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Generating functions and statistics on spaces of maximal tori in classical Lie groups

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Abstract. In this paper we use generating function methods to obtain new asymptotic results about spaces of F-stable maximal tori in $\mathrm{GL}_n(\overline{F_q})$, $\mathrm{Sp}_{2n}(\overline{F_q})$, and $\mathrm{SO}_{2n+1}(\overline{F_q})$. We recover stability results of Church–Ellenberg–Farb and Jiménez Rolland–Wilson for "polynomial" statistics on these spaces, and we compute explicit formulas for their stable values. We derive a double generating function for the characters of the cohomology of flag varieties in type B/C, which we use to obtain analogs in type B/C of results of Chen: we recover "twisted homological stability" for the spaces of maximal tori in $\mathrm{Sp}_{2n}(\mathbb{C})$ and $\mathrm{SO}_{2n+1}(\mathbb{C})$, and we compute a generating function for their "stable twisted Betti numbers". We also give a new proof of a result of Lehrer using symmetric function theory.

Contents

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1. Introduction

In recent work, Church, Ellenberg, and Farb [ChuEF14] and Jiménez Rolland and Wilson [JRW16] proved asymptotic stability results for certain 'polynomial' statistics on the set of maximal tori in $\mathrm{GL}_n(\overline{F_q})$, $\mathrm{Sp}_{2n}(\overline{F_q})$, and $\mathrm{SO}_{2n+1}(\overline{F_q})$ that are stable under the action of the Frobenius morphism F. In this paper we use generating function techniques to give new proofs of these results, and moreover compute explicit formulas for the limiting values of these statistics. We then compute a double generating function for the characters of the cohomology of flag varieties in type B/C. With these computations we derive analogs in type B/C of results of Chen [Che16] in type A: we prove "twisted homological stability" for the spaces of maximal tori in $\mathrm{Sp}_{2n}(\mathbb{C})$ and $\mathrm{SO}_{2n+1}(\mathbb{C})$, and we give a generating function for the "stable twisted Betti numbers". The form of this generating function implies that the stable Betti numbers are quasipolynomial and satisfy linear recurrence relations.

1.1. Statistics on maximal tori in classical Lie groups. The paper [ChuEF14] proved results on maximal tori in type A, and [JRW16] on maximal tori in types B and C, by using the following formulas. These formulas, first obtained by Lehrer, relate the representation theory of the Lie groups' associated *coinvariant algebras* with point-counts on the set of F-stable maximal tori. Each F-stable maximal torus T of $\mathrm{GL}_n(\overline{F_q})$ is naturally associated to a conjugacy class of the symmetric group S_n; this correspondence is explained in Carter [Car85, Chapter 3] or Niemeyer–Praeger [NP10, Sections 2 and 3]. Hence any class function χ of S_n can be interpreted as a function on maximal tori, with $\chi(T)$ defined to be the value of χ in the corresponding S_n-conjugacy class. Similarly, for each F-stable maximal torus T of $\mathrm{Sp}_{2n}(\overline{F_q})$ or $\mathrm{SO}_{2n+1}(\overline{F_q})$ there is an associated conjugacy class of the hyperoctahedral group B_n. Lehrer proved the following.

Theorem 1.1 (Statistics on maximal tori [Leh92, Corollary 1.10])**.**

(i) (Type A). *Let χ be a class function on S_n and let $T(n,q)$ denote the set of F-stable maximal tori of $\mathrm{GL}_n(\overline{F_q})$. Then*

$$\frac{1}{q^{n^2-n}} \sum_{T \in T(n,q)} \chi(T) = \sum_{i=0}^{\binom{n}{2}} q^{-i} \langle \chi, R_n^i \rangle_{S_n} \tag{1}$$

where R_n^i is the i^{th} graded piece of the coinvariant algebra R_n^* in type A, and $\langle -, - \rangle_{S_n}$ is the standard inner product on S_n class functions.

(ii) (Type B/C). *Let χ be a class function on B_n and let $T(n,q)$ denote the set of F-stable maximal tori of $\mathrm{SO}_{2n+1}(\overline{F_q})$ or $\mathrm{Sp}_{2n}(\overline{F_q})$. Then*

$$\frac{1}{q^{2n^2}} \sum_{T \in T(n,q)} \chi(T) = \sum_{i=0}^{n^2} q^{-i} \langle \chi, R_n^i \rangle_{B_n}, \tag{2}$$

where R_n^i is the i^{th} graded piece of the coinvariant algebra R_n^* in type B/C, and $\langle -, - \rangle_{B_n}$ is the standard inner product on B_n class functions.

It will always be clear from context whether $T(n,q)$ refers to type A or to type B/C. In Section 2 we give a new proof of Lehrer's result in type A, Theorem 1.1(i), using symmetric function theory. See Church–Ellenberg–Farb [ChuEF14, Theorem 5.3] for a proof using the Grothendieck–Lefschetz Theorem.

To obtain their asymptotic stability results on polynomial statistics on the space of maximal tori, Church–Ellenberg–Farb [ChuEF14] and Jiménez Rolland–Wilson [JRW16] study the right-hand side of the formulas in Theorem 1.1 as n tends to infinity. They use results from the field of *representation stability* to show that for sequences of characters χ_n that are determined by a *character polynomial* (Definitions 1.2 and 1.4), the quantities $\langle \chi_n, R_n^i \rangle_{S_n}$ and $\langle \chi_n, R_n^i \rangle_{B_n}$ stabilize for n sufficiently large relative to i. They then prove that the stable values are subexponential in i, and so conclude that the series on the right-hand side of Equations (1) and (2) must converge.

In this paper we study instead the left-hand side of the formulas in Theorem 1.1, using generating function techniques analogous to the work of Fulman [Ful16] and later Chen [Che16]. In Theorem 1.6 we give explicit formulas for the limiting values of the left-hand sides of these equations for sequences of characters defined by a character polynomial.

1.2. Character polynomials in type A and B/C. Character polynomials for the symmetric groups are implicit in the work of Murnaghan and Specht, and are studied explicitly by Macdonald [Mac95, I.7.14]. Here we recall their definitions and their analogs in type B/C .

