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Rank 4 premodular categories

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Appendix with César Galindo, Siu-Hung Ng, Julia Plavnik, Eric Rowell, and Zhenghan Wang

Abstract. We consider the classification problem for rank 4 premodular categories. We uncover a formula for the 2^{nd} Frobenius–Schur indicator of a premodular category, and complete the classification of rank 4 premodular categories (up to Grothendieck equivalence). In the appendix we show rank finiteness for premodular categories.

Contents

1. Introduction

The theory of fusion categories is a natural generalization of representation theory — not only of finite groups, but of Lie groups and Hopf algebras

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and so, in some sense, their classification began with the classification of groups and their representations. At the time of this writing, a complete classification has only been completed for rank 2 and 3 fusion categories [19, 21]. While the classification problem for fusion categories is largely believed to be intractable, several natural structures can be imposed on fusion categories to make them more amenable to study.

One such structure is that of braiding. This gives rise to a kind of commutativity and forces the underlying Grothendieck semiring to be commutative. On the other hand, one might expect that the two natural notions of dimension in the theory coincide, leading to pseudo-unitary fusion categories. If study is restricted to pseudo-unitary fusion categories, then it is known that the category is also spherical [8]. The appearance of a spherical structure is perhaps not surprising as there are no known examples of nonspherical fusion categories at this time.

Even with the addition of these structures, a full classification is believed to be out of reach as it would include a classification of finite groups. However, these categories admit a stratification by degeneracy of the S-matrix into symmetric, properly premodular, and modular categories. The representation categories fall naturally in the symmetric case and in fact completely fill it out [6]. At the other end of the spectrum, a large amount of work has gone into understanding modular categories spurred by their relationship to rational conformal field theories, quantum computation, link invariants, and 3-manifold invariants [28, 25, 1]. However, recently premodular categories have been shown to provide the algebraic underpinnings of $(3+1)$-dimensional topological quantum field theories and thereby govern topological insulators and some high-T_c superconductors [27]. In addition to their innate uses, premodular categories give rise to modular categories through the double construction.

Classification of premodular categories has been completed for rank 2 and 3 [19, 20] and in this paper we extend the classification to rank 4. Since the techniques commonly applied in the modular setting do not apply in the premodular setting new tools are developed. Specifically, the following formula for the 2^{nd} Frobenius–Schur indicator for a self-dual object is determined in terms of the premodular datum.

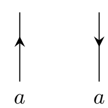
$$\nu_2\left(X_a\right) = \frac{1}{D^2}\sum_{b,c} N_{bc}^{a} d_b d_c \left(\frac{\theta_b}{\theta_c}\right)^2 - \theta_a \sum_{\gamma\in\mathcal{C}'\setminus\mathbb{I}} d_\gamma \operatorname{Tr}\left(R_\gamma^{aa}\right).$$

We will begin by reviewing the theory of modular and premodular categories. Having dispensed with these preliminaries, a formula for the 2^{nd} Frobenius–Schur indicator will be derived in the premodular setting. As an application of this indicator, the rank 4 premodular categories will then be classified. In conjunction with [23], this will complete the classification of rank 4 premodular and modular categories. Finally, in Appendix A we prove that there are only finitely many premodular categories of fixed rank, up to equivalence.

2. Preliminaries

A premodular category $\mathcal{C}$ is a braided, balanced, and fusion category. Furthermore, if the S-matrix is invertible then $\mathcal{C}$ is said to be modular. Every premodular category $\mathcal{C}$ is a ribbon category and as such enjoys a graphical calculus. A brief account of this calculus in addition to some salient algebraic relations will be given and further detail can be found in [1, 12, 25].

2.1. Pivotal structure and dimensions. By virtue of being a fusion category, $\mathcal{C}$ is semisimple and we will denote the isomorphism classes of the simple objects by $\mathbb{I} = X_0, \ldots, X_{n-1}$ where n is known as the *rank* of $\mathcal{C}$. Furthermore, $\mathcal{C}$ is balanced and hence pivotal. This structure manifests itself through a duality $*$ acting by $X_a^* = X_{a^*}$. Such a duality induces an involution on the labeling set for the simple objects and can be encoded by the *charge conjugation matrix* $C_{ab} = \delta_{ab^*}$. Graphically, a nontrivial simple object X_a is denoted by an upward arrow and its dual by a downward arrow,

(2.1)  .

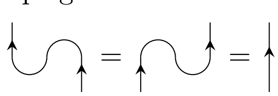
For the trivial object, $X_0 = \mathbb{I}$, no arrow is drawn. Note that for a self-dual object the arrow may be safely omitted. The pivotal structure of $\mathcal{C}$ further provides a collection of evaluation and co-evaluation maps

$$ev_X : X^* \otimes X \to \mathbb{I}, \tag{2.2}$$
$$coev_X : \mathbb{I} \to X \otimes X^*.$$

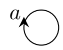
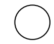
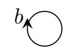
These maps are given by the cup and cap

(2.3) $coev =$  $\quad ev =$  .

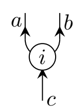
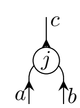
Compatibility of such maps give rise to the allowed graphical moves:

(2.4)  .

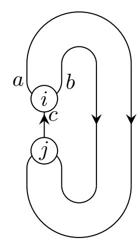
A pivotal category also comes equipped with a family of natural isomorphisms $j_X : X \to X^{**}$. The presence of these maps give rise to two canonical traces called left and right pivotal traces [17]. In a spherical category, these traces coincide and so, for $f \in \mathrm{End}_{\mathcal{C}}(X)$, one simply writes $\mathrm{Tr}_{\mathcal{C}}(f)$. By the coherence theorems, it is known that every premodular category is equivalent to a strict premodular category and so we will, without loss of generality, restrict our attention to strict categories. One benefit of focusing on strict categories is that the isomorphisms j_X can be removed, which greatly simplifies the graphical calculus. For instance, taking the trace of id_{X_a} allows one to define the *dimension* of X_a and the *global dimension*, D^2. These dimensions are graphically given by

(2.5) $\dim(X_a) = d_a =$  , $D^2 = \dim(\mathcal{C}) =$  $:= \sum_{b \in \mathrm{Irr}(\mathcal{C})} d_b$  .

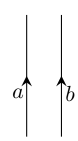
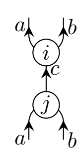
2.2. Fusion and splitting spaces. $\mathbb{C}$-linearity of $\mathcal{C}$ endows $\mathrm{Hom}_{\mathcal{C}}(V, W)$ with the structure of a complex vector space for all objects V and W in $\mathcal{C}$. However, certain families of Hom-spaces are distinguished due to semisimplicity, they are the *fusion spaces* $V_{ab}^c = \mathrm{Hom}_{\mathcal{C}}(X_a \otimes X_b, X_c)$ and the *splitting spaces* $V_c^{ab} = \mathrm{Hom}_{\mathcal{C}}(X_c, X_a \otimes X_b)$. In the course of this work a basis of the splitting space will be denoted by $\left\{\psi_{c,i}^{ab}\right\}$ and the dual basis of the fusion space is given by $\left\{\psi_{ab,j}^{c} = \left(\psi_{c,j}^{ab}\right)^{\dagger}\right\}$. These bases are graphically depicted by

(2.6)  and  ,

respectively. The normalization of these bases will always be such that

(2.7) $\theta(a, b, c)\,\delta_{ij} =$ 

where $\theta(a, b, c) = \sqrt{d_a d_b d_c}$ is the *theta symbol*. Further note that this normalization is consistent with the graphical dimensions given in Equation (2.5), i.e., $b = a^*$ and $c = 0$. This particular symbol appears in the decomposition of $id_{X_a \otimes X_a}$ as

(2.8)  $= \sum_{c \in \mathrm{Irr}\,\mathcal{C}} \sum_{i \in V_c^{ab}} \sum_{j \in V_{ab}^c} \frac{d_c}{\theta(a, b, c)}$  .

The dimension of the fusion space $\mathrm{Hom}_{\mathcal{C}}(X_a \otimes X_b, X_c)$, N_{ab}^c, gives the multiplicity of X_c appearing in $X_a \otimes X_b$, and is called a *fusion coefficient*. The fusion coefficients are generally collected into *fusion matrices*

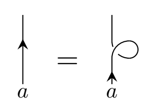
$$(N_a)_{bc} = N_{ab}^c$$

and furnish a representation of the Grothendieck semiring $Gr(\mathcal{C})$ [10]. Since the fusion coefficients are nonnegative integers, the Frobenius–Perron Theorem can be applied to deduce the existence of a largest eigenvalue of

N_a, such an eigenvalue is called the *Frobenius–Perron dimension* or *FP-dimension* of X_a and is denoted $\mathrm{FPdim}(X_a)$. One says that a premodular category is *pseudo-unitary* if $\mathrm{FPdim}(X_a) = \dim(X_a)$ for all a. The *global FP-dimension* of the category is defined by $\mathrm{FPdim}(\mathcal{C}) = \sum_a \mathrm{FPdim}(X_a)^2$. If the global FP-dimension is an integer, the category is said to be *weakly integral* and if $\mathrm{FPdim}(X_a) \in \mathbb{Z}$ for all a then one says $\mathcal{C}$ is *integral.* Finally, duality and braiding endow the fusion matrices with the following symmetries [1]:

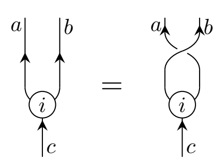
$$N_{ab}^{c} = N_{ba}^{c} = N_{ac^*}^{b^*} = N_{a^*b^*}^{c^*} \tag{2.9}$$
$$N_{ab^*}^{0} = 1, \quad N_{a^*} = N_a^T, \quad N_a N_b = N_b N_a.$$

2.3. Spherical structure. The braiding and spherical structure give rise to canonical elements $\theta_a \in \mathrm{End}_{\mathcal{C}}(X_a)$ called *twists.* Since $\mathrm{End}_{\mathcal{C}}(X_a)$ is one dimensional, the twists are scalar multiples of the identity, also denoted θ_a. Graphically, we have

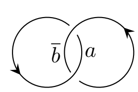
θ_a  .

