A note on lattices of z-ideals of f-rings

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ABSTRACT. The lattice of z-ideals of the ring $C(X)$ of real-valued continuous functions on a completely regular Hausdorff space X has been shown by Martínez and Zenk to be a complete Heyting algebra with certain properties. We show that these properties are due only to the fact that $C(X)$ is an f-ring with bounded inversion. This we do by studying lattices of algebraic z-ideals of abstract f-rings with bounded inversion.

CONTENTS

1. Introduction

It has been remarked by Subramanian [13] that many results in the ring $C(X)$ of real-valued continuous functions on a completely regular Hausdorff space X are mainly due to the fact that $C(X)$ is an f-ring. Of course $C(X)$ is not just any f-ring; it also has some special properties such as bounded inversion, which means that every element above the identity (relative to its natural partial order) is invertible. It also has the property that the sum of two z-ideals in it is a z-ideal [3]. In the course of this introduction we shall use terms which (although standard) we will recall, but only in subsequent sections.

Our aim in this paper is to show that it is because $C(X)$ is an f-ring with bounded inversion that the lattice $\mathscr{C}_z(X)$ of z-ideals of $C(X)$ is a normal

coherent Yosida frame. This we do by showing that if A is any f-ring with bounded inversion, then the lattice $\mathrm{ZId}(A)$ of z-ideals of A (here we mean z-ideals in the algebraic sense) is a normal coherent Yosida frame (Theorems 3.2 and 3.5). We hasten to mention that an ideal of $C(X)$ is a z-ideal algebraically if and only if it is z-ideal in the usual topological sense.

We show that $\mathrm{ZId}(A)$ is a frame by constructing a nucleus,

$$\mathfrak{z}\colon R\mathfrak{L}(A) \to R\mathfrak{L}(A),$$

on the frame $R\mathfrak{L}(A)$ of radical ℓ-ideals of A whose image is precisely $\mathrm{ZId}(A)$. As it turns out, this nucleus induces a codense coherent map

$$R\mathfrak{L}(A) \to \mathrm{ZId}(A),$$

which is dense precisely when the Jacobson radical and the nilradical of A coincide (Proposition 3.10). Furthermore, it leads to a natural transformation $R\mathfrak{L} \to \mathrm{ZId}$ (Proposition 4.2).

2. Preliminaries

2.1. Rings and f-rings. All rings (including f-rings) in this paper are commutative with identity 1. We recall that an f-*ring* is a lattice-ordered ring A which satisfies the condition that, for any $a, b \in A$ and any $c \geq 0$ in A,

$$(a \wedge b)c = (ac) \wedge (bc).$$

By a *positive* element of an f-ring A we mean an element $a \geq 0$, and we set

$$A^+ = \{a \in A \mid a \geq 0\}.$$

Squares are positive in f-rings. An f-ring has *bounded inversion* if every element $a \geq 1$ is invertible. A ring is *semiprime* if it has no nonzero nilpotent element. By the *Jacobson radical* of a ring we mean the intersection of all its maximal ideals. We write $\mathrm{Max}(A)$ for the set of all maximal ideals of a ring A. For any $a \in A$, we set

$$\mathfrak{M}(a) = \{M \in \mathrm{Max}(A) \mid a \in M\}.$$

An ideal I of A is a z-*ideal* if, for any $a, b \in A$, $\mathfrak{M}(a) = \mathfrak{M}(b)$ and $a \in I$ imply $b \in I$. See [11] for a detailed study of z-ideals in rings. Every maximal ideal is a z-ideal, and every minimal prime ideal is a z-ideal. Intersections of z-ideals are z-ideals, and the Jacobson radical of A, here denoted $\mathrm{Jac}(A)$, is a z-ideal contained in every z-ideal. An ideal of $C(X)$ is a z-ideal in the usual sense if and only if it is a z-ideal in the algebraic sense as just defined.

2.2. Algebraic frames. The lattices we shall deal with are of a special kind that includes lattices of open sets of a topological space. They are called frames. To recall, a *frame* is a complete lattice L such that the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge x \mid x \in S\}$$

holds for every $a \in L$ and $S \subseteq L$. We denote by 0 the zero of a frame, and by 1 its unit. A *frame homomorphism* is a mapping $h \colon L \to M$ between frames that preserves finite meets (including the unit), and arbitrary joins (including the zero). The resulting category is denoted by Frm. Our references for frames are [5] and [12].

