Quillen adjunctions induce adjunctions of quasicategories

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ABSTRACT. We prove that a Quillen adjunction of model categories (of which we do not require functorial factorizations and of which we only require finite bicompleteness) induces a canonical adjunction of underlying quasicategories.

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0. Introduction

Background and motivation. Broadly speaking, the methods of abstract homotopy theory can be divided into two types: those that work *internally* to a given homotopy theory, and those that work *externally* with all homotopy theories at once. By far the most prominent method of the first type is the theory of model categories, introduced by Quillen in his seminal work [Qui67]. On the other hand, there are now a plethora of models for "the homotopy theory of homotopy theories", all of them equivalent in an essentially unique manner (reviewed briefly in §1); for the moment, we will refer to such objects collectively as "∞-categories".

However, there is some apparent overlap between these two situations: model categories do not exist in isolation, but can be related by Quillen adjunctions and Quillen equivalences. We are thus led to a natural question.

Question 0.1. What ∞-categorical phenomena do Quillen adjunctions and Quillen equivalences encode?

Of course, one expects that Quillen adjunctions induce "adjunctions of ∞-categories" and that Quillen equivalences induce "equivalences of ∞-categories". However, it turns out that actually making these statements precise is a subtle task. On the other hand, it is made easier by imposing various additional assumptions or by settling for more modest conclusions, and hence there already exist an assortment of partial results in this direction. We defer a full history to §A; the state of affairs can be summarized as follows.

- Quillen equivalences are known to induce weak equivalences of $s\mathcal{S}$et-enriched categories (where $s\mathcal{S}$et denotes the category of simplicial sets, and by "weak equivalence" we mean in the Bergner model structure).
- Quillen adjunctions are known to induce adjunctions of homotopy categories, and are more-or-less known to induce adjunctions of $\mathrm{ho}(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})$-enriched homotopy categories (where $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ denotes the category of simplicial sets equipped with the standard Kan–Quillen model structure).
- Quillen adjunctions between model categories that admit suitable co/fibrant replacement functors are more-or-less known to induce adjunctions of quasicategories.
- Simplicial Quillen adjunctions between simplicial model categories are known to induce adjunctions of quasicategories, and moreover certain Quillen adjunctions can be replaced by simplicial Quillen adjunctions of simplicial model categories.

Thus, in order to fully unify the internal and external approaches to abstract homotopy theory, it remains to show that an arbitrary Quillen adjunction induces an adjunction of ∞-categories. The purpose of the present paper is to prove this assertion when we take the term "∞-category" to mean "quasicategory".[1]

Since model categories have figured so foundationally into much of the development of axiomatic homotopy theory, it seems that Quillen adjunctions are generally viewed as such basic and fundamental objects that they hardly merit further interrogation. However, inasmuch as there is a far deeper understanding today of "the homotopy theory of homotopy theories" than existed in 1967, we consider it to be a worthwhile endeavor to settle this matter once and for all.

Remark 0.2. An adjunction of $s\mathcal{S}$et-enriched categories induces an adjunction of quasicategories (see [Lur09, Corollary 5.2.4.5]), but the converse is presumably false: an adjunction of $s\mathcal{S}$et-enriched categories is by its very nature extremely rigid — making reference to simplicial sets up to isomorphism, with no mention of their ambient model structure —, whereas an

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[1]This result has been asserted in the literature, but the argument given there falls short of proving the actual claim; see Remark 2.3.

adjunction of quasicategories is a much more flexible notion and incorporates a wealth of homotopy coherence data. (Of course, both of these notions are strictly stronger than that of an adjunction of $\mathrm{ho}(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})$-enriched categories.)

In fact, the datum of "an adjunction of quasicategories" only specifies the actual adjoint functors themselves up to contractible spaces of choices.[2] This situation may appear somewhat abstruse to those not familiar with the theory of quasicategories, but in our view, quasicategories were never really meant to be worked with at the simplex-by-simplex level anyways: they function best when manipulated via (quasicategorical) universal properties, the praxis of which is actually quite similar to that of 1-categories in many ways. So, all in all, we view this situation primarily as a reaffirmation of the philosophy of quasicategories: that it's too much to demand strict composition in the first place, and that working with rigid models can obscure the essential features of the true and underlying mathematics.

1. Notation and conventions

1.1. Specific categories. As we have already indicated, we write $s\mathcal{S}\mathrm{et}$ for the category of simplicial sets. Of course, this is because we write $\mathcal{S}\mathrm{et}$ for the category of sets and we write $c(-)$ and $s(-)$ for categories of co/simplicial objects; hence, we will write $ss\mathcal{S}\mathrm{et}$ for the category of bisimplicial sets. We also write $\mathcal{C}\mathrm{at}$ for the category of categories, $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$ for the category of relative categories, and $\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$ for the category of $s\mathcal{S}\mathrm{et}$-enriched categories. We will write $\mathrm{N} : \mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}$ for the usual nerve functor.

We will consider categories as special instances of both relative categories and $s\mathcal{S}\mathrm{et}$-enriched categories: for the former we consider $\mathcal{C}\mathrm{at} \subset \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$ by endowing our categories with the *minimal* relative structure (in which only the identity maps are marked as weak equivalences), and for the latter we consider $\mathcal{C}\mathrm{at} \subset \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$ via the inclusion $\mathcal{S}\mathrm{et} \subset s\mathcal{S}\mathrm{et}$ of sets as discrete simplicial sets.

1.2. Specific model categories. As we have already indicated, we will model "spaces" using the standard *Kan–Quillen model structure* $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$, while to model "the homotopy theory of homotopy theories", we will make use of all four of

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[2]We refer the reader to [Lur09, §5.2] for a thorough exposition of the theory of adjunctions of quasicategories.

- the *Rezk model structure* (a/k/a the "complete Segal space" model structure) $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ of [Rez01, Theorem 7.2],
- the *Barwick–Kan model structure* $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\mathrm{BK}}$ of [BarK12b, Theorem 6.1],
- the *Bergner model structure* $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ of [Ber07, Theorem 1.1], and
- the *Joyal model structure* $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ of [Lur09, Theorem 2.2.5.1].

As explained in [BarSP], these are all equivalent in an essentially unique way, though the meanings of the phrases "equivalent" and "essentially unique" here are both slightly subtle.

We will use the following equivalences between these models for "the homotopy theory of homotopy theories".

- The Barwick–Kan model structure is defined by lifting the cofibrantly generated model structure $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ along an adjunction $ss\mathcal{S}\mathrm{et} \rightleftarrows \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$ (so that the right adjoint creates the fibrations and weak equivalences), which then becomes a Quillen equivalence (see [BarK12b, Theorem 6.1]). Then, the *Rezk nerve* functor

  $$\mathrm{N}^{\mathrm{R}} : \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \to ss\mathcal{S}\mathrm{et}$$

  of [Rez01, 3.3] (there called the "classification diagram" functor) admits a natural weak equivalence in $s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}}$ to this right Quillen equivalence (see [BarK12b, Lemma 5.4]). Thus, in light of the left Bousfield localization

  $$s(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{Reedy}} \rightleftarrows ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}},$$

  we see that the Rezk nerve defines a relative functor

  $$\mathrm{N}^{\mathrm{R}} : \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\mathrm{BK}} \to ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$$

  which creates the weak equivalences in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\mathrm{BK}}$.

  For any relative category $(\mathcal{R}, \mathbf{W}_{\mathcal{R}})$, we will write

  $$\mathrm{Fun}([n], \mathcal{R})^{\mathbf{W}} \subset \mathrm{Fun}([n], \mathcal{R})$$

  for the wide subcategory on the componentwise weak equivalences, the nerve of which is precisely $\mathrm{N}^{\mathrm{R}}(\mathcal{R})_n$.
- The *hammock localization* functor

  $$\mathscr{L}^{H} : \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$$

  of [DwK80a, 2.1] defines a weak equivalence in $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\mathrm{BK}}$ on the underlying relative categories of the model categories $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\mathrm{BK}}$ and $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ (see [BarK12a, Theorem 1.7]).
- The *homotopy-coherent nerve* functor

  $$\mathrm{N}^{\mathrm{hc}} : \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \to s\mathcal{S}\mathrm{et}$$

of [Lur09, Definition 1.1.5.5] (there called the "simplicial nerve" functor, originally defined in [Cor82], there called the "nerf homotopiquement cohérent" functor) defines a right Quillen equivalence

$$(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}} \to s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$$

(see [Lur09, Theorem 2.2.5.1]).

Since the model category $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ is cofibrantly generated, it comes naturally equipped with a fibrant replacement functor. However, it will be convenient for us to use one which does not change the objects. Thus, for definiteness we define

$$\mathbb{R}_{\mathrm{Bergner}} : (\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}} \to (\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$$

to be the functor given by applying Kan's Ex^∞ functor locally, i.e., to each hom-object. (Note that Ex^∞ preserves finite products, being a filtered colimit of right adjoints.) We now define the *underlying quasicategory* functor to be the composite

$$\mathrm{u.q.} : \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \xrightarrow{\mathscr{L}^H} \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \xrightarrow{\mathbb{R}_{\mathrm{Bergner}}} \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \xrightarrow{\mathrm{N}^{\mathrm{hc}}} s\mathcal{S}\mathrm{et}.$$

As N^{hc} is a right Quillen functor, this does indeed take values in quasicategories (and defines a relative functor $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\mathrm{BK}} \to s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$).

1.3. General model categories. A model category $\mathcal{C}$ comes equipped with various attendant subcategories, for which we must fix some notation. We write

- $\mathbf{W}_{\mathcal{C}} \subset \mathcal{C}$ for the subcategory of weak equivalences,
- $\mathcal{C}^c, \mathcal{C}^f, \mathcal{C}^{cf} \subset \mathcal{C}$ for the full subcategories of cofibrant, fibrant, and bifibrant objects, respectively,
- $\mathbf{W}_{\mathcal{C}}^c = \mathcal{C}^c \cap \mathbf{W}_{\mathcal{C}} \subset \mathcal{C}$ and $\mathbf{W}_{\mathcal{C}}^f = \mathcal{C}^f \cap \mathbf{W}_{\mathcal{C}} \subset \mathcal{C}$,

and similarly for other model categories. We will use the arrows $\rightarrowtail$ and $\twoheadrightarrow$ to denote cofibrations and fibrations, respectively, and we will decorate an arrow with the symbol $\approx$ to denote that it is a weak equivalence.

1.4. Foundations. We will ignore all set-theoretic issues. These are irrelevant to our aims, and in any case they can be dispensed with by appealing to the usual device of *Grothendieck universes* (see [Lur09, §1.2.15]).

2. The main theorem

Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction of model categories. Note that the functors F and G do not generally define functors of underlying relative categories: they do not generally preserve weak equivalences. Nevertheless, all is not lost: the inclusions

$$(\mathcal{C}^c, \mathbf{W}_{\mathcal{C}}^c) \hookrightarrow (\mathcal{C}, \mathbf{W}_{\mathcal{C}}) \quad \text{and} \quad (\mathcal{D}, \mathbf{W}_{\mathcal{D}}) \hookrightarrow (\mathcal{D}^f, \mathbf{W}_{\mathcal{D}}^f)$$

are weak equivalences in $\mathcal{R}el\mathcal{C}at_{BK}$ (as is proved in Lemma 2.8), and moreover by Kenny Brown's lemma (or rather its immediate consequence [Hir03, Corollary 7.7.2]), the composites

$$F^c : \mathcal{C}^c \hookrightarrow \mathcal{C} \xrightarrow{F} \mathcal{D} \quad \text{and} \quad \mathcal{C} \xleftarrow{G} \mathcal{D} \hookleftarrow \mathcal{D}^f : G^f$$

do preserve weak equivalences. Of course, this presents a problem: these two functors no longer have opposite sources and targets! Despite this, we have the following theorem, which is the main result of this paper.

Theorem 2.1. *The functors F^c and G^f induce a canonical adjunction between the underlying quasicategories* $\mathrm{u.q.}(\mathcal{C})$ *and* $\mathrm{u.q.}(\mathcal{D})$, *informally denoted by*

$$\mathrm{u.q.}(F^c) : \mathrm{u.q.}(\mathcal{C}) \rightleftarrows \mathrm{u.q.}(\mathcal{D}) : \mathrm{u.q.}(G^f).$$

Recall that an adjunction of quasicategories is determined by a map $\mathcal{M} \to \Delta^1$ of simplicial sets which is simultaneously a cocartesian fibration and a cartesian fibration: the left adjoint is then its unstraightening $\mathcal{M}_0 \to \mathcal{M}_1$ as a *cocartesian* fibration, while the right adjoint is its unstraightening $\mathcal{M}_0 \leftarrow \mathcal{M}_1$ as a *cartesian* fibration. Thus, the first step in proving Theorem 2.1 is to obtain a cocartesian fibration over Δ^1 which models F^c and a cartesian fibration over Δ^1 which models G^f. We will actually define these as $s\mathcal{S}et$-enriched categories over [1], relying on a recognition result ([Lur09, Proposition 5.2.4.4]) to deduce that these induce co/cartesian fibrations of quasicategories over Δ^1.

Construction 2.2. We define the object $\mathrm{cocart}(\mathscr{L}^H(F^c)) \in (\mathcal{C}at_{s\mathcal{S}et})_{/[1]}$ as follows:

- the fiber over $0 \in [1]$ is $\mathscr{L}^H(\mathcal{C}^c)$;
- the fiber over $1 \in [1]$ is $\mathscr{L}^H(\mathcal{D})$;
- for any $x \in \mathscr{L}^H(\mathcal{C}^c)$ and any $y \in \mathscr{L}^H(\mathcal{D})$, we set

$$\mathrm{hom}_{\mathrm{cocart}(\mathscr{L}^H(F^c))}(y, x) = \varnothing, \quad \text{and}$$
$$\mathrm{hom}_{\mathrm{cocart}(\mathscr{L}^H(F^c))}(x, y) = \mathrm{hom}_{\mathscr{L}^H(\mathcal{D})}(F^c(x), y).$$

Composition within the fibers is immediate, and is otherwise given by composition in $\mathscr{L}^H(\mathcal{D})$.