The celebrated Vafa Theorem tells us these twists are roots of unity [26]. For convenience, the twists are collected into the diagonal matrix $T_{ab} = \delta_{ab}\theta_b$ called the *T-matrix.*

2.4. Braiding. The braiding in $\mathcal{C}$ is given by elements $R_{ab} \in \mathrm{Hom}_{\mathcal{C}}(X_a \otimes X_b, X_b \otimes X_a)$. Coupling these maps with the splitting spaces, one can define the R-matrices $(R_c)_{ab} = R_c^{ab}$, where R_c^{ab} is obtained by "braiding X_a with X_b in the X_c channel." In fact, the bases of the splitting space V_c^{ab} can be chosen to diagonalize R_c^{ab} by $R_c^{ab}\psi_{c,i}^{ab} = R_{c,i}^{ab}\psi_{c,i}^{ab}$ [12]. Pictorially, this is given by

$R_{c,i}^{ab}$  .

These braidings give rise to a family of natural isomorphisms $c_{ab} = R_{ba}R_{ab}$ in $\mathrm{End}_{\mathcal{C}}(X_a \otimes X_b)$ which can be traced to define the S-matrix

(2.10) $\tilde{s}_{ab} = \mathrm{Tr}_{\mathcal{C}}\left(c_{a\bar{b}}\right) =$  .

2.5. Algebraic identities. The S-matrix is highly symmetric and, in fact we have

$$\tilde{s}_{a^*b}^{*} = \tilde{s}_{ab} = \tilde{s}_{ba} = \tilde{s}_{a^*b^*}, \quad \tilde{s}_{a0} = d_a. \tag{2.11}$$

In the course of this work the tuple $\left(\tilde{S},\ T,\ N_0,\ \ldots,\ N_n\right)$ will be referred to as premodular datum. Perhaps not surprisingly, the matrices comprising premodular datum are strongly related. For instance, an elementary application of the graphical calculus leads to the *balancing relation* [1]

$$\tilde{s}_{ab} = \theta_a^{-1}\theta_b^{-1}\sum_c N_{a^*b}^c \theta_c d_c. \tag{2.12}$$

Additionally, one can show that the columns of S-matrix are eigenvectors of the fusion matrices. In a modular category, this leads to the well-known Verlinde Formula, while in the premodular setting it is shown in [14] that

$$\tilde{s}_{ab}\tilde{s}_{ac} = d_a \sum_\ell N_{bc}^\ell \tilde{s}_{a\ell}. \tag{2.13}$$

It can further be shown that the S- and T-matrices are related by

$$\begin{aligned} \left(\tilde{S}T\right)^3 &= p^+\tilde{S}^2, \\ \left(\tilde{S}T^{-1}\right)^3 &= p^-\tilde{S}^2 C, \end{aligned} \tag{2.14}$$

where $C_{a,b} = \delta_{a,b^*}$ is the *charge conjugation* matrix, and $p^\pm$ are the *Gauss sums*:

$$p^\pm = \sum_a \theta_a^\pm d_a^2. \tag{2.15}$$

If $\det\left(\tilde{S}\right) \neq 0$ then $\mathcal{C}$ is said to be modular and the additional identities

$$\tilde{S}\tilde{S}^\dagger = D^2\mathbb{I} \quad \text{and} \quad p^+p^- = D^2, \tag{2.16}$$

are acquired, from which it is clear that $\tilde{S}$ and T furnish a projective representation of the modular group $\mathrm{SL}(2,\mathbb{Z})$.

$\mathcal{C}$ is said to be *symmetric* if $\tilde{s}_{ab} = d_a d_b$ for all a and b. One can view symmetric categories as completely degenerate premodular categories while modular categories are completely nondegenerate. It is between these two extremes that we will be focusing our attention and so we define a *properly premodular category* $\mathcal{C}$ to be a premodular category that is neither symmetric nor modular. In this way, symmetric, properly premodular, and modular categories partition the class of premodular categories.

2.6. The Müger center and finiteness. The braiding can be used to define the *Müger center* of a premodular category by [14]

$$\mathcal{C}' = \left\{X \in \mathcal{C} \mid c_{X,Y} = id_{X\otimes Y},\ \forall Y \in \mathcal{C}\right\}. \tag{2.17}$$

The elements of the center are often called *central* or *transparent* [14, 3].[1] This center constitutes a full symmetric ribbon subcategory of $\mathcal{C}$ which is

[1]In the course of this work, simple objects in the Müger center will be indexed by Greek letters to distinguish them from simple objects in $\mathcal{C}$ which will be indexed by lower case Latin letters.

trivial if and only if $\mathcal{C}$ is modular. In fact, if $\mathcal{C}$ is not modular then some column of the S-matrix is a multiple of the first [3]. Thus a premodular category $\mathcal{C}$ is symmetric if $\mathcal{C} = \mathcal{C}'$, $\mathcal{C}$ is modular if $\mathcal{C}' = \text{Vec}$, and $\mathcal{C}$ is properly premodular otherwise.

Given these abstract constructions one might wonder if premodular categories exist and indeed they do; for instance, quantum groups lead not only to modular, but also to properly premodular categories [22]. Given their existence, a classification program has been taken up. In [19], [21], and [20], Ostrik has classified all fusion categories of ranks 2 and 3 and all premodular categories of rank 3. However, until the time of this writing it was unknown whether or not there are finitely many premodular categories of fixed rank, up to equivalence.

Such a problem is referred to as a *rank finiteness problem.* In [23] the rank finiteness problem was posed for modular categories while in [19] it was posed for fusion categories. Over the years progress has been made in various directions. For instance, direct classification of (pre)modular categories demonstrate the conjecture in low rank, while [8] showed rank finiteness for bounded FP-dimension and weakly integral categories. In a recent paper, [4], the rank finiteness problem was solved for modular categories. The proof for modular categories demonstrated connections between number theory and modular categories and heavily relied on the Frobenius–Schur indicators via the Cauchy Theorem for Modular Categories. In this paper we will extend the rank 4 premodular classification which depends strongly on Frobenius–Schur indicators. This suggests that they are fundamental to the theory of premodular categories. Finally, in Appendix A we will settle the rank finiteness problem for premodular categories.

3. Frobenius–Schur indicators

As alluded to in the literature, e.g., [7], the study of fusion categories is the correct generalization of the study of the representation theory of finite groups. Each finite group, G, gives rise to a fusion category whose objects are the representations of G and whose morphisms are intertwiners [7]. With this connection, it is natural to ask if the techniques used in the study of finite group representations can be generalized to arbitrary fusion categories and often they can. For instance, the class equation was generalized in [8], a rigorous study of Frobenius–Schur indicators was undertaken in [17, 18], and the Cauchy Theorem was fully extended to modular categories in [4].

In the classical theory of the representations of finite groups one can form the n^{th}-Frobenius–Schur indicator from the characters for any $n \in \mathbb{N}$. The 0^{th} Frobenius–Schur indicator gives the dimension of the representation, the 1^{st} indicator detects if the representation is the trivial representation. The 2^{nd} indicator of an irreducible representation is 1, 0, or -1 depending on if the representation is real, complex, or quaternionic. Frobenius–Schur indicators have also been developed for and applied to semisimple Hopf algebras [13, 11].

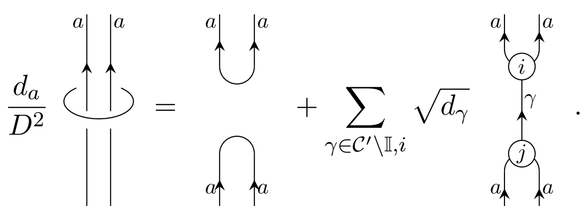
The 2^{nd} Frobenius–Schur indicator in the context of fusion categories was first computed by physicists studying rational conformal field theories [2]. The study of Frobenius–Schur indicators was furthered by Siu-Hung Ng and Peter Schauenburg who applied the graphical calculus and categorical considerations to derive graphical expressions for the n^{th} Frobenius–Schur indicators of pivotal, spherical, and modular categories. In the modular case, they recovered Bantay's result and found similar formulas for computing the n^{th} indicator of a modular category in terms of the modular datum. If the modularity assumption is dropped it is not known how to compute the n^{th} indicator strictly in terms of the premodular datum; that is without recourse to the graphical calculus. In this section, we will determine the following formula for the 2^{nd} Frobenius–Schur indicator of a premodular category:

$$\nu_2\left(X_a\right) = \frac{1}{D^2}\sum_{b,c} N_{bc}^{a} d_b d_c \left(\frac{\theta_b}{\theta_c}\right)^2 - \theta_a \sum_{\gamma\in\mathcal{C}'\setminus\mathbb{I}} d_\gamma \operatorname{Tr}\left(R_\gamma^{aa}\right).$$

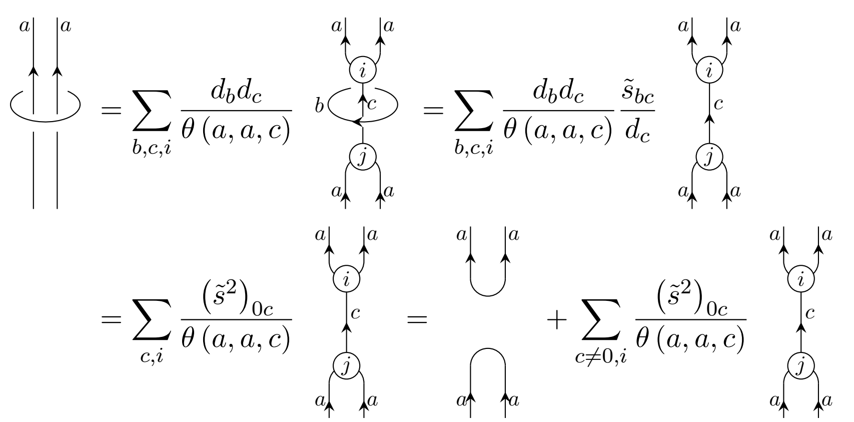
If the modularity condition is enforced, one sees that $\mathcal{C}' = \{\mathbb{I}\}$ and so the above formula recovers Bantay's result.