We write $\mathfrak{k}(A)$ for the set of compact elements of a frame A. If $\mathfrak{k}(A)$ generates A, in the sense that every element of A is the join of compact elements below it, then A is said to be *algebraic*. An algebraic frame A is said to have the *finite intersection property* (FIP) if $a \wedge b \in \mathfrak{k}(A)$ for all $a, b \in \mathfrak{k}(A)$. A compact algebraic frame with FIP is called *coherent*, as is a frame homomorphism $\phi \colon A \to B$ between coherent frames that takes compact elements to compact elements. As usual, CohFrm denotes the category of coherent frames with coherent frame homomorphisms. A coherent frame is called a *Yosida frame* [10] if every compact element in it is a meet of maximal elements.

3. Frames of z-ideals of f-rings

As stated in the Introduction, we aim to show in this section that some of the properties of the lattices $\mathscr{C}_z(X)$ are mainly due to the fact that $C(X)$ is an f-ring with bounded inversion. Recall that an ideal I of an f-ring A (or ℓ-ring, for that matter) is an *ℓ-ideal* if, for any $a, b \in A$,

$$|a| \leq |b| \quad \text{and} \quad b \in I \quad \Longrightarrow \quad a \in I.$$

As remarked in [7], if A is an f-ring with bounded inversion, then every z-ideal in A is an ℓ-ideal. In fact, an f-ring A has bounded inversion precisely when every z-ideal in A is an ℓ-ideal. Indeed, if every z-ideal is an ℓ-ideal, then every maximal ideal is an ℓ-ideal, which is known to be equivalent to having the bounded inversion property. We denote by $\mathrm{ZId}(A)$ the lattice of z-ideals of A.

Banaschewski shows in [2, Proposition 2.4] that the lattice $R\mathfrak{L}(A)$ of radical ℓ-ideals of any f-ring A is a coherently normal frame. A frame A is *coherently normal* if it is coherent, and for each compact c in A, the frame $\downarrow c$ is normal. We show that for any f-ring A with bounded inversion, $\mathrm{ZId}(A)$ is a quotient of $R\mathfrak{L}(A)$. This we do by defining a nucleus $\mathfrak{z}$ on $R\mathfrak{L}(A)$ for which $\mathrm{Fix}(\mathfrak{z}) = \mathrm{ZId}(A)$. For $a \in A$, set

$$M(a) = \bigcap \{M \in \mathrm{Max}(A) \mid a \in M\}.$$

The convention, as usual, is that if a belongs to no maximal ideal (which is the case precisely when a is invertible) then $M(a)$ is the entire ring. Note that, being an intersection of z-ideals (recall that maximal ideals are z-ideals), $M(a)$ is a z-ideal, and is, in fact, the smallest z-ideal containing a. It is clear that $M(a^2) = M(a)$ since our maximal ideals are prime. Also, $M(0) = \mathrm{Jac}(A)$. Observe that, in any ring A, if $I \subseteq A$ is an ideal, then

$$I \text{ is a } z\text{-ideal} \quad \Longleftrightarrow \quad \forall a \in I, \ M(a) \subseteq I.$$

We need the following lemma, which we state more generally that will be needed for our purposes.

Lemma 3.1. *Let A be an f-ring with bounded inversion, and let $a, b \in A$.*

(a) $M(a) \cap M(b) = M(ab)$.
(b) *If $a \geq 0$ and $b \geq 0$, then $M(ab) = M(a \wedge b)$.*
(c) *If $a \geq 0$ and $b \geq 0$, then $M(a) + M(b) \subseteq M(a + b)$.*

Proof. (a) Let $x \in M(a) \cap M(b)$. If N is a maximal ideal containing ab, then N contains a or it contains b, by primeness. In either case, $x \in N$ since x belongs to every maximal ideal containing a, and to every maximal ideal containing b. Thus, $x \in M(ab)$, showing that $M(a) \cap M(b) \subseteq M(ab)$. On the other hand, ab belongs to every ideal that contains a, and every ideal that contains b. Therefore $ab \in M(a) \cap M(b)$, and hence $M(ab) \subseteq M(a) \cap M(b)$ since $M(a) \cap M(b)$ is a z-ideal. Thus, $M(ab) = M(a) \cap M(b)$.

