial r}}\eta = 0$, the form η is the extension of $\eta|_S$ to $C(S)$. The vector field ξ is tangent to the hypersurface $\{r = c\}$ for each positive constant c. In particular, ξ is considered as the vector field on S and satisfies $g(\xi, \xi) = 1$ and $L_\xi g = 0$. Hence we shall not distinguish between (η, ξ) on $C(S)$ and the restriction $(\eta|_S, \xi|_S)$ on S. Then the form η is a contact 1-form on S:

$$\eta \wedge (d\eta)^n \neq 0$$

since ω is nondegenerate. Equation (1) implies that

$$\eta(\xi) = 1, \quad i_\xi d\eta = 0, \tag{2}$$

on S. For a contact form η, a vector field ξ on S satisfying Equation (2) is unique, and called the *Reeb vector field*. We define the contact subbundle $D \subset TS$ by $D = \ker \eta$. Then the tangent bundle TS has the orthogonal decomposition

$$TS = D \oplus \langle \xi \rangle$$

where $\langle \xi \rangle$ is the line bundle generated by ξ. We define a section Φ of $\mathrm{End}(TS)$ by setting $\Phi|_D = J|_D$ and $\Phi|_{\langle \xi \rangle} = 0$. One can see that

$$\Phi^2 = -\mathrm{id} + \xi \otimes \eta, \tag{3}$$

$$d\eta(\Phi X, \Phi Y) = d\eta(X, Y), \tag{4}$$

for any $X, Y \in TS$. The Riemannian metric g satisfies

$$g(X, \Phi Y) = d\eta(X, Y) \tag{5}$$

for any $X, Y \in TS$.

A *contact metric structure* (ξ, η, Φ, g) on S consists of a contact form η, Reeb vector field ξ, a section Φ of $\mathrm{End}(TS)$ and a Riemannian metric g that satisfy Equations (3), (4) and (5). Moreover, a contact metric structure (ξ, η, Φ, g) is called a *K-contact structure* on S if ξ is a Killing vector field with respect to g. The section Φ of a K-contact structure (ξ, η, Φ, g) defines an almost CR structure $(D, \Phi|_D)$ on S. As we saw above, any Sasaki manifold (S, g) has a K-contact structure (ξ, η, Φ, g) with the integrable CR structure $(D, \Phi|_D = J|_D)$ on S. Conversely, if we have such a structure (ξ, η, Φ, g) on S, then $(\overline{g}, \frac{1}{2}d(r^2\eta))$ is a Kähler structure on the cone $C(S)$, hence (S, g) is a Sasaki manifold. We call a K-contact structure (ξ, η, Φ, g) with the integrable CR structure $(D, \Phi|_D)$ a *Sasaki structure* on S.

2.2. The Reeb foliation. Let (ξ, η, Φ, g) be a Sasaki structure on S. Then the Reeb vector field ξ generates a foliation $\mathcal{F}_\xi$ of codimension $2n$ on S. The foliation $\mathcal{F}_\xi$ is called a *Reeb foliation*. A Reeb foliation $\mathcal{F}_\xi$ is *quasi-regular* if any orbit of the Reeb vector field ξ is compact. Each orbit is associated with a locally free S^1-action. If the S^1-action is free, $\mathcal{F}_\xi$ is called *regular*. If $\mathcal{F}_\xi$ is not quasi-regular, it is called *irregular*.

A differential form ϕ on S is called *basic* if

$$i_v \phi = 0, \quad L_v \phi = 0,$$

for any $v \in \Gamma(\langle \xi \rangle)$. Let $\wedge_B^k$ be the sheaf of basic k-forms on the foliated manifold $(S, \mathcal{F}_\xi)$. It is easy to see that for a basic form ϕ the derivative $d\phi$ is also basic. Thus the exterior derivative d induces the operator

$$d_B = d|_{\wedge_B^k} : \wedge_B^k \to \wedge_B^{k+1}$$

by the restriction. The corresponding complex $(\wedge_B^*, d_B)$ associates the cohomology group $H_B^*(S)$ which is called the *basic de Rham cohomology group*. If $\mathcal{F}_\xi$ is a transversely holomorphic foliation, the associate transverse complex structure I on $(S,\mathcal{F}_\xi)$ gives rise to the decomposition $\wedge^k_B\otimes\mathbb{C}=\oplus_{r+s=k}\wedge^{r,s}_B$ in the same manner as complex geometry, and we have operators

$$\partial_B:\wedge^{p,q}_B\to\wedge^{p+1,q}_B$$
$$\overline{\partial}_B:\wedge^{p,q}_B\to\wedge^{p,q+1}_B.$$

We denote by $H^{p,*}_B(S)$ the cohomology of the complex $(\wedge^{p,*}_B,\overline{\partial}_B)$ which is called the *basic Dolbeault cohomology group*.

On the cone $C(S)$, a foliation $\mathcal{F}_{\langle\xi,r\frac{\partial}{\partial r}\rangle}$ is induced by the vector bundle $\langle\xi,r\frac{\partial}{\partial r}\rangle$ generated by ξ and $r\frac{\partial}{\partial r}$. Let $\widetilde{\phi}$ be a basic form on $(C(S),\mathcal{F}_{\langle\xi,r\frac{\partial}{\partial r}\rangle})$, that is, $i_v\widetilde{\phi}=0$ and $L_v\widetilde{\phi}=0$ for any $v\in\Gamma(\langle\xi,r\frac{\partial}{\partial r}\rangle)$. Then the restriction $\widetilde{\phi}|_S$ of $\widetilde{\phi}$ to S is also basic on $(S,\mathcal{F}_\xi)$. Conversely, for any basic form ϕ on $(S,\mathcal{F}_\xi)$, the trivial extension $\widetilde{\phi}$ of ϕ to $C(S)=\mathbb{R}_{>0}\times S$ is a basic form on $(C(S),\mathcal{F}_{\langle\xi,r\frac{\partial}{\partial r}\rangle})$. In this paper, we identify a basic form ϕ on $(S,\mathcal{F}_\xi)$ with the extension $\widetilde{\phi}$ on $(C(S),\mathcal{F}_{\langle\xi,r\frac{\partial}{\partial r}\rangle})$.