Similarly, we define the object $\mathrm{cart}(\mathscr{L}^H(G^f)) \in (\mathcal{C}at_{s\mathcal{S}et})_{/[1]}$ as follows:

- the fiber over $0 \in [1]$ is $\mathscr{L}^H(\mathcal{C})$;
- the fiber over $1 \in [1]$ is $\mathscr{L}^H(\mathcal{D}^f)$;
- for any $x \in \mathscr{L}^H(\mathcal{C})$ and any $y \in \mathscr{L}^H(\mathcal{D}^f)$, we set

$$\mathrm{hom}_{\mathrm{cart}(\mathscr{L}^H(G^f))}(y, x) = \varnothing, \quad \text{and}$$
$$\mathrm{hom}_{\mathrm{cart}(\mathscr{L}^H(G^f))}(x, y) = \mathrm{hom}_{\mathscr{L}^H(\mathcal{C})}(x, G^f(y)).$$

Again composition within the fibers is immediate, but this time it is otherwise given by composition in $\mathscr{L}^H(\mathcal{C})$.

Remark 2.3. It is actually not so hard to show using [Lur09, Proposition 5.2.4.4] that $\mathrm{cocart}(\mathscr{L}^H(F^c))$ gives rise to an adjunction of quasicategories whose left adjoint is u.q.(F^c).[3] In fact, this is essentially the content of the proof of [Hin14, Proposition 1.5.1]. Dually, we obtain that $\mathrm{cart}(\mathscr{L}^H(G^f))$ gives rise to an adjunction of quasicategories whose right adjoint is u.q.(G^f). However, a priori there is no reason that these two adjunctions must agree! Indeed, a left adjoint has a contractible space of right adjoints and vice versa, but this does *not* imply that a given left adjoint and a given right adjoint with opposite sources and targets must actually form an adjoint pair.

We introduce the following intermediate object to show the adjunctions of quasicategories determined by $\mathrm{cocart}(\mathscr{L}^H(F^c))$ and $\mathrm{cart}(\mathscr{L}^H(G^f))$ actually agree.

Construction 2.4. We define the object $(\mathcal{C}^c + \mathcal{D}^f, \mathbf{W}_{\mathcal{C}^c + \mathcal{D}^f}) \in \mathcal{R}el\mathcal{C}at_{/[1]}$ as follows:

- the fiber over $0 \in [1]$ is $(\mathcal{C}^c, \mathbf{W}_{\mathcal{C}}^c)$;
- the fiber over $1 \in [1]$ is $(\mathcal{D}^f, \mathbf{W}_{\mathcal{D}}^f)$;
- for any $x \in \mathcal{C}^c$ and any $y \in \mathcal{D}^f$, we set $\hom_{\mathcal{C}^c + \mathcal{D}^f}(y, x) = \varnothing$ and
$$\hom_{\mathcal{C}^c + \mathcal{D}^f}(x, y) = \hom_{\mathcal{C}}(x, G(y)) \cong \hom_{\mathcal{D}}(F(x), y),$$
  declaring none of these maps to be weak equivalences.

Composition within fibers is immediate, and is otherwise given by composition in either $\mathcal{C}^c$ or $\mathcal{D}^f$, whichever contains two of the three objects involved. We will depict arrows living over the unique nonidentity map in $[1]$ by

$$x \rightsquigarrow y,$$

and we will refer to such arrows as *bridge arrows*, or simply as *bridges*.

Applying the hammock localization functor to Construction 2.4 gives rise to an object
$$\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f) \in (\mathcal{C}at_{s\mathcal{S}et})_{/[1]},$$
and this will be what connects the two objects of Construction 2.2. In order to see how this works, we must examine the $s\mathcal{S}et$-enriched category $\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f)$. First of all, its fiber over $0 \in [1]$ is precisely $\mathscr{L}^H(\mathcal{C}^c)$, while its fiber over $1 \in [1]$ is precisely $\mathscr{L}^H(\mathcal{D}^f)$. On the other hand, for $x \in \mathcal{C}^c$ and $y \in \mathcal{D}^f$, the simplicial set $\hom_{\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f)}(x, y)$ has as its n-simplices the reduced hammocks of width n in the relative category $(\mathcal{C}^c + \mathcal{D}^f, \mathbf{W}_{\mathcal{C}^c + \mathcal{D}^f})$;

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[3]Used in such a way, [Lur09, Proposition 5.2.4.4] becomes a quasicategorical analog of the dual of [MacL98, Chapter IV, §1, Theorem 2(ii)]: given a functor $F_1 : \mathcal{C}_1 \to \mathcal{C}_2$ of small 1-categories, a right adjoint is freely generated by choices, for all $c_2 \in \mathcal{C}_2$, of objects $F_2(c_2) \in \mathcal{C}_1$ and morphisms $F_1(F_2(c_2)) \to c_2$ in $\mathcal{C}_2$ inducing natural isomorphisms $\hom_{\mathcal{C}_1}(-, F_2(c_2)) \xrightarrow{\cong} \hom_{\mathcal{C}_2}(F_1(-), c_2)$. (Of course, both the category of right adjoints to F_1 and the category of such data are (-1)-connected groupoids in any case; we are making the evil assertion that they are actually *equal*.)

since none of the bridge arrows are weak equivalences, such a hammock must
be of the form
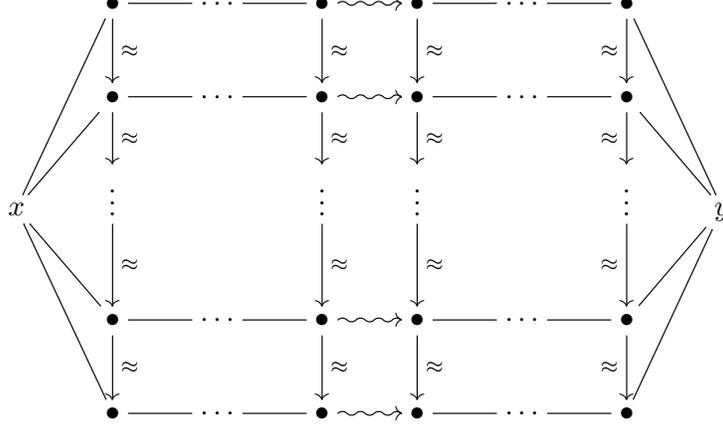
(where everything to the left of the column of bridge arrows lies in $\mathcal{C}^c$, while
everything to the right of the column of bridge arrows lies in $\mathcal{D}^f$). Hence,
there are two maps

$$\hom_{\mathrm{cocart}(\mathscr{L}^H(F^c))}(x,y) \leftarrow \hom_{\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f)}(x,y) \rightarrow \hom_{\mathrm{cart}(\mathscr{L}^H(G^f))}(x,y),$$

in which the left-pointing arrow is obtained by applying the relative functor
$\mathcal{C}^c \xrightarrow{F^c} \mathcal{D}$ to the "left half" of the above hammock, while the right-pointing
arrow is obtained by applying the relative functor $\mathcal{C} \xleftarrow{G^f} \mathcal{D}^f$ to the "right
half" of the above hammock. In fact, it is not hard to see that this respects
composition of hammocks, and hence we obtain a diagram

$$\mathrm{cocart}(\mathscr{L}^H(F^c)) \leftarrow \mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f) \rightarrow \mathrm{cart}(\mathscr{L}^H(G^f))$$

in $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{/[1]}$.

The main ingredient in the proof of Theorem 2.1 is the following result.

Proposition 2.5. *The horizontal maps*

$$\mathrm{cocart}(\mathscr{L}^H(F^c)) \xleftarrow{\approx} \mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f) \xrightarrow{\approx} \mathrm{cart}(\mathscr{L}^H(G^f))$$
$$\searrow \qquad \downarrow \qquad \swarrow$$
$$[1]$$

are weak equivalences in $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}.$

We defer the proof of Proposition 2.5 to §3. In essence, we will show that
the relative category $(\mathcal{C}^c + \mathcal{D}^f)$ admits a "homotopical three-arrow calculus",
and then we will use this to show that the hom-objects in $\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f)$ can
be computed using the co/simplicial resolutions of [DwK80b]. For an object

$x \in \mathcal{C}^c$ equipped with a cosimplicial resolution $x^\bullet \in c(\mathcal{C}^c)$ and an object $y \in \mathcal{D}^f$ equipped with a simplicial resolution $y_\bullet \in s(\mathcal{D}^f)$, the isomorphisms

$$\hom_{\mathcal{D}}^{\mathrm{lw}}(F^c(x^\bullet), y_\bullet) \cong \hom_{(\mathcal{C}^c + \mathcal{D}^f)}^{\mathrm{lw}}(x^\bullet, y_\bullet) \cong \hom_{\mathcal{C}}^{\mathrm{lw}}(x^\bullet, G^f(y_\bullet))$$

of bisimplicial sets (where the superscript "lw" stands for "levelwise") will, in light of the above observations and upon passing to diagonals, yield weak equivalences

$$\hom_{\mathscr{L}^H(\mathcal{D})}(F(x), y) \xleftarrow{\approx} \hom_{\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f)}(x, y) \xrightarrow{\approx} \hom_{\mathscr{L}^H(\mathcal{C})}(x, G(y))$$

$$\|\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\|$$

$$\hom_{\mathrm{cocart}(\mathscr{L}^H(F^c))}(x, y) \qquad\qquad\qquad\qquad \hom_{\mathrm{cart}(\mathscr{L}^H(G^f))}(x, y)$$

in $s\mathcal{S}et_{\mathrm{KQ}}$ (which are appropriately compatible with the given maps of hom-objects).

Using Proposition 2.5, we now prove the main theorem.

Proof of Theorem 2.1. Recall from [Lur09, Definition 5.2.2.1] that an adjunction of quasicategories is a map $\mathcal{M} \to \Delta^1$ which is simultaneously a cocartesian fibration and a cartesian fibration, along with weak equivalences in $s\mathcal{S}et_{\mathrm{Joyal}}$ from the participating quasicategories to the respective fibers $\mathcal{M}_0$ and $\mathcal{M}_1$. (The unstraightening of $\mathcal{M} \to \Delta^1$ as a cocartesian fibration yields the left adjoint, while its unstraightening as a cartesian fibration yields the right adjoint; meanwhile, the requested weak equivalences allow us to consider these as functors between the participating quasicategories.)

We argue using the recognition result [Lur09, Proposition 5.2.4.4] for when the functor

$$N^{\mathrm{hc}} : ((\mathcal{C}at_{s\mathcal{S}et})_{\mathrm{Bergner}})_{/[1]} \to (s\mathcal{S}et_{\mathrm{Joyal}})_{/\Delta^1}$$

takes a fibrant object $\mathcal{E} \twoheadrightarrow [1]$ to a cartesian fibration

$$N^{\mathrm{hc}}(p) : N^{\mathrm{hc}}(\mathcal{E}) \to N^{\mathrm{hc}}([1]) \cong \Delta^1.$$

Namely, the following condition is both necessary and sufficient:

(∗) For each object $e_1 \in \mathcal{E}_1$, there exists an object $e_0 \in \mathcal{E}_0$ and a morphism $e_0 \to e_1$ in $\mathcal{E}$ inducing a weak equivalence

$$\hom_{\mathcal{E}_0}(e, e_0) = \hom_{\mathcal{E}}(e, e_0) \xrightarrow{\approx} \hom_{\mathcal{E}}(e, e_1)$$

in $s\mathcal{S}et_{\mathrm{KQ}}$ for every $e \in \mathcal{E}_0$.

(The morphism $e_0 \to e_1$ will then determine a cartesian edge of $N^{\mathrm{hc}}(\mathcal{E})$.) This recognition result immediately implies that the map

$$\mathbb{R}_{\mathrm{Bergner}}(\mathrm{cart}(\mathscr{L}^H(G^f))) \to [1]$$

induces a cartesian fibration corresponding to u.q.(G^f), while its evident dual implies that the map

$$\mathbb{R}_{\mathrm{Bergner}}(\mathrm{cocart}(\mathscr{L}^H(F^c))) \to [1]$$

induces a cocartesian fibration corresponding to u.q.(F^c).[4] Moreover, the criterion $(*)$ is clearly invariant under weak equivalence between fibrant objects in $((\mathcal{C}at_{s\mathcal{S}et})_{\mathrm{Bergner}})_{/[1]}$. Thus, it follows from Proposition 2.5 that the map

$$\mathbb{R}_{\mathrm{Bergner}}(\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f)) \to [1]$$

in $\mathcal{C}at_{s\mathcal{S}et}$ induces both a cartesian fibration and a cocartesian fibration of quasicategories: that is, it induces an adjunction

$$\mathrm{u.q.}(\mathcal{C}^c + \mathcal{D}^f) \to \Delta^1$$

of quasicategories, whose left adjoint can be identified with u.q.(F^c) and whose right adjoint can be identified with u.q.(G^f).

Finally, the fact that the inclusions

$$(\mathcal{C}^c, \mathbf{W}_{\mathcal{C}}^c) \hookrightarrow (\mathcal{C}, \mathbf{W}_{\mathcal{C}}) \quad \text{and} \quad (\mathcal{D}^f, \mathbf{W}_{\mathcal{D}}^f) \hookrightarrow (\mathcal{D}, \mathbf{W}_{\mathcal{D}})$$

are weak equivalences in $\mathcal{R}el\mathcal{C}at_{\mathrm{BK}}$ (and hence induce weak equivalences in $s\mathcal{S}et_{\mathrm{Joyal}}$ of underlying quasicategories) follows from Lemma 2.8 below. Hence, choosing retractions in $s\mathcal{S}et_{\mathrm{Joyal}}$ as indicated (the simplicial sets of which are contractible Kan complexes), we obtain an adjunction of quasi-categories

which might denoted informally as

$$\mathrm{u.q.}(F^c) : \mathrm{u.q.}(\mathcal{C}) \rightleftarrows \mathrm{u.q.}(\mathcal{D}) : \mathrm{u.q.}(G^f),$$

precisely as claimed.   □

Remark 2.6. In general, the property of being a co/cartesian fibration over $S \in s\mathcal{S}et$ is not invariant under weak equivalence between inner fibrations in $(s\mathcal{S}et_{\mathrm{Joyal}})_{/S}$. However, it becomes invariant in the special case that $S = \Delta^1$, a fact we've exploited in the proof of Theorem 2.1 (through our usage of [Lur09, Proposition 5.2.4.4]). Roughly speaking, this follows from the paucity of nondegenerate edges in Δ^1. Indeed, recall that given an inner fibration $X \to S$:

---

[4]Note that fibrancy in $((\mathcal{C}at_{s\mathcal{S}et})_{\mathrm{Bergner}})_{/[1]}$ is created in $(\mathcal{C}at_{s\mathcal{S}et})_{\mathrm{Bergner}}$ (see [Lur09, Theorem A.3.2.24(2)]).