(b) Since $ab = (a \vee b)(a \wedge b)$, $ab \in M(a \wedge b)$, hence $M(ab) \subseteq M(a \wedge b)$. Also, $0 \leq (a \wedge b)^2 \leq ab \in M(ab)$, which implies $M(a \wedge b) = M((a \wedge b)^2) \subseteq M(ab)$. The claimed equality follows.

(c) Since $0 \leq a \leq a + b$ and $M(a + b)$ is an ℓ-ideal (as it is a z-ideal) containing $a + b$, we have $a \in M(a + b)$. Hence $M(a) \subseteq M(a + b)$. Similarly, $M(b) \subseteq M(a + b)$, whence $M(a) + M(b) \subseteq M(a + b)$. $\square$

Theorem 3.2. *For any f-ring A with bounded inversion, $\mathrm{ZId}(A)$ is a frame. In fact, it is a quotient of $R\mathfrak{L}(A)$.*

Proof. Define a map $\mathfrak{z} \colon R\mathfrak{L}(A) \to R\mathfrak{L}(A)$ by

$$\mathfrak{z}(I) = \bigvee_{R\mathfrak{L}(A)} \{M(a) \mid a \in I\} = \bigcup \{M(a) \mid a \in I\}.$$

To see that the join is the union, observe that the collection $\{M(a) \mid a \in I\}$ is up-directed (henceforth abbreviated "directed") because for any $u, v \in I$, $u^2 + v^2 \in I$ and $M(u) \cup M(v) \subseteq M(u^2 + v^2)$, as one deduces from Lemma 3.1(c). Clearly, $I \subseteq \mathfrak{z}(I)$ for any $I \in R\mathfrak{L}(A)$. Observe also that $\mathfrak{z}$ is order-preserving. Now for any $I, J \in R\mathfrak{L}(A)$,

$$\begin{aligned}
\mathfrak{z}(I) \cap \mathfrak{z}(J) &= \bigvee_{a \in I} M(a) \wedge \bigvee_{b \in J} M(b) \\
&= \bigvee \{M(a) \cap M(b) \mid a \in I, b \in J\} \\
&\leq \bigvee \{M(c) \mid c \in I \cap J\} \qquad \text{by Lemma 3.1(a)} \\
&= \mathfrak{z}(I \cap J).
\end{aligned}$$

Hence $\mathfrak{z}$ preserves meets. To show idempotency, we need to show that, for any $I \in R\mathfrak{L}(A)$, $\mathfrak{z}(\mathfrak{z}(I)) \subseteq \mathfrak{z}(I)$. Let $u \in \mathfrak{z}(\mathfrak{z}(I))$. Pick $v \in \mathfrak{z}(I)$ such that $u \in M(v)$. Since $\mathfrak{z}(I)$ is a z-ideal — being a directed union of z-ideals — and since $v \in \mathfrak{z}(I)$, it follows that $M(v) \subseteq \mathfrak{z}(I)$, and hence $u \in \mathfrak{z}(I)$. Consequently $\mathfrak{z}(\mathfrak{z}(I)) \subseteq \mathfrak{z}(I)$, and hence equality. Thus, $\mathfrak{z}$ is a nucleus.

We have already commented that each $\mathfrak{z}(I)$, for $I \in R\mathfrak{L}(A)$, is a z-ideal. On the other hand, if J is a z-ideal, then $J \in R\mathfrak{L}(A)$, and $\mathfrak{z}(J) = J$. Therefore the image of $\mathfrak{z}$ is precisely $\mathrm{ZId}(A)$, as desired. $\square$

Remark 3.3. We mentioned in the Preliminaries that the Jacobson radical is the smallest z-ideal in any ring. This is reconfirmed (in the present case) by the fact that $0_{\mathrm{ZId}(A)} = \mathfrak{z}(0_{R\mathfrak{L}(A)})$, and the zero of $R\mathfrak{L}(A)$ is $\sqrt{0}$, the nilradical of A. Simple calculations show that $\mathfrak{z}(\sqrt{0}) = \mathrm{Jac}(A)$.