Examination of Ng and Schauenburg's proof presented in [17] reveals that modularity is only used indirectly when invoking [1, Corollary 3.1.11]. This corollary can be modified to give a starting place for computing the 2^{nd} indicator in the premodular setting.

Proposition 3.1. *If $\mathcal{C}$ is premodular and X_a is self-dual then*

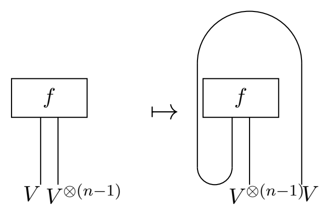


Proof. Applying Equation (2.8) and [1, Lemma 3.1.4] we have

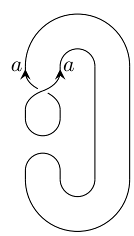


Since the columns of the columns of the S-matrix are eigenvectors of the fusion matrices we know that $\left(\tilde{s}^2\right)_{\gamma 0} = d_\gamma D^2$ if $X_\gamma \in \mathcal{C}'$ and 0 otherwise; this observation gives the desired result. □

Recall from [17] that the n^{th} Frobenius–Schur indicator is defined by $\nu_n(X) = Tr\left(E_X^{(n)}\right)$, where $E_X^{(n)}$ is given by

$E_X^n :$  .

Applying techniques from [17] and our bases for the splitting and fusion spaces, to this definition, we find that if X_a is self-dual, then the 2^{nd} Frobenius–Schur indicator is given by

(3.1) $$\nu_2(X_a) = \frac{\theta_a}{d_a}$$  .

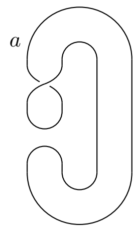
Otherwise we define it to be zero. Here the factor $\frac{1}{d_a}$ appears due to renormalization of the basis elements of $\mathrm{Hom}_{\mathcal{C}}\left(X^{\otimes 2}, \mathbb{I}\right)$ and $\mathrm{Hom}_{\mathcal{C}}\left(\mathbb{I}, X^{\otimes 2}\right)$ to have norm 1. With this definition and proposition in place we can prove the following theorem.

Theorem 3.2. *If $\mathcal{C}$ is a premodular category and X_a is a simple self-dual object then*

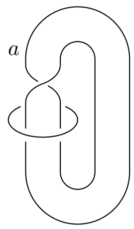
$$\nu_2(X_a) = \frac{1}{D^2}\sum_{b,c} N_{bc}^a d_b d_c \left(\frac{\theta_b}{\theta_c}\right)^2 - \theta_a \sum_{\gamma \in \mathcal{C}' \setminus \mathbb{I}} d_\gamma \operatorname{Tr}\left(R_\gamma^{aa}\right).$$

Proof. The proof proceeds by applying Proposition 3.1 to Equation (3.1) and then making use of the graphical calculus. To simplify notation we observe that since X_a is self-dual the arrow on the ribbon corresponding to this object can be safely removed.

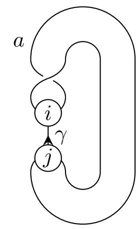
$$\nu_2\left(X_a\right)=\frac{\theta_a}{d_a}$$



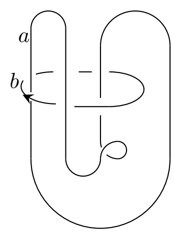
$$=\frac{\theta_a}{d_a}\frac{d_a}{D^2}$$



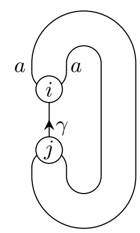
$$-\frac{\theta_a}{d_a}\sum_{\gamma\in\mathcal{C}'\setminus\mathbb{I},i,j}\sqrt{d_\gamma}$$



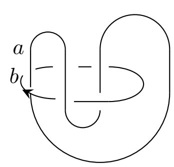
$$=\frac{\theta_a}{D^2}\sum_b d_b$$



$$-\frac{\theta_a}{d_a}\sum_{\gamma\in\mathcal{C}'\setminus\mathbb{I},i,j}\sqrt{d_\gamma}R^{aa}_{\gamma,i}$$

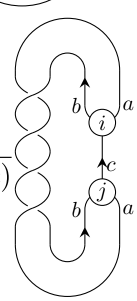


$$=\frac{\theta_a^2}{D^2}\sum_b d_b$$



$$-\frac{\theta_a}{d_a}\sum_{\gamma\in\mathcal{C}'\setminus\mathbb{I},i,j}\sqrt{d_\gamma}R^{aa}_{\gamma,i}\theta\left(a,a,\gamma\right)\delta_{ij}$$

$$=\frac{\theta_a^2}{D^2}\sum_{b,c,i,j}\frac{d_b d_c}{\theta\left(a,b,c\right)}$$



$$-\theta_a\sum_{\gamma\in\mathcal{C}'\setminus\mathbb{I}}d_\gamma\,\mathrm{Tr}\left(R^{aa}_\gamma\right)$$

$$=\frac{\theta_a^2}{D^2}\sum_{b,c,i,j}\frac{d_b d_c\left(R^{ab}_{c,i}R^{ba}_{c,i}\right)^2}{\theta\left(a,b,c\right)}\theta\left(a,b,c\right)\delta_{ij}-\theta_a\sum_{\gamma\in\mathcal{C}'\setminus\mathbb{I}}d_\gamma\,\mathrm{Tr}\left(R^{aa}_\gamma\right)$$

$$=\frac{\theta_a^2}{D^2}\sum_{b,c,i}d_b d_c\left(R^{ab}_{c,i}R^{ba}_{c,i}\right)^2-\theta_a\sum_{\gamma\in\mathcal{C}'\setminus\mathbb{I}}d_\gamma\,\mathrm{Tr}\left(R^{aa}_\gamma\right).$$

Applying Equation (216) of Appendix E in [12] and noting that $\left(\tilde{s}^2\right)_{\gamma 0} = d_\gamma D^2$ for $X_\gamma \in \mathcal{C}'$ gives

$$\nu_2\left(X_a\right) = \frac{\theta_a^2}{D^2}\sum_{b,c,i} d_b d_c \left(\frac{\theta_c}{\theta_a\theta_b}\right)^2 - \theta_a \sum_{\gamma\in\mathcal{C}'\setminus\mathbb{I}} d_\gamma \operatorname{Tr}\left(R_\gamma^{aa}\right).$$

Making use of Equation (2.9) we have $N_{ab}^c = N_{ba}^c = N_{bc^*}^a = N_{c^*b}^a$. However, $\theta_{b^*} = \theta_b$ and $d_{b^*} = d_b$ so

$$\nu_2\left(X_a\right) = \frac{1}{D^2}\sum_{b,c} N_{bc^*}^a d_b d_{c^*} \left(\frac{\theta_{c^*}}{\theta_b}\right)^2 - \theta_a \sum_{\gamma\in\mathcal{C}'\setminus\mathbb{I}} d_\gamma \operatorname{Tr}\left(R_\gamma^{aa}\right).$$

Reindexing the first sum gives the desired result. □

Since the R-matrices appear in this indicator, it is of limited computational use. However, one can show that the two sums of Theorem 3.2 are both rational integers. To do this, we first recall that the Müger center of $\mathcal{C}$ is a ribbon fusion category over $\mathbb{C}$ with fusion rules and twists descending from $\mathcal{C}$. Moreover, $c_{W,V} \circ c_{V,W} = id_{V\otimes W}$ on $\mathcal{C}'$ by its definition. So applying [17, Proposition 6.1], we can deduce that if $X_\gamma \in \mathcal{C}'$ then $\theta_\gamma = \pm 1$. However, $\theta_a R_{c,i}^{aa} = \pm\sqrt{\theta_c}$ and so, if $X_\gamma \in \mathcal{C}'$, we deduce that $\theta_a R_{\gamma,i}^{aa} \in \{\pm 1, \pm i\}$, which leads to the following corollary.

Corollary 3.3. *If $\mathcal{C}$ is premodular and $X_a \in \mathcal{C}$ simple, then*

$$\frac{1}{D^2}\sum_{b,c} N_{bc}^a d_b d_c \left(\frac{\theta_b}{\theta_c}\right)^2$$

is real and if X_a is self-dual then it is a rational integer.