Definition 1.2 (The functions X_r; character polynomials in type A)**.** For a permutation $\sigma \in S_n$ and integer $r \geq 1$, let $X_r(\sigma)$ denote the number of cycles of size r in the cycle decomposition of σ. A polynomial in the functions X_r is called a *character polynomial* for S_n; a character polynomial determines a class function on S_n for each $n \geq 0$. Character polynomials over $\mathbb{Q}$ form a graded ring with generator X_r in degree r.

Definition 1.3 (The S_n character polynomials $\binom{X}{\lambda}$)**.** Given a partition λ, let $|\lambda|$ be the size of λ, and let $n_r(\lambda)$ denote the number of parts of λ of

length r. In this notation, define a character polynomial $\binom{X}{\lambda}$ of degree $|\lambda|$ by

$$\binom{X}{\lambda}(\sigma) := \prod_{r=1}^{|\lambda|} \binom{X_r(\sigma)}{n_r(\lambda)} \qquad \text{for all } \sigma \in S_n, \text{ for all } n \in \mathbb{Z}_{\geq 0}.$$

Character polynomials of the form $\binom{X}{\lambda}$ span the space of character polynomials with rational coefficients.

Recall that the conjugacy classes of B_n are parameterized by *double partitions* of n, that is, ordered pairs of partitions (μ, λ) such that $|\mu|+|\lambda| = n$. We adopt the convention that the parts of μ encode the lengths of the positive cycles of a signed permutation, and the parts of λ encode the lengths of the negative cycles.

Definition 1.4 (The functions X_r and Y_r; character polynomials in type B/C)**.** For $\sigma \in B_n$, let $X_r(\sigma)$ be the number of positive r-cycles of σ, and let $Y_r(\sigma)$ be the number of negative r-cycles. A *character polynomial* for B_n is a polynomial in the graded ring $\mathbb{Q}[X_1, Y_1, X_2, Y_2, \ldots]$. Its degree is defined by assigning $\deg X_r = \deg Y_r = r$ for each $r \geq 1$.

Definition 1.5 (The B_n character polynomials $\binom{X}{\mu}\binom{Y}{\lambda}$)**.** The space of rational character polynomials in type B/C is spanned by character polynomials of the following form. Given a double partition (μ, λ), define a degree $|\mu|+|\lambda|$ character polynomial $\binom{X}{\mu}\binom{Y}{\lambda}$ by the formula

$$\binom{X}{\mu}\binom{Y}{\lambda}(\sigma) := \prod_{r=1}^{|\lambda|+|\mu|} \binom{X_r(\sigma)}{n_r(\mu)}\binom{Y_r(\sigma)}{n_r(\lambda)} \qquad \text{for all } \sigma \in B_n, n \in \mathbb{Z}_{\geq 0}.$$

1.3. Asymptotic results for polynomial statistics on F-stable maximal tori. Given an F-stable maximal torus T in $\mathrm{GL}_n(\overline{F_q})$ and a character polynomial P, the value $P(T)$ is given by evaluating P on the S_n-conjugacy class associated to T. Specifically, the statistic $X_r(T)$ counts the number of r-dimensional irreducible F-stable subtori of T; see Church–Ellenberg–Farb [ChuEF14, Section 5].

For an F-stable maximal torus T in $\mathrm{SO}_{2n+1}(\overline{F_q})$ or $\mathrm{Sp}_{2n}(\overline{F_q})$ we call the associated double partition (μ_T, λ_T) the *type* of T. The value of a character polynomial P in T is given by the value of P on the conjugacy class (μ_T, λ_T). The statistic $X_r(T)$ (respectively, $Y_r(T)$) counts the number of r-dimensional F-stable subtori that are irreducible over F_q and that split (respectively, do not split) over F_{q^r}. See Jiménez Rolland–Wilson [JRW16, Section 4.1.4] for a more detailed explanation of this interpretation.

In types A, B, and C, we say that a character polynomial P defines a *polynomial statistic* on the corresponding sequence of spaces of maximal tori. In Theorem 1.6 we give formulas for the limits of the expected values of polynomial statistics on F-stable maximal tori in type A and B/C. To state this theorem, define the following notation.

For a partition λ of n, let

$$z_\lambda = \prod_{r=1}^{|\lambda|} n_r(\lambda)!\, r^{n_r(\lambda)}$$

so z_λ is defined such that $\frac{n!}{z_\lambda}$ is the number of permutations in S_n of cycle type λ. Analogously, for a partition μ, let

$$v_\mu = \prod_{r=1}^{|\mu|} n_r(\mu)!\, (2r)^{n_r(\mu)}$$

so $\frac{2^n n!}{v_\mu v_\lambda}$ is the number of signed permutations in B_n of signed cycle type (μ, λ).

Theorem 1.6 (Asymptotics for polynomial statistics on maximal tori)**.**

(i) (Type A). *Let λ be a fixed partition. Let $T(n,q)$ denote the set of F-stable maximal tori of $\mathrm{GL}_n(\overline{F_q})$. Then*

$$\lim_{n\to\infty} \frac{1}{q^{n^2-n}} \sum_{T\in T(n,q)} \binom{X}{\lambda}(T) = \frac{1}{z_\lambda} \prod_{r=1}^{|\lambda|} \left(\frac{q^r}{q^r-1}\right)^{n_r(\lambda)}.$$

(ii) (Type B/C). *Fix partitions μ and λ. Let $T(n,q)$ denote the set of F-stable maximal tori of $\mathrm{SO}_{2n+1}(\overline{F_q})$ or $\mathrm{Sp}_{2n}(\overline{F_q})$. Then*

$$\lim_{n\to\infty} \frac{1}{q^{2n^2}} \sum_{T\in T(n,q)} \binom{X}{\mu}\binom{Y}{\lambda}(T)$$

$$= \frac{1}{v_\mu v_\lambda} \prod_{r=1}^{|\mu|} \left(\frac{q^r}{q^r-1}\right)^{n_r(\mu)} \prod_{r=1}^{|\lambda|} \left(\frac{q^r}{q^r+1}\right)^{n_r(\lambda)}.$$

Theorem 1.6(i) is proved in Section 2 and Theorem 1.6(ii) in Section 3.