2.3. Transverse Kähler structures. Let $\mathcal{F}$ be a foliation of codimension $2n$ on S. In order to characterize transverse structures on $(S,\mathcal{F})$, we consider the quotient bundle $Q=TS/F$ where F is the line bundle associated by the foliation $\mathcal{F}$. We define an action of $\Gamma(F)$ to any section $I\in\Gamma(\mathrm{End}(Q))$ as follows:

$$(L_vI)(u)=L_v(I(u))-I(L_vu)$$

for $v\in\Gamma(F)$ and $u\in\Gamma(Q)$. If I is a complex structure of Q, i.e., $I^2=-\mathrm{id}_Q$, and satisfies that $L_vI=0$ for any $v\in\Gamma(F)$, then a tensor $N_I\in\Gamma(\otimes^2Q^*\otimes Q)$ can be defined by

$$N_I(u,w)=[Iu,Iw]_Q-[u,w]_Q-I[u,Iw]_Q-I[Iu,w]_Q$$

for $u,w\in\Gamma(Q)$, where $[u,w]_Q$ denotes the bracket $\pi[\widetilde{u},\widetilde{w}]$ for each lift $\widetilde{u}$ and $\widetilde{w}$ by the quotient map $\pi:TS\to Q$. A section $I\in\Gamma(\mathrm{End}(Q))$ is a *transverse complex structure on* $(S,\mathcal{F})$ if I is a complex structure of Q such that $L_vI=0$ for any $v\in\Gamma(F)$ and $N_I=0$. If a basic 2-form ω^T satisfies $d\omega^T=0$ and $(\omega^T)^n\neq0$, then we call the form ω^T a *transverse symplectic structure* on $(S,\mathcal{F})$. We can consider the basic form ω^T as a tensor of $\wedge^2Q^*$. The pair (ω^T,I) is called a *transverse Kähler structure* on $(S,\mathcal{F})$ if the 2-tensor $\omega^T(\cdot,I\cdot)$ is positive on Q and $\omega^T(I\cdot,I\cdot)=\omega^T(\cdot,\cdot)$ holds.

Let (ξ,η,Φ,g) be a Sasaki structure and $\mathcal{F}_\xi$ the Reeb foliation on S. We can consider Φ as a section of $\mathrm{End}(Q)$ since $\Phi|_{\langle\xi\rangle}=0$. Then Φ is a transverse complex structure on $(S,\mathcal{F}_\xi)$ by the integrability of the CR structure $\Phi|_D$. Moreover, the pair $(\Phi,\frac{1}{2}d\eta)$ is a transverse Kähler structure with the transverse Kähler metric $g^T(\cdot,\cdot)=\frac{1}{2}d\eta(\cdot,\Phi\cdot)$ on $(S,\mathcal{F}_\xi)$. The transverse Ricci form ρ^T is a basic d-closed $(1,1)$-form on $(S,\mathcal{F}_\xi)$ and defines a $(1,1)$-basic Dolbeault cohomology class $[\rho^T]\in H^{1,1}_B(S)$ as in the Kähler case. The basic class $[\frac{1}{2\pi}\rho^T]$ in $H^{1,1}_B(S)$ is called the *basic first Chern class* on $(S, \mathcal{F}_\xi)$ and is denoted by $c_1^B(S)$ (for short, we write it c_1^B). We say the basic first Chern class is positive if c_1^B is represented by a transverse Kähler form.

We provide a new Sasaki structure fixing the Reeb vector field ξ and varying η as follows. We define $\widetilde{\eta}$ by

$$\widetilde{\eta} = \eta + 2d_B^c\phi$$

for a basic function ϕ on $(S, \mathcal{F}_\xi)$, where $d_B^c = \sqrt{-1}\,(\overline{\partial}_B - \partial_B)$. It implies that

$$d\widetilde{\eta} = d\eta + 2d_B d_B^c\phi = d\eta + 2\sqrt{-1}\,\partial_B\overline{\partial}_B\phi.$$

If we choose a small ϕ such that $\widetilde{\eta} \wedge (d\widetilde{\eta})^n \neq 0$, then $\frac{1}{2}d\widetilde{\eta}$ is a transverse Kähler form for the same transverse complex structure Φ. Putting

$$\tilde{r} = r \exp\phi,$$

then we obtain

$$\tilde{r}\frac{\partial}{\partial\tilde{r}} = r\frac{\partial}{\partial r}$$

on the cone $C(S)$. It implies that the holomorphic structure J on $C(S)$ is unchanged. The function $\frac{1}{2}\tilde{r}^2$ on $C(S)$ is a new Kähler potential, that is $\frac{1}{2}d(\tilde{r}^2\widetilde{\eta}) = \frac{\sqrt{-1}}{2}dd^c\tilde{r}^2$, since

$$\widetilde{\eta} = \eta + 2d_B^c\phi = 2d^c\log\tilde{r}.$$

Thus the deformation

$$\eta \to \widetilde{\eta} = \eta + 2d_B^c\phi \tag{6}$$

gives a new Sasaki structure with the same Reeb vector field, the same transverse complex structure and the same holomorphic structure of $C(S)$. Conversely, such a Sasaki structure is given by the deformation (6), by using the transverse $\partial\overline{\partial}$-lemma proved in [7]. The deformations (6) are called *transverse Kähler deformations.*

2.4. Sasaki–Einstein structures and weighted Calabi–Yau structures. In this section, we assume that S is a compact manifold. We provide the definition of Sasaki–Einstein manifolds.

Definition 2.2. A Sasaki manifold (S, g) is *Sasaki–Einstein* if the metric g is Einstein.

Let (ξ, η, Φ, g) be a Sasaki structure on S. Then the Ricci tensor Ric of g has following relations:

$$\mathrm{Ric}(u, \xi) = 2n\eta(u),\ u \in TS,$$
$$\mathrm{Ric}(u, v) = \mathrm{Ric}^T(u, v) - 2g(u, v),\ u, v \in D,$$

where Ric^T is the Ricci tensor of g^T. Thus the Einstein constant of a Sasaki–Einstein metric g has to be $2n$, that is, $\mathrm{Ric} = 2ng$. It follows from the above equations that the Einstein condition $\mathrm{Ric} = 2ng$ is equal to the transverse

Einstein condition $\mathrm{Ric}^T = 2(n+1)g^T$. Moreover, the cone metric $\overline{g}$ is Ricci-flat on $C(S)$ if and only if g is Einstein with the Einstein constant $2n$ on S (we refer to Lemma 11.1.5 in [2]). Hence we can characterize the Sasaki–Einstein condition as follows:

Proposition 2.3. *Let (S, g) be a Sasaki manifold of dimension $2n+1$. Then the following conditions are equivalent.*

(a) *(S, g) is a Sasaki–Einstein manifold.*
(b) *$(C(S), \overline{g})$ is Ricci-flat, that is,* $\mathrm{Ric}_{\overline{g}} = 0$.
(c) *g^T is transverse Kähler–Einstein with* $\mathrm{Ric}^T = 2(n+1)g^T$.