Proof. Applying [17], we know that $\nu_2\left(X_a\right) \in \{-1, 0, 1\}$. Coupling this observation with the aforementioned fact that $\theta_a R_{\gamma,i}^{aa} \in \{\pm 1, \pm i\}$ for $X_\gamma \in \mathcal{C}'$, we can conclude that

$$\frac{1}{D^2}\sum_{b,c} N_{bc}^a d_b d_c \left(\frac{\theta_b}{\theta_c}\right)^2 \in \mathbb{Z}\left[i\right].$$

However, $N_{bc}^a = N_{cb}^a$, $d_b \in \mathbb{R}$, and $\overline{\theta_b} = \theta_b^{-1}$ for all a, b, c. So for any a we have that $\frac{1}{D^2}\sum_{b,c} N_{ab}^c d_b d_c \left(\frac{\theta_c}{\theta_b}\right)^2$ is invariant under complex conjugation. Consequently $\frac{1}{D^2}\sum_{b,c} N_{bc}^a d_b d_c \left(\frac{\theta_b}{\theta_c}\right)^2 \in \mathbb{Z}\left[i\right] \cap \mathbb{R} = \mathbb{Z}$. □

Remark 3.4. One can apply this corollary to show that the Müger center of a premodular category is integral as follows. Recall from [17, Section 6], that if $\alpha, \beta \in \mathcal{C}'$ then $\theta_{\alpha\otimes\beta} = \theta_\alpha \otimes \theta_\beta$ so $\theta_{\alpha\otimes\beta}^2 = \theta_\alpha^2 \otimes \theta_\beta^2$. Consequently,

$\sum_{\beta\in C'} N_{\alpha\gamma}^{\beta}\theta_\beta^2 d_\beta = \theta_\alpha^2\theta_\gamma^2 d_\alpha d_\gamma$ which can be rearranged to give

$$\sum_{\beta\in\mathcal{C}'} N_{\alpha\gamma}^{\beta} d_\beta d_\gamma \left(\frac{\theta_\beta}{\theta_\gamma}\right)^2 = \theta_\alpha^2 d_\alpha d_\gamma^2.$$

Summing over $\gamma \in \mathcal{C}'$ and reindexing gives

$$\theta_\alpha d_\alpha = \frac{1}{D_{\mathcal{C}'}^2} \sum_{\beta,\gamma\in\mathcal{C}'} N_{\beta\gamma}^{\alpha} d_\beta d_\gamma \left(\frac{\theta_\beta}{\theta_\gamma}\right)^2 \in \mathbb{Z}.$$

This is equivalent to saying that the Müger center is an integral subcategory of $\mathcal{C}$. Since the Müger center is a symmetric category and hence necessarily Grothendieck-equivalent to a representation category of a finite group, we know that it is integral. However, this does provide a new (to this author) route to this result.

Examination of Theorem 3.2 reveals that R_c^{aa} enters into the formula for the second indicator. Since the R-matrices involve square roots of the twists, we have that R_c^{ab} is a $2N^{\text{th}}$ root of unity where $N = \operatorname{ord}(T)$. Coupling this observation with Frobenius–Schur exponent of [17] motivates the following conjecture.

Conjecture 3.5. *If $\mathcal{C}$ is premodular, X_a is a simple object and $N = \operatorname{ord}(T)$, then $d_a \in \mathbb{Z}[\zeta_{2N}]$.*

This result is reminiscent of the Ng–Schauenburg Theorem for modular categories, which tells us that for any simple object X_a, $d_a \in \mathbb{Z}[\zeta_N]$ where $N = \operatorname{ord}(T)$ [17]. One might wonder if this theorem holds in the premodular setting despite the appearance of the R-matrices. However, examination of the premodular category $\mathcal{C}(sl(2), 8)_{ad}$ reveals that the Ng–Schauenburg Theorem fails, but that Conjecture 3.5 holds. Preliminary results indicate that more complicated combinations of the R-matrices may appear in higher indicators so more work is needed before the techniques of Ng and Schauenburg can be applied to Conjecture 3.5. However, this conjecture has been verified for premodular categories of rank < 5.

4. Rank 4 premodular categories

To classify all rank 4 premodular categories, we would need to determine the premodular datum — $\left(\tilde{S}, T, N_0, \ldots, N_n\right)$ — in addition to the R- and F-matrices. However, Ocneanu Rigidity tells us that there are only finitely many braided fusion categories realizing a given fusion ring and so it suffices to understand only the premodular datum. When classifying modular categories, one has a full range of Galois techniques available in addition to the divisibility of dimensions and the universal grading group. However, in the premodular setting, all of these techniques fail. Indeed, examination of $\mathcal{C}(sl(2), 8)_{ad}$ reveals that the universal grading group need not be isomorphic to $\mathcal{C}_{pt}$, the full subcategory generated by the invertible objects. This category further illustrates that the Ng–Schauenburg Theorem fails.[2] If we instead consider $\mathcal{C}(sl(2), 6)_{ad}$, then we see that the square of the dimensions of the simple objects need not divide the categorical dimension. Finally, the tensor category $\mathrm{Fib} \times \mathrm{Rep}(\mathbb{Z}_2)$ reveals that the Galois techniques fail in the premodular setting.

Given the failure of many of the techniques used in modular classification, what is left? To perform low rank premodular classification, people have, in the past, examined the double $\mathcal{Z}(\mathcal{C})$ as a module category [19]. However, in the rank 4 case, this approach is infeasible due to the number of simple objects. To overcome these difficulties, we will make use of the equations governing the premodular datum as well as cyclotomic and number theoretic techniques; the minimal modularization developed by Bruguières; and the 2^{nd} Frobenius–Schur indicators.

Recalling our partition of premodular categories into symmetric, properly premodular, and modular, we will discuss each of these classes in turn. We begin with the symmetric case, which is readily dealt with using the classification due to [6].

Proposition 4.1. *If $\mathcal{C}$ is a rank 4 symmetric category, then it is Grothendieck equivalent to* $\mathrm{Rep}(G)$ *where G is $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, D_{10}, or $\mathfrak{A}_4$.*

Continuing onto the well understood setting of modular categories. We recall that much of the classification has been completed in [23]. The omissions will be filled in and the classification completed in the following result.[3]

Proposition 4.2. *If $\mathcal{C}$ is a rank 4 modular category then it is Galois conjugate to a modular category from* [23] *or has S-matrix*

$$\begin{pmatrix} 1 & -1 & \overline{\tau} & \tau \\ -1 & 1 & -\tau & -\overline{\tau} \\ \overline{\tau} & -\tau & -1 & -1 \\ \tau & -\overline{\tau} & -1 & -1 \end{pmatrix},$$

where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden mean and $\overline{\tau} = \frac{1-\sqrt{5}}{2}$ is its Galois conjugate.

Proof. Using an argument due to V. Ostrik, [10, Appendix A], it suffices to consider Galois groups such that the column of the S-matrix corresponding to the FP-dimension and the 0-column reside in distinct Galois orbits and neither are fixed. Since the Galois group of a rank 4 modular category is an abelian subgroup of $\mathfrak{S}_4$, we see that, up to relabelling, the only Galois group that we need to consider is $\langle (0,1)(2,3) \rangle$. This is precisely case 5 of [23]. Applying the standard Galois techniques present in [23] leads to[4]

$$\tilde{S} = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ d_1 & \epsilon_0 & \epsilon_3 d_3 & \epsilon_0 \epsilon_3 d_2 \\ d_2 & \epsilon_3 d_3 & s_{22} & s_{23} \\ d_3 & \epsilon_0 \epsilon_3 d_2 & s_{23} & \epsilon_0 s_{22} \end{pmatrix}.$$

[2]The dimensions of the simple objects need not live in the cyclotomic extension of $\mathbb{Q}$ generated by the twists.

[3]The author would like to thank Eric Rowell for suggesting this approach.

[4]Here we index from 0 rather than 1 as in [23].

Where $\epsilon_j^2 = 1$ for all j, d_j are the categorical dimensions, and s_{22} and s_{23} are unknown S-matrix entries. Since $\epsilon_0 = \pm 1$ we consider these two cases separately.

Case 1. $\epsilon_0 = 1$.

Orthogonality of the first two columns of $\tilde{S}$ gives $d_1 = -\epsilon_3 d_2 d_3$. Applying our Galois element to this equation gives that $\epsilon_3 = -1$. Next, orthogonality of the last column with the others gives us that $\tilde{s}_{23}\tilde{s}_{22} = -d_2 d_3$ and $\tilde{s}_{22} = -1$ or $\tilde{s}_{22} = d_3^2$. We now examine these two subcases separately.

Case 1.1. $s_{22} = d_3^2$.

Applying the orthogonality of the first and the fourth columns of the S-matrix we find that $d_3 = \pm d_2$, we can apply the Verlinde formula and this relation to compute $N_{11}^3 = d_3 - \frac{1}{d_3}$ and so $d_3 = \frac{n \pm \sqrt{4+n^2}}{2}$ for some $n \in \mathbb{N}$. Examining the remaining N_{1j}^k we find that either $n = 0$ or $d_2 = d_3$. However, if $n = 0$, we have $d_a = \pm 1$ for all a. Since rank 4 pointed modular categories have been classified we may assume $d_2 = d_3$. Under this assumption the S-matrix takes the form

$$\tilde{S} = \begin{pmatrix} 1 & d_3^2 & d_3 & d_3 \\ d_3^2 & 1 & -d_3 & -d_3 \\ d_3 & -d_3 & d_3^2 & -1 \\ d_3 & -d_3 & -1 & d_3^2 \end{pmatrix}.$$

Applying the balancing relation — Equation (2.12), and the Verlinde formula, we find
$-1 = \tilde{s}_{23} = \frac{\left(n \pm \sqrt{4+n^2}\right)^2 \theta_1}{4\theta_2\theta_3}$. Taking the modulus of both sides and recalling that $|\theta_a| = 1$ gives the equation $4 = \left(n \pm \sqrt{4+n^2}\right)^2$, whose only solution over $\mathbb{N}$ is $n = 0$ and so we have that $\mathcal{C}$ is pointed.