In particular, Theorem 1.6 implies that the left-hand side of both equations converge for any character polynomial and any field F_q. Hence Theorem 1.6 recovers a primary result of Church, Ellenberg, and Farb in type A, a consequence of [ChuEF14, Theorem 5.6]. In types B/C, it recovers a primary result of Jiménez Rolland and Wilson, [JRW16, Theorem 4.3]. Theorem 1.6 improves upon both convergence results by providing explicit formulas for the limits. In Examples 2.3 and 3.6 below we use Theorem 1.6 to recompute the explicit asymptotic identities from [ChuEF14, Section 1] and [JRW16, Table 1].

1.4. Cohomology of flag varieties and maximal tori in type B/C. Chen [Che16] uses generating function techniques to count F-stable maximal tori of $\mathrm{GL}_n(\overline{F_q})$ and uses the formula in Theorem 1.1(i) to extract topological information about the complex flag manifolds in type A, and the space of maximal tori in the complex group $\mathrm{GL}_n(\mathbb{C})$. In Section 3 we use Theorem 1.1(ii) and follow his approach to study the cohomology of

generalized flag manifolds and the spaces of maximal tori of $\mathrm{Sp}_{2n}(\mathbb{C})$ and $\mathrm{SO}_{2n+1}(\mathbb{C})$.

In what follows we denote by R_n^* the *complex coinvariant algebra of type B/C*. Consider the action of B_n on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ by permuting and negating the n variables x_i. The coinvariant algebra R_n^* is defined as the quotient $R_n^* \cong \mathbb{C}[x_1, \ldots, x_n]/I_n$ by the homogeneous ideal I_n generated by the B_n-invariant polynomials with constant term zero.

Borel [Bor53] proved that R_n^* is isomorphic as a graded $\mathbb{C}[B_n]$-algebra to the cohomology of the *generalized complete complex flag manifolds* in type B and C. In type C, the flag manifold is the variety of complete flags equal to their symplectic complements. The flag manifold in type B is the variety of complete flags equal to their orthogonal complements. The cohomology groups of these flag manifolds are supported in even cohomological degree, and the isomorphism from R_n^* to the cohomology ring multiplies the grading by two.

Wilson [Wil14] uses the theory of *$FI_{\mathcal{W}}$-modules* to prove that, in each cohomological degree, the characters of the cohomology of these flag varieties are given by a character polynomial — independent of n — for all n sufficiently large.

Theorem 1.7 (Existence of character polynomials for R_n^i [Wil14, Corollary 6.5])**.** *For each $i \geq 0$, there exists a unique hyperoctahedral character polynomial Q_i such that $\chi_{R_n^i} = Q_i$ for all n sufficiently large (depending on i). Moreover the degree of the polynomial Q_i is at most i.*

Following a suggestion made by Chen, we obtain in Section 3.1 a generating function for the B_n characters of R_n^i.

Theorem 1.8 (A generating function for the B_n characters of R_n^i)**.** *Fix an integer $n \geq 1$. For all signed permutations $\sigma \in B_n$,*

$$\sum_{i=0}^{n^2} \chi_{R_n^i}(\sigma) z^i = \frac{(1-z^2)(1-z^4)\cdots(1-z^{2n})}{\prod_{r=1}^{n}(1-z^r)^{X_r(\sigma)}(1+z^r)^{Y_r(\sigma)}}.$$

Using this result we can recover Theorem 1.7 and give a generating function for the polynomials Q_i. Chen and Specter [CheS16] compute the corresponding result in type A.

Theorem 1.9 (Character polynomials for R_n^i in type B/C)**.** *Define character polynomials Q_i by the generating function*

$$\begin{aligned}\sum_{i=0}^{\infty} Q_i\, t^i &= \prod_{k=1}^{\infty} \frac{(1-t^{2k})}{(1-t^k)^{X_k}(1+t^k)^{Y_k}} \\ &= \prod_{k=1}^{\infty} \left(\sum_{j=0}^{\infty} \binom{1-X_k}{j} (-t^k)^j \right) \left(\sum_{j=0}^{\infty} \binom{1-Y_k}{j} (t^k)^j \right).\end{aligned}$$

Then the character of the B_n-representation R_n^i, the i^{th} graded piece of the coinvariant algebra R_n^* in type B/C, is given by

$$\chi_{R_n^i} = Q_i \quad \textit{for all} \quad i \leq 2n+1.$$

Equivalently, in fixed degree i, the character of the group R_n^i is given by the character polynomial Q_i for all $n \geq \lceil \frac{i-1}{2} \rceil$. Theorem 1.9 is proved in Section 3.

Remark 1.10. Theorem 1.9 significantly improves upon Wilson's result Theorem 1.7: not only does Theorem 1.9 present an explicit generating function for the character polynomials, it also gives a quantitative range $i \leq 2n+1$ for when the equality $\chi_{R_n^i} = Q_i$ holds. The bound $i \leq 2n+1$ is sharp; see Remark 3.7. Moreover, from the form of the generating function we can infer that the polynomial Q_i includes the terms $X_i - Y_i$, hence $\deg(Q_i) = i$. Thus Theorem 1.9 implies that the bound on the degree of the character polynomials Q_i in Theorem 1.7 is also sharp.