We remark that Sasaki–Einstein manifolds have finite fundamental groups from Mayer's theorem. From now on, we assume that S is simply connected. Any Sasaki–Einstein manifold associates a transverse Kähler–Einstein structure with positive basic first Chern class $c_1^B = \frac{n+1}{2\pi}[d\eta] \in H_B^{1,1}(S)$. Thus $c_1^B > 0$ and $c_1(D) = 0$ are necessary conditions for a Sasaki metric to admit a deformation of transverse Kähler structures to a Sasaki–Einstein metric. The following lemma is formalized in [8]:

Lemma 2.4 ([8]). *A Sasaki manifold (S, g) satisfies $c_1^B > 0$ and $c_1(D) = 0$ if and only if there exists a holomorphic section Ω of $K_{C(S)}$ with*

$$L_{r\frac{\partial}{\partial r}}\Omega = (n+1)\Omega \quad \textit{and} \quad \Omega \wedge \overline{\Omega} = e^h c_{n+1}\omega^{n+1}$$

for a basic function h on $C(S)$, where $\omega = \frac{1}{2}d(r^2\eta)$ and

$$c_{n+1} = \frac{1}{(n+1)!}(-1)^{\frac{n(n+1)}{2}}\left(\frac{2}{\sqrt{-1}}\right)^{n+1}.$$

Definition 2.5. A pair $(\Omega, \omega) \in \wedge^{n+1} \otimes \mathbb{C} \oplus \wedge^2$ is called a *weighted Calabi–Yau structure* on $C(S)$ if Ω is a holomorphic section of $K_{C(S)}$ and ω is a Kähler form satisfying the equation

$$\Omega \wedge \overline{\Omega} = c_{n+1}\omega^{n+1}$$

where $c_{n+1} = \frac{1}{(n+1)!}(-1)^{\frac{n(n+1)}{2}}(\frac{2}{\sqrt{-1}})^{n+1}$ and

$$L_{r\frac{\partial}{\partial r}}\Omega = (n+1)\Omega,$$
$$L_{r\frac{\partial}{\partial r}}\omega = 2\omega.$$

If there exists a weighted Calabi–Yau structure (Ω, ω) on $C(S)$, then it is unique up to change $\Omega \to e^{\sqrt{-1}\theta}\Omega$ of a phase $\theta \in \mathbb{R}$. Proposition 2.3 and Lemma 2.4 imply the following:

Proposition 2.6. *A Riemannian metric g on S is Sasaki–Einstein if and only if there exists a weighted Calabi–Yau structure (Ω, ω) on $C(S)$ such that $\overline{g}$ is the Kähler metric.*

3. Special Legendrian submanifolds

We assume that (S, g) is a smooth compact Riemannian manifold of dimension $(2n+1)$ which is greater than or equal to five. Let $(C(S), \overline{g})$ be the metric cone of (S, g).

3.1. Special Legendrian submanifolds and special Lagrangian cones. We assume that (S, g) is a simply connected Sasaki–Einstein manifold and fix a weighted Calabi–Yau structure (Ω, ω) on $C(S)$ such that $\overline{g}$ is the Kähler metric. The real part $(e^{\sqrt{-1}\,\theta}\Omega)^{\mathrm{Re}}$ of $e^{\sqrt{-1}\,\theta}\Omega$ is a calibration whose calibrated submanifolds are called *θ-special Lagrangian submanifolds*. We consider such submanifolds of cone type. For any submanifold L in S, the cone

$$C(L) = \mathbb{R}_{>0} \times L$$

is a submanifold in $C(S)$. We identify L with the hypersurface $\{1\} \times L$ in $C(L)$. Then L is considered as the link $C(L) \cap S$.

Definition 3.1. A submanifold L in S is *special Legendrian* if and only if the cone $C(L)$ is a θ-special Lagrangian submanifold in $C(S)$ for a phase θ.

A θ-special Lagrangian cone $C(L)$ is a minimal submanifold in $C(S)$, that is, the mean curvature vector field $\widetilde{H}$ of $C(L)$ vanishes. The mean curvature vector field H of the link L in S satisfies that

$$\widetilde{H}_{(r,x)} = \frac{1}{r^2} H_x$$

at $(r, x) \in \mathbb{R}_{>0} \times S = C(S)$. Hence any special Legendrian submanifold L is also minimal. Conversely, we assume that L is a connected oriented minimal Legendrian submanifold in S. Then the cone $C(L)$ is minimal Lagrangian. There exists a function θ on $C(L)$ such that $*(\Omega|_{C(L)}) = e^{\sqrt{-1}\,\theta}$ where $*$ is the Hodge operator with respect to the metric $\overline{g}|_L$ on L induced by $\overline{g}$. We have

$$X(\theta) = -\omega(\widetilde{H}, X)$$

for any vector filed X on $C(S)$ tangent to $C(L)$ (Lemma 2.1 [22]). It yields that θ is constant. Thus, the cone $C(L)$ is special Lagrangian with respect to a weighted Calabi–Yau structure $(e^{\sqrt{-1}\,\theta}\Omega, \omega)$ for a phase θ. Hence $C(L)$ is θ-special Lagrangian, and the link $L = C(L) \cap S$ is a special Legendrian submanifold. We obtain the following (for the case of the sphere S^{2n+1}, we refer to Proposition 26 [12]):

Proposition 3.2. *A connected oriented Legendrian submanifold in S is minimal if and only if it is special Legendrian.*

Let (ξ, η, Φ, g) be the corresponding Sasaki structure on S. We also denote by η the extension to $C(S)$. We provide a characterization of special Lagrangian cones in $C(S)$.

Proposition 3.3. *An $(n+1)$-dimensional closed submanifold $\widetilde{L}$ in $C(S)$ is a special Lagrangian cone if and only if $\Omega^{\mathrm{Im}}|_{\widetilde{L}} = 0$ and $\eta|_{\widetilde{L}} = 0$.*

Proof. It suffices to show that $\widetilde{L}$ is a Lagrangian cone if and only if $\eta|_{\widetilde{L}} = 0$ since a special Lagrangian submanifold is characterized by a Lagrangian submanifold where Ω^{Im} vanishes. If $\widetilde{L}$ is a Lagrangian cone, then the vector field $r\frac{\partial}{\partial r}$ is tangent to $\widetilde{L}$. The vector fields ξ and $r\frac{\partial}{\partial r}$ span a symplectic subspace of $T_pC(S)$ with respect to ω_p at each point $p \in C(S)$. We can obtain $\eta|_{\widetilde{L}} = 0$ since $\eta = i_{r\frac{\partial}{\partial r}}\omega$ and $\omega|_{\widetilde{L}} = 0$.