Remark 3.4. We could also have defined the nucleus $\mathfrak{z}$ on $\mathrm{RId}(A)$, the frame of radical ideals of A [1], to arrive at the conclusion that $\mathrm{ZId}(A)$ is a quotient of $\mathrm{RId}(A)$. Of course the question of whether $\mathrm{RId}(A) = R\mathfrak{L}(A)$ for f-rings with bounded inversion is unanswered. See Remark 3.3 in [2] for instances where the answer is affirmative.

Next, we show that $\mathrm{ZId}(A)$ is, in fact, a normal coherent frame Yosida frame, and we describe its compact elements. A frame homomorphism $h \colon L \to M$ is *codense* if, for any $a \in L$, $h(a) = 1$ implies $a = 1$. Recall from [2, Lemma 1.2] that any codense image of a normal frame is normal. We shall write $\mathfrak{z}_A \colon R\mathfrak{L}(A) \to \mathrm{ZId}(A)$ for the frame homomorphism emanating from the nucleus constructed in the proof of Proposition 3.2. That is,

$$\mathfrak{z}_A(I) = \bigvee_{\mathrm{ZId}(A)} \{M(a) \mid a \in I\} = \bigcup \{M(a) \mid a \in I\}.$$

Theorem 3.5. *If A is an f-ring with bounded inversion, then $\mathrm{ZId}(A)$ is a normal coherent Yosida frame, and the set of its compact elements is*

$$\mathfrak{k}(\mathrm{ZId}(A)) = \{M(a) \mid a \in A\} = \{M(a) \mid a \in A^+\}.$$

Proof. The equality of the last two sets above follows from the fact that $M(a) = M(a^2)$ for every $a \in A$, and squares are positive in any f-ring. Let I be a compact element in $\mathrm{ZId}(A)$. The equality

$$I = \bigvee_{\mathrm{ZId}(A)} \{M(u) \mid u \in I\}$$

implies that there are finitely many elements $u_1, \dots, u_n$ in I such that

$$I = M(u_1) \vee \cdots \vee M(u_n) = M(u_1^2) \vee \cdots \vee M(u_n^2).$$

We claim that, for any finitely many $a_1, \dots, a_n$ in A^+,

$$(\dagger) \qquad M(a_1) \vee \cdots \vee M(a_n) = M(a_1 + \cdots + a_n).$$

That the left hand side of ($\dagger$) is contained in the right hand side follows from Lemma 3.1. To see the other inclusion, let J be a z-ideal containing each of the ideals $M(a_i)$. Then $a_1 + \cdots + a_n \in J$, so that $M(a_1 + \cdots + a_n) \subseteq J$ since J is a z-ideal. This says $M(a_1 + \cdots + a_n) \subseteq J$ is the least upper bound for the set $\{M(a_1), \dots, M(a_n)\}$ in the frame $\mathrm{ZId}(A)$, which is exactly what ($\dagger$) says. Consequently, every compact element of $\mathrm{ZId}(A)$ is of the form $M(a)$

for some $a \in A$. Next, let $b \in A$, and consider any directed family $\{I_\alpha\}$ of elements of $\mathrm{ZId}(A)$ with $M(b) \subseteq \bigcup_\alpha I_\alpha$. Then $b \in I_{\alpha_0}$, for some index α_0. Since I_{α_0} is a z-ideal, $M(b) \subseteq I_{\alpha_0}$. Therefore $M(b)$ is compact. In all then,

$$\mathfrak{k}(\mathrm{ZId}(A)) = \{M(a) \mid a \in A\} = \{M(a) \mid a \in A^+\};$$

and the ideals $M(a)$, $a \in A$, generate $\mathrm{ZId}(A)$, showing that $\mathrm{ZId}(A)$ is an algebraic frame. That $\mathrm{ZId}(A)$ is compact follows easily from the fact that A has identity. Finally, from Lemma 3.1(a) we have that the meet of two compact elements is compact, hence $\mathrm{ZId}(A)$ is coherent.