Case 1.2. $\tilde{s}_{22} = -1$.

In this case, we apply the Verlinde formula to compute N_{11}^2 and N_{11}^3 which leads to

$$d_2 = \frac{1}{2}\left(n \pm \sqrt{4+n^2}\right) \quad \text{and} \quad d_3 = \frac{1}{2}\left(m \pm \sqrt{4+m^2}\right)$$

for some $m, n \in \mathbb{N}$. The balancing equation for $\tilde{s}_{23}$ gives that $\theta_1 = \theta_2\theta_3$ which then leads to

$$d_2 = \pm\sqrt{\frac{-1+\theta_2-\theta_2^2}{\theta_2}}, \quad d_3 = \pm\sqrt{\frac{-1+\theta_3-\theta_3^2}{\theta_3}}$$

by the balancing relation for $\tilde{s}_{22}$ and $\tilde{s}_{33}$. However, these results imply that θ_2 and θ_3 satisfy degree 4 integral polynomials and are roots of unity. Applying the inverse Euler (totient) phi function, we see that θ_2, θ_3 are $\pm i$ or primitive 5^{th} roots of unity and so $d_2, d_3 \in \{\pm 1, \pm\tau, \pm\overline{\tau}\}$ where τ

is the golden mean $\frac{1}{2}\left(1+\sqrt{5}\right)$ and $\overline{\tau}$ is its Galois conjugate. Simple computer search leads to 48 $\left(\tilde{S},T\right)$ combinations. Twelve of the S-matrices are distinct with half of them Galois conjugate to the other half. Of these remaining six, two can be removed by relabeling. Thus, we have the following four S-matrices and their Galois conjugates:

$$\begin{pmatrix} 1 & -1 & \overline{\tau} & \tau \\ -1 & 1 & -\tau & -\overline{\tau} \\ \overline{\tau} & -\tau & -1 & -1 \\ \tau & -\overline{\tau} & -1 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & -\tau & \tau \\ -1 & -1 & -\tau & -\tau \\ -\tau & -\tau & 1 & 1 \\ \tau & -\tau & 1 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & \tau & \tau \\ 1 & -1 & -\tau & \tau \\ \tau & -\tau & 1 & -1 \\ \tau & \tau & -1 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & \tau^2 & \tau & \tau \\ \tau^2 & 1 & -\tau & -\tau \\ \tau & -\tau & -1 & \tau^2 \\ \tau & -\tau & \tau^2 & -1 \end{pmatrix}.$$

The second matrix can be discarded since there is no rank 2 modular category with S-matrix $\left(\begin{smallmatrix} 1 & -1 \\ -1 & -1 \end{smallmatrix}\right)$.[5] The last two matrices are pseudo-unitary and hence appear in [23] which leaves only the first S-matrix which corresponds to $\mathrm{Fib}\boxtimes\overline{\mathrm{Fib}}$.

Case 2. $\epsilon_0 = -1$.

By resolving the labeling ambiguity present between the 2 and 3 labels we can take $\epsilon_3 = 1$. There are now two subcases:

Case 2.1. $|d_1| \geq 1$.

Following the procedure of [23], we find that $d_1 = \frac{1}{2}\left(n \pm \sqrt{n^2+4}\right)$ and $\exists a, b \in \mathbb{Q}$ and $r, s \in \mathbb{Z}$ such that

$$\begin{aligned} r &= 2b + an, & s &= bn - 2a, \\ d_2 &= ad_1 + b, & d_3 &= bd_1 - a, \\ D^2 &= \left(1+d_1^2\right)\left(1+a^2+b^2\right). \end{aligned}$$

Additionally, their techniques lead to $|d_1|^4 \leq 1 + 5|d_1| + 8|d_1|^2 + 5|d_1|^3$. Coupling these results with $|d_1| \geq 1$ gives that $1 \leq |d_1| \leq \psi$, where ψ is a root of $x^4 - 5x^3 - 8x^2 - 5x - 1$, and is approximately given by 6.38048. Thus $-7 < d_1 < 7$. We also find that

$$r^2 + s^2 \leq \left(n^2+4\right) \frac{4|d_1|^3 + 5|d_1|^2 + 4|d_1| + 1}{|d_1|^2\left(1+|d_1|^2\right)}. \tag{4.1}$$

Given a bound on d_1 we now have a bound on a sum of squares of integers and hence we can exhaust all possibilities. To do this we proceed in two subcases:

Case 2.1.1. $n > 0$.

The fact that $d_1 = \frac{1}{2}\left(n + \sqrt{n^2+4}\right)$ implies $1 \leq n \leq 6$ and we have the case considered in [23].

[5] To see this, note that $N_{11}^1 = 0$ by dimension count and the other fusion coefficients are determined by Equation (2.9). However, these fusion coefficients violate the Verlinde formula.

In particular, we may apply inequality (4.1), these bounds for n, and our formula for d_1 to produce a list of triples (n, r, s). Just as in [23] we may enforce integrality of d_2d_3/d_1, $d_3/d_2-d_2/d_3$, $\tilde{s}_{22}/d_2+\tilde{s}_{23}/d_3$, $\tilde{s}_{23}/d_2-\tilde{s}_{22}/d_3$, and $\tilde{s}_{22}\tilde{s}_{23}/\left(d_2d_3\right)$. This leads to 24 possible triples (n, r, s). The Verlinde formula provides enough integrality conditions to further reduce these 24 triples to 8. Of these 8, only $(n, r, s) = (1, -2, -1)$ or $(1, 2, 1)$ are compatible with the balancing equation and the twists being roots of unity. In these cases one finds, $d_1 = \tau$, $d_3 = \pm\tau$ and $d_2 = \pm 1$. However, these lead to relabelings of the S-matrices from Case 1.

Case 2.1.2. $n < 0$.

Proceeding as in Case 2.1.1, we find, by computer search, that there are 446 possible triples (n, r, s) of which only 24 pass the integrality tests of [23]. Applying the Verlinde formula to determine the fusion rules in these cases, we find that all of these either violate the integrality or nonnegativity of the fusion coefficients.

Case 2.2. $|d_1| < 1$.

Applying our Galois element, we see that $\sigma\left(d_1\right) = -\frac{1}{d_1}$. Setting $\delta_a = \sigma\left(d_a\right)$, we find a category $\hat{\mathcal{C}}$, which is Galois conjugate to $\mathcal{C}$; whence if $\hat{\mathcal{C}}$ does not exist, then neither does $\mathcal{C}$. However, $|\delta_1| > 1$ and, since Galois conjugation preserves all categorical identities used in Case 2.1, we see that we must have $\delta_3 = \delta_2\delta_1$, $\delta_2 = \pm 1$ and $\delta_1 = \tau$. However, this is the same conclusion as in Case 2.1.1. Ergo, $\mathcal{C}$ must be Galois conjugate to one of the Case 2.1.1 results. Since these were conjugate to the categories determined in [23], we can conclude that $\mathcal{C}$ has an S-matrix Galois conjugate to one appearing in Case 1. □

Having dispensed with the symmetric and modular cases, we find that it is useful to stratify the properly premodular categories by self-duality and symmetric subcategory. It is known that that every properly premodular category has a symmetric subcategory [14]. Since the rank has been fixed the possible symmetric subcategories can be completely determined.

Proposition 4.3. *If $\mathcal{C}$ is a rank 4 nonpointed properly premodular category, then there are four cases:*

(1) *$\mathcal{C}$ is self-dual and has a symmetric subcategory Grothendieck equivalent to* $\operatorname{Rep}\left(\mathfrak{S}_3\right)$.
(2) *$X_1^* = X_2$ and generate a symmetric subcategory of $\mathcal{C}$ Grothendieck equivalent to* $\operatorname{Rep}\left(\mathbb{Z}/3\mathbb{Z}\right)$.
(3) *$\mathcal{C}$ is self-dual and has a symmetric subcategory Grothendieck equivalent to* $\operatorname{Rep}\left(\mathbb{Z}/2\mathbb{Z}\right)$.
(4) *$\mathbb{I}$ and X_1 generate a symmetric subcategory of $\mathcal{C}$ Grothendieck equivalent to* $\operatorname{Rep}\left(\mathbb{Z}/2\mathbb{Z}\right)$. *Moreover, $X_2^* = X_3$.*

In each case, the symmetric subcategory is the Müger center.

Proof. We know from [14] Corollary 2.16 and comments in the introduction,[6] that since $\mathcal{C}$ is nonsymmetric and nonmodular, then it must have a nontrivial symmetric subcategory of rank 2 or 3. Rank 3 symmetric subcategories are known to be Grothendieck equivalent to $\operatorname{Rep}(\mathbb{Z}/3\mathbb{Z})$ or $\operatorname{Rep}(\mathfrak{S}_3)$ [20]. Rank 2 proceeds similarly and leads to $\operatorname{Rep}(\mathbb{Z}/2\mathbb{Z})$.

In the rank 3 case, we take $X_0 = \mathbb{I}$, X_1, and X_2 to be representatives of distinct simple isomorphism classes that generate the symmetric subcategory, while, in rank 2, we take $X_0 = \mathbb{I}$ and X_1 to be the representative generators. The result then follows immediately by standard representation theory. □

Classification of the properly premodular categories now proceeds by cases. The categories with high rank symmetric subcategories are, perhaps not surprisingly, easier to deal with since more of the datum is predetermined. As such, we will proceed through $\operatorname{Rep}(\mathfrak{S}_3)$ and $\operatorname{Rep}(\mathbb{Z}/3\mathbb{Z})$ first and then discuss the $\operatorname{Rep}(\mathbb{Z}/2\mathbb{Z})$ cases.