Conversely, if $\widetilde{L}$ satisfies $\eta|_{\widetilde{L}} = 0$, then $\widetilde{L}$ is a Lagrangian submanifold since $\omega|_{\widetilde{L}} = \frac{1}{2}d(r^2\eta|_{\widetilde{L}}) = 0$. In order to see that $\widetilde{L}$ is a cone, we consider the set

$$I_p = \{a \in \mathbb{R}_{>0} \mid \lambda_a p \in \widetilde{L}\}$$

for each $p \in \widetilde{L}$. The set I_p is a closed subset of $\mathbb{R}_{>0}$ since $\widetilde{L}$ is closed. On the other hand, the vector field $r\frac{\partial}{\partial r}$ has to be tangent to $\widetilde{L}$ since $\widetilde{L}$ is Lagrangian and $\eta|_{\widetilde{L}} = 0$. The vector field $r\frac{\partial}{\partial r}$ is the infinitesimal transformation of the action λ. Therefore I_p is open, and so $I_p = \mathbb{R}_{>0}$ for each point $p \in \widetilde{L}$. Hence $\widetilde{L}$ is a cone, and it completes the proof. □

Many compact special Lagrangian submanifolds are obtained as the fixed point sets of anti-holomorphic involutions of compact Calabi–Yau manifolds. Bryant constructs special Lagrangian tori in Calabi–Yau 3-folds by the method [4]. We apply the method to find special Legendrian submanifolds in Sasaki–Einstein manifolds. An anti-holomorphic involution τ of $C(S)$ is a diffeomorphism $\tau : C(S) \to C(S)$ with $\tau^2 = \mathrm{id}$ and $\tau_* \circ J = -J \circ \tau_*$ where J is the complex structure on $C(S)$ induced by the Sasaki structure.

Proposition 3.4. *We assume there exists an anti-holomorphic involution τ of $C(S)$ such that $\tau^* r = r$. If the set $\mathrm{fix}(\tau)$ is not empty, then the link $\mathrm{fix}(\tau) \cap S$ is a special Legendrian submanifold in S.*

Proof. Let (Ω, ω) be a weighted Calabi–Yau structure on $C(S)$ such that $\omega = \frac{1}{2}d(r^2\eta)$. Then we have

$$\tau^*\eta = \tau^* \circ d^c \log r = -d^c \circ \tau^* \log r = -d^c \log r = -\eta$$

since $\tau^* \circ d^c = -d^c \circ \tau^*$ and $\tau^* r = r$. It yields that $\tau^*\omega = -\omega$ and τ is an isometry. There exists a holomorphic function f on $C(S)$ such that

$$\overline{\tau^*\Omega} = f\Omega. \tag{7}$$

The Lie derivative $L_{r\frac{\partial}{\partial r}}$ satisfies

$$L_{r\frac{\partial}{\partial r}} \circ \tau^* = \tau^* \circ L_{r\frac{\partial}{\partial r}} \quad \text{and} \quad L_{r\frac{\partial}{\partial r}}\Omega = (n+1)\Omega.$$

We also have $L_\xi\Omega = \sqrt{-1}\,(n+1)\Omega$. Taking the Lie derivative $L_{r\frac{\partial}{\partial r}}$ on Equation (7), then we obtain that $L_{r\frac{\partial}{\partial r}}f = 0$ and $L_\xi f = 0$. Thus f is the pull-back of a basic and transversely holomorphic function on S. Hence f is constant. Moreover, Equation (7) implies that $f = e^{2\sqrt{-1}\,\theta}$ for a real constant θ since the map τ is an isometry. We denote by Ω_θ the holomorphic $(n+1)$-form $e^{\sqrt{-1}\,\theta}\Omega$. Then (Ω_θ, ω) is a weighted Calabi–Yau structure on $C(S)$ such that

$$\tau^*\Omega_\theta = \overline{\Omega}_\theta.$$

The set $\mathrm{fix}(\tau)$ is an $(n+1)$-dimensional closed submanifold, if it is not empty, since τ is an isometric and anti-holomorphic involution. We denote the manifold $\mathrm{fix}(\tau)$ by $\widetilde{L}$. Since τ is the identity map on $\widetilde{L}$, we have

$$\Omega_\theta|_{\widetilde{L}} = \tau^*\Omega_\theta|_{\widetilde{L}} = \overline{\Omega}_\theta|_{\widetilde{L}}.$$

It yields that $\Omega_\theta^{\mathrm{Im}}|_{\widetilde{L}} = 0$. Therefore, Proposition 3.3 implies that $\widetilde{L}$ is a θ-special Lagrangian cone in $C(S)$, and the link $\widetilde{L} \cap S$ is a special Legendrian submanifold. □

A real form of a Kähler manifold is a totally geodesic Lagrangian submanifold [20]. We can generalize Proposition 3.4 to Sasaki manifolds which are not necessarily Einstein and simply connected as follows:

Proposition 3.5. *Let (S, g) be a Sasaki manifold. We assume there exists an anti-holomorphic involution τ of $C(S)$ such that $\tau^*r = r$. If the set* $\mathrm{fix}(\tau)$ *is not empty, then the link* $\mathrm{fix}(\tau) \cap S$ *is a totally geodesic Legendrian submanifold in S.*

Proof. We remark that τ satisfies $\tau^*\eta = -\eta$ and $\tau^*\omega = -\omega$. The fixed point set $\mathrm{fix}(\tau)$ of the anti-symplectic involution τ is a Lagrangian submanifold in $C(S)$ if it is not empty. Moreover, any closed Lagrangian submanifold where η vanishes is a cone as in the proof of Proposition 3.3. It follows from $\eta|_{\widetilde{L}} = 0$ that $\mathrm{fix}(\tau)$ is a Lagrangian cone in $C(S)$. Thus the link $\mathrm{fix}(\tau) \cap S$ is a Legendrian submanifold. The restriction $\tau|_S$ of τ to S induces a map from S to itself since τ preserves a level set of r, and so $\mathrm{fix}(\tau) \cap S$ is the fixed point set $\mathrm{fix}(\tau|_S)$ of $\tau|_S$. The set $\mathrm{fix}(\tau|_S)$ is totally geodesic since the map $\tau|_S$ is an isometric involution on (S, g). Hence $\mathrm{fix}(\tau) \cap S$ is a totally geodesic Legendrian submanifold. □

Remark 3.6. Tomassini and Vezzoni introduced special Legendrian submanifolds in contact Calabi–Yau manifolds which are contact manifolds with transversely Calabi–Yau foliations [23]. A contact Calabi–Yau manifold is a Sasaki manifold with a transversely null Kähler–Einstein structure. Hence it is not Sasaki–Einstein.