Since each $M(a)$ is an intersection of maximal ideals of A, and maximal ideals of A are exactly the maximal elements of the frame $\mathrm{ZId}(A)$, it follows that $\mathrm{ZId}(A)$ is a Yosida frame. We prove its normality by showing that the surjective frame homomorphism $\mathfrak{z}_A \colon R\mathfrak{L}(A) \to \mathrm{ZId}(A)$ is codense. This will establish normality of $\mathrm{ZId}(A)$ because $R\mathfrak{L}(A)$ is normal. Consider then any $J \in R\mathfrak{L}(A)$ such that

$$\mathfrak{z}_A(J) = \bigcup \{M(u) \mid u \in J\} = 1_{\mathrm{ZId}(A)} = A.$$

There is therefore an element $u \in J$ such that $1 \in M(u)$. But now this implies u is invertible, and so $J = A = 1_{R\mathfrak{L}(A)}$, hence $\mathrm{ZId}(A)$ is a codense image of a normal frame, which makes it normal. $\qquad\square$

Remark 3.6. If we assume that the sum of two z-ideals in A is a z-ideal then an argument similar to that employed by Banaschewski [2] to show that $\mathfrak{L}(A)$ is coherently normal can be modified, using $M(a)$ in place of the principal ℓ-ideal generated by a, to show that $\mathrm{ZId}(A)$ is coherently normal.

We shall now identify the primes (i.e., the meet-irreducible elements) of $\mathrm{ZId}(A)$ with a view to characterizing those A for which $\mathrm{ZId}(A)$ is regular. Since $\mathrm{ZId}(A)$ is coherent, we know from Martínez and Zenk [8, Theorem 2.4(a)] that $\mathrm{ZId}(A)$ is regular if and only if every prime element in this frame is minimal prime. So it behoves us to describe the primes of this frame.

Lemma 3.7. *If A is an f-ring with bounded inversion, then the prime elements of $\mathrm{ZId}(A)$ are precisely the prime z-ideals of A.*

Proof. Let P be meet-irreducible in $\mathrm{ZId}(A)$. We must show that P is a prime ideal in A. Consider $ab \in P$. Then $M(a) \cap M(b) = M(ab) \subseteq P$ since P is a z-ideal. Since P is meet-irreducible in $\mathrm{ZId}(A)$, we have $M(a) \subseteq P$ or $M(b) \subseteq P$. Therefore $a \in P$ or $b \in P$, and so P is a prime ideal in A. On the other hand, if P is a prime z-ideal in A, and $I \cap J \subseteq P$ for some $I, J \in \mathrm{ZId}(A)$, then $IJ \subseteq I \cap J \subseteq P$, so that $I \subseteq P$ or $J \subseteq P$ by primeness, showing that P is meet-irreducible in $\mathrm{ZId}(A)$. $\qquad\square$

Remark 3.8. For an f-ring A with bounded inversion we may define the *Z-spectrum* of A as the space of its prime z-ideals with the hull-kernel topology. By this lemma, it is then the frame-theoretic spectrum of $\mathrm{ZId}(A)$.

Recall that a ring A is *von Neumann regular* if for every $a \in A$ there exists $b \in A$ such that $a = a^2 b$. It is well known that a semiprime ring is von Neumann regular if and only if every prime ideal in it is minimal prime. Since maximal ideals and minimal prime ideals are z-ideals, and since every prime ideal contains a minimal prime ideal, and is contained in a maximal ideal, we deduce from the theorem of Martínez and Zenk cited above the following result.