Expanding the generating function in Theorem 1.9, we recover and extend the list of explicit character polynomials given by Wilson [Wil14, Section 6].

$$\begin{aligned}
&1 + \Big(X_1 - Y_1\Big)t \\
&+ \Big(-1 + \tfrac{1}{2}X_1 + \tfrac{1}{2}Y_1 + \tfrac{1}{2}X_1^2 + \tfrac{1}{2}Y_1^2 - X_1Y_1 + X_2 - Y_2\Big)t^2 \\
&+ \Big(-\tfrac{2}{3}X_1 + \tfrac{2}{3}Y_1 + \tfrac{1}{2}X_1^2 - \tfrac{1}{2}Y_1^2 + \tfrac{1}{6}X_1^3 - \tfrac{1}{6}Y_1^3 - \tfrac{1}{2}X_1^2Y_1 \\
&\quad + X_1X_2 + Y_1Y_2 + \tfrac{1}{2}X_1Y_1^2 - X_2Y_1 - X_1Y_2 + X_3 - Y_3\Big)t^3 \\
&+ \Big(-1 - \tfrac{1}{4}X_1 - \tfrac{1}{4}Y_1 - \tfrac{1}{24}X_1^2 - \tfrac{1}{24}Y_1^2 + \tfrac{7}{12}X_1Y_1 + \tfrac{3}{2}Y_2 - \tfrac{1}{2}X_2 + \tfrac{1}{4}X_1^3 \\
&\quad + \tfrac{1}{4}Y_1^3 - \tfrac{1}{4}X_1^2Y_1 - \tfrac{1}{4}X_1Y_1^2 + \tfrac{1}{2}X_1X_2 - \tfrac{1}{2}X_1Y_2 - \tfrac{1}{2}Y_1Y_2 + \tfrac{1}{2}X_2Y_1 \\
&\quad + \tfrac{1}{24}X_1^4 + \tfrac{1}{24}Y_1^4 - \tfrac{1}{6}X_1^3Y_1 - \tfrac{1}{6}X_1Y_1^3 + \tfrac{1}{4}X_1^2Y_1^2 + \tfrac{1}{2}X_1^2X_2 - \tfrac{1}{2}X_1^2Y_2 \\
&\quad + \tfrac{1}{2}X_2Y_1^2 - \tfrac{1}{2}Y_1^2Y_2 - \tfrac{1}{2}Y_1^2Y_2 - X_1X_2Y_1 + X_1Y_1Y_2 + \tfrac{1}{2}X_2^2 + \tfrac{1}{2}Y_2^2 \\
&\quad + X_1X_3 - X_1Y_3 - X_3Y_1 + Y_1Y_3 - X_2Y_2 + X_4 - Y_4\Big)t^4 + \cdots
\end{aligned}$$

1.5. Stable twisted Betti numbers.

Definition 1.11 (Twisted Betti numbers in type B/C). Let $T(n,\mathbb{C})$ denote the space of maximal tori in $\mathrm{Sp}_{2n}(\mathbb{C})$ or (equivalently) $\mathrm{SO}_{2n+1}(\mathbb{C})$. For a B_n-representation V_n, the *twisted Betti numbers of* $T(n,\mathbb{C})$ *in* V_n are the values

$$\dim_{\mathbb{C}} H^{2i}(T(n,\mathbb{C}), V_n) = \langle R_n^i, V_n \rangle_{B_n}.$$

For a virtual B_n-representation V_n, we again call the values $\langle R_n^i, V_n \rangle_{B_n} \in \mathbb{C}$ the twisted Betti numbers of $T(n,\mathbb{C})$ in V_n.

Theorem 1.9 and Jiménez Rolland–Wilson [JRW16, Proposition 3.1] imply that if P is any hyperoctahedral character polynomial, then the inner product $\langle P, R_n^i \rangle_{B_n}$ is independent of n for $n \geq \deg(P) + i$. We conclude the following theorem.

Corollary 1.12 (Twisted homological stability for complex tori)**.** *Let P be a hyperoctahedral character polynomial and V_n^P the associated sequence of virtual representations of B_n. For every $i \geq 0$ and all $n \geq \deg(P) + i$,*

$$\langle V_n^P, R_n^i \rangle_{B_n} = \langle V_{n+1}^P, R_{n+1}^i \rangle_{B_{n+1}}.$$

For genuine representations V_n^P this implies in particular

$$\dim_{\mathbb{C}} \left(H^{2i}(T(n, \mathbb{C}), V_n^P) \right) = \dim_{\mathbb{C}} \left(H^{2i}(T(n+1, \mathbb{C}), V_{n+1}^P) \right).$$

We remark that the stable range in Corollary 1.12 relies on the result of Theorem 1.9 that the character of R_n^i agrees with its character polynomial for all $i \leq 2n + 1$, and in particular for all $i \leq n$.

If P is the character polynomial associated to a sequence of irreducible B_n-representations (as constructed in Wilson [Wil15, Theorem 4.11]), Corollary 1.12 can be interpreted as the statement that the cohomology groups of the flag varieties are representation stable in the sense of Church–Farb [ChuF13, Definition 1.1], a consequence of Wilson [Wil14, Corollary 6.5].

Finally, in Section 3 we prove a type B/C analog of results of Chen [Che16, Theorem 1 (II) and Corollary 2 (II)]. First we give a double generating function for the twisted Betti numbers of $T(n, \mathbb{C})$.

Theorem 1.13 (The twisted Betti numbers $\left\langle \binom{X}{\mu}\binom{Y}{\lambda}, R_n^i \right\rangle_{B_n}$)**.** *Fix a double partition (μ, λ). Let*

$$\beta_i(n) = \left\langle \binom{X}{\mu}\binom{Y}{\lambda}, R_n^i \right\rangle_{B_n}.$$

Then

$$\sum_{n=0}^{\infty} \sum_{i=0}^{n^2} \frac{\beta_i(n) z^i u^n}{(1-z^2)(1-z^4)\cdots(1-z^{2n})}$$
$$= \frac{1}{v_\mu v_\lambda} \prod_{r=1}^{|\mu|} \left(\frac{u^r}{1-z^r} \right)^{n_r(\mu)} \prod_{r=1}^{|\lambda|} \left(\frac{u^r}{1+z^r} \right)^{n_r(\lambda)} \prod_{r=1}^{\infty} \frac{1}{1-uz^{2r-2}}.$$

We obtain a generating function for the stable twisted Betti numbers of $T(n, \mathbb{C})$. From the form of this generating function we can deduce properties of these sequences such as linear recurrence relationships and *quasipolynomiality*.