3.2. Toric Sasaki manifolds. In this section, we consider the toric Sasaki manifolds. We refer to [6], [10] and [17] for some facts of toric Sasaki manifolds. We provide the definition of toric Sasaki manifolds.

Definition 3.7. A Sasaki manifold (S, g) is *toric* if there exists an effective action of an $(n+1)$-torus $\mathbb{T}^{n+1} = G$ preserving the Sasaki structure such that the Reeb vector field ξ is an element of the Lie algebra $\mathfrak{g}$ of G. Equivalently, a toric Sasaki manifold (S, g) is a Sasaki manifold whose metric cone $(C(S), \overline{g})$ is a toric Kähler cone.

We define the moment map

$$\tilde{\mu} : C(S) \to \mathfrak{g}^*$$

of the action G on $C(S)$ by

$$\langle \tilde{\mu}, \zeta \rangle = \frac{1}{2} r^2 \eta(X_\zeta) \tag{8}$$

for any $\zeta \in \mathfrak{g}$, where X_ζ is the vector field on $C(S)$ induced by $\zeta \in \mathfrak{g}$. Let $G_{\mathbb{C}} = (\mathbb{C}^*)^{n+1}$ denote the complexification of G. The action $G_{\mathbb{C}}$ on the cone $C(S)$ is holomorphic and has an open dense orbit. The restriction of $\tilde{\mu}$ to S is a moment map of the action G on S. Equation (8) implies that $\tilde{\mu}(S) = \{y \in \mathfrak{g}^* \mid \langle y, \xi \rangle = \frac{1}{2}\}$. The hyperplane $\{y \in \mathfrak{g}^* \mid \langle y, \xi \rangle = \frac{1}{2}\}$ is called the *characteristic hyperplane* [1]. We define $C(\tilde{\mu})$ by

$$C(\tilde{\mu}) = \tilde{\mu}(C(S)) \cup \{0\}.$$

Then we obtain

$$C(\tilde{\mu}) = \{t\xi \in \mathfrak{g}^* \mid \xi \in \tilde{\mu}(S),\ t \in [0, \infty)\}.$$

The cone $C(\tilde{\mu})$ is called the *moment cone* of the toric Sasaki manifold.

We provide the definition of a *good rational polyhedral cone* which is due to Lerman [17]:

Definition 3.8. Let $\mathbb{Z}_{\mathfrak{g}}$ be the integral lattice of $\mathfrak{g}$, which is the kernel of the exponential map $\exp : \mathfrak{g} \to G$. A subset C of $\mathfrak{g}^*$ is a *rational polyhedral cone* if there exist an integer $d \geq n + 1$ and vectors $\lambda_i \in \mathbb{Z}_{\mathfrak{g}}$, $i = 1, \ldots, d$, such that

$$C = \{y \in \mathfrak{g}^* \mid \langle y, \lambda_i \rangle \geq 0 \text{ for } i = 1, \ldots, d\}.$$

The set $\{\lambda_i\}$ is *minimal* if

$$C \neq \{y \in \mathfrak{g}^* \mid \langle y, \lambda_i \rangle \geq 0 \text{ for } i \neq j\}$$

for any j, and is *primitive* if there does not exist an integer $n_i (\geq 2)$ and $\lambda_i' \in \mathbb{Z}_{\mathfrak{g}}$ such that $\lambda_i = n_i \lambda_i'$ for each i. A rational polyhedral cone C such that $\{\lambda_i\}$ is minimal and primitive is called *good* if C has nonempty interior and satisfies the following condition: if

$$\{y \in C \mid \langle y, \lambda_{i_j} \rangle = 0 \text{ for } j = 1, \ldots, k\}$$

is nonempty face of C for some $\{i_1, \ldots, i_k\} \subset \{1, \ldots, d\}$, then $\{\lambda_{i_1}, \ldots, \lambda_{i_k}\}$ is linearly independent over $\mathbb{Z}$ and

$$\left\{ \sum_{j=1}^{k} a_j \lambda_{i_j} \,\middle|\, a_j \in \mathbb{R} \right\} \cap \mathbb{Z}_{\mathfrak{g}} = \left\{ \sum_{j=1}^{k} m_j \lambda_{i_j} \,\middle|\, m_j \in \mathbb{Z} \right\}.$$

Any moment cone of compact toric Sasaki manifolds of dim ≥ 5 is a good rational polyhedral cone which is strongly convex, that is, the cone does not contain nonzero linear subspace (cf. Proposition 4.38. [2]). Conversely, given a strongly convex good rational polyhedral cone we can obtain a toric Sasaki manifold by Delzant construction.

Proposition 3.9 (cf. [6], [17], [18])**.** *If C is a strongly convex good rational polyhedral cone and ξ is an element of*

$$C_0^* = \{\xi \in \mathfrak{g} \ \mid \ \langle v, \xi \rangle > 0, \ {}^\forall v \in C\},$$

then there exists a connected toric Sasaki manifold S with the Reeb vector field ξ such that the moment cone is C.

Outline of the proof. Let $\{e_1, \ldots, e_d\}$ be the canonical basis of $\mathbb{R}^d$. The basis generates the lattice $\mathbb{Z}^d$. Let $\beta : \mathbb{R}^d \to \mathfrak{g}$ be the linear map defined by

$$\beta(e_i) = \lambda_i$$

for $i = 1, \ldots, d$. Since the polyhedral cone C has nonempty interior, there exists a basis $\{\lambda_{i_1}, \cdots, \lambda_{i_{n+1}}\}$ of $\mathfrak{g}$ over $\mathbb{R}$. Thus the map β is surjective. The map β induces the map $\widetilde{\beta}$ from $\mathbb{T}^d \cong \mathbb{R}^d/\mathbb{Z}^d$ to $G \cong \mathfrak{g}/\mathbb{Z}_{\mathfrak{g}}$. Let K denote the kernel of $\widetilde{\beta}$. Then we have

$$0 \to K \xrightarrow{\widetilde{\iota}} \mathbb{T}^d \xrightarrow{\widetilde{\beta}} G \to 0$$

where $\widetilde{\iota}$ is the natural monomorphism. The group K is a compact abelian subgroup of $\mathbb{T}^d$ and represented by $K = \left\{[a] \in \mathbb{T}^d \mid \sum_{i=1}^d a_i \lambda_i \in \mathbb{Z}_{\mathfrak{g}}\right\}$ where $[a]$ denotes the equivalent class of $a \in \mathbb{R}^d$. Let $\mathfrak{k}$ denote the Lie algebra of K. Then $\mathfrak{k}$ is equal to $\ker \beta$. Thus we obtain the exact sequence