- an edge $\Delta^1 \to X$ is *cartesian* (with respect to $X \to S$) if it satisfies some universal property defined in terms of all of X and S (see [Lur09, Definition 2.4.1.1 and Remark 2.4.1.9]);
- an edge of $\Delta^1 \to X$ is *locally cartesian* if the induced edge

$$\Delta^1 \to \Delta^1 \times_S X$$

  is cartesian with respect to the inner fibration $\Delta^1 \times_S X \to \Delta^1$ (see [Lur09, Definition 2.4.1.11]);
- the map $X \to S$ is a (resp. *locally*) *cartesian fibration* if it has a sufficient supply of (resp. locally) cartesian edges (see [Lur09, Definitions 2.4.2.1 and 2.4.2.6]);
- a locally cartesian fibration is a cartesian fibration if and only if the locally cartesian edges are "closed under composition" in the strongest possible sense (see [Lur09, Proposition 2.4.2.8]);
- if an edge of X maps to an equivalence in S, then that edge is cartesian if and only if it is also an equivalence (see [Lur09, Proposition 2.4.1.5]);
- in light of the universal property defining cartesian edges, pre- or post-composing a cartesian edge in X with an equivalence which projects to a degenerate edge in S clearly yields another cartesian edge.

(The notions of cartesian fibrations and of locally cartesian fibrations are the quasicategorical analogs of the 1-categorical notions of "Grothendieck fibrations" and "Grothendieck prefibrations".)

Remark 2.7. One might also wonder about the possibility of using the objects $\mathrm{cocart}(F^c) = (\mathcal{C}^c + \mathcal{D})$ and $\mathrm{cart}(G^f) = (\mathcal{C} + \mathcal{D}^f)$ of $\mathcal{R}el\mathcal{C}at_{/[1]}$ in order to prove Theorem 2.1. In fact, it is not so hard to show that the inclusions $\mathrm{cocart}(F^c) \hookleftarrow (\mathcal{C}^c + \mathcal{D}^f) \hookrightarrow \mathrm{cart}(G^f)$ are weak equivalences in $\mathcal{R}el\mathcal{C}at_{\mathrm{BK}}$, and moreover there are natural maps $\mathscr{L}^H(\mathrm{cocart}(F^c)) \to \mathrm{cocart}(\mathscr{L}^H(F^c))$ and $\mathscr{L}^H(\mathrm{cart}(G^f)) \to \mathrm{cart}(\mathscr{L}^H(G^f))$ in $\mathcal{C}at_{s\mathcal{S}et}$, but it is essentially no easier to show that these latter maps are weak equivalences in $(\mathcal{C}at_{s\mathcal{S}et})_{\mathrm{Bergner}}$ than it is to prove Proposition 2.5.

We end this section with a result used in the proof of the main theorem, which we learned from Horel and which now appears in a joint paper of his as [BaHH, Proposition 2.4.9].[5]

Lemma 2.8. *The inclusions*

$$(\mathcal{C}^c, \mathbf{W}_{\mathcal{C}}^c) \hookrightarrow (\mathcal{C}, \mathbf{W}_{\mathcal{C}}) \quad and \quad (\mathcal{D}^f, \mathbf{W}_{\mathcal{D}}^f) \hookrightarrow (\mathcal{D}, \mathbf{W}_{\mathcal{D}})$$

are weak equivalences in $\mathcal{R}el\mathcal{C}at_{\mathrm{BK}}$.

---

[5]The authors of [BaHH] in turn credit Cisinski for their proof. They actually work in the more general setting of "weak fibration categories", and in their proof they replace the appeal to [Hin05, Theorem A.3.2(1)] with an appeal to work of Cisinski's.

Proof. We will prove the second of these two dual statements, which we will accomplish by showing that the map $(\mathcal{D}^f, \mathbf{W}_{\mathcal{D}}^f) \hookrightarrow (\mathcal{D}, \mathbf{W}_{\mathcal{D}})$ induces a weak equivalence in $s(s\mathcal{S}et_{\mathrm{KQ}})_{\mathrm{Reedy}}$ upon application of the functor

$$\mathrm{N}^{\mathrm{R}} : \mathcal{R}el\mathcal{C}at \to ss\mathcal{S}et.$$

In other words, we will show that for all $n \geq 0$, the inclusion

$$\mathrm{Fun}([n], \mathcal{D}^f)^{\mathbf{W}} \hookrightarrow \mathrm{Fun}([n], \mathcal{D})^{\mathbf{W}}$$

induces a weak equivalence in $s\mathcal{S}et_{\mathrm{KQ}}$ upon application of the functor

$$\mathrm{N} : \mathcal{C}at \to s\mathcal{S}et.$$

For this, let us equip $\mathrm{Fun}([n], \mathcal{D})$ with the projective model structure, which exists since it coincides with the Reedy model structure when we consider $[n]$ as a Reedy category with no nonidentity degree-lowering maps. Then, the above inclusion is precisely the inclusion

$$\mathbf{W}_{\mathrm{Fun}([n],\mathcal{D})}^f \hookrightarrow \mathbf{W}_{\mathrm{Fun}([n],\mathcal{D})},$$

which induces a weak equivalence on nerves by combining the duals of [Qui73, Theorem A] and [Hin05, Theorem A.3.2(1)].   □

Remark 2.9. Lemma 2.8 actually goes back to [DwK80b, Proposition 5.2], but the proof given there relies on a claim whose proof is omitted, namely that the relative category $(\mathcal{C}^c, \mathbf{W}_{\mathcal{C}}^c)$ admits a "homotopy calculus of left fractions" as in [DwK80a, 6.1(ii)] (see [DwK80b, 8.2(ii)]). We have not been able to prove this result ourselves, so we provide this alternative proof for completeness.

3. Model diagrams and the proof of Proposition 2.5

In this section, we prove Proposition 2.5 (the main ingredient in the proof of Theorem 2.1), which asserts the existence of a diagram

$$\mathrm{cocart}(\mathscr{L}^H(F^c)) \xleftarrow{\approx} \mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f) \xrightarrow{\approx} \mathrm{cart}(\mathscr{L}^H(G^f))$$
$$\searrow \qquad \downarrow \qquad \swarrow$$
$$[1]$$

in $(\mathcal{C}at_{s\mathcal{S}et})_{\mathrm{Bergner}}$: this is what connects the made-to-order relative category $(\mathcal{C}^c + \mathcal{D}^f)$ of Construction 2.4 with the actual relative functors

$$F^c : \mathcal{C}^c \hookrightarrow \mathcal{C} \xrightarrow{F} \mathcal{D}$$

and

$$\mathcal{C} \xleftarrow{G} \mathcal{D} \hookleftarrow \mathcal{D}^f : G^f$$

of interest. (Recall from the proof of Theorem 2.1 that the former ultimately gives rise to our desired adjunction of quasicategories.) It follows directly from Lemma 2.8 that these horizontal maps are weak equivalences

in $(\mathcal{C}at_{s\mathcal{S}et})_{\mathrm{Bergner}}$ when we restrict to the fibers over $0 \in [1]$ and $1 \in [1]$; the real difficulty lies in showing that we also obtain weak equivalences on hom-objects (in $s\mathcal{S}et_{\mathrm{KQ}}$) when our source lies over $0 \in [1]$ and our target lies over $1 \in [1]$.

To accomplish this, we first provide an alternative description of such "bridge-containing" hom-objects in $\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f)$. Fix a source object $x \in \mathcal{C}^c \subset (\mathcal{C}^c + \mathcal{D}^f)$ and a target object $y \in \mathcal{D}^f \subset (\mathcal{C}^c + \mathcal{D}^f)$. Then, we prove as Proposition 3.16(2) that the relative category $(\mathcal{C}^c + \mathcal{D}^f)$ admits a *homotopical three-arrow calculus with respect to x and y* (see Definition 3.14). This notion is a slight variant on Dwyer–Kan's definition of a "homotopy calculus of fractions", and it affords the same conclusion: that a much smaller simplicial set — that of *three-arrow zigzags* from x to y — maps to $\hom_{\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f)}(x, y)$ as a weak equivalence in $s\mathcal{S}et_{\mathrm{KQ}}$ (see Proposition 3.15).

However, a three-arrow zigzag in $(\mathcal{C}^c + \mathcal{D}^f)$ from $x \in \mathcal{C}^c$ to $y \in \mathcal{D}^f$ will be of the form

$$x \xleftarrow{\;\approx\;} \bullet \rightsquigarrow \bullet \xleftarrow{\;\approx\;} y$$

(with the middle arrow a bridge), and these are still not in any sense visibly equivalent to the corresponding bridge-containing hom-objects in $\mathrm{cocart}(\mathscr{L}^H(F^c))$ or $\mathrm{cart}(\mathscr{L}^H(G^f))$: the latter are respectively given by three-arrow zigzags in $\mathcal{D}$ or $\mathcal{C}$ (since model categories admit homotopical three-arrow calculi (with respect to any choices of source and target objects)), whereas the above-depicted three-arrow zigzag lies partly in $\mathcal{C}^c$ and partly in $\mathcal{D}^f$. To complete the connection (and prove the above weak equivalences in $(\mathcal{C}at_{s\mathcal{S}et})_{\mathrm{Bergner}}$), we will appeal to Dwyer–Kan's theory of *co/simplicial resolutions*: the classical story concerns the computation of hom-objects in the hammock localization of a model category, and we will show that it can be made to apply to $(\mathcal{C}^c + \mathcal{D}^f)$ as well. For this, we will moreover need to pass from three-arrow zigzags to *special* three-arrow zigzags, i.e., those of the form

$$x \xleftarrow{\;\approx\;}\!\!\!\leftarrow \bullet \rightsquigarrow \bullet \leftarrow\!\!\!\xleftarrow{\;\approx\;} y,$$

which we accomplish as Proposition 3.11(2).

This section is organized as follows. First, in §3.1, we introduce the formalism of *model diagrams*, which provides a language for manipulating diagrams of a specified shape inside of a model category (or inside of a more general category equipped with three analogously-named distinguished subcategories, such as $(\mathcal{C}^c + \mathcal{D}^f)$). Then, in §3.2, we prove that three-arrow zigzags (and special three-arrow zigzags) do indeed compute the bridge-containing hom-objects in $\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f)$. Finally, in §3.3 we prove Proposition 2.5.

3.1. Model diagrams. To prove statements about categories of diagrams in model categories, we provide a general framework for parametrizing them.

Definition 3.1. A *model diagram* is a category $\mathcal{I}$ equipped with three wide subcategories $\mathbf{W}_{\mathcal{I}}, \mathbf{C}_{\mathcal{I}}, \mathbf{F}_{\mathcal{I}} \subset \mathcal{I}$ such that $\mathbf{W}_{\mathcal{I}}$ satisfies the two-out-of-three axiom.[6] These assemble into a category, denoted $\mathcal{M}$odel, whose morphisms are simply those functors of underlying categories that respect the defining subcategories. For $\mathcal{I}, \mathcal{J} \in \mathcal{M}$odel, we denote by $\mathrm{Fun}(\mathcal{I}, \mathcal{J})^{\mathbf{W}}$ the category whose objects are morphisms of model diagrams and whose morphisms are natural *weak equivalences* between them. We will consider relative categories (and in particular, categories) as equipped with the *minimal* model diagram structure (in which $\mathbf{C}$ and $\mathbf{F}$ consist only of the identity maps).

Remark 3.2. Among the axioms for a model category, all but the limit axiom (so the two-out-of-three, retract, lifting, and factorization axioms) can be encoded by requiring that the underlying model diagram has the extension property with respect to certain maps of model diagrams.

Variant 3.3. A *decorated model diagram* is a model diagram with some subdiagrams decorated as colimit or limit diagrams.[7] For instance, if we define $\mathcal{I}$ to be the "walking pullback square", then for any other model diagram $\mathcal{J}$, we let $\mathrm{hom}^{\star}_{\mathcal{M}\mathrm{odel}}(\mathcal{I}, \mathcal{J}) \subset \mathrm{hom}_{\mathcal{M}\mathrm{odel}}(\mathcal{I}, \mathcal{J})$ and $\mathrm{Fun}^{\star}(\mathcal{I}, \mathcal{J})^{\mathbf{W}} \subset \mathrm{Fun}(\mathcal{I}, \mathcal{J})^{\mathbf{W}}$ denote the subobjects spanned by those morphisms $\mathcal{I} \to \mathcal{J}$ of model diagrams which select a pullback square in $\mathcal{J}$.

For the most part, we will only use this variant on Definition 3.1 for pushout and pullback squares. In fact, all but one of the model diagrams that we will decorate in this way will only have a single square anyways, and so in the interest of easing our Ti*k*Zographical burden, we will simply superscript these model diagrams with either "p.o." or "p.b." (as in the proof of Proposition 3.11 below). The other one (which will appear in the proof of Proposition 3.16) will be endowed with sufficiently clear ad hoc notation.

However, this formalism also allows us to require that certain objects are sent to cofibrant or fibrant objects, by decorating a new object as initial/terminal and then marking its maps to/from the other objects as co/fibrations. Rather than write this explicitly, we will abbreviate the notation for this procedure by superscripting objects by c, f, or cf (to indicate that we wish these to select cofibrant, fibrant, or bifibrant objects, respectively).

Note that the constructions $\mathrm{hom}^{\star}_{\mathcal{M}\mathrm{odel}}(\mathcal{I}, \mathcal{J})$ and $\mathrm{Fun}^{\star}(\mathcal{I}, \mathcal{J})^{\mathbf{W}}$ are not generally functorial in the target $\mathcal{J}$. On the other hand, they are functorial for *some* maps in the source $\mathcal{I}$. We will refer to such maps as *decoration-respecting*. These define a category $\mathcal{M}$odel*. (Note the distinction between

---

[6]The assumption that $\mathbf{W}_{\mathcal{I}}$ satisfies the two-out-of-three axiom is probably superfluous, since we'll generally be mapping into model diagrams whose weak equivalences already have this property (namely the model category $\mathcal{D}$ as well as $(\mathcal{C}^c + \mathcal{D}^f)$ and its cousins). Nevertheless, it seems like a good idea to include it, just to be safe.