Corollary 1.14 (Generating functions for stable Betti numbers)**.** *Given a double partition (μ, λ), let*

$$\beta_i = \lim_{n \to \infty} \left\langle \binom{X}{\mu}\binom{Y}{\lambda}, R_n^i \right\rangle_{B_n},$$

where R_n^i is the i^{th} graded piece of the coinvariant algebra R_n^* in type B/C. Then

$$\sum_{i=0}^{\infty} \beta_i z^i = \frac{1}{v_\mu v_\lambda} \prod_{r=1}^{|\mu|} \left(\frac{1}{1-z^r}\right)^{n_r(\mu)} \prod_{r=1}^{|\lambda|} \left(\frac{1}{1+z^r}\right)^{n_r(\lambda)}.$$

Since character polynomials of the form $\binom{X}{\mu}\binom{Y}{\lambda}$ form an additive basis for the space of all hyperoctahedral character polynomials, the result of Corollary 1.14 is sufficient to derive a generating function for the twisted stable Betti numbers associated to any given character polynomial. Because the generating function in Corollary 1.14 is rational with denominator of degree $|\mu| + |\lambda|$, we obtain the following consequence.

Corollary 1.15 (Linear recurrence for stable Betti numbers)**.** *Given a hyperoctahedral character polynomial P, let*

$$\beta_i = \lim_{n\to\infty} \langle P, R_n^i \rangle_{B_n},$$

where R_n^i is the i^{th} graded piece of the coinvariant algebra R_n^* in type B/C. Then there exist integers $d_1, \cdots, d_N$ such that for all $i \geq N$,

$$\beta_i = d_1\beta_{i-1} + d_2\beta_{i-2} + \cdots + d_N\beta_{i-N}.$$

When $P = \binom{X}{\mu}\binom{Y}{\lambda}$, then $N = \deg(P)$. In general $N \leq 2(\deg(P))^2$.

To state the next consequence of Corollary 1.14, we recall the definition of a quasipolynomial from Stanley [Sta97, Section 4.4].

Definition 1.16 (Quasipolynomials; quasiperiods)**.** A function $p(t)$ is *quasipolynomial of degree d* if it can be expressed in the form

$$p(t) = c_d(t)t^d + c_{d-1}(t)t^{d-1} + \cdots + c_1(t)t + c_0(t)$$

where the coefficients $c_i(t)$ are periodic functions of t with integer periods, and $c_d(t)$ is not identically zero. Equivalently, the function $p(t)$ is quasipolynomial if for some $M \geq 1$ there are polynomials $p_0(t), p_1(t), \ldots, p_{M-1}(t)$, such that

$$p(t) = p_i(t) \qquad \text{for } t \equiv i \pmod{M}.$$

The integer M is called a *quasiperiod* of $p(t)$.

Corollary 1.17 (Stable Betti numbers are quasipolynomial)**.** *Given a hyperoctahedral character polynomial P, again let*

$$\beta_i = \lim_{n\to\infty} \langle P, R_n^i \rangle_{B_n},$$

where R_n^i is the i^{th} graded piece of the coinvariant algebra R_n^* in type B/C. Then for $i \geq 1$ the Betti numbers β_i are quasipolynomial in i, with degree at most $\deg(P) - 1$, and have a quasiperiod at most the least common multiple of $\{2k \mid 1 \leq k \leq \deg(P)\}$. If the character polynomial P has constant term zero, then the above statement holds for all $i \geq 0$.

Let $\mathbb{C}^n$ denote the canonical representation of B_n by signed permutation matrices. To illustrate Corollaries 1.14, 1.15, and 1.17, Example 1.18 gives the character polynomial for the sequence of B_n-representations $\{\mathrm{Sym}^2\mathbb{C}^n\}_n$ and their associated stable twisted Betti numbers.

Example 1.18. ($\mathrm{Sym}^2\mathbb{C}^n$).

Character polynomial:

$$X_1 + \binom{X_1}{2} + Y_1 + \binom{Y_1}{2} + X_2 - Y_2 - X_1Y_1$$
$$= \binom{X}{\square} + \binom{X}{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + \binom{Y}{\square} + \binom{Y}{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + \binom{X}{\square\square} - \binom{Y}{\square\square} - \binom{X}{\square}\binom{Y}{\square}.$$

Betti numbers:

$$\sum_{i=0}^{\infty} \beta_i z^i = \frac{-z^4 + z^2 + 1}{(1-z^2)^2(1+z^2)}$$
$$= 1 + 2z^2 + 2z^4 + 3z^6 + 3z^8 + 4z^{10} + 4z^{12} + \cdots + \left\lfloor \frac{d+3}{2} \right\rfloor z^{2d} + \cdots .$$

Recurrence: $\beta_d = \beta_{d-2} + \beta_{d-4} - \beta_{d-6}$ for $d \geq 7$.

Quasipolynomiality: For $d \geq 0$,

$$\beta_d = \begin{cases} \frac{d+4}{4} & d \equiv 0 \pmod 4 \\ \frac{d+6}{4} & d \equiv 2 \pmod 4 \\ 0 & d \equiv 1, 3 \pmod 4. \end{cases}$$

Additional examples of stable twisted Betti numbers are given in Section 3.2, with their associated generating functions, recurrence relations, and quasipolynomials.

2. Maximal tori in the general linear groups

In this section, we let $T(n,q)$ denote the set of F-stable maximal tori of $\mathrm{GL}_n(\overline{F_q})$. Fulman [Ful16, Theorem 3.2] proved the following:

$$1 + \sum_{n=1}^{\infty} \frac{u^n}{|\mathrm{GL}_n(F_q)|} \sum_{T \in T(n,q)} \prod_{i=1}^{\infty} x_i^{X_i(T)} = \prod_{k=1}^{\infty} \exp\left[\frac{x_k u^k}{(q^k-1)k}\right]. \tag{3}$$

Chen [Che16] deduced from (3) that for any partition λ,

$$\sum_{n=0}^{\infty} \frac{u^n}{|\mathrm{GL}_n(F_q)|} \sum_{T \in T(n,q)} \binom{X}{\lambda}(T) = \frac{1}{z_\lambda} \prod_{k=1}^{|\lambda|} \left(\frac{u^k}{q^k-1}\right)^{n_k(\lambda)} \prod_{r=1}^{\infty} \frac{1}{(1-u/q^r)}. \tag{4}$$

Here if λ is the empty partition, we take the constant $n = 0$ term on the left hand side to be 1; otherwise, the constant term is 0.