$$0 \to \mathfrak{k} \xrightarrow{\iota} \mathbb{R}^d \xrightarrow{\beta} \mathfrak{g} \to 0 \tag{9}$$

where ι is the natural inclusion. The action of $\mathbb{T}^d$ on $\mathbb{C}^d$ is given by

$$[a] \circ (z_1, \ldots, z_d) = (e^{2\pi\sqrt{-1}\, a_1} z_1, \ldots, e^{2\pi\sqrt{-1}\, a_d} z_d)$$

for $[a] = [a_1, \ldots, a_d] \in \mathbb{T}^d \cong \mathbb{R}^d/\mathbb{Z}^d$ and $(z_1, \ldots, z_d) \in \mathbb{C}^d$. This action preserves the standard Kähler form on $\mathbb{C}^d$. The corresponding moment map

$$\mu_0 : \mathbb{C}^d \to (\mathbb{R}^d)^*$$

is given by

$$\mu_0(z) = \sum_{j=1}^{d} |z_j|^2 e_j^*$$

for $z \in \mathbb{C}^d$ where $\{e_1^*, \ldots, e_d^*\}$ is the dual basis to $\{e_1, \ldots, e_d\}$. Choose a basis $\{v_1, \ldots, v_k\}$ of $\mathfrak{k}$ where $k = \dim \mathfrak{k} = d - n - 1$, and then there exists an integer $k \times d$-matrix (a_{ij}) such that $\iota(v_i) = \sum_{j=1}^d a_{ij} e_j$ for $i = 1, \ldots, k$. We also consider the following exact sequence

$$0 \to \mathfrak{g}^* \xrightarrow{\beta^*} (\mathbb{R}^d)^* \xrightarrow{\iota^*} \mathfrak{k}^* \to 0 \tag{10}$$

which is the dual sequence to (9). We define a map

$$\mu : \mathbb{C}^d \to \mathfrak{k}^*$$

by $\mu = \iota^* \circ \mu_0$. This map μ is a moment map of the action of K on $\mathbb{C}^d$ and given by

$$\mu(z) = \sum_{i=1}^{k} \left(\sum_{j=1}^{d} a_{ij} |z_j|^2 \right) v_i^*$$

for $z \in \mathbb{C}^d$ where $\{v_1^*, \dots, v_k^*\}$ is the dual basis to $\{v_1, \dots, v_k\}$. It follows from the exact sequence (10) that $\mu_0(\mu^{-1}(0)) \subset \beta^* \mathfrak{g}^* \simeq \mathfrak{g}^*$. Hence we have the map $\mu_0|_{\mu^{-1}(0)} : \mu^{-1}(0) \to \mathfrak{g}^*$. Moreover, it induces a map $\widetilde{\mu}$ from the quotient space $(\mu^{-1}(0) \backslash \{0\})/K$ to $\mathfrak{g}^*$:

$$\widetilde{\mu} : (\mu^{-1}(0) \backslash \{0\})/K \to \mathfrak{g}^*. \tag{11}$$

The map $\widetilde{\mu}$ is a moment map of the action $G = \mathbb{T}^d/K$ on $(\mu^{-1}(0) \backslash \{0\})/K$. The image of $\widetilde{\mu}$ is equal to C since the image $\mu_0(\mu^{-1}(0))$ is precisely $\beta^*(C) \simeq C$.

We define ξ_0 by the element

$$\xi_0 = \sum_{i=1}^{n+1} \lambda_i \tag{12}$$

of $\mathfrak{g}$. We provide a Kähler metric on $(\mu^{-1}(0) \backslash \{0\})/K$ by the Kähler reduction. Then the function

$$F_0(z) = \langle \widetilde{\mu}(z), \xi_0 \rangle$$

is a Kähler potential on $(\mu^{-1}(0) \backslash \{0\})/K$. We define r_0 by the function

$$r_0 = \sqrt{2F_0}$$

on $(\mu^{-1}(0) \backslash \{0\})/K$. It yields that the Kähler potential is $\frac{1}{2} r_0^2 = F_0$ and the manifold

$$S = (\mu^{-1}(0) \cap S^{2d-1})/K$$

is the hypersurface $\{r_0 = 1\}$ in $(\mu^{-1}(0) \backslash \{0\})/K$. The cone $C(S)$ of S is obtained as $(\mu^{-1}(0) \backslash \{0\})/K$:

$$C(S) = (\mu^{-1}(0) \backslash \{0\})/K.$$

The manifold S admits a Sasaki structure with the Reeb vector filed ξ_0 such that the following embedding from S into $C(S)$ is isometric:

$$S = \{r_0 = 1\} \subset C(S).$$

Given an element $\xi \in C_0^*$, we can obtain a Kähler potential F_ξ defined by

$$F_\xi(z) = \langle \widetilde{\mu}(z), \xi \rangle$$

for $z \in C(S)$ (see (61) in [8]). We denote by H_ξ the hypersurface

$$H_\xi = \mu_0^{-1} \left(\left\{ y \in \mathfrak{g}^* \,\middle|\, \langle y, \xi \rangle = \frac{1}{2} \right\} \right)$$

in $\mathbb{C}^d$, which is the inverse image of the characteristic hyperplane by μ_0. We define a nonnegative function r on $C(S)$ by

$$r = \sqrt{2F_\xi}.$$

Then the manifold

$$S_\xi = (\mu^{-1}(0) \cap H_\xi)/K$$

is the hypersurface $\{r = 1\}$ in $C(S)$. We remark that S_ξ is also the inverse image of the characteristic hyperplane by $\widetilde{\mu}$:

$$S_\xi = \widetilde{\mu}^{-1}\left(\left\{ y \in \mathfrak{g}^* \;\middle|\; \langle y, \xi\rangle = \frac{1}{2} \right\}\right).$$

Thus it follows from $S = S_{\xi_0}$ that there exists a diffeomorphism $S \simeq S_\xi$. By the diffeomorphism, S has a Sasaki structure such that ξ is the Reeb vector field and can be isometrically embedded into $C(S)$ as the hypersurface $\{r = 1\}$. □

Toric Sasaki manifolds are constructed by a strongly convex good rational polyhedral cone C and a Reeb vector field $\xi \in C_0^*$, and then the Kähler potential can be taken by F_ξ as in the proof of Proposition 3.9. Any toric Sasaki structure with the same Reeb vector field ξ and the same holomorphic structure on $C(S)$ is given by deformations of transverse Kähler structures (See Section 2.3). Martelli, Sparks and Yau proved the following:

Lemma 3.10 ([18]). *The moduli space of toric Kähler cone metrics on $C(S)$ is*

$$C_0^* \times \mathcal{H}^1(C)$$

where $\xi \in C_0^*$ is the Reeb vector field and $\mathcal{H}^1(C)$ denotes the space of homogeneous degree one functions on C such that each element ϕ is smooth up to the boundary and $\sqrt{-1}\,\partial\overline{\partial}(F_\xi \exp 2\widetilde{\mu}^*\phi)$ is positive definite on $C(S)$.