[7]These are closely related to what are now called "sketches", originally introduced in [Ehr68].

$\hom_{\mathcal{M}odel^\star}$ and $\hom^\star_{\mathcal{M}odel}$.) We consider $\mathcal{M}odel \subset \mathcal{M}odel^\star$ simply by considering undecorated model diagrams as being trivially decorated. We will not need a general theory for understanding which maps of decorated model diagrams are decoration-respecting; rather, it will suffice to observe once and for all

- that objects marked as co/fibrant must be sent to the same, and
- that given a square which is decorated as a pushout or pullback square, it is decoration-respecting to either
  - take it to another similarly decorated square, or
  - collapse it onto a single edge (since a square in which two parallel edges are identity maps is both a pushout and a pullback).

Note that if the source of a map of decorated model diagrams is actually undecorated, then the map is automatically decoration-respecting; in other words, we must only check that maps in which the *source* is decorated are decoration-respecting.

Remark 3.4. Of course, adding in this variant allows us to also demand finite bicompleteness of a model diagram via lifting conditions, and hence all of the axioms for a model diagram to be a model category can now be encoded in this language.

We will be concerned with diagrams in model categories which connect specified "source" and "target" objects. We thus introduce the following variant.

Variant 3.5. A *doubly-pointed model diagram* is a model diagram $\mathcal{I}$ equipped with a map $\mathrm{pt} \sqcup \mathrm{pt} \to \mathcal{I}$. The two inclusions $\mathrm{pt} \hookrightarrow \mathrm{pt} \sqcup \mathrm{pt}$ select objects $s, t \in \mathcal{I}$, which we call the *source* and the *target*. These assemble into the evident category, which we denote by $\mathcal{M}odel_{**} = \mathcal{M}odel_{\mathrm{pt} \sqcup \mathrm{pt}/}$. Of course, there is a forgetful functor $\mathcal{M}odel_{**} \to \mathcal{M}odel$, which we will occasionally implicitly use. For $\mathcal{I}, \mathcal{J} \in \mathcal{M}odel_{**}$, we denote by

$$\mathrm{Fun}_{**}(\mathcal{I}, \mathcal{J})^{\mathbf{W}} \subset \mathrm{Fun}(\mathcal{I}, \mathcal{J})^{\mathbf{W}}$$

the (not generally full) subcategory whose objects are those morphisms of model diagrams which preserve the double-pointing, and whose morphisms are those natural weak equivalences whose components at s and t are respectively id_s and id_t. We will refer to such a morphism as a *doubly-pointed natural weak equivalence*. If we have chosen "source" and "target" objects in a model diagram, we will use these to consider the model diagram as doubly-pointed without explicitly mentioning it. Of course, we may decorate a doubly-pointed model diagram as in Variant 3.3.

We will furthermore be interested in the following special case of Variant 3.5.

Variant 3.6. We define a *model word* to be a word $\underline{\mathbf{m}}$ in any the symbols describing a morphism in a model diagram or their inverses (e.g., $\mathbf{W}$,

$(\mathbf{W} \cap \mathbf{F})^{-1}$, $(\mathbf{W} \cap \mathbf{C}))$, or in the symbol $\mathbf{A}$ (for "any arbitrary arrow") or its inverse. We will write $\mathbf{A}^{\circ n}$ to denote n consecutive copies of the symbol $\mathbf{A}$ (for any $n \geq 0$). We can extract a doubly-pointed model diagram from a model word, which for our sanity we will carry out by reading *forwards*. So for instance, the model word $\underline{\mathbf{m}} = [\mathbf{C}; (\mathbf{W} \cap \mathbf{F})^{-1}; \mathbf{A}]$ defines the doubly-pointed model diagram

$$s \rightarrowtail \bullet \xleftarrow{\approx} \bullet \longrightarrow t.$$

We denote this object again by $\underline{\mathbf{m}} \in \mathcal{M}\mathrm{odel}_{**}$.

Remark 3.7. Restricting to those model words in the symbols $\mathbf{A}$ and $\mathbf{W}^{-1}$ and the *order-preserving* maps between them, we recover the category of "zigzag types", i.e., the opposite of the category $\mathbf{II}$ of [DwK80a, 4.1]. In this way, we consider $\mathbf{II}^{op} \subset \mathcal{M}\mathrm{odel}_{**}$ as a (nonfull) subcategory. For any relative category $(\mathcal{R}, \mathbf{W}_{\mathcal{R}}) \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$ and any objects $x, y \in \mathcal{R}$, by [DwK80a, Proposition 5.5] we have an isomorphism

$$\mathrm{colim}_{\underline{\mathbf{m}} \in \mathbf{II}} \, \mathrm{N}\left(\mathrm{Fun}_{**}(\underline{\mathbf{m}}, \mathcal{R})^{\mathbf{W}}\right) \xrightarrow{\cong} \mathrm{hom}_{\mathscr{L}^H(\mathcal{R})}(x, y)$$

of simplicial sets, induced by the maps

$$\mathrm{N}\left(\mathrm{Fun}_{**}(\underline{\mathbf{m}}, \mathcal{R})^{\mathbf{W}}\right) \to \mathrm{hom}_{\mathscr{L}^H(\mathcal{R})}(x, y)$$

given by reducing the hammocks involved (as described in [DwK80a, 2.1]).

Definition 3.8. We will use the abbreviations $\underline{\mathbf{3}} = [\mathbf{W}^{-1}; \mathbf{A}; \mathbf{W}^{-1}]$ and $\underline{\tilde{\mathbf{3}}} = [(\mathbf{W} \cap \mathbf{F})^{-1}; \mathbf{A}; (\mathbf{W} \cap \mathbf{C})^{-1}]$; these model words correspond to the doubly-pointed model diagrams

$$s \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} t$$

and

$$s \xleftarrow{\approx} \bullet \longrightarrow \bullet \xleftarrow{\approx} t,$$

respectively. We refer to diagrams of shape $\underline{\mathbf{3}}$ as *three-arrow zigzags*, and we refer to diagrams of shape $\underline{\tilde{\mathbf{3}}}$ as *special three-arrow zigzags*.

3.2. From special three-arrow zigzags to hammocks. Note that there is a unique map $\underline{\mathbf{3}} \to \underline{\tilde{\mathbf{3}}}$ in $\mathcal{M}\mathrm{odel}_{**}$. In [DwK80b, 7.2(ii)], Dwyer–Kan indicate how to prove that for any $x, y \in \mathcal{D}$, the induced composite

$$\mathrm{N}\left(\mathrm{Fun}_{**}(\underline{\tilde{\mathbf{3}}}, \mathcal{D})^{\mathbf{W}}\right) \to \mathrm{N}\left(\mathrm{Fun}_{**}(\underline{\mathbf{3}}, \mathcal{D})^{\mathbf{W}}\right) \to \mathrm{hom}_{\mathscr{L}^H(\mathcal{D})}(x, y)$$

is a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. In order to prove Proposition 2.5, in this subsection we will show that these two maps are again weak equivalences if we replace $\mathcal{D}$ by $(\mathcal{C}^c + \mathcal{D}^f)$ and take $x \in \mathcal{C}^c$ and $y \in \mathcal{D}^f$. The arguments for $(\mathcal{C}^c + \mathcal{D}^f)$ are patterned on those for $\mathcal{D}$, and so for the sake of exposition we will re-prove that case in tandem.

3.2.1. From $\tilde{\underline{\mathbf{3}}}$-shaped zigzags to $\underline{\mathbf{3}}$-shaped zigzags.

In this subsubsection, we prove that the map

$$\mathrm{N}\left(\mathrm{Fun}_{**}(\tilde{\underline{\mathbf{3}}}, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right) \to \mathrm{N}\left(\mathrm{Fun}_{**}(\underline{\mathbf{3}}, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right)$$

is a weak equivalence in $s\mathcal{S}et_{\mathrm{KQ}}$. We begin with some preliminary results.

Lemma 3.9. *Choose any doubly-pointed model diagram $\mathcal{I} \in \mathcal{M}odel_{**}$, select a weak equivalence in $\mathcal{I}$ by choosing a map $[\mathbf{W}] \to \mathcal{I}$ in $\mathcal{M}odel$, and define $\mathcal{J} \in \mathcal{M}odel_{**}$ by taking a pushout*

$$
\begin{array}{ccc}
[\mathbf{W}] & \longrightarrow & \mathcal{I} \\
\downarrow & & \downarrow \\
[(\mathbf{W} \cap \mathbf{C}); (\mathbf{W} \cap \mathbf{F})] & \longrightarrow & \mathcal{J}
\end{array}
$$

in $\mathcal{M}odel$ (where the left map is the unique map in $\mathcal{M}odel_{**}$, and $\mathcal{J}$ is doubly-pointed via the composition $\mathrm{pt} \sqcup \mathrm{pt} \to \mathcal{I} \to \mathcal{J}$). Then, the map $\mathcal{I} \to \mathcal{J}$ induces

(1) *for any $x, y \in \mathcal{D}$, a weak equivalence*

$$\mathrm{N}\left(\mathrm{Fun}_{**}(\mathcal{J}, \mathcal{D})^{\mathbf{W}}\right) \overset{\approx}{\to} \mathrm{N}\left(\mathrm{Fun}_{**}(\mathcal{I}, \mathcal{D})^{\mathbf{W}}\right),$$

and

(2) *for any $x, y \in (\mathcal{C}^c + \mathcal{D}^f)$, a weak equivalence*

$$\mathrm{N}\left(\mathrm{Fun}_{**}(\mathcal{J}, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right) \overset{\approx}{\to} \mathrm{N}\left(\mathrm{Fun}_{**}(\mathcal{I}, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right).$$

Proof. This is a mild generalization of [DwK80b, 8.1], and the proof adapts readily.[8,9] The following observations may be helpful.

- The proof is unaffected by whether or not the map $[\mathbf{W}] \to \mathcal{I}$ selects an identity map, and by whether or not it hits one or both of the objects $s, t \in \mathcal{I}$.
- The proof does not require that the map $[\mathbf{W}] \to \mathcal{I}$ be "free" (i.e., obtained by taking a pushout $[\mathbf{W}] \leftarrow \mathrm{pt} \sqcup \mathrm{pt} \to \mathcal{I}'$), although we will actually only need this special case.
- For item (2), note that all of the computations happen in one fiber or the other (since none of the bridge arrows are weak equivalences), and that the necessary simplicial resolution will automatically lie in the relevant subcategory $\mathcal{C}^c \subset \mathcal{C}$ or $\mathcal{D}^f \subset \mathcal{D}$ (in fact, it will even consist of *bifibrant* objects). $\quad\square$

---

[8]In [DwK80b, 6.7], which constructs (special) simplicial resolutions, the factorization of the latching-to-matching map which produces the next simplicial level should be as $\overset{\approx}{\rightarrowtail}\twoheadrightarrow$, not $\overset{\approx}{\rightarrowtail}\overset{\approx}{\twoheadrightarrow}$.

[9]We have greatly expanded on the proof of [DwK80b, 8.1] (while generalizing it from 1-categories to ∞-categories) in our proof of [MazG, Lemma 4.24]; the reader may find this expansion useful in verifying that the former does indeed adapt readily from model words to more general doubly-pointed model diagrams.

Lemma 3.10. *Let* $\mathcal{I}, \mathcal{J} \in \mathrm{Model}^\star_{**}$ *be decorated doubly-pointed model diagrams, let* $\alpha, \beta : \mathcal{I} \rightrightarrows \mathcal{J}$ *be parallel morphisms in* $\mathrm{Model}^\star_{**}$, *and let* $\gamma : \alpha \to \beta$ *be a doubly-pointed natural weak equivalence. Then for any* $\mathcal{E} \in \mathrm{Model}_{**}$, γ *induces a natural transformation between the induced parallel maps*

$$\alpha^*, \beta^* : \mathrm{Fun}^\star_{**}(\mathcal{J}, \mathcal{E})^\mathbf{W} \rightrightarrows \mathrm{Fun}^\star_{**}(\mathcal{I}, \mathcal{E})^\mathbf{W}$$

in $\mathcal{C}\mathrm{at}$.

Proof. This is immediate from the definitions. $\square$

We can now prove the main result of this subsubsection.

Proposition 3.11. *The unique map* $\underline{\mathbf{3}} \to \tilde{\underline{\mathbf{3}}}$ *in* Model_{**} *induces*
  (1) *for any* $x, y \in \mathcal{D}$, *a weak equivalence*
  $$\mathrm{N}\left(\mathrm{Fun}_{**}(\tilde{\underline{\mathbf{3}}}, \mathcal{D})^\mathbf{W}\right) \overset{\approx}{\to} \mathrm{N}\left(\mathrm{Fun}_{**}(\underline{\mathbf{3}}, \mathcal{D})^\mathbf{W}\right),$$
  and
  (2) *for any* $x \in \mathcal{C}^c$ *and* $y \in \mathcal{D}^f$, *a weak equivalence*
  $$\mathrm{N}\left(\mathrm{Fun}_{**}(\tilde{\underline{\mathbf{3}}}, (\mathcal{C}^c + \mathcal{D}^f))^\mathbf{W}\right) \overset{\approx}{\to} \mathrm{N}\left(\mathrm{Fun}_{**}(\underline{\mathbf{3}}, (\mathcal{C}^c + \mathcal{D}^f))^\mathbf{W}\right).$$

Proof. We first address item (1), which is somewhat simpler to prove. For this, we factor the unique map $\underline{\mathbf{3}} \to \tilde{\underline{\mathbf{3}}}$ in Model_{**} through a sequence

$$\underline{\mathbf{3}} \xrightarrow{\varphi_1} \mathcal{I}_1 \xrightarrow{\varphi_2} \mathcal{I}_2 \xrightarrow{\varphi_3} \mathcal{I}_3 \xrightarrow{\varphi_4} \mathcal{I}_4 \xrightarrow{\varphi_5} \mathcal{I}_5 \xrightarrow{\varphi_6} \tilde{\underline{\mathbf{3}}}$$

of maps in $\mathrm{Model}^\star_{**}$, given by

in which

- the maps φ_1, φ_2, φ_4, and φ_5 are the evident inclusions, and
- the maps φ_3 and φ_6 are given by collapsing vertically.