To analyze these formulas, we use the following elementary and well-known lemma. We adopt the following notation: given a power series $f(u)$, we let $[u^n]f(u)$ denote the coefficient of u^n.

Lemma 2.1 (The limit of a series' coefficients)**.** *If the Taylor series of $f(u)$ around 0 converges at $u = 1$, then*

$$\lim_{n\to\infty}[u^n]\frac{f(u)}{1-u} = f(1).$$

Proof. Write the Taylor expansion $f(u) = \sum_{n=0}^{\infty} a_n u^n$. Then observe that

$$[u^n]\frac{f(u)}{1-u} = \sum_{i=0}^{n} a_i. \qquad \square$$

We can now prove Theorem 1.6(i), which gives formulas for the stable values of polynomial statistics on the space of F-stable maximal tori in $\mathrm{GL}_n(\overline{F_q})$. Recall that the number of such tori is q^{n^2-n}.

Proof of Theorem 1.6(i). It follows from (4) that

$$\frac{1}{q^{n^2-n}}\sum_{T\in T(n,q)}\binom{X}{\lambda}(T)$$

is equal to

$$\frac{|\mathrm{GL}_n(F_q)|}{q^{n^2-n}}[u^n]\frac{1}{z_\lambda}\prod_{k=1}^{|\lambda|}\left(\frac{u^k}{q^k-1}\right)^{n_k(\lambda)}\cdot\prod_{r=1}^{\infty}\frac{1}{1-u/q^r}.$$

Since $[u^n]f(u) = [u^n]\dfrac{f(uq)}{q^n}$ for any function f, this expression is equal to

$$\begin{aligned}&\frac{|\mathrm{GL}_n(F_q)|}{q^{n^2}}[u^n]\frac{1}{z_\lambda}\prod_{k=1}^{|\lambda|}\left(\frac{u^kq^k}{q^k-1}\right)^{n_k(\lambda)}\cdot\frac{1}{1-u}\prod_{r=1}^{\infty}\frac{1}{1-u/q^r}\\ &= (1-1/q)\cdots(1-1/q^n)[u^n]\frac{1}{z_\lambda}\prod_{k=1}^{|\lambda|}\left(\frac{u^kq^k}{q^k-1}\right)^{n_k(\lambda)}\cdot\frac{1}{1-u}\prod_{r=1}^{\infty}\frac{1}{1-u/q^r}.\end{aligned}$$

The result now follows by taking the limit as n tends to infinity, and using Lemma 2.1. To see that Lemma 2.1 is applicable, note that the Taylor series of $\prod_{r\geq 1}\frac{1}{1-u/q^r}$ around 0 converges at $u = 1$. Indeed, this Taylor series is given by part 2 of Lemma 3.3, and

$$(1-1/q)\cdots(1-1/q^n) \geq 1 - \frac{1}{q} - \frac{1}{q^2}$$

for all n, as is well known from Euler's pentagonal number theorem (page 11 of [And98]). $\square$

Remark 2.2. We remark that there is an alternate approach to proving Theorem 1.6(i). Chen computed a generating function for the stable twisted Betti numbers for maximal tori in type A [Che16, Proof of Corollary (II)]. Combining this with [ChuEF14, Theorem 5.6] proves the result.

Theorem 1.6(i) gives an efficient way to recover some computations of Church–Ellenberg–Farb [ChuEF14].

Example 2.3 ([ChuEF14, Theorems 5.9 & 5.10])**.**

(a) $$\lim_{n\to\infty} \frac{1}{q^{n^2-n}} \sum_{T\in T(n,q)} X_1(T) = \frac{q}{q-1}.$$

(b) $$\lim_{n\to\infty} \frac{1}{q^{n^2-n}} \sum_{T\in T(n,q)} \left[\binom{X_1}{2}(T) - X_2(T)\right] = \frac{1}{2}\left(\frac{q}{q-1}\right)^2 - \frac{1}{2}\frac{q^2}{(q^2-1)}$$
$$= \frac{1}{q(1-1/q)(1-1/q^2)}.$$

2.1. A new proof of Lehrer's theorem. In what follows we give a new proof of Theorem 1.1(i) using symmetric function theory. It should be possible to extend these methods to types B/C, but the argument is more involved and we do not pursue it here. For the remainder of this section, R_n^i denotes the ith graded piece of the coinvariant algebra R_n^* in type A. All necessary background in symmetric function theory can be found in Macdonald [Mac95] or Stanley [Sta99]. The following two lemmas will be crucial. Lemma 2.4 is stated in Fulman [Ful16] and is immediate from Springer–Steinberg [SpS70, Section 2.7].

Lemma 2.4 (Enumerating tori in $\mathrm{GL}_n(\overline{F_q})$ of a given type [SpS70, Ful16])**.** *Fix a partition λ of n. The number of F-stable maximal tori of $\mathrm{GL}_n(\overline{F_q})$ of type λ is equal to*

$$\frac{|\mathrm{GL}_n(F_q)|}{z_\lambda \prod_{r=1}^n (q^r-1)^{n_r(\lambda)}}.$$

Here $n_r(\lambda)$ is the number of parts of λ of size r, and $z_\lambda = \prod_{r=1}^{|\lambda|} r^{n_r(\lambda)} n_r(\lambda)!$

Recall that a *standard tableau* of *shape* λ is a bijective labeling of the boxes of the Young diagram for λ by the numbers $1, \cdots, n$ with the property that in each row and in each column the labels are increasing. The *descent set* of such a tableau is the set of numbers $i \in \{1, \cdots, n-1\}$ for which the box labeled $(i+1)$ is in a lower row than the box labeled i. The *major index* $\mathrm{maj}(Y)$ of a tableau Y is the sum of the numbers in its descent set. We let $f_{\lambda,i}$ be the number of standard Young tableaux of shape λ and major index i.