We identify an element ϕ of $\mathcal{H}^1(C)$ with the pull-back $\widetilde{\mu}^*\phi$ by the moment map $\widetilde{\mu}$ as in (11). Then, for any element $(\xi, \phi) \in C_0^* \times \mathcal{H}^1(C)$ we can define the function r on $C(S)$ by

$$r = \sqrt{2F_\xi}\exp\phi.$$

Let $S_{\xi,\phi}$ denote the hypersurface $\{r = 1\}$ in $C(S)$:

$$S_{\xi,\phi} = \{r = 1\}.$$

It is easy to see that $S = S_{\xi_0,0}$ where ξ_0 is given by (12). There exists a diffeomorphism $S \simeq S_{\xi,\phi}$ for any $(\xi, \phi) \in C_0^* \times \mathcal{H}^1(C)$. By the diffeomorphism, S admits a Sasaki structure with the Reeb vector field ξ and the Kähler potential $\frac{1}{2}r^2$ on $C(S)$ and can be isometrically embedded into $C(S)$ as $\{r = 1\}$. Thus the deformation

$$(\xi, \phi) \to (\xi', \phi')$$

of $C_0^* \times \mathcal{H}^1(C)$ induces a deformation of Sasaki structures on S. These deformations are called *deformations of toric Sasaki structures* on S.

3.3. Main theorems. Let (S, g) be a toric Sasaki manifold. The metric cone $C(S)$ is given by the Kähler quotient $C(S) = (\mu^{-1}(0)\backslash\{0\})/K$ for the moment map $\mu : \mathbb{C}^d \to \mathfrak{k}^*$ as in the proof of Proposition 3.9. Then there exists an anti-holomorphic involution τ on $C(S)$ as follows. We consider the anti-holomorphic involution $\widetilde{\tau} : \mathbb{C}^d \to \mathbb{C}^d$ defined by

$$\widetilde{\tau}(z) = \overline{z}$$

for $z \in \mathbb{C}^d$. The inverse image $\mu^{-1}(0)$ is invariant under the map $\widetilde{\tau}$. Thus $\widetilde{\tau}$ induces a diffeomorphism of $\mu^{-1}(0)$. Moreover, $\widetilde{\tau}$ maps a K-orbit to another K-orbit. Hence we can define a map $\tau : C(S) \to C(S)$ by

$$\tau[z] = [\widetilde{\tau}(z)] = [\overline{z}]$$

for $[z] \in (\mu^{-1}(0)\backslash\{0\})/K = C(S)$. The map τ is an anti-holomorphic involution of $C(S)$. We recall that the group $G_{\mathbb{C}}$ acts holomorphically on the cone $C(S)$ with an open dense orbit. We denote by X_0 the open dense orbit of $G_{\mathbb{C}}$. Since the orbit X_0 is identified with $(\mathbb{C}^*)^{n+1}$, we can give a coordinate $w = (w_1, \ldots, w_{n+1})$ on X_0 as $u_i = e^{w_i}$ for any

$$u = (u_1, \ldots, u_{n+1}) \in X_0 \subset (\mathbb{C}^*)^{n+1}.$$

Then the map τ is given by

$$\tau(w) = \overline{w}$$

on the coordinate (X_0, w) on $C(S)$. Hence the set $\mathrm{fix}(\tau)$ is nonempty.

Theorem 3.11. *Let (S, g) be a compact simply connected toric Sasaki–Einstein manifold. Then the link* $\mathrm{fix}(\tau) \cap S$ *is a special Legendrian submanifold.*

Proof. Let S be a toric Sasaki–Einstein manifold with the Sasaki structure induced by the element $(\xi, \phi) \in C_0^* \times \mathcal{H}^1(C)$. Then the Kähler potential on $C(S)$ is

$$\frac{1}{2}r^2 = F_\xi \exp 2\phi.$$

It follows from $\tau^*\tilde{\mu} = \tilde{\mu}$ that F_ξ and ϕ are also τ-invariant. It gives rise to

$$\tau^* r = r.$$

Hence, Proposition 3.4 implies that the link $\mathrm{fix}(\tau) \cap S$ is a special Legendrian submanifold in S. It completes the proof. □

Remark 3.12. In the case that S is not simply connected, the canonical line bundle $K_{C(S)}$ is not necessarily trivial. However, the l-th power $K_{C(S)}^l$ of $K_{C(S)}$ is trivial for some integer l. Hence, we can remove the condition that S is simply connected in Theorem 3.11 by considering nowhere vanishing holomorphic sections of $K_{C(S)}^l$ instead of $K_{C(S)}$. Then we need to define a special Lagrangian submanifold in $C(S)$ as a Lagrangian submanifolds whose l-th covering is a special Lagrangian submanifold in the l-th covering of $C(S)$.

In the proof of Theorem 3.11, we only need the Einstein condition of (S, g) to use Proposition 3.4. By applying Proposition 3.5 instead of Proposition 3.4, we obtain the following:

Theorem 3.13. *Let S be a compact toric Sasaki manifold. Then the link* $\mathrm{fix}(\tau) \cap S$ *is a totally geodesic Legendrian submanifold in S.*

3.4. Covering spaces over the link $\mathbf{fix}(\boldsymbol{\tau}) \cap \boldsymbol{S}$. In this section, we will see that the special Legendrian submanifold in Theorem 3.11 is given by a base space of a finite covering map (we also refer to [10]).