We now prove that each of these maps induces a weak equivalence upon application of $\mathrm{N}\left(\mathrm{Fun}_{**}^{\star}(-, \mathcal{D})^{\mathbf{W}}\right)$. The arguments can be grouped as follows.

- The fact that the maps φ_1 and φ_4 induce weak equivalences follows from Lemma 3.9(1).
- The maps φ_2 and φ_5 induce acyclic fibrations in $s\mathcal{S}et_{\mathrm{KQ}}$, since:
    - $\mathcal{D}$ is finitely bicomplete.
    - Limits and colimits are unique up to unique isomorphism.
    - The subcategories $(\mathbf{W} \cap \mathbf{C})_\mathcal{D}, (\mathbf{W} \cap \mathbf{F})_\mathcal{D} \subset \mathcal{D}$ are respectively closed under pushout and pullback.
  (See, e.g., [Lur09, Proposition 4.3.2.15].)
- Note that the maps φ_3 and φ_6 admit respective sections ψ_3 and ψ_6 in $\mathcal{M}odel_{**}^{\star}$. Moreover, there are evident doubly-pointed natural weak equivalences $\mathrm{id}_{\mathcal{I}_2} \to \psi_3\varphi_3$ and $\psi_6\varphi_6 \to \mathrm{id}_{\mathcal{I}_5}$. Hence, it follows from Lemma 3.10 that φ_3 and φ_6 induce homotopy equivalences in $s\mathcal{S}et_{\mathrm{KQ}}$.

The proof of item (2) is similar, but requires some modification: we will show that the above sequence induces weak equivalences in $s\mathcal{S}et_{\mathrm{KQ}}$ upon application of $\mathrm{N}\left(\mathrm{Fun}_{**}(-, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right)$ (note the lack of decorations).[10]

First of all, note that since we have assumed that $x \in \mathcal{C}^c$ and $y \in \mathcal{D}^f$, then all maps to $(\mathcal{C}^c + \mathcal{D}^f)$ in $\mathcal{M}odel_{**}$ from all doubly-pointed model diagrams in the above sequence must take the unmarked arrows to bridge arrows, with everything to the left mapping into $\mathcal{C}^c$ and everything to the right mapping into $\mathcal{D}^f$.

We now show that restriction along φ_2 induces a weak equivalence

$$\mathrm{N}\left(\mathrm{Fun}_{**}(\mathcal{I}_2, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right) \overset{\approx}{\to} \mathrm{N}\left(\mathrm{Fun}_{**}(\mathcal{I}_1, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right).$$

For this, we use the analogously defined object $(\mathcal{C}^c + \mathcal{D}) \in \mathcal{M}odel_{**}$, and we define a diagram

$$\mathcal{I}_{2a} \xrightarrow{\kappa_1} \mathcal{I}_{2b} \xleftarrow{\kappa_2} \mathcal{I}_{2c} \xrightarrow{\kappa_3} \mathcal{I}_{2d} \xleftarrow{\kappa_4} \mathcal{I}_{2e}$$

---

[10]In fact, the following argument also works for $\mathcal{D}$, but its additional complexity would just have been confusingly unnecessary if we had provided it above.

in $\mathcal{M}\mathrm{odel}^{\star}_{\ast\ast}$, given by
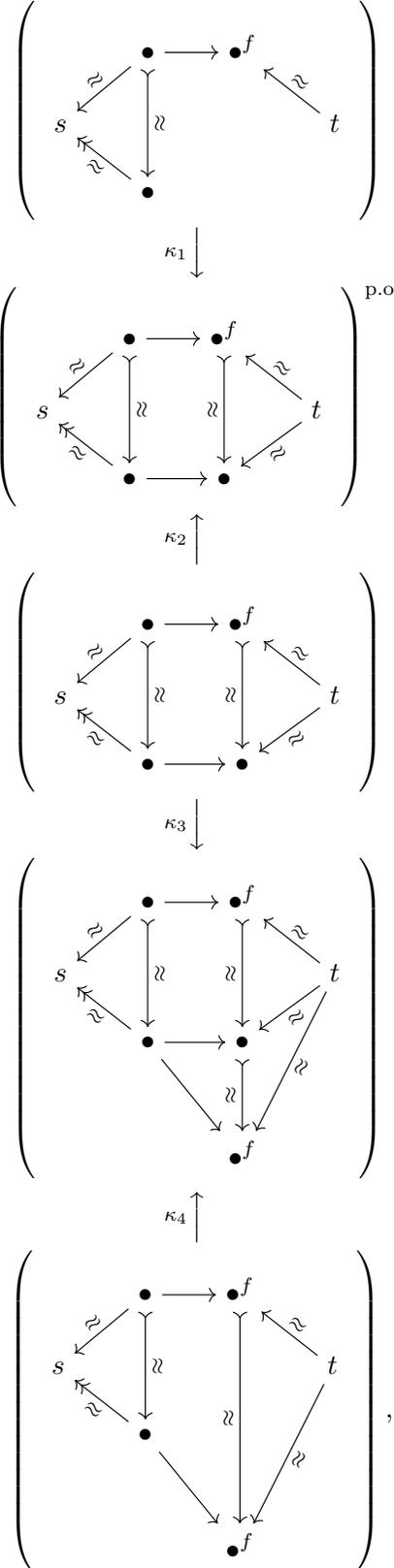
in which all maps are the evident inclusions. Then, we proceed by the following arguments.

- There is an evident isomorphism

$$\mathrm{Fun}_{**}(\mathcal{I}_1, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}} \cong \mathrm{Fun}_{**}^\star(\mathcal{I}_{2a}, (\mathcal{C}^c + \mathcal{D}))^{\mathbf{W}}.$$

- The map κ_1 induces an acyclic fibration

$$\mathrm{N}\left(\mathrm{Fun}_{**}^\star(\mathcal{I}_{2b}, (\mathcal{C}^c + \mathcal{D}))^{\mathbf{W}}\right) \xrightarrow{\overset{\approx}{\twoheadrightarrow}} \mathrm{N}\left(\mathrm{Fun}_{**}^\star(\mathcal{I}_{2a}, (\mathcal{C}^c + \mathcal{D}))^{\mathbf{W}}\right)$$

  in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ for the same reasons that φ_2 and φ_5 induced them above.

- The map κ_2 induces a weak equivalence

$$\mathrm{N}\left(\mathrm{Fun}_{**}^\star(\mathcal{I}_{2b}, (\mathcal{C}^c + \mathcal{D}))^{\mathbf{W}}\right) \xrightarrow{\approx} \mathrm{N}\left(\mathrm{Fun}_{**}^\star(\mathcal{I}_{2c}, (\mathcal{C}^c + \mathcal{D}))^{\mathbf{W}}\right)$$

  in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ since it is the nerve of a functor which admits a right adjoint. (Using the dual of the characterization of [MacL98, Chapter IV, §1, Theorem 2(ii)], to obtain such a right adjoint suffices to choose a coreflection in $\mathrm{Fun}_{**}^\star(\mathcal{I}_{2b}, (\mathcal{C}^c + \mathcal{D}))^{\mathbf{W}}$ of each object of $\mathrm{Fun}_{**}^\star(\mathcal{I}_{2c}, (\mathcal{C}^c + \mathcal{D}))^{\mathbf{W}}$, along with the corresponding component of the counit: to obtain a coreflection we can take a pushout of the span defined by our object, and the corresponding component of the counit will then be the canonical map.)

- The map κ_3 induces a weak equivalence

$$\mathrm{N}\left(\mathrm{Fun}_{**}^\star(\mathcal{I}_{2d}, (\mathcal{C}^c + \mathcal{D}))^{\mathbf{W}}\right) \xrightarrow{\approx} \mathrm{N}\left(\mathrm{Fun}_{**}^\star(\mathcal{I}_{2c}, (\mathcal{C}^c + \mathcal{D}))^{\mathbf{W}}\right)$$

  by Lemma 3.12 below.

- The inclusion κ_4 admits a retraction λ_4, given by taking the additional object in $\mathcal{I}_{2d}$ to the bottommost object in $\mathcal{I}_{2e}$. Moreover, there is an evident doubly-pointed natural weak equivalence $\mathrm{id}_{\mathcal{I}_{2d}} \to \kappa_4\lambda_4$. Hence, it follows from Lemma 3.10 that κ_4 induces a homotopy equivalence

$$\mathrm{N}\left(\mathrm{Fun}_{**}^\star(\mathcal{I}_{2d}, (\mathcal{C}^c + \mathcal{D}))^{\mathbf{W}}\right) \xrightarrow{\approx} \mathrm{N}\left(\mathrm{Fun}_{**}^\star(\mathcal{I}_{2e}, (\mathcal{C}^c + \mathcal{D}))^{\mathbf{W}}\right)$$

  in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$.

- There is an evident isomorphism

$$\mathrm{Fun}_{**}^\star(\mathcal{I}_{2e}, (\mathcal{C}^c + \mathcal{D}))^{\mathbf{W}} \cong \mathrm{Fun}_{**}(\mathcal{I}_2, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}.$$

As these weak equivalences are compatible with the map

$$\mathrm{N}\left(\mathrm{Fun}_{**}(\mathcal{I}_2, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right) \to \mathrm{N}\left(\mathrm{Fun}_{**}(\mathcal{I}_1, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right)$$

induced by restriction along φ_2 (in the sense that adding in the evident inclusion $\mathcal{I}_{2a} \to \mathcal{I}_{2e}$ in $\mathcal{M}\mathrm{odel}_{**}^\star$, to which when we apply $\mathrm{N}\left(\mathrm{Fun}_{**}^\star(-, (\mathcal{C}^c + \mathcal{D}))^{\mathbf{W}}\right)$ yields an isomorphic map to the one given by applying

$$\mathrm{N}\left(\mathrm{Fun}_{**}(-, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right)$$

to $\mathcal{I}_1 \xrightarrow{\varphi_2} \mathcal{I}_2$, yields a commutative diagram in $\mathcal{M}\mathrm{odel}_{**}$), we see that it is indeed a weak equivalence as well.

From here, a nearly identical dual argument (this time using $(\mathcal{C} + \mathcal{D}^f) \in$ Model_{**}) shows that restriction along φ_5 also induces a weak equivalence

$$\mathrm{N}\left(\mathrm{Fun}_{**}(\mathcal{I}_5, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right) \overset{\approx}{\to} \mathrm{N}\left(\mathrm{Fun}_{**}(\mathcal{I}_4, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right).$$

So we have proved that upon application of $\mathrm{N}\left(\mathrm{Fun}_{**}(-, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right)$, the maps φ_2 and φ_5 induce weak equivalences in $s\mathcal{S}et_{\mathrm{KQ}}$. That φ_1 and φ_4 induce weak equivalences follows from Lemma 3.9(2), and that φ_3 and φ_6 induce weak equivalences follows from the same argument as given above in the proof of item (1). □

We now prove a lemma that was used in the proof of Proposition 3.11.

Lemma 3.12. *Choose any decorated doubly-pointed model diagram* $\mathcal{I} \in$ $\mathrm{Model}_{**}^{\star}$, *choose an object* $a \in \mathcal{I}$ *which is connected by a zigzag in* $\mathbf{W}_{\mathcal{I}}$ *to the object* $t \in \mathcal{I}$, *and use it to define* $\mathcal{J} \in \mathrm{Model}_{**}^{\star}$ *by freely adjoining a new fibrant object and an acyclic cofibration* $a \overset{\approx}{\rightarrowtail} \bullet^f$ *to it from* a.[11] *Then, assuming that the target object of* $(\mathcal{C}^c + \mathcal{D})$ *lives in the subcategory* $\mathcal{D} \subset (\mathcal{C}^c + \mathcal{D})$, *the evident inclusion* $\mathcal{I} \overset{\varphi}{\to} \mathcal{J}$ *induces a weak equivalence*

$$\mathrm{N}\left(\mathrm{Fun}_{**}^{\star}(\mathcal{J}, (\mathcal{C}^c + \mathcal{D}))^{\mathbf{W}}\right) \overset{\approx}{\to} \mathrm{N}\left(\mathrm{Fun}_{**}^{\star}(\mathcal{I}, (\mathcal{C}^c + \mathcal{D}))^{\mathbf{W}}\right)$$

in $s\mathcal{S}et_{\mathrm{KQ}}$.

Proof. To ease notation, let us write this map as $\mathrm{N}(\mathbf{B}_1) \to \mathrm{N}(\mathbf{B}_2)$. Then, appealing to the dual of [Qui73, Theorem A], it suffices to prove that for any $b \in \mathbf{B}_2$, the comma category

$$\mathbf{B}_3 = \mathbf{B}_1 \times_{\mathbf{B}_2} (\mathbf{B}_2)_{b/}$$

has weakly contractible nerve.

For this, define the subcategory $\mathbf{B}_3' \subset \mathbf{B}_3$ on those objects $(c, b \to \varphi^*(c)) \in$ $\mathbf{B}_3$ such that the doubly-pointed natural weak equivalence $b \to \varphi^*(c)$ is actually id_b. Then $\mathbf{B}_3'$ is isomorphic to the category

$$(\mathbf{W}_{\mathcal{D}}^f)_{b(a)\downarrow} = \mathbf{W}_{\mathcal{D}}^f \times_{\mathbf{W}_{\mathcal{D}}} (\mathbf{W}_{\mathcal{D}})_{b(a)\downarrow}$$

of acyclic cofibrations from $b(a)$ to a fibrant object, which has weakly contractible nerve by applying the dual of [Qui73, Theorem A] to Lemma 3.13 (since $\mathrm{N}(\boldsymbol{\Delta}^{op})$ is weakly contractible).[12] On the other hand, to show that the map $\mathrm{N}(\mathbf{B}_3') \to \mathrm{N}(\mathbf{B}_3)$ is a weak equivalence, again appealing to [Qui73, Theorem A], it suffices to prove that for any $(c, b \to \varphi^*(c)) \in \mathbf{B}_3$, the comma category

$$\mathbf{B}_4 = \mathbf{B}_3' \times_{\mathbf{B}_3} (\mathbf{B}_3)_{/(c,b \to \varphi^*(c))}$$

has weakly contractible nerve.

---

[11]That is, if $\mathcal{I}$ already has an object marked as terminal then we add a new object equipped with a fibration to it; otherwise we add both.

[12]This statement also follows from applying the dual of [Hin05, Theorem A.3.2(2)] to the initial object of the model category $\mathcal{D}_{b(a)/}$.