Corollary 3.9. *If A is a semiprime f-ring with bounded inversion, then $\mathrm{ZId}(A)$ is regular if and only if A is von Neumann regular.*

We now say more about the homomorphism $\mathfrak{z}_A \colon R\mathfrak{L}(A) \to \mathrm{ZId}(A)$. Let us recall from [2] that the compact elements of $R\mathfrak{L}(A)$ are precisely the radicals of the principal ℓ-ideals of positive elements of A. For any $a \in A$, write

$$[a] = \{x \in A \mid |x| \leq s|a| \text{ for some } s \geq 0\},$$

for the principal ℓ-ideal of A generated by a. The radical of an ideal J will here be written as $\sqrt{J}$. It is an ℓ-ideal if J is an ℓ-ideal. This is all in [2], and it is observed there that the set of compact elements of $R\mathfrak{L}(A)$ is

$$\mathfrak{k}(R\mathfrak{L}(A)) = \{\sqrt{[a]} \mid a \in A^+\}.$$

Let us note, for use in the upcoming proof, that if A has bounded inversion, then for any $0 \leq a \leq b$, $M(a) \subseteq M(b)$, and for any $c \in A$, $M(c) = M(|c|)$.

Proposition 3.10. *Let $\mathfrak{z}_A \colon R\mathfrak{L}(A) \to \mathrm{ZId}(A)$ be as above for an f-ring A with bounded inversion.*

- *(a) $\mathfrak{z}_A$ is a codense coherent map.*
- *(b) $\mathfrak{z}_A$ is dense if and only if the Jacobson radical and the nilradical of A coincide.*
- *(c) $\mathfrak{z}_A$ is injective if and only if every radical ℓ-ideal of A is a z-ideal.*

Proof. (a) We have already shown the codensity of this map. Turning to its coherence, we claim that, for any $a \in A^+$, $\mathfrak{z}_A(\sqrt{[a]}) = M(a)$. Since $a \in \sqrt{[a]}$, we immediately have $M(a) \subseteq \mathfrak{z}_A(\sqrt{[a]})$. Now let $x \in \mathfrak{z}_A(\sqrt{[a]})$. Then there is a non-negative integer n such that $x^n \in [a]$; so, there is an $s \in A^+$ such that $|x|^n = |x^n| \leq sa$. Thus,

$$M(x) = M(|x|) = M(|x|^n) \subseteq M(sa) = M(s) \cap M(a) \subseteq M(a),$$

so that $\mathfrak{z}_A(\sqrt{[a]}) \subseteq M(a)$, and hence equality. This tells us that $\mathfrak{z}_A$ is coherent.

(b) Since the zero of $R\mathfrak{L}(A)$ is the nilradical, and the zero of $\mathrm{ZId}(A)$ is the Jacobson radical, $\mathfrak{z}_A$ is dense if and only if the nucleus $\mathfrak{z}$ sends $\sqrt{0}$ to $\sqrt{0}$. This in turn holds if and only if $\sqrt{0} = \mathrm{Jac}(A)$.

(c) Any frame homomorphism is injective if and only if its right adjoint is surjective. The right adjoint of $\mathfrak{z}_A$ is the inclusion map. The result follows from this. $\qquad\square$

For the next result we assume that the sum of two z-ideals is a z-ideal. So we are still dealing with f-rings that include function rings. Note that the binary join in $R\mathfrak{L}(A)$ is given by $I \vee J = \sqrt{I + J}$. Recall that a frame homomorphism $h\colon L \to M$ is called *closed* precisely when, for every $a, b \in L$ and any $u \in M$,

$$h(a) \le h(b) \vee u \quad \implies \quad a \le b \vee h_*(u).$$

A closed frame homomorphism whose right adjoint preserves directed joins is called a *proper map*.

Although we can obviate use of the following lemma in the proof of our intended result, we shall present it because it also brings to the fore a notewrothy fact about sums of z-ideals.

Lemma 3.11. *If the sum of two z-ideals in any ring A is a z-ideal, then the sum of any collection of z-ideals of A is a z-ideal.*

Proof. A straightforward induction shows that the sum of any finitely many z-ideals is a z-ideal. Now let $\{I_\alpha\}$ be a collection of z-ideals of A. Let $a \in \sum_\alpha I_\alpha$, and pick finitely many indices $\alpha_1, \ldots, \alpha_n$ and elements $a_{\alpha_i} \in I_{\alpha_i}$ such that $a = a_{\alpha_1} + \cdots + a_{\alpha_n}$. Since $I_{\alpha_1} + \cdots + I_{\alpha_n}$ is a z-ideal containing a, we have $M(a) \subseteq I_{\alpha_1} + \cdots + I_{\alpha_n} \subseteq \sum_\alpha I_\alpha$, as desired. $\square$