Proposition 4.4. *There is no rank* 4 *nonpointed properly premodular category with* $\mathcal{C}'$ *Grothendieck equivalent to* $\operatorname{Rep}(\mathfrak{S}_3)$.

Proof. Applying the known representation theory of $\mathfrak{S}_3$, Equation (2.9) and dimension counts, we find

$$N_1 = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \quad N_2 = \begin{pmatrix} 0&0&1&0\\ 0&0&1&0\\ 1&1&1&0\\ 0&0&0&2 \end{pmatrix} \quad N_3 = \begin{pmatrix} 0&0&0&1\\ 0&0&0&1\\ 0&0&0&2\\ 1&1&2&M \end{pmatrix}.$$

Recall that $\tilde{s}_{ab} = d_a d_b$ for $0 \le a, b \le 2$ by [14, Proposition 2.5]. Coupling this with Equation (2.12), we find $\theta_1 = \theta_2 = 1$. Denoting θ_3 by θ, this gives

$$\tilde{S} = \begin{pmatrix} 1 & 1 & 2 & \frac{M\pm\sqrt{24+M^2}}{2} \\ 1 & 1 & 2 & \frac{M\pm\sqrt{24+M^2}}{2} \\ 2 & 2 & 4 & M\pm\sqrt{24+M^2} \\ \frac{M\pm\sqrt{24+M^2}}{2} & \frac{M\pm\sqrt{24+M^2}}{2} & M\pm\sqrt{24+M^2} & \frac{12+\left(M\pm\sqrt{24+M^2}\right)M\theta}{2\theta^2} \end{pmatrix}.$$

Since $\frac{\tilde{s}_{33}}{\tilde{s}_{03}}$ must satisfy the characteristic polynomial of N_3, we can deduce that θ must be a primitive root of unity satisfying a degree integral 3 polynomial. Employing the inverse of Euler's totient function, we find that $\theta = \pm 1$ and $M = 0$. Thus $d = \pm\sqrt{6}$. Having removed the free parameters from this datum, we are in a position to prove that such a category cannot exist. In this case the Müger center, $\operatorname{Rep}(\mathfrak{S}_3)$, constitutes a Tannakian subcategory of $\mathcal{C}$. By [15] and [7, Remark 5.10], we can form the de-equivariantization, $\mathcal{C}_{\mathfrak{S}_3}$, which is a braided $\mathfrak{S}_3$-crossed fusion category. However, $\operatorname{FPdim}(\mathcal{C}_{\mathfrak{S}_3}) = \frac{1}{6}\operatorname{FPdim}(\mathcal{C})$, $\dim(\mathcal{C}_{\mathfrak{S}_3}) = \frac{1}{6}\dim(\mathcal{C}) = 2$, and $\operatorname{FPdim}(\mathcal{C}_{\mathfrak{S}_3}) = 2$ [7]. Thus $\mathcal{C}_{\mathfrak{S}_3}$ is weakly integral braided $\mathfrak{S}_3$-crossed fusion

[6] $\mathcal{C}' = \mathcal{Z}_2(\mathcal{C})$ is a canonical full symmetric subcategory of $\mathcal{C}$.

category and we may apply [8, Corollary 8.30] to deduce that $\mathcal{C}_{\mathfrak{S}_3}$ is equivalent to $\operatorname{Rep}(\mathbb{Z}/2\mathbb{Z})$ and hence pointed. Consequently, $\mathcal{C}$ is group-theoretical and in particular integral, contradicting $d = \pm\sqrt{6}$ [15, 7]. □

Proposition 4.5. *If $\mathcal{C}$ is a nonpointed properly premodular category such that $\langle X_0, X_1, X_2\rangle = \mathcal{C}'$ is Grothendieck equivalent to* $\operatorname{Rep}(\mathbb{Z}/3\mathbb{Z})$*, then:*

$$\tilde{S} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \\ 3 & 3 & 3 & -3 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$N_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad N_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad N_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix},$$

and $\mathcal{C}$ is realized by $\mathcal{C}(\mathfrak{sl}(2), 6)_{ad}$.

Proof. Applying Proposition 4.3, we know that $\mathcal{C}$ is self-dual and so applying the representation theory of $\mathbb{Z}/3\mathbb{Z}$ and Equation (2.9), we find that the fusion matrices are determined up to N_{33}^3. Making use of Equation (2.12), the fact that $\tilde{S} = \tilde{S}^T$, and the fact that in a properly premodular category some column of $\tilde{S}$ is a multiple of the first, one finds that

$$\tilde{S} = \begin{pmatrix} 1 & 1 & 1 & d_3 \\ 1 & 1 & 1 & d_3 \\ 1 & 1 & 1 & d_3 \\ d_3 & d_3 & d_3 & \frac{3+d_3 N_{33}^3\theta_3}{\theta_3^2} \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \theta_3 \end{pmatrix}.$$

By dimension count, we see that $d_3 = \frac{1}{2}\left(N_{33}^3 \pm \sqrt{12 + N_{33}^3}\right)$. So it remains to determine N_{33}^3 and θ_3. For notational brevity, we let $M = N_{33}^3$. Applying Equation (2.13) we find that

$$(4.2)\quad (\theta_3 - 1)\left(18\theta_3\left(\theta_3^2 + \theta_3 + 1\right) + \theta_3^2 M^4 + 3\theta_3(\theta_3+1)(\theta_3+2)M^2 + 18\right)$$
$$= \pm(\theta_3 - 1)\left(3\theta_3\left(\theta_3^2 + \theta_3 + 2\right)\sqrt{M^2+12}M + \theta_3^2\sqrt{M^2+12}M^3\right).$$

We first note that if $\theta_3 = 1$, then $\mathcal{C} = \mathcal{C}'$ contradicting the nonsymmetric assumption. Thus, θ_3 satisfies a degree 6 integral polynomial. However, θ_3 is a root of unity, so applying the inverse Euler phi function to determine a list of potential values for θ_3. Combing the possible cases, one finds $N_{33}^3 \in \{0, 2\}$ and $\theta_3 \in \{\pm i, -1\}$. Applying Corollary 3.3 with $a = 3$, we find that only $N_{33}^3 = 2$ gives a rational integer. Evaluating Equation (4.2) at $N_{33}^3 = 2$ reveals that $\theta = -1$ is the only solution.[7] □

Having dispensed with the large symmetric subcategories, we need to consider the case that $\operatorname{Rep}(\mathbb{Z}/2\mathbb{Z})$ appears as a symmetric subcategory. We first consider the non-self-dual case which can be dealt with by cyclotomic/number theoretic techniques.

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[7] If one proceeds without appealing to the Frobenius–Schur indicators then the Tambara-Yamagami with dimensions $1, 1, 1, \sqrt{3}$ appear. This can of course be excluded since such categories do not admit a braiding [24].

Proposition 4.6. *There is no rank 4 nonpointed properly premodular category such that* $\langle X_0, X_1\rangle = \mathcal{C}'$ *is Grothendieck equivalent to* $\mathrm{Rep}\,(\mathbb{Z}/2\mathbb{Z})$*, and* $X_2^* = X_3$.

Proof. Given the standard representation theory of $\mathbb{Z}/2\mathbb{Z}$ and the equation (2.9), we immediately obtain:

$$N_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & N_{32}^1 & N_{33}^1 \\ 0 & 0 & N_{33}^1 & N_{32}^1 \end{pmatrix} \quad N_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & N_{32}^1 & N_{33}^1 \\ 0 & N_{33}^1 & N_{33}^3 & N_{33}^2 \\ 1 & N_{32}^1 & N_{33}^3 & N_{33}^3 \end{pmatrix} \quad N_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & N_{33}^1 & N_{32}^1 \\ 1 & N_{32}^1 & N_{33}^3 & N_{33}^3 \\ 0 & N_{33}^1 & N_{33}^2 & N_{33}^3 \end{pmatrix}.$$

Demanding that the fusion matrices mutually commute reveals that either N_{32}^1 or N_{33}^1 is 0 and the other is 1. Hence, the proof bifurcates into two cases.

Case 1. $N_{32}^1 = 1$ *and* $N_{33}^1 = 0$.

Returning to the commutativity of the fusion matrices, we are reduced to one equation:

$$2 = \left(N_{33}^2\right)^2 - \left(N_{33}^3\right)^2 = \left(N_{33}^2 - N_{33}^3\right)\left(N_{33}^2 + N_{33}^3\right).$$

Of course the fusion coefficients are nonnegative integers and so

$$N_{33}^2 - N_{33}^3 = 1 \quad \text{and} \quad N_{33}^2 + N_{33}^3 = 2.$$

Of course this system has no solution in $\mathbb{Z}$.

Case 2. $N_{32}^1 = 0$ *and* $N_{33}^1 = 1$.

In this case the commutativity of the fusion matrices reveals that $N_{33}^2 = N_{33}^3$, which we will simply call M for brevity. Applying Equation (2.12), and dimension count, we can determine the S-matrix to be

$$\tilde{S} = \begin{pmatrix} 1 & M\pm\sqrt{1+M^2} \\ M\pm\sqrt{1+M^2} & \frac{1+2\left(M\pm\sqrt{1+M^2}\right)M\theta}{\theta^2} \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Where $\theta := \theta_2 = \theta_3$ and $\theta_1 = 1$, which follows from the fact that some column of the S-matrix must be a multiple of the first [3]. However, $\frac{\tilde{s}_{22}}{\tilde{s}_{02}}$ must satisfy the characteristic polynomial of N_2, which factors into two quadratics. Inserting this quotient into the factors, we find that θ must satisfy either a degree 4 or degree 8 polynomial over $\mathbb{Z}$. Since θ is a primitive root of unity we can apply the inverse Euler phi function to bound the degree of the minimal polynomial of θ. Proceeding through all cases, we find that $M = 0$ and $\mathcal{C}$ is pointed. □

While this cyclotomic analysis has been quite fruitful, the remaining, properly premodular case proves to be resistant and so other approaches are necessary. We begin by recalling that every fusion category admits a (possibly trivial) grading. Since the category has small rank, the grading possibilities allow for further stratification of the problem.