Now, an object of $\mathbf{B}_4$ is given by the pair of an object $(c', b \overset{=}{\to} \varphi^*(c')) \in \mathbf{B}_3'$ and a morphism $(c', b \overset{=}{\to} \varphi^*(c')) \to (c, b \to \varphi^*(c))$ in $\mathbf{B}_3$. Unwinding the definitions, we see that the data of such an object is precisely that of a factorization of the composite $b(a) \overset{=}{\to} c'(a) \overset{\approx}{\to} c(a) \overset{\approx}{\rightarrowtail} c(\bullet^f)$ in $\mathcal{D} \subset (\mathcal{C}^c + \mathcal{D})$ through some composite $b(a) \overset{=}{\to} c'(a) \overset{\approx}{\rightarrowtail} c'(\bullet^f) \overset{\approx}{\to} c(\bullet^f)$, i.e., the specification of the upper right composite in a commutative square

$$
\begin{array}{ccc}
b(a) & \overset{\approx}{\rightarrowtail} & c'(\bullet^f) \\
{\scriptstyle \wr}\big\downarrow & & \big\downarrow{\scriptstyle \wr} \\
c(a) & \underset{\approx}{\rightarrowtail} & c(\bullet^f)
\end{array}
$$

in $\mathcal{D}$. Hence the category $\mathbf{B}_4$ is isomorphic to the category

$$
\mathbf{W}^{cf}_{\mathcal{D}_{b(a)/}} \times_{\mathbf{W}_{\mathcal{D}_{b(a)/}}} \left(\mathbf{W}_{\mathcal{D}_{b(a)/}}\right)_{/c(\bullet^f)}
$$

of left replacements of the fibrant object $c(\bullet^f) \in \mathcal{D}_{b(a)/}$ by a bifibrant object, the nerve of which is weakly contractible by [Hin05, Theorem A.3.2(2)]. $\square$

We now prove a lemma that was used in the proof of Lemma 3.12.

Lemma 3.13. *Any object $y \in \mathcal{D}$ admits a special simplicial replacement $y_\bullet \in s(\mathbf{W}^f_{\mathcal{D}})$ (as in [DwK80b, 4.3 and Remark 6.8]), and for any such choice, the corresponding map $\mathbf{\Delta}^{op} \overset{y_\bullet}{\longrightarrow} (\mathbf{W}^f_{\mathcal{D}})_{y\downharpoonleft}$ is homotopy right cofinal.*

Proof. The first statement is just [DwK80b, Proposition 4.5 and 6.7]. The second statement follows from combining [DwK80b, Proposition 6.11] with the following general fact: if a composite of functors is homotopy right cofinal and the second functor is fully faithful, then the first functor is also homotopy right cofinal. $\square$

3.2.2. From $\underline{\mathbf{3}}$-shaped zigzags to hammocks. In this subsection, we prove that the map

$$
\mathrm{N}\left(\mathrm{Fun}_{**}(\underline{\mathbf{3}}, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right) \to \hom_{\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f)}(x, y)
$$

is a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. Purely as a matter of terminology, we begin with a slight variant on [LowMG15, Definition 4.1], which is in turn a slight variant on the original definition of a "homotopy calculus of fractions" given in [DwK80a, 6.1(i)].

Definition 3.14. Let $(\mathcal{R}, \mathbf{W}_{\mathcal{R}})$ be a relative category, and let $x, y \in \mathcal{R}$. We say that the relative category $(\mathcal{R}, \mathbf{W}_{\mathcal{R}})$ admits a *homotopical three-arrow calculus* with respect to x and y if for all $i, j \geq 1$, the evident map

$$
\mathrm{N}\left(\mathrm{Fun}_{**}([\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}], \mathcal{R})^{\mathbf{W}}\right)
$$
$$
\to \mathrm{N}\left(\mathrm{Fun}_{**}([\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{W}^{-1}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}], \mathcal{R})^{\mathbf{W}}\right)
$$

is a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$.

This notion is useful for the following reason (which is where it gets its name).

Proposition 3.15. *If* $(\mathcal{R}, \mathbf{W}_\mathcal{R})$ *admits a homotopical three-arrow calculus with respect to* x *and* y, *then the map*

$$\mathrm{N}\left(\mathrm{Fun}_{**}(\underline{\mathbf{3}}, \mathcal{R})^{\mathbf{W}}\right) \to \hom_{\mathscr{L}^H(\mathcal{R})}(x, y)$$

is a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$.

Proof. This is essentially [DwK80a, Proposition 6.2]; that the proof carries over to the present setting is justified in [LowMG15, Theorem 4.5].[13]   □

Using this language, we can now prove the main result of this subsubsection.

Proposition 3.16. *The doubly-pointed relative categories*

(1) $(\mathcal{D}, \mathbf{W}_\mathcal{D})$, *where* $x, y \in \mathcal{D}$ *are any objects, and*
(2) $(\mathcal{C}^c + \mathcal{D}^f, \mathbf{W}_{\mathcal{C}^c + \mathcal{D}^f})$, *where* $x \in \mathcal{C}^c$ *and* $y \in \mathcal{D}^f$

admit homotopical three-arrow calculi.

Proof. Again, we begin with item (1) since it is somewhat simpler to prove. In this case, we define a diagram

$$[\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{W}^{-1}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}] \xrightarrow{\rho_1} \mathcal{J}_1 \xrightarrow{\rho_2} \mathcal{J}_2 \xleftarrow{\rho_3} [\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}]$$

in $\mathcal{M}\mathrm{odel}^\star_{**}$, given by the evident inclusions

$$\overset{\rho_3}{\longleftarrow} \left(\quad \overset{s}{\nwarrow} \overset{t}{\nearrow} \quad \right)$$

$$= [\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}].$$

We claim that this induces a diagram of weak equivalences in $s\mathcal{S}et_{\mathrm{KQ}}$ upon application of $\mathrm{N}\left(\mathrm{Fun}^{\star}_{**}(-, \mathcal{D})^{\mathbf{W}}\right)$ in a way compatible with the map in Definition 3.14 (in the sense that adding in the evident map which corepresents it gives a commutative diagram in $\mathcal{M}odel^{\star}_{**}$). We proceed by the following arguments.

- The map ρ_1 induces a weak equivalence by Lemma 3.9(1).
- The map ρ_2 induces an acyclic fibration in $s\mathcal{S}et_{\mathrm{KQ}}$ by repeatedly applying the argument for why the maps φ_2 and φ_5 induce them in the proof of Proposition 3.11(1).
- The inclusion ρ_3 admits a retraction σ_3 in $\mathcal{M}odel^{\star}_{**}$, given by collapsing the whole top row of $\mathcal{J}_2$ (besides the objects s and t) down onto the lower row, so that the "middle" backwards weak equivalence in the top row gets sent to the identity map on the "middle" object of the bottom row. If we define $\tau_3 \in \hom_{\mathcal{M}odel^{\star}_{**}}(\mathcal{J}_2, \mathcal{J}_2)$ to be the morphism which collapses the "left half" of $\mathcal{J}_2$ (besides the object s) down onto the lower row and leaves the "right half" unchanged, then there is an evident span

$$\mathrm{id}_{\mathcal{J}_2} \leftarrow \tau_3 \rightarrow \rho_3\sigma_3$$

  of doubly-pointed natural weak equivalences. Hence, it follows from Lemma 3.10 that ρ_3 induces a homotopy equivalence in $s\mathcal{S}et_{\mathrm{KQ}}$.

Thus, $\mathcal{D}$ admits a homotopical three-arrow calculus (with respect to any double-pointing).

For item (2), as in the proof of Proposition 3.11(2) we again modify the proof of item (1) by applying the functor $\mathrm{N}\left(\mathrm{Fun}_{**}(-, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right)$ to the given diagram (i.e., by ignoring decorations).

The first thing to observe here is that since no bridge arrows are weak equivalences, then the path components of both the source and the target of

$$\mathrm{N}\left(\mathrm{Fun}_{**}([\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}], (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right)$$

$$\rightarrow \mathrm{N}\left(\mathrm{Fun}_{**}([\mathbf{W}^{-1}; \mathbf{A}^{\circ i}; \mathbf{W}^{-1}; \mathbf{A}^{\circ j}; \mathbf{W}^{-1}], (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right)$$

decompose according to where the (necessarily unique) bridge arrow lies among the $(i + j)$ possibilities, and moreover the map respects these decompositions. For each $k \in \{1, \ldots, i + j\}$, let us use the ad hoc notation $\mathrm{Fun}^{\{k\}}_{**}(-, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}$ to denote the respective subcategories on those

zigzags where the kth copy of $\mathbf{A}$ gets sent to a bridge arrow; it suffices to show that the functor $\mathrm{N}\left(\mathrm{Fun}_{**}^{\{k\}}(-,(\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right)$ induces a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ for each k. We will focus on the case that $k \leq i$; the case that $k \geq i+1$ will follow from a completely dual argument.

- The fact that ρ_1 induces a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ follows from Lemma 3.9(2).
- To see that ρ_2 induces a weak equivalence, we will recycle arguments from the proof of Proposition 3.11(2). Unlike in the proof of item (1), we will need to separate the multiple steps in which we build the map $\mathcal{J}_1 \xrightarrow{\rho_2} \mathcal{J}_2$ into various cases. Note that by our assumption that $k \leq i$, the objects of $\mathrm{Fun}_{**}^{\{k\}}(\mathcal{J}_1, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}$ all select maps $\mathcal{J}_1 \to (\mathcal{C}^c + \mathcal{D}^f)$ such that the composite in $\mathcal{M}\mathrm{odel}$ with the evident inclusion

$$[(\mathbf{W} \cap \mathbf{F})^{-1}; (\mathbf{W} \cap \mathbf{C})^{-1}] \to \mathcal{J}_1 \to (\mathcal{C}^c + \mathcal{D}^f)$$

  lands in $\mathcal{D}^f \subset (\mathcal{C}^c + \mathcal{D}^f)$.
  - We begin by adding the new acyclic cofibrations one by one, moving to the right and working in $\mathcal{D}^f \subset (\mathcal{C}^c + \mathcal{D}^f)$. That these induce weak equivalences in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ upon applying

    $$\mathrm{N}\left(\mathrm{Fun}_{**}(-,(\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right)$$

    follows from a nearly identical argument to the one that the same functor induces one upon application to φ_2.
  - Then, we begin adding the new acyclic fibrations, moving to the left and working in $\mathcal{D}^f \subset (\mathcal{C}^c + \mathcal{D}^f)$ until we reach the bridge arrow. In this case, we can work entirely within $(\mathcal{C}^c + \mathcal{D}^f)$ (i.e., without using the auxiliary object $(\mathcal{C}^c + \mathcal{D}) \in \mathcal{M}\mathrm{odel}_{**}$) since the pullback of an acyclic fibration among fibrant objects will automatically be fibrant, and we can use completely dual arguments to the ones used to show that κ_1 and κ_2 induce weak equivalences to see that this again induces a weak equivalence.
  - Once we hit the bridge arrow and thereafter, we continue adding the new acyclic fibrations. But now, since we will be adding objects that are in $\mathcal{C}^c$ that we will originally construct via pullback in $\mathcal{C}$, we use the full strength of the argument that the functor $\mathrm{N}\left(\mathrm{Fun}_{**}(-,(\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right)$ induces a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$ upon application to φ_5.
- The fact that ρ_3 induces a weak equivalence follows from the same argument as given above in the proof of item (1).

Thus, $(\mathcal{C}^c + \mathcal{D}^f)$ admits a homotopical three-arrow calculus with respect to any $x \in \mathcal{C}^c$ and $y \in \mathcal{D}^f$. $\qquad\square$

3.3. The proof of Proposition 2.5. We now give a proof of the main ingredient in the proof of the main theorem, which is based on the arguments of [Man99, §7].

Proof of Proposition 2.5. We will prove that the map

$$\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f) \to \mathrm{cocart}(\mathscr{L}^H(F^c))$$

is a weak equivalence in $((\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}})_{/[1]}$; that the map

$$\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f) \to \mathrm{cart}(\mathscr{L}^H(G^f))$$

is also a weak equivalence will follow from a completely dual argument.

First of all, over $0 \in [1]$ this map is an isomorphism $\mathscr{L}^H(\mathcal{C}^c) \xrightarrow{\cong} \mathscr{L}^H(\mathcal{C}^c)$, while over $1 \in [1]$ this map is given by the inclusion $\mathscr{L}^H(\mathcal{D}^f) \hookrightarrow \mathscr{L}^H(\mathcal{D})$, which is a weak equivalence by Lemma 2.8. We claim that for any $x \in \mathcal{C}^c$ and any $y \in \mathcal{D}^f$, the induced map

$$\hom_{\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f)}(x, y) \to \hom_{\mathrm{cocart}(\mathscr{L}^H(F^c))}(x, y) = \hom_{\mathscr{L}^H(\mathcal{D})}(F^c(x), y)$$

is a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. From here it will follow easily that the induced map on homotopy categories (i.e., under the functor

$$\mathrm{ho} : \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \to \mathcal{C}\mathrm{at},$$

given locally by the product-preserving functor $\pi_0 : s\mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$) will be an equivalence of categories with target the analogously defined object

$$\mathrm{cocart}(\mathrm{ho}(F^c)) \in \mathcal{C}\mathrm{at}_{/[1]}.$$

Hence, we will have shown that the map $\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f) \to \mathrm{cocart}(\mathscr{L}^H(F^c))$ is indeed a weak equivalence in $((\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}})_{/[1]}$.