Proposition 3.12. *If the sum of two z-ideals in an f-ring A with bounded inversion is a z-ideal, then $\mathfrak{z}_A$ is a proper map. Furthermore, if the nilradical and the Jacobson radical of A coincide, then the right adjoint of $\mathfrak{z}_A$ is a frame homomorphism.*

Proof. We shall abbreviate $\mathfrak{z}_A$ as $\mathfrak{z}$, so that $\mathfrak{z}_*$ abbreviates the right adjoint of $\mathfrak{z}_A$. We show first that $\mathfrak{z}$ is a closed map. Consider $I, J \in R\mathfrak{L}(A)$ and $K \in \mathrm{ZId}(A)$ such that $\mathfrak{z}(I) \subseteq \mathfrak{z}(J) \vee K$ in $\mathrm{ZId}(A)$. Let $u \in I$. Then

$$M(u) \le \mathfrak{z}(I) \le \bigvee_{\mathrm{ZId}(A)} \{M(v) \mid v \in J\} \vee K,$$

which, by compactness, implies there is a $w \in J$ such that

$$M(u) \le M(w) \vee K = M(w) + K$$

in $\mathrm{ZId}(A)$. Since $M(w)$ and K are z-ideals, $M(u) + K$ is a z-ideal, by hypothesis, and hence a radical ℓ-ideal in A, meaning that the join $M(w) \vee K$ in $R\mathfrak{L}(A)$ is $\sqrt{M(w) + K} = M(w) + K$. Thus, with the join below calculated in $R\mathfrak{L}(A)$, we have

$$u \in M(u) \le M(w) \vee K \le J \vee K = J \vee \mathfrak{z}_*(K),$$

which shows that $I \le J \vee \mathfrak{z}_*(K)$, whence $\mathfrak{z}_A$ is closed. Next, we show that $\mathfrak{z}_*$ preserves directed joins. In fact, we show that $\mathfrak{z}_*$ preserves all joins. So

let $\{I_\alpha\} \subseteq \mathrm{ZId}(A)$. Then, by Lemma 3.11,

$$\mathfrak{z}_* \left(\bigvee_{\mathrm{ZId}(A)} I_\alpha \right) = \mathfrak{z}_* \left(\sum_\alpha I_\alpha \right) = \sum_\alpha I_\alpha$$

$$= \sqrt{\sum_\alpha I_\alpha} = \sqrt{\sum_\alpha \mathfrak{z}_*(I_\alpha)} = \bigvee_{R\mathfrak{L}(A)} \mathfrak{z}_*(I_\alpha).$$

Therefore $\mathfrak{z}_A$ is a proper map.

The latter part of the proposition follows from Proposition 3.10(b) because then $\mathfrak{z}_*$ sends the zero element to the zero element. $\square$

4. Functoriality emanating from z-ideals

In this last section we say a word about functoriality. Let BfRng denote the category whose objects are f-rings with bounded inversion, and whose morphisms are ℓ-ring homomorphisms. We aim to show that the assignment $A \mapsto \mathrm{ZId}(A)$ is the object part of a functor BfRng $\to$ CohFrm, and that the association $A \mapsto \mathfrak{z}_A \colon R\mathfrak{L} \to \mathrm{ZId}$ is a natural transformation. We need a lemma.

Lemma 4.1. *For any $\phi \colon A \to B$ in* BfRng*, the map*

$$\tau \colon \mathfrak{k}(\mathrm{ZId}(A)) \to \mathfrak{k}(\mathrm{ZId}(B))$$

given by $M(a) \mapsto M(\phi(a))$ for $a \in A^+$ is a lattice homomorphism.