Proposition 4.7. *If $\mathcal{C}$ is a self-dual rank 4 nonpointed properly premodular category $\langle X_0, X_1\rangle = \mathcal{C}'$ is Grothendieck equivalent to* $\operatorname{Rep}(\mathbb{Z}/2\mathbb{Z})$*, then there are three cases:*

(1) *$\mathcal{C}$ admits a universal $\mathbb{Z}/2\mathbb{Z}$ grading.*
(2) *$\mathcal{C}$ does not admit a universal $\mathbb{Z}/2\mathbb{Z}$ grading and $X_1 \otimes X_2 = X_2$.*
(3) *$\mathcal{C}$ does not admit a universal $\mathbb{Z}/2\mathbb{Z}$ grading and $X_1 \otimes X_2 = X_3$.*

Proof. If $\mathcal{C}$ admits a nontrivial universal grading, then it must be by $\mathbb{Z}/2\mathbb{Z}$. On the other hand, if $\mathcal{C}$ does not admit a universal grading, then $\mathcal{C}_{ad} = \mathcal{C}$ [7]. Since X_1 generates $\mathcal{C}' \cong \operatorname{Rep}(\mathbb{Z}/2\mathbb{Z})$, we can conclude that if $\mathcal{C}_{ad} = \mathcal{C}$ then either $X_1 \otimes X_2 = X_2$ or $X_1 \otimes X_2 = X_3$. □

With this proposition in hand we again proceed by cases. First, we consider with the relatively simple case: $\mathcal{C}$ admits a universal $\mathbb{Z}/2\mathbb{Z}$ grading.

Proposition 4.8. *Suppose $\mathcal{C}$ is a self-dual rank 4 nonpointed properly premodular category admitting a universal $\mathbb{Z}_2$ grading such that $\mathcal{C}'$ is Grothendieck equivalent to* $\operatorname{Rep}(\mathbb{Z}_2)$*, then $\mathcal{C}$ is a Deligne product of the* Fib *with* $\operatorname{Rep}(\mathbb{Z}/2\mathbb{Z})$ *or* sVec*.*

Proof. Dimension count coupled with the representation theory of $\mathbb{Z}/2\mathbb{Z}$ completely determines the fusion relations up to N_{22}^2. However, we can apply [20] to conclude that $N_{22}^2 \in \{0,1\}$. $N_{22}^2 = 0$ leaves a pointed category and so we must have $N_{22}^2 = 1$, and $d := d_2 = d_3 = \frac{1\pm\sqrt{5}}{2}$. Applying Equation (2.12) and the fact that a column of the S-matrix must be a multiple of the first we find that $\theta_1 = \pm 1$, $\theta := \theta_2 = \theta_1\theta_3$, and

$$\tilde{S} = \begin{pmatrix} 1 & 1 & d & d \\ 1 & 1 & d & d \\ d & d & \frac{1+d\theta}{\theta^2} & \frac{1+d\theta}{\theta^2} \\ d & d & \frac{1+d\theta}{\theta^2} & \frac{1+d\theta}{\theta^2} \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \theta & 0 \\ 0 & 0 & 0 & \pm\theta \end{pmatrix}.$$

Since the normalized columns of the S-matrix are characters of the fusion ring, it must be that $\frac{1+d\theta}{d\theta^2}$ is a simultaneous root of the characteristic polynomials of N_2 and N_3. This gives the desired result. □

Finally, we come to the last two cases where $\mathcal{C}'$ is Grothendieck equivalent to $\operatorname{Rep}(\mathbb{Z}_2)$ and the universal grading group is trivial. These are by far the most complicated cases. To dispense with the first case we make use of the minimal modularization [3].

Proposition 4.9. *Suppose $\mathcal{C}$ is a self-dual, rank 4, nonpointed, properly premodular category such that $\mathcal{C}'$ is Grothendieck equivalent to* $\operatorname{Rep}(\mathbb{Z}/2\mathbb{Z})$*, $\mathcal{C}$ does not admit a nontrivial universal grading, and $X_1 \otimes X_2 = X_2$, then*

$$\tilde{S} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & \frac{2+2\theta}{\theta^3} & \frac{2+2\theta^2}{\theta} \\ 2 & 2 & \frac{2+2\theta^2}{\theta} & \frac{2+2\theta}{\theta^3} \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \theta & 0 \\ 0 & 0 & 0 & \theta^{-1} \end{pmatrix}$$

$$N_1 = \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \qquad N_2 = \begin{pmatrix} 0&0&1&0\\0&0&1&0\\1&1&0&1\\0&0&1&1 \end{pmatrix} \qquad N_3 = \begin{pmatrix} 0&0&0&1\\0&0&0&1\\0&0&1&1\\1&1&1&0 \end{pmatrix},$$

and θ is a primitive 5^{th} root of unity. Such datum are realized by $\mathcal{C} = \mathcal{C}(\mathfrak{so}(5), 10)_{ad}$.

Proof. The representation theory of $\mathbb{Z}/2\mathbb{Z}$, dimension count, Equation (2.9), and Equation (2.12) give

$$N_1 = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} \quad N_2 = \begin{pmatrix} 0&0&1&0 \\ 0&0&1&0 \\ 1&1&N_{22}^2&N_{32}^2 \\ 0&0&N_{32}^2&N_{33}^2 \end{pmatrix} \quad N_3 = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&1 \\ 0&0&N_{32}^2&N_{33}^2 \\ 1&1&N_{33}^2&N_{33}^3 \end{pmatrix}$$

$$\tilde{S} = \begin{pmatrix} 1&1&d_2&d_3 \\ 1&1&d_2&d_3 \\ d_2&d_2&\frac{2+d_2N_{22}^2\theta_2+d_3N_{32}^2\theta_3}{\theta_2^2}&\frac{d_2N_{32}^2\theta_2+d_3N_{33}^2\theta_3}{\theta_2\theta_3} \\ d_3&d_3&\frac{d_2N_{32}^2\theta_2+d_3N_{33}^2\theta_3}{\theta_2\theta_3}&\frac{2+d_2N_{33}^2\theta_2+d_3N_{33}^3\theta_3}{\theta_3^2} \end{pmatrix} \quad T = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&\theta_2&0 \\ 0&0&0&\theta_3 \end{pmatrix}.$$

Applying [3, Proposition 4.2], we can deduce that $\mathcal{C}$ admits a modularization $\hat{\mathcal{C}}$. We can now apply [3] Proposition 4.4 and the equivalence between Bruguières modularization and the de-equivariantization to deduce that $\hat{\mathcal{C}}$ is a rank 5 modular category with simple isomorphism classes of simple objects $\mathbb{I}, Y_1, Y_2, Z_1, Z_2$ such that $Y_i^* \in \{Y_1, Y_2\}$ and $Z_i^* \in \{Z_1, Z_2\}$. Applying the classification of rank 5 modular categories, [5], we can conclude that $\hat{\mathcal{C}}$ is pointed and hence $d_2 = \pm 2$ and $d_3 = \pm 2$. Dimension count then allows us to eliminate all fusion coefficients except for N_{33}^3. Applying [16, Theorem 4.2], we know that $\mathcal{C}$ is Grothendieck equivalent to $\operatorname{Rep}(D_n)$ and is group-theoretical. This gives $d_2 = d_3 = 2$, and determines the fusion coefficients. Applying Equations (2.13) and (2.14), we find $\theta_3 = \theta_2^{-1}$ and that θ_2 is a primitive 5th root of unity. □

The final case requires not only the minimal modularization of Bruguières but also the second Frobenius–Schur indicators.

Proposition 4.10. *Suppose $\mathcal{C}$ is a self-dual rank 4 nonpointed properly premodular category such that $\mathcal{C}'$ is Grothendieck equivalent to* $\operatorname{Rep}(\mathbb{Z}/2\mathbb{Z})$, *$\mathcal{C}$ does not admit a nontrivial universal grading, and $X_1 \otimes X_2 = X_3$ then*

$$\tilde{S} = \begin{pmatrix} 1 & 1\pm\sqrt{2} \\ 1\pm\sqrt{2} & -1 \end{pmatrix} \otimes \begin{pmatrix} 1&1 \\ 1&1 \end{pmatrix} \qquad T = \begin{pmatrix} 1&0&0&0 \\ 0&-1&0&0 \\ 0&0&i&0 \\ 0&0&0&-i \end{pmatrix}$$

$$N_2 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&1&1 \\ 0&1&1&1 \end{pmatrix} \qquad N_3 = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&1&1 \\ 1&0&1&1 \end{pmatrix},$$

such a datum is realized by $\mathcal{C} = \mathcal{C}(\mathfrak{sl}(2), 8)_{ad}$ and its conjugates.