So, let $x \in \mathcal{C}^c$ and $y \in \mathcal{D}^f$. Let $x^\bullet \in c(\mathbf{W}^c_{\mathcal{C}})$ be a special cosimplicial resolution of x, and let $y_\bullet \in s(\mathbf{W}^f_{\mathcal{D}})$ be a special simplicial resolution of y (as in [DwK80b, 4.3 and Remark 6.8]; the existence of these resolutions is guaranteed by [DwK80b, Proposition 4.5 and 6.7]). Let us also define a simplicial set $M_\bullet \in s\mathcal{S}\mathrm{et}$ with

$$M_n = \coprod_{(\alpha,\beta) \in \hom_{\mathcal{C}\mathrm{at}}([n], \mathbf{\Delta} \times \mathbf{\Delta}^{op})} \hom_{(\mathcal{C}^c + \mathcal{D}^f)}(x^{\alpha(n)}, y_{\beta(0)})$$

$$\cong \coprod_{(\alpha,\beta) \in \hom_{\mathcal{C}\mathrm{at}}([n], \mathbf{\Delta} \times \mathbf{\Delta}^{op})} \hom_{\mathcal{D}}(F^c(x^{\alpha(n)}), y_{\beta(0)})$$

and with structure maps as in [Man99, §7]. Then, considering

$$(\mathcal{C}^c + \mathcal{D}^f) \in \mathcal{M}\mathrm{odel}_{**}$$

via the objects $x, y \in (\mathcal{C}^c + \mathcal{D}^f)$ and considering $\mathcal{D} \in \mathcal{M}\mathrm{odel}_{**}$ via the objects $F^c(x), y \in \mathcal{D}$, we claim that we obtain a commutative diagram

$$\text{diag}\left(\hom^{\text{lw}}_{(\mathcal{C}^c + \mathcal{D}^f)}(x^\bullet, y_\bullet)\right) \xrightarrow{\ \cong\ } \text{diag}\left(\hom^{\text{lw}}_{\mathcal{D}}(F^c(x^\bullet), y_\bullet)\right)$$

$$M_\bullet \xrightarrow{\ =\ } M_\bullet$$

$$\mathrm{N}\left(\mathrm{Fun}_{**}(\tilde{\underline{\mathbf{3}}}, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right) \longrightarrow \mathrm{N}\left(\mathrm{Fun}_{**}(\tilde{\underline{\mathbf{3}}}, \mathcal{D})^{\mathbf{W}}\right)$$

$$\mathrm{N}\left(\mathrm{Fun}_{**}(\underline{\mathbf{3}}, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right) \longrightarrow \mathrm{N}\left(\mathrm{Fun}_{**}(\underline{\mathbf{3}}, \mathcal{D})^{\mathbf{W}}\right)$$

$$\hom_{\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f)}(x, y) \longrightarrow \hom_{\mathscr{L}^H(\mathcal{D})}(F^c(x), y)$$

in $s\mathcal{S}et_{\mathrm{KQ}}$ (where the maps involving $M_\bullet$ are as described in [Man99, §7]), from which it will follow that the bottom map is indeed a weak equivalence as well.[14] We argue as follows.

- The first right vertical arrow is a weak equivalence by [Man99, Proposition 7.2], and hence (using the evident fact that the top horizontal map is indeed an isomorphism) we obtain that the first pair of vertical arrows are weak equivalences.[15]
- The second right vertical arrow is a weak equivalence by [Man99, Proposition 7.3]. The second left vertical arrow is a weak equivalence by a similar argument; we modify the one given there as follows.
  - We redefine $N_\bullet \in s\mathcal{S}et$ to be the simplicial replacement of the functor
    $$((\mathbf{W}_{\mathcal{C}}^c)_{\downarrow x})^{op} \times (\mathbf{W}_{\mathcal{D}}^f)_{y\updownarrow} \xrightarrow{\hom_{(\mathcal{C}^c + \mathcal{D}^f)}(-,-)} \mathcal{S}et.$$
  - The functor $\mathbf{\Delta}^{op} \xrightarrow{y_\bullet} (\mathbf{W}_{\mathcal{D}}^f)_{y\updownarrow}$ is again homotopy right cofinal by Lemma 3.13; the functor $\mathbf{\Delta} \xrightarrow{x^\bullet} (\mathbf{W}_{\mathcal{C}}^c)_{\downarrow x}$ is again homotopy left cofinal by its dual.
  - We redefine $P_{\bullet\bullet} \in ss\mathcal{S}et$ analogously to how we redefined $N_\bullet \in s\mathcal{S}et$ (i.e., requiring the chosen objects of $(\mathbf{W}_{\mathcal{C}})_{\downarrow x}$ to be cofibrant and requiring the chosen objects of $(\mathbf{W}_{\mathcal{D}})_{y\updownarrow}$ to be fibrant).

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[14]There is a small mistake in the description of the first pair of vertical arrows in [Man99, §7]: in the notation there, the map $f \in \hom_{\mathbf{\Delta}}([m], [p_m])$ should be given by $i \mapsto f_m \circ \cdots \circ f_{i+1}(p_i)$ for $i < m$ and $m \mapsto p_m$, and the map $g \in \hom_{\mathbf{\Delta}}([m], [q_0])$ should be given by $0 \mapsto 0$ and $i \mapsto g_1 \circ \cdots \circ g_i(0)$ for $i > 0$.

[15]Of course, the uppermost square is unnecessary from a strictly logical point of view, but it clarifies the connection between our proof and co/simplicial resolutions.

– Let us clarify why the asserted maps from $\mathrm{diag}(P_{\bullet\bullet})$ are weak equivalences in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$.[16]

* To see that the map

$$\mathrm{diag}(P_{\bullet\bullet}) \to \mathrm{N}\left(\mathrm{Fun}_{**}(\tilde{\underline{\mathbf{3}}}, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right)$$

is a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$, let us define the object $\mathrm{const}\left(\mathrm{N}\left(\mathrm{Fun}_{**}(\tilde{\underline{\mathbf{3}}}, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right)\right) \in ss\mathcal{S}\mathrm{et}$ by precomposition with the projection $\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathbf{\Delta}^{op}$ to the second factor. This admits an evident map

$$P_{\bullet\bullet} \to \mathrm{const}\left(\mathrm{N}\left(\mathrm{Fun}_{**}(\tilde{\underline{\mathbf{3}}}, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right)\right)$$

which yields the original map when we apply

$$\mathrm{diag} : ss\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}.$$

By [GJ99, Chapter IV, Proposition 1.9] it suffices to show that this is a levelwise weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. In level n, this is given by the map

$$P_{\bullet,n} \to \mathrm{N}\left(\mathrm{Fun}_{**}(\tilde{\underline{\mathbf{3}}}, (\mathcal{C}^c + \mathcal{D}^f))^{\mathbf{W}}\right)_n,$$

whose target is a discrete (i.e., constant) simplicial set. Moreover, the fiber over any point of the target is the nerve of a category with an initial object, and hence it is weakly contractible in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$.

* To see that the map $\mathrm{diag}(P_{\bullet\bullet}) \to N_\bullet$ is a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$, let us define $N_{\bullet\bullet} \in ss\mathcal{S}\mathrm{et}$ by

$$N_{m,n} = \coprod_{(\alpha,\beta) \in \mathrm{hom}_{\mathcal{C}\mathrm{at}}\left([n], (\mathbf{W}_{\mathcal{C}}^c)_{\downarrow x}\right) \times \mathrm{hom}_{\mathcal{C}\mathrm{at}}\left([m], (\mathbf{W}_{\mathcal{D}}^f)_{y\downarrow}\right)} \mathrm{hom}_{(\mathcal{C}^c + \mathcal{D}^f)}(\alpha(n), \beta(0)).$$

Note that this has $\mathrm{diag}(N_{\bullet\bullet}) \cong N_\bullet$, and moreover it admits an evident map $P_{\bullet\bullet} \to N_{\bullet\bullet}$ which yields the original map when we apply $\mathrm{diag} : ss\mathcal{S}\mathrm{et} \to s\mathcal{S}\mathrm{et}$. Hence, again by [GJ99, Chapter IV, Proposition 1.9], it suffices to show that for each $n \geq 0$, the map $P_{n,\bullet} \to N_{n,\bullet}$ is a weak equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$. In fact, it is not hard to see that this last map admits a section which defines a homotopy equivalence in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$.

• The third pair of vertical arrows are weak equivalences by Proposition 3.11.

• The fourth pair of vertical arrows are weak equivalences by Propositions 3.15 and 3.16.

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[16]We work in the modified situation of $(\mathcal{C}^c + \mathcal{D}^f)$, but the clarifications equally well clarify the argument given in the original proof of [Man99, Proposition 7.3]; these clarifications actually come from private correspondence with Mandell regarding the original proof.

Thus, the map $\mathscr{L}^H(\mathcal{C}^c + \mathcal{D}^f) \to \mathrm{cocart}(\mathscr{L}^H(F^c))$ is indeed a weak equivalence in $((\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}})_{/[1]}$. $\qquad\qquad\square$

Appendix A. A history of partial answers to Question 0.1

In this appendix, we survey the results that are either explicitly stated in the existing literature or can be extracted therefrom surrounding the question of providing external, homotopy-theoretic meaning to the notions of Quillen adjunctions and Quillen equivalences.

A.1. Derived adjunctions and derived equivalences. Quillen [Qui67] proved the following results (which appear together as [Qui67, Chapter I, §4, Theorem 3]).

- A Quillen adjunction induces a canonical adjunction between homotopy categories, called the *derived adjunction* of the Quillen adjunction.
- In the special case of a Quillen equivalence, the derived adjunction actually defines an equivalence of categories, called the *derived equivalence* of the Quillen equivalence.

A.2. Enhancements to ho($s\mathcal{S}\mathrm{et}_{\mathbf{KQ}}$)-enriched categories. Dwyer–Kan [DwK80a] introduced their *hammock localization* construction, which takes any relative category — and hence in particular a model category — and yields a category enriched in simplicial sets. As $s\mathcal{S}\mathrm{et}$-enriched categories provide a model for "the homotopy theory of homotopy theories", this laid the foundations for the following enhancements of Quillen's results that they proved (which appear together by combining [DwK80b, Propositions 5.4 and 4.4]).[17,18]

- A Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ induces weak equivalences

$$\mathrm{hom}_{\mathscr{L}^H(\mathcal{C})}(x, G(y)) \approx \mathrm{hom}_{\mathscr{L}^H(\mathcal{D})}(F(x), y)$$

  in $s\mathcal{S}\mathrm{et}_{\mathbf{KQ}}$ for every $x \in \mathcal{C}$ and every $y \in \mathcal{D}$.
- A Quillen equivalence $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ induces weak equivalences $\mathscr{L}^H(F^c) : \mathscr{L}^H(\mathcal{C}^c) \xrightarrow{\approx} \mathscr{L}^H(\mathcal{D}^c)$ and $\mathscr{L}^H(\mathcal{C}^f) \xleftarrow{\approx} \mathscr{L}^H(\mathcal{D}^f) : \mathscr{L}^H(G^f)$ in $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$. As illustrated in Figure 1, it follows that $\mathscr{L}^H(\mathcal{C})$ and $\mathscr{L}^H(\mathcal{D})$ are weakly equivalent objects of $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$.

Note that the first result does *not* posit the existence of any sort of adjunction. Indeed, this is a very subtle issue. What we have so far is the diagram in $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ of Figure 1, in which the fact that the indicated inclusions are weak equivalences follows from [DwK80b, Proposition 5.2] (or Lemma 2.8, see Remark 2.9).

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[17]There is a small typo in the statement of [DwK80b, Proposition 5.4]: it should involve a *cosimplicial* resolution of the source and a *simplicial* resolution of the target.

[18]The proof of [DwK80b, Proposition 4.4] contains a mistake, which is both explained and corrected in [Dug09] and is corrected in [Man99, §7].

$$\begin{array}{ccc}
\mathscr{L}^H(\mathcal{C}^c) & \xrightarrow{\;\mathscr{L}^H(F^c)\;} & \mathscr{L}^H(\mathcal{D}^c) \\
{\scriptstyle\approx}\downarrow & & \downarrow{\scriptstyle\approx} \\
\mathscr{L}^H(\mathcal{C}) & & \mathscr{L}^H(\mathcal{D}), \\
{\scriptstyle\approx}\downarrow & & \uparrow{\scriptstyle\approx} \\
\mathscr{L}^H(\mathcal{C}^f) & \xleftarrow{\;\mathscr{L}^H(G^f)\;} & \mathscr{L}^H(\mathcal{D}^f)
\end{array}$$
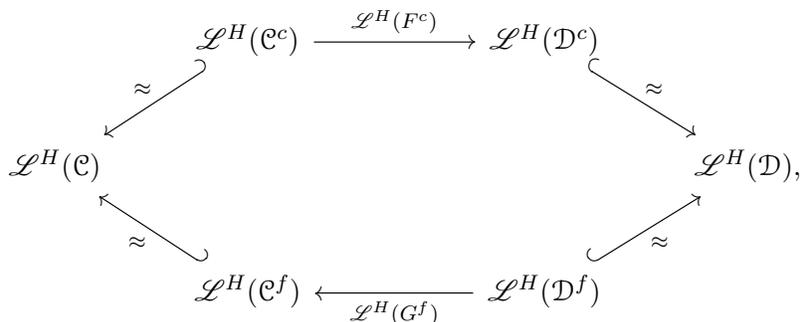
FIGURE 1. The hexagon in $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ arising from a Quillen adjunction.

Now, a weak equivalence in $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ induces an equivalence of $\mathrm{ho}(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})$-enriched categories.[19] Hence, if we apply the "enriched homotopy category" functor $\mathrm{ho}^{\mathrm{enr}} : \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}} \to \mathcal{C}\mathrm{at}_{\mathrm{ho}(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})}$ to the above diagram, we can choose enriched inverse equivalences to the upper-left and lower-right inclusions, and then the upper and lower composites will respectively be candidates for the left and right adjoints of a $\mathrm{ho}(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})$-enriched adjunction between $\mathrm{ho}^{\mathrm{enr}}(\mathscr{L}^H(\mathcal{C}))$ and $\mathrm{ho}^{\mathrm{enr}}(\mathscr{L}^H(\mathcal{D}))$.

However, things are still not so clean as this. The weak equivalences between corresponding hom-objects in $\mathscr{L}^H(\mathcal{C})$ and $\mathscr{L}^H(\mathcal{D})$ pass through the co/simplicial resolutions of [DwK80b, 4.3], and apparently nowhere in the literature are these shown to give *functorially* weakly equivalent simplicial sets to the hom-objects in the hammock localizations, at least not in full generality. In fact, the main purpose of [Low] is to show that these weak equivalences are indeed functorial (in $\mathrm{ho}(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})$) when the model category admits functorial factorizations (although Low mentions in that paper that he intends to return to the general case in future work).