Proof. Since the zeros of these lattices are the Jacobson radicals of the rings in question, this map sends zero to zero. It also clearly sends the top element to the top element. By Lemma 3.1,

$$\tau\big(M(a) \wedge M(b)\big) = \tau\big(M(ab)\big) = M(\phi(ab))$$
$$= M(\phi(a)\phi(b))$$
$$= M(\phi(a)) \wedge M(\phi(b))$$
$$= \tau\big(M(a)\big) \wedge \tau\big(M(b)\big).$$

Finally, as noted in (†) in the proof of Theorem 3.5, for any $a, b \in A^+$, $M(a \vee b) = M(a+b)$. So, since ϕ sends positive elements to positive elements, we have

$$\tau\big(M(a) \vee M(b)\big) = \tau\big(M(a + b)\big)$$
$$= M(\phi(a) + \phi(b))$$
$$= M(\phi(a)) \vee M(\phi(b)) = \tau\big(M(a)\big) \vee \tau\big(M(b)\big).$$

Therefore τ is a lattice homomorphism. $\square$

As a consequence of this lemma, given $\phi\colon A \to B$ in BfRng, there is a unique coherent map $\bar{\phi}\colon \mathrm{ZId}(A) \to \mathrm{ZId}(B)$ that extends ϕ (see [5, p. 64]). Explicitly, $\bar{\phi}$ maps as follows:

$$\bar{\phi}(J) = \bigvee_{\mathrm{ZId}(B)} \{M(\phi(a)) \mid a \in J\} = \bigcup\{M(\phi(a)) \mid a \in J\}.$$

Recall from [2] that $R\mathfrak{L}\colon \mathsf{BfRng} \to \mathsf{CohFrm}$ is a functor whose action on a morphism $\phi\colon A \to B$ is

$$R\mathfrak{L}(\phi)(J) = \sqrt{\bigcup\{[\phi(a)] \mid a \in J\}},$$

so that for any $a \in A^+$, $R\mathfrak{L}(\phi)(\sqrt{[a]}) = \sqrt{\phi(a)}$.

Proposition 4.2. *The mappings $A \mapsto \mathrm{ZId}(A)$ and $\phi \mapsto \bar{\phi}$ define a functor*

$$\mathrm{ZId}\colon \mathsf{BfRng} \to \mathsf{CohFrm}$$

such that $A \mapsto \mathfrak{z}_A\colon R\mathfrak{L} \to \mathrm{ZId}$ is a natural transformation.

Proof. One checks routinely that

$$\mathrm{ZId}(\mathrm{id}_A) = \mathrm{id}_{\mathrm{ZId}(A)} \quad \text{and} \quad \mathrm{ZId}(\psi\phi) = \mathrm{ZId}(\psi)\,\mathrm{ZId}(\phi)$$

for any $\phi\colon A \to B$ and $\psi\colon B \to C$ in BfRng. To prove that

$$A \mapsto \mathfrak{z}_A\colon R\mathfrak{L} \to \mathrm{ZId}$$

is a natural transformation, we need to show that, for any $\phi\colon A \to B$ in BfRng, the square

$$
\begin{array}{ccc}
R\mathfrak{L}(A) & \xrightarrow{\ \mathfrak{z}_A\ } & \mathrm{ZId}(A) \\
{\scriptstyle R\mathfrak{L}(\phi)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ZId}(\phi)} \\
R\mathfrak{L}(B) & \xrightarrow{\ \mathfrak{z}_B\ } & \mathrm{ZId}(B)
\end{array}
$$

commutes. Since $R\mathfrak{L}(A)$ is generated by its compact elements, it suffices to show that $\mathrm{ZId}(\phi)\mathfrak{z}_A$ and $\mathfrak{z}_B R\mathfrak{L}(\phi)$ agree on the compact elements of $R\mathfrak{L}(A)$. Consider then $\sqrt{[a]}$ for $a \in A^+$. We observed in the proof of Proposition 3.10 that $\mathfrak{z}_A(\sqrt{[a]}) = M(a)$. Thus,

$$\mathrm{ZId}(\phi)\mathfrak{z}_A(\sqrt{[a]}) = \mathrm{ZId}(\phi)\big(M(a)\big) = M(\phi(a)),$$

and

$$\mathfrak{z}_B R\mathfrak{L}(\phi)(\sqrt{[a]}) = \mathfrak{z}_B(\sqrt{[\phi(a)]}) = M(\phi(a)),$$

which proves the result. $\qquad\square$

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