We recall the exact sequence

$$0 \to K \xrightarrow{\widetilde{\iota}} \mathbb{T}^d \xrightarrow{\widetilde{\beta}} \mathbb{T}^{n+1} \to 0$$

is associated with a strongly convex good rational polyhedral cone C as in Section 3.2. This sequence equips the following sequence

$$0 \to \mathfrak{k} \xrightarrow{\iota} \mathbb{R}^d \xrightarrow{\beta} \mathbb{R}^{n+1} \to 0.$$

We consider each element λ_i of the set $\{\lambda_1, \dots, \lambda_d\}$ as a vector of $\mathbb{R}^{n+1}$. Then the map β is represented by

$$(\lambda_1 \cdots \lambda_d) : \mathbb{R}^d \to \mathbb{R}^{n+1}.$$

where $(\lambda_1 \cdots \lambda_d)$ is the integer $(n+1) \times d$ matrix. By choosing a basis of $\mathfrak{k}$, the map ι is represented by the $d \times k$ matrix

$$A = {}^t(a_{ij}) : \mathbb{R}^k \to \mathbb{R}^d$$

where each component a_{ij} is an integer and ${}^tB$ means the transpose of a matrix B.

In order to analyse $\mathrm{fix}(\tau) \cap S$, we define a map

$$\mu_{\mathbb{R}} : \mathbb{R}^d \to \mathfrak{k}^*$$

by the restriction of the moment map $\mu : \mathbb{C}^d \to \mathfrak{k}^*$ to $\mathbb{R}^d = \mathrm{fix}(\widetilde{\tau}) \cap \mathbb{C}^d$. The map $\mu_{\mathbb{R}}$ is represented by

$$\mu_{\mathbb{R}}(x) = \sum_{i=1}^{k} \left(\sum_{j=1}^{d} a_{ij} x_j^2 \right) v_i^*$$

for $x \in \mathbb{R}^d$ since $\mu(z) = \sum_i (\sum_{j=1}^d a_{ij} |z_j|^2) v_i^*$ for $z \in \mathbb{C}^d$. The inverse image $\mu_{\mathbb{R}}^{-1}(0)$ is precisely $\mathrm{fix}(\widetilde{\tau}) \cap \mu^{-1}(0)$:

$$\mu_{\mathbb{R}}^{-1}(0) = \mathrm{fix}(\widetilde{\tau}) \cap \mu^{-1}(0).$$

The set $\mathrm{fix}(\tau)$ is the image of $\mathrm{fix}(\widetilde{\tau}) \cap \mu^{-1}(0)$ by the quotient map

$$\pi' : \mu^{-1}(0) \backslash \{0\} \to (\mu^{-1}(0) \backslash \{0\})/K. \tag{13}$$

Hence we have the 2^k-fold map

$$\pi' : \mu_{\mathbb{R}}^{-1}(0) \backslash \{0\} \to \mathrm{fix}(\tau)$$

with the deck transformation $\{a \in K \mid a^2 = 1\}$. We also consider the quotient map

$$\pi : \mu^{-1}(0) \cap H_\xi \to (\mu^{-1}(0) \cap H_\xi)/K = S_\xi.$$

which is the restriction of (13) to $\mu^{-1}(0) \cap H_\xi$. Then $\mathrm{fix}(\tau) \cap S_\xi$ is the base space of the 2^k-fold map

$$\pi : \mu_{\mathbb{R}}^{-1}(0) \cap H_\xi \to \mathrm{fix}(\tau) \cap S_\xi.$$

If we take an element $\xi = \sum_{j=1}^d b_j \lambda_j$ of C_0^*, then $\mathrm{fix}(\tau) \cap S_\xi$ is the quotient space of

$$\mu_{\mathbb{R}}^{-1}(0) \cap H_\xi = \left\{ x \in \mathbb{R}^d \;\middle|\; \begin{array}{l} \sum_{j=1}^d a_{ij} x_j^2 = 0,\ j = 1, \dots, k \\ \sum_{j=1}^d b_j x_j^2 = 1 \end{array} \right\}$$

by the action of the deck transformation.

3.5. The Sasaki–Einstein manifold $\boldsymbol{Y_{p,q}}$. In this section, we provide an example of special Legendrian submanifolds in $Y_{p,q}$. Gauntlett, Martelli, Sparks and Waldram provided an explicit toric Sasaki–Einstein metric $g_{p,q}$ on $S^2 \times S^3$ [9]. For relatively prime nonnegative integers p and q with $p > q$, the inward pointing normals to the polyhedral cone C can be taken to be

$$\lambda_1 = {}^t(1, 0, 0),\ \lambda_2 = {}^t(1, p - q - 1, p - q),\ \lambda_3 = {}^t(1, p, p),\ \lambda_4 = {}^t(1, 1, 0).$$

Then we obtain the representation matrix $A = {}^t(a_{ij})$ as

$$A = {}^t(-p - q,\ p,\ -p + q,\ p).$$

By the calculation in [18], the Reeb vector field $\xi_{\min}$ of the toric Sasaki–Einstein metric is given by

$$\xi_{\min} = (3,\ \frac{1}{2}(3p - 3q + l^{-1}),\ \frac{1}{2}(3p - 3q + l^{-1}))$$

where $l^{-1} = \frac{1}{q}(3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2})$. Thus we can obtain

$$\mu_{\mathbb{R}}^{-1}(0) \cap H_{\xi_{\min}} = \left\{ x \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} p x_2^2 + p x_4^2 = (p + q) x_1^2 + (p - q) x_3^2 \\ (3p + 3q - l^{-1}) x_1^2 + (3p - 3q + l^{-1}) x_3^2 = 2p \end{array} \right\},$$

which is diffeomorphic to $S^1 \times S^1 = \{x \in \mathbb{R}^4 \mid x_1^2 + x_3^2 = x_2^2 + x_4^2 = 1\}$. The deck transformations induces an action on $S^1 \times S^1$ given by

$$\begin{cases} \{\mathrm{id} \times \mathrm{id} \times \mathrm{id} \times \mathrm{id},\ (-\mathrm{id}) \times \mathrm{id} \times (-\mathrm{id}) \times \mathrm{id}\}, & p\text{: even, } q\text{: odd}, \\ \{\mathrm{id} \times \mathrm{id} \times \mathrm{id} \times \mathrm{id},\ \mathrm{id} \times (-\mathrm{id}) \times \mathrm{id} \times (-\mathrm{id})\}, & p\text{: odd, } q\text{: odd}, \\ \{\mathrm{id} \times \mathrm{id} \times \mathrm{id} \times \mathrm{id},\ (-\mathrm{id}) \times (-\mathrm{id}) \times (-\mathrm{id}) \times (-\mathrm{id})\}, & p\text{: odd, } q\text{: even}. \end{cases}$$

The quotient space of $S^1 \times S^1$ by the action is also $S^1 \times S^1$ for each (p, q). Therefore the link $\mathrm{fix}(\tau) \cap Y_{p,q}$ is also diffeomorphic to $S^1 \times S^1$. Hence we obtain:

Theorem 3.14. *There exists a special Legendrian torus $S^1 \times S^1$ in $Y_{p,q}$.*

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