Proof. Applying dimension count, Equation (2.9), and the usual representation theory for $\mathbb{Z}/2\mathbb{Z}$, we can determine the fusion rules up to two parameters:

$$N_1 = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix} \quad N_2 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&N&M \\ 0&1&M&N \end{pmatrix} \quad N_3 = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&M&N \\ 1&0&N&M \end{pmatrix}.$$

Furthermore, we can deduce that $M, N \neq 0$ lest we reduce to the fusion rules of Proposition 4.8 or a pointed category. Next, we may use Equation (2.12), dimension count, and that $\tilde{s}_{ij} = \lambda \tilde{s_{i0}}$ for some j and some $\lambda \in \mathbb{C}^{\times}$, to find the S- and T-matrices:

$$\tilde{S} = \begin{pmatrix} 1 & \frac{N+M+\epsilon\sqrt{4+(M+N)^2}}{2} \\ \frac{N+M+\epsilon\sqrt{4+(M+N)^2}}{2} & \frac{2+(N\theta+\delta M\theta)\left(N+M+\epsilon\sqrt{4+(M+N)^2}\right)}{2\theta^2} \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \theta & 0 \\ 0 & 0 & 0 & \delta\theta \end{pmatrix},$$

where $\epsilon, \delta = \pm 1$. We treat $\delta = 1$ and $\delta = -1$ in separate cases.

Case 1. $\delta = 1$.

Here we can apply [3] to deduce that $\mathcal{C}$ is modularizable. Letting

$$H : \mathcal{C} \to \hat{\mathcal{C}}$$

denote its minimal modularization then we have $X_2 \in M_{\mathcal{C}} X_3$ and so

$$H(X_2) \cong H(X_3).$$

Furthermore, $\|Stab_{M_{\mathcal{C}}} X\| = 1$ for all simple X and thus,

$$\dim H(X_2) = \dim(X_2).$$

Consequently, the trivial object in $\hat{\mathcal{C}}$ as well as $H(X_2)$ account for $1 + d^2$ of the dimension of $\hat{\mathcal{C}}$. However, $\dim \hat{\mathcal{C}} = \frac{1}{2}\dim(\mathcal{C}) = 1 + d^2$ and so $\hat{\mathcal{C}}$ is a rank 2 modular category with simple objects $\mathbb{I}$ and $H(X_2)$. Such categories have been classified in [23] and are the Semion and the Fibonacci. In these situations, we find either that $\mathcal{C}$ is pointed or that $M = N = 0$ and so we can exclude the case of $\delta = 1$.

Case 2. $\delta = -1$.

A straightforward application of (2.13) and (2.14) in a computer algebra system is used to further reduces the solution space. Discarding any solutions where either M or N is 0 or $\mathcal{C}$ is symmetric leaves 7 possible families of solutions. One of these families contains a Pythagorean triple with 1 which forces $N < 0$ and hence can be discarded. Two of the other families of solutions have M and N related by

$$M = \frac{-N\theta^2 \pm \sqrt{-\theta\left(1+\theta^2\right)^2\left(1-\left(1+N^2\right)\theta+\theta^2\right)}}{\theta\left(1+\theta\left(\theta-1\right)\right)}.$$

Since $\theta \neq 0$, this can be arranged into a monic integral degree 6 polynomial θ. Since θ is a root of unity we can apply the inverse Euler phi function to find a possible list of values for θ. Direct calculation reveals that none of these roots of unity can satisfy this polynomial in a manner consistent with $M, N > 0$.

The remaining four families can be reduced by resolving a labeling ambiguity to give

$$\tilde{S} = \begin{pmatrix} 1 & N+\epsilon\sqrt{1+N^2} \\ N+\epsilon\sqrt{1+N^2} & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$N_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & N & N \\ 0 & 1 & N & N \end{pmatrix} \qquad N_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & N & N \\ 1 & 0 & N & N \end{pmatrix}.$$

Applying Corollary 3.3 to X_2, we find that $N \pm \frac{N^2-1}{\sqrt{N^2+1}} \in \mathbb{Z}$. Denoting this integer by L and simplifying we find

$$4 = \left(N^2+1\right)\left(3+L^2-2LN\right)$$

However, $N^2+1 \neq 0$ and so, reducing modulo N^2+1, we find that $4 \equiv 0 \mod N^2+1$.

This only occurs for $N \in \{-1, 0, 1\}$. Since $N = 0$ leads to $\mathcal{C}$ being pointed and we know $N \geq 0$, we can conclude that $N = 1$. □

The results of this section can be compiled to give the following theorem.

Theorem 4.11. *If $\mathcal{C}$ is a nonpointed rank 4 premodular category, then exactly one of the following is true.*

(1) *$\mathcal{C}$ is symmetric and is Grothendieck equivalent to* $\mathrm{Rep}(G)$ *where G is $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, D_{10}, or $\mathfrak{A}_4$.*

(2) *$\mathcal{C}$ is properly premodular and is Grothendieck equivalent to a Galois conjugate of one of the following: $\mathcal{C}(sl(2), 8)_{ad}$, $\mathcal{C}(sl(2), 6)_{ad}$, $\mathcal{C}(so(5), 10)_{ad}$,* $\mathrm{Fib} \boxtimes \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$, *or* $\mathrm{Fib} \boxtimes \mathrm{sVec}$.

(3) *$\mathcal{C}$ is modular and is Galois conjugate to a modular category from* [23] *or has S-matrix*

$$\begin{pmatrix} 1 & -1 & \overline{\tau} & \tau \\ -1 & 1 & -\tau & -\overline{\tau} \\ \overline{\tau} & -\tau & -1 & -1 \\ \tau & -\overline{\tau} & -1 & -1 \end{pmatrix},$$

where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden mean and $\overline{\tau} = \frac{1-\sqrt{5}}{2}$ is its Galois conjugate.

Appendix A. Premodular rank finiteness

By: Paul Bruillard, César Galindo, Siu-Hung Ng, Julia Plavnik, Eric Rowell, and Zhenghan Wang.

In this appendix we will consider the rank finiteness problem for premodular categories. First we will review the rank finiteness result for modularizable premodular categories presented in [4]. The remaining case will be reduced to the situation in which the premodular category is slightly degenerate at which point the general rank finiteness result will be produced by an inductive argument.

First we recall from [4]:

Lemma A.1. *There are finitely many modularizable premodular categories of rank r, up to equivalence.*

By this lemma it suffices to consider only premodular categories which are not modularizable. Such categories are characterized by the presence of sVec.

Lemma A.2. *If $\mathcal{C}$ is a premodular category and* $\mathrm{Rep}\,(G)$ *is a maximal Tannakian subcategory in $\mathcal{C}$', then $\mathcal{C}$ is nonmodularizable if and only if* $(\mathcal{C}_G)' \cong \mathrm{sVec}$

Proof. First we will show sufficiency. To this end we assume that $\mathcal{C}$ is not modularizable. Now let $\mathcal{E} \cong \mathrm{Rep}\,(G)$ be the maximal Tannakian subcategory of $\mathcal{C}'$. Then by [7] we have $\mathcal{C}_G = \mathcal{C} \boxtimes_{\mathcal{E}} \mathrm{Vec}$. Applying [7, Proposition 4.30(iii)] we see that $(\mathcal{C} \boxtimes_{\mathcal{E}} \mathrm{Vec})' = \mathcal{C}' \boxtimes_{\mathcal{E}} \mathrm{Vec} = (\mathcal{C}')_G$. Referring to section 5 of [7] we see that $(\mathcal{C}')_G$ is the core of of $\mathcal{C}'$ corresponding to $\mathcal{E}$ because Müger center of $\mathcal{E}$ in $\mathcal{C}'$ is $\mathcal{C}'$. We further observe that $(\mathcal{C}')_G$ is weakly anisotropic and hence is either Vec or sVec. Since $\mathcal{C}$ is not modularizable we can conclude that $\mathcal{C}_G$ is not modular and hence $(\mathcal{C}')_G \cong \mathrm{sVec}$.

To show necessity we note that if $\mathcal{C}$ is modularizable, then $\mathcal{C}' \cong \mathrm{Rep}\,(G)$ and $(\mathcal{C}_G)' \cong \mathrm{Vec} \ncong \mathrm{sVec}$. □

With this characterization in hand, we are positioned to show rank finiteness for nonmodularizable premodular categories.

Lemma A.3. *There are finitely many nonmodularizable premodular categories of rank at most r up to equivalence.*

Proof. The proof proceeds by induction on the rank. The base case is handled by the low rank classification of premodular categories. So we assume that: *there are finitely many nonmodularizable premodular categories of rank at most $r-1$.*

Suppose $\mathcal{C}$ is a nonmodularizable premodular category of rank r. Then, by Lemma A.2 and the induction hypothesis, we may assume, without loss of generality, that $\mathcal{C}' \cong \mathrm{sVec}$.

In this case we know that if $\chi \in \mathrm{sVec} \subset \mathcal{C}$, then $\chi \notin \mathcal{C}_{ad}$ by [9]. Thus $\mathcal{C}_{ad}$ is a premodular subcategory of $\mathcal{C}$ of strictly smaller rank. Now note that if $\mathcal{C}_{ad}$ is modularizable, then it has bounded global dimension by [4]. On the other hand, if $\mathcal{C}_{ad}$ is nonmodularizable, then it has bounded dimension by the induction hypothesis.

Next note, by [7, Proposition 2.3] that $\mathcal{C}$ is faithfully graded by the universal grading group and the trivial component under this grading is $\mathcal{C}_{ad}$. Since the universal grading group has bounded order and the grading is faithful, we can conclude that $\mathcal{C}$ has bounded dimension. Finiteness then follows by standard arguments, e.g., [4, 8]. □

Lemmas A.1 and A.3 can be collected into the following theorem.

Theorem A.4. *There are finitely many premodular categories of rank r, up to equivalence.*

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