But even if these weak equivalences were shown to be functorial, we still would not immediately obtain a $\mathrm{ho}(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})$-enriched adjunction. Rather, we would need to choose our enriched inverse equivalences to be enriched *adjoint* equivalences, in order to select preferred and functorial isomorphisms in $\mathrm{ho}^{\mathrm{enr}}(\mathscr{L}^H(\mathcal{C}))$ and $\mathrm{ho}^{\mathrm{enr}}(\mathscr{L}^H(\mathcal{D}))$ (via the unit or counit) between objects and their images under the retractions.[20]

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[19]Equivalences of enriched categories are precisely the enriched functors which are essentially surjective on objects and induce isomorphisms on hom-objects (see [Kel05, §1.11]).

[20]Adjoint equivalences would be guaranteed by the existence of functorial factorizations in $\mathcal{C}$ and $\mathcal{D}$ (or even just functorial cofibrant replacement in $\mathcal{C}$ and functorial fibrant replacement in $\mathcal{D}$), but such assumptions are unnecessary since we are ultimately only working at the $\mathrm{ho}(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})$-enriched level anyways: just as in ordinary category theory, an enriched functor is an enriched equivalence if and only if it admits an enriched adjoint equivalence (again see [Kel05, §1.11]).

A.3. Enhancements to quasicategories. It has been established in the literature that certain Quillen adjunctions satisfying additional hypotheses induce adjunctions of quasicategories.

A.3.1. Model categories with functorial replacements. Lurie proves as [Lur09, Proposition 5.2.2.8] that a pair of functors between quasicategories are adjoints if and only if there exists a "unit transformation" with the expected behavior at the level of $\mathrm{ho}(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})$-enriched homotopy categories (see [Lur09, Definition 5.2.2.7]).

However, it is a subtle matter to obtain such a unit transformation. Note that a Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ gives rise to a unit transformation $\mathrm{id}_{\mathcal{C}} \to GF$ of endofunctors on the underlying category $\mathcal{C}$, but its target GF will not generally be a relative endofunctor. The standard fix is to take cofibrant replacements in $\mathcal{C}$ before applying F and fibrant replacements in $\mathcal{D}$ before applying G. Of course, in order to obtain a unit transformation, these replacements must be functorial. Let us assume we are in the usual situation in which such replacement functors exist, namely that they are obtained as special cases of functorial factorizations; we denote them by

$$\mathbb{Q}^{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^c \hookrightarrow \mathcal{C} \quad \text{and} \quad \mathbb{R}^{\mathcal{D}} : \mathcal{D} \to \mathcal{D}^f \hookrightarrow \mathcal{D},$$

and we denote their corresponding replacement transformations by

$$\mathbb{Q}^{\mathcal{C}} \xrightarrow{q^{\mathcal{C}}} \mathrm{id}_{\mathcal{C}} \quad \text{and} \quad \mathrm{id}_{\mathcal{D}} \xrightarrow{r^{\mathcal{D}}} \mathbb{R}^{\mathcal{D}}.$$

Now, we are interested in obtaining a unit map for the relative endofunctor $G\mathbb{R}^{\mathcal{D}}F\mathbb{Q}^{\mathcal{C}}$ on $(\mathcal{C}, \mathbf{W}_{\mathcal{C}})$, at least at the level of its underlying quasicategory. The first thing to note here is that we cannot proceed by passing through hammock localizations, since the functor $\mathscr{L}^H : \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$ does not preserve natural transformations.[21] On the other hand, the relative functor $\mathrm{N}^{\mathrm{R}} : \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at} \to ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ preserves products (being pointwise corepresented), and from this it is not hard to see that it preserves natural transformations and takes natural weak equivalences to natural equivalences (in the evident internal sense in $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$); since the model category $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ is compatibly cartesian closed (see [Rez01, Theorem 7.2]) and all its objects are cofibrant, we view this as an acceptable substitute. Hence, up to the contractible ambiguity in the various functors between models for "the homotopy theory of homotopy theories", we may consider a natural transformation or natural weak equivalence between relative functors

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[21]Rather, given $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}$ and a morphism $F_1 \to F_2$ in $\mathrm{Fun}(\mathcal{C}_1, \mathcal{C}_2)^{\mathbf{W}}$, for any $x, y \in \mathcal{C}_1$ we obtain a natural cospan of weak equivalences

$$\hom_{\mathscr{L}^H(\mathcal{C}_2)}(F_1(x), F_1(y)) \xrightarrow{\approx} \hom_{\mathscr{L}^H(\mathcal{C}_2)}(F_1(x), F_2(y)) \xleftarrow{\approx} \hom_{\hom(\mathcal{C}_2)}(F_2(x), F_2(y))$$

in $s\mathcal{S}\mathrm{et}_{\mathrm{KQ}}$, and combining this with the span

$$\hom_{\mathscr{L}^H(\mathcal{C}_2)}(F_1(x), F_1(y)) \leftarrow \hom_{\mathscr{L}^H(\mathcal{C}_1)}(x, y) \to \hom_{\mathscr{L}^H(\mathcal{C}_2)}(F_2(x), F_2(y))$$

yields a square which commutes up to a specified homotopy (see [DwK80a, Propositions 3.5 and 3.3]).

between relative categories as giving natural transformations and natural equivalences between the corresponding functors between their underlying quasicategories.

From here, the most direct way to proceed would be to obtain the unit map from the natural zigzag

$$x \xleftarrow[\approx]{q_x^{\mathcal{C}}} \mathbb{Q}^{\mathcal{C}}(x) \xrightarrow{\eta_{\mathbb{Q}^{\mathcal{C}}(x)}} G\left(F\left(\mathbb{Q}^{\mathcal{C}}(x)\right)\right) \xrightarrow{G\left(r^{\mathcal{D}}_{F(\mathbb{Q}^{\mathcal{C}}(x))}\right)} G\left(\mathbb{R}^{\mathcal{D}}\left(F\left(\mathbb{Q}^{\mathcal{C}}(x)\right)\right)\right)$$

in $\mathcal{C}$ in which the backwards arrow is a weak equivalence; assembling these across all $x \in \mathcal{C}$, we obtain a span

$$\mathrm{id}_{\mathcal{C}} \xleftarrow[\approx]{q^{\mathcal{C}}} \mathbb{Q}^{\mathcal{C}} \xrightarrow{G(r^{\mathcal{D}})\circ\eta} G\mathbb{R}^{\mathcal{D}}F\mathbb{Q}^{\mathcal{C}}$$

between relative endofunctors on $(\mathcal{C}, \mathbf{W}_{\mathcal{C}})$ in which the backwards arrow is a natural weak equivalence. Passing through $ss\mathcal{S}\mathrm{et}_{\mathrm{Rezk}}$ as discussed above (and implicitly identifying the two different ways of passing from $\mathcal{R}\mathrm{el}\mathcal{C}\mathrm{at}_{\mathrm{BK}}$ to $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$), we obtain a span

$$\mathrm{id}_{\mathrm{u.q.}(\mathcal{C})} \xleftarrow[\sim]{\mathrm{u.q.}(q^{\mathcal{C}})} \mathrm{u.q.}\left(\mathbb{Q}^{\mathcal{C}}\right) \xrightarrow{\mathrm{u.q.}(G(r^{\mathcal{D}})\circ\eta)} \mathrm{u.q.}\left(G\mathbb{R}^{\mathcal{D}}F\mathbb{Q}^{\mathcal{C}}\right)$$

in which the backwards arrow is an equivalence.[22] Hence, we can obtain a candidate unit transformation $\mathrm{id}_{\mathrm{u.q.}(\mathcal{C})} \to \mathrm{u.q.}\left(G\mathbb{R}^{\mathcal{D}}F\mathbb{Q}^{\mathcal{C}}\right)$, which one might then hope to verify satisfies the hypotheses of [Lur09, Definition 5.2.2.7] using e.g., the co/simplicial resolutions of [DwK80b]. Of course, this requires knowing that the hom-objects obtained from co/simplicial resolutions are indeed *functorially* weakly equivalent to the hom-objects in the hammock localizations, but at least this follows from [Low] in the case that $\mathcal{C}$ and $\mathcal{D}$ both admit functorial factorizations, as mentioned above.

Remark A.1. This approach would also work if the cofibrant replacement functor $\mathbb{Q}^{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ were augmented (instead of coaugmented), and in fact we would also obtain a candidate unit transformation if the fibrant replacement functor $\mathbb{R}^{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ were coaugmented (instead of augmented). On the other hand, because of the way model categories are set up, it seems that such replacement functors do not arise very frequently in practice.

Remark A.2. Of course, if all objects of $\mathcal{C}$ are cofibrant then the identity functor can serve as a cofibrant replacement functor; a dual observation holds for $\mathcal{D}$.

Remark A.3. Actually, slightly more cleverly, we can use a similar argument to the one given above to obtain a natural transformation between the standard inclusion $\mathcal{C}^c \hookrightarrow \mathcal{C}$ and the composite

$$\mathcal{C}^c \xrightarrow{F^c} \mathcal{D}^c \hookrightarrow \mathcal{D} \xrightarrow{\mathbb{R}^{\mathcal{D}}} \mathcal{D}^f \xrightarrow{G} \mathcal{C};$$

---

[22]Note that we are now working *internally* to a quasicategory, namely the quasicategory of endofunctors of u.q.($\mathcal{C}$).

this yields a natural transformation between functors from u.q.($\mathcal{C}^c$) to u.q.($\mathcal{C}$), and (horizontally) precomposing with a retraction to the acyclic cofibration u.q.($\mathcal{C}^c$) $\overset{\approx}{\rightarrowtail}$ u.q.($\mathcal{C}$) in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$ yields a candidate unit transformation, all without requiring that $\mathcal{C}$ admit any sort of cofibrant replacement functor. Dually, one can obtain a candidate counit transformation if one assumes that $\mathcal{C}$ has a cofibrant replacement functor but without assuming that $\mathcal{D}$ admit any sort of fibrant replacement functor.

Remark A.4. Instead of assuming the existence of appropriate co/fibrant replacement functors, one might alternatively extract retractions

$$\mathrm{u.q.}(\mathcal{C}) \overset{\approx}{\to} \mathrm{u.q.}(\mathcal{C}^c) \quad \text{and} \quad \mathrm{u.q.}(\mathcal{D}) \overset{\approx}{\to} \mathrm{u.q.}(\mathcal{D}^f)$$

in $s\mathcal{S}\mathrm{et}_{\mathrm{Joyal}}$, i.e., at the level of underlying quasicategories. However, it appears that the original adjunction $F \dashv G$ will be entirely lost by this point, and hence that one cannot hope to provide the desired unit transformation in full generality using this approach.

Remark A.5. Lurie proves that a Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ between combinatorial model categories gives rise to a colimit-preserving functor u.q.(F^c) : u.q.($\mathcal{C}^c$) $\to$ u.q.($\mathcal{D}^c$) of quasicategories (see [Lur14, Corollary 1.3.4.26]). He also proves that these quasicategories are presentable, and hence deduces from the adjoint functor theorem that this must therefore be a left adjoint, and moreover that one can obtain a right adjoint from the composite

$$\mathcal{D}^c \hookrightarrow \mathcal{D} \xrightarrow{\mathbb{R}^{\mathcal{D}}} \mathcal{D}^f \xrightarrow{G} \mathcal{C} \xrightarrow{\mathbb{Q}^{\mathcal{C}}} \mathcal{C}^c$$

(see [Lur14, Remark 1.3.4.27]). Of course, since combinatorial model categories automatically admit replacement functors, this result can also be recovered from the preceding argument.

A.3.2. Simplicial model categories. Dwyer–Kan prove that given a simplicial model category $\mathcal{C}_\bullet$, the two possible notions of "underlying homotopy theory" agree: the full $s\mathcal{S}\mathrm{et}$-enriched subcategory $\mathcal{C}_\bullet^{cf}$ of bifibrant objects is equivalent (via a zigzag of weak equivalences in $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$) to the hammock localization $\mathscr{L}^H(\mathcal{C})$ of the underlying model category (see [DwK80b, Proposition 4.8]).[23] This paved the way for the following enhancement of their results.

First of all, Lurie proves as [Lur09, Proposition 5.2.4.6] — and Riehl–Verity later re-prove as [RV15, Theorem 6.2.1] — that a simplicial Quillen adjunction of simplicial model categories $F_\bullet : \mathcal{C}_\bullet \rightleftarrows \mathcal{D}_\bullet : G_\bullet$ (that is, an enriched adjunction in $\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}}$ which is moreover a Quillen adjunction on underlying model categories) induces an adjunction between the quasicategories

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[23]Note that in the statement of [DwK80b, Proposition 4.8], the right arrow should also be labeled as a weak equivalence in $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$, there called simply a "weak equivalence".

$\mathrm{N}^{\mathrm{hc}}(\mathcal{C}^{cf}_\bullet)$ and $\mathrm{N}^{\mathrm{hc}}(\mathcal{D}^{cf}_\bullet)$.[24] (Note that the objects $\mathcal{C}^{cf}_\bullet, \mathcal{D}^{cf}_\bullet \in (\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$ are already fibrant, and hence do not require fibrant replacement.)

Moreover, there are various results concerning the replacement of model categories and of Quillen equivalences by simplicial ones.

- In [Dug01], Dugger shows that a model category which is left proper and is additionally either cellular or combinatorial admits a left Quillen equivalence to a simplicial model category (see [Dug01, Theorem 1.2 or 6.1]).
- In [RSS01], Rezk–Schwede–Shipley work with model categories that are left proper, cofibrantly generated (under a slightly stronger definition than the usual one, see [RSS01, Definition 8.1]), and satisfy their "realization axiom" (see [RSS01, Axiom 3.4]), and prove:
  - Every such model category admits a left Quillen equivalence to a simplicial model category (see [RSS01, Theorem 3.6]).
  - A Quillen adjunction between such model categories induces a simplicial Quillen adjunction between their replacements by simplicial model categories (see [RSS01, Proposition 6.1]).

Whenever these results can be used to upgrade a Quillen adjunction to a simplicial Quillen adjunction (see [BluR14, §A] for an expanded summary of these techniques), then by combining Lurie's result with the Dwyer–Kan result cited earlier (that Quillen equivalences induce weak equivalences in $(\mathcal{C}\mathrm{at}_{s\mathcal{S}\mathrm{et}})_{\mathrm{Bergner}}$), we obtain from the original Quillen adjunction an adjunction of underlying quasicategories.

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[24]Neither of these sources defines model categories to come equipped with functorial factorizations, although Lurie assumes bicompleteness (whereas Riehl–Verity only assume finite bicompleteness). On the other hand, Lurie's proof readily adapts to the more general case.

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