For example, consider the Young diagram for the partition $\lambda = (2,1,1)$ and its three associated standard tableaux

$$Y_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad Y_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad Y_3 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

Their descent sets are $\{2,3\}$, $\{1,3\}$, and $\{1,2\}$, respectively. Hence

$$\mathrm{maj}(Y_1) = 5 \qquad \mathrm{maj}(Y_2) = 4, \qquad \text{and} \qquad \mathrm{maj}(Y_3) = 3.$$

In this example,

$$f_{(2,1,1),i} = \begin{cases} 1, & i = 3,4,5 \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.5 is due to Stanley, Lusztig, and Kraskiewicz–Weyman. See Reutenauer [Reu93, Theorem 8.8] for a proof.

Lemma 2.5 (Decomposing the S_n-representations R_n^i [Reu93, Theorem 8.8])**.** *The multiplicity of the irreducible representation of S_n indexed by λ in R_n^i is equal to $f_{\lambda,i}$.*

We now prove Theorem 1.1(i).

Proof of Theorem 1.1(i). It suffices to prove Theorem 1.1(i) in the special case that χ is the class function which is equal to p_μ on elements of type μ, where p_μ denotes the power sum symmetric function corresponding to a partition μ. Indeed, taking the coefficient of the Schur function s_λ in p_μ gives the irreducible character value χ_μ^λ of the symmetric group, and these are a basis for the space of class functions.

By Lemma 2.4, the left hand side of (1) is equal to

$$\frac{1}{q^{n^2-n}} \sum_{|\mu|=n} \frac{|\mathrm{GL}_n(F_q)|}{z_\mu \prod_{k=1}^n (q^k-1)^{n_k(\mu)}} p_\mu.$$

On the other hand, since

$$p_\mu = \sum_{|\lambda|=n} \chi_\mu^\lambda s_\lambda,$$

it follows from Lemma 2.5 that the right hand side of (1) is equal to

$$\begin{aligned} \sum_{i=0}^{\binom{n}{2}} q^{-i} \left\langle \sum_{|\lambda|=n} \chi^\lambda s_\lambda, \sum_{|\lambda|=n} f_{\lambda,i} \chi^\lambda \right\rangle &= \sum_{i=0}^{\binom{n}{2}} q^{-i} \sum_{|\lambda|=n} f_{\lambda,i} s_\lambda \\ &= \sum_{|\lambda|=n} s_\lambda \sum_{\mathrm{sh}(Y)=\lambda} \frac{1}{q^{\mathrm{maj}(Y)}}, \end{aligned}$$

where the sum is over standard Young tableaux Y of shape λ. By Stanley [Sta99, p. 363] this is equal to

$$\sum_{|\lambda|=n} s_\lambda s_\lambda(1, 1/q, 1/q^2, \cdots) \prod_{i=1}^{n}(1 - 1/q^i)$$
$$= \frac{|\mathrm{GL}_n(F_q)|}{q^{n^2-n}} \sum_{|\lambda|=n} s_\lambda s_\lambda(1/q, 1/q^2, \cdots).$$

By Macdonald [Mac95, Section 1.4], this is

$$\frac{|\mathrm{GL}_n(F_q)|}{q^{n^2-n}} \sum_{|\mu|=n} \frac{1}{z_\mu} p_\mu p_\mu(1/q, 1/q^2, \cdots)$$
$$= \frac{1}{q^{n^2-n}} \sum_{|\mu|=n} \frac{|\mathrm{GL}_n(F_q)|}{z_\mu \prod_{k=1}^{n}(q^k - 1)^{n_k(\mu)}} p_\mu.$$

Thus we have shown that the two sides of the formula in Theorem 1.1(i) are equal, completing the proof. □

3. Maximal tori in symplectic and special orthogonal groups

In this section we use generating function techniques to study F-stable maximal tori in $\mathrm{Sp}_{2n}(\overline{F_q})$, and (equivalently) in $\mathrm{SO}_{2n+1}(\overline{F_q})$. We prove Theorem 1.6(ii) and use it to recover results from Jiménez Rolland–Wilson [JRW16]. In the second half of the section, we derive analogs of results from Chen's lovely paper [Che16] for the special orthogonal and symplectic groups, as well as analogs of results in a personal communication from Chen and Specter [CheS16].

The next two results are symplectic versions of results from Fulman [Ful16].

Lemma 3.1 (Enumerating tori in $\mathrm{Sp}_{2n}(\overline{F_q})$ of a given type)**.** *Let (μ, λ) be an ordered pair of partitions satisfying $|\mu| + |\lambda| = n$. Let $n_r(\mu)$ denote the number of parts of μ of size r, and let $n_r(\lambda)$ denote the number of parts of λ of size r. Then the number of F-stable maximal tori of $\mathrm{Sp}_{2n}(\overline{F_q})$ of type (μ, λ) is equal to*

$$\frac{|\mathrm{Sp}_{2n}(F_q)|}{\prod_{r=1}^{n} n_r(\mu)!\, n_r(\lambda)!\, (2r)^{n_r(\mu)+n_r(\lambda)} \prod_{r=1}^{n}(q^r - 1)^{n_r(\mu)}(q^r + 1)^{n_r(\lambda)}}.$$

Proof. This is immediate from Springer–Steinberg [SpS70, Section 2.7], together with the fact that the centralizer size of an element of B_n of type (μ, λ) is equal to

$$\prod_{r=1}^{n} n_r(\mu)!\, n_r(\lambda)!\, (2r)^{n_r(\mu)+n_r(\lambda)}.$$ □

In this section, $T(n,q)$ denotes the set of F-stable maximal tori T of $\mathrm{Sp}_{2n}(\overline{F_q})$ (or $\mathrm{SO}_{2n+1}(\overline{F_q})$).

Theorem 3.2 (A generating function for statistics on $T(n,q)$)**.**

$$1+\sum_{n=1}^{\infty}\frac{u^n}{|\mathrm{Sp}_{2n}(F_q)|}\sum_{T\in T(n,q)}\prod_{r=1}^{n}x_r^{X_r(T)}y_r^{Y_r(T)}$$
$$=\prod_{k=1}^{\infty}\exp\left[\frac{x_k u^k}{(q^k-1)2k}+\frac{y_k u^k}{(q^k+1)2k}\right].$$

Proof. Lemma 3.1 implies that the left-hand side of the theorem is equal to

$$1+\sum_{n=1}^{\infty}\sum_{\substack{(\mu,\lambda)\\ |\mu|+|\lambda|=n}}\frac{u^n\prod_{r=1}^{n}x_r^{n_r(\mu)}y_r^{n_r(\lambda)}}{\prod_r n_r(\mu)!\,n_r(\lambda)!\,(2r)^{n_r(\mu)+n_r(\lambda)}(q^r-1)^{n_r(\mu)}(q^r+1)^{n_r(\lambda)}}$$

By the Taylor expansion of the exponential function, this is equal to

$$\prod_{k=1}^{\infty}\exp\left[\frac{x_k u^k}{(q^k-1)2k}+\frac{y_k u^k}{(q^k+1)2k}\right].$$

□

The following lemma will be helpful in manipulating the generating function of Theorem 3.2.

Lemma 3.3 (Two generating function identities)**.**

(i) $\displaystyle\prod_{i=1}^{\infty}\exp\left[\frac{1}{(q^i-1)}\frac{u^i}{i}\right]=\prod_{r=1}^{\infty}\frac{1}{1-u/q^r}.$

(ii) $\displaystyle\prod_{r=1}^{\infty}\frac{1}{(1-u/q^r)}=1+\sum_{n=1}^{\infty}\frac{u^n}{q^n(1-1/q)\cdots(1-1/q^n)}.$

Proof. The first assertion is proved in the proof of Fulman [Ful16, Theorem 3.4]. The second assertion is classical and goes back to Euler; see Andrews [And98, Corollary 2.2]. □

As a corollary, we recover Steinberg's enumeration of the number of F-stable maximal tori of $\mathrm{Sp}_{2n}(\overline{F_q})$ [Ste68, Corollary 14.16]. Recall that given a power series $f(u)$, we let $[u^n]f(u)$ denote the coefficient of u^n.

Corollary 3.4 (Enumerating tori in $\mathrm{Sp}_{2n}(\overline{F_q})$ [Ste68, Corollary 14.16])**.** *The number of F-stable maximal tori of $\mathrm{Sp}_{2n}(\overline{F_q})$ is q^{2n^2}.*

Proof. By setting all variables $x_r = y_r = 1$ in Theorem 3.2, we find that the number of maximal tori is equal to

$$|\mathrm{Sp}_{2n}(F_q)|\,[u^n]\prod_{r=1}^{\infty}\exp\left[\frac{u^r}{(q^r-1)2r}+\frac{u^r}{(q^r+1)2r}\right]$$
$$=|\mathrm{Sp}_{2n}(F_q)|\,[u^n]\prod_{r=1}^{\infty}\exp\left[\frac{u^r q^r}{r(q^{2r}-1)}\right].$$

Since $[u^n]f(u) = q^n[u^n]f(u/q)$ for any function f, this coefficient is equal to

$$|\mathrm{Sp}_{2n}(F_q)|\,q^n[u^n]\exp\left[\sum_{r=1}^{\infty}\frac{u^r}{r(q^{2r}-1)}\right].$$

By Lemma 3.3(i), this is

$$|\mathrm{Sp}_{2n}(F_q)|\,q^n[u^n]\prod_{r=1}^{\infty}\frac{1}{(1-u/q^{2r})}$$

and by Lemma 3.3(ii), this expression equals

$$|\mathrm{Sp}_{2n}(F_q)|\,q^n\frac{1}{q^{2n}(1-1/q^2)\cdots(1-1/q^{2n})}.$$

Finally, we use the identity

$$|\mathrm{Sp}_{2n}(F_q)| = q^{n^2}\prod_{i=1}^{n}(q^{2i}-1)$$

to conclude that the number of F-stable maximal tori is q^{2n^2} as claimed. □

Table 1 of the paper Jiménez Rolland–Wilson [JRW16] gives explicit expressions for the average values of the following statistics on $T(n,q)$:

$$X_1, \qquad X_1+Y_1, \qquad \binom{X_1+Y_1}{2}-(X_2+Y_2), \qquad X_2-Y_2$$

These average values can be derived more efficiently from the generating function given in Theorem 3.5, and their limiting values are special cases of Theorem 1.6(ii). Theorem 3.5, which we now prove, is an analog of Chen's Equation (4) from Section 2.

Theorem 3.5 (Average values of the statistic $\binom{X}{\mu}\binom{Y}{\lambda}$ on $T(n,q)$)**.** *Fix a double partition (μ,λ). Then*

$$\sum_{n=0}^{\infty}\frac{u^n}{|\mathrm{Sp}_{2n}(F_q)|}\sum_{T\in T(n,q)}\binom{X}{\mu}\binom{Y}{\lambda}(T)$$

is equal to

$$\frac{1}{v_\mu v_\lambda}\prod_{r=1}^{|\mu|}\left(\frac{u^r}{q^r-1}\right)^{n_r(\mu)}\prod_{r=1}^{|\lambda|}\left(\frac{u^r}{q^r+1}\right)^{n_r(\lambda)}\prod_{k=1}^{\infty}\frac{1}{1-u/q^{2k-1}}.$$

If both μ and λ are the empty partitions, we define the constant $n = 0$ term to be 1; otherwise, the constant term is 0.

Proof. We take the generating function of Theorem 3.2, and for each $r \geq 1$ we differentiate $n_r(\mu)$ many times with respect to the variable x_r, and $n_r(\lambda)$ many times with respect to y_r. We then set all variables $x_r = y_r = 1$. The result is that the quantity

$$\sum_{T \in T(n,q)} \binom{X}{\mu}\binom{Y}{\lambda}(T)$$

is equal to

$$\frac{|\mathrm{Sp}_{2n}(F_q)|}{v_\mu v_\lambda}[u^n]\prod_{r=1}^{|\mu|}\left(\frac{u^r}{q^r-1}\right)^{n_r(\mu)}\prod_{r=1}^{|\lambda|}\left(\frac{u^r}{q^r+1}\right)^{n_r(\lambda)}$$
$$\cdot\prod_{k=1}^{\infty}\exp\left[\frac{u^k}{(q^k-1)2k}+\frac{u^k}{(q^k+1)2k}\right].$$

By Lemma 3.3(i), this is

$$\frac{|\mathrm{Sp}_{2n}(F_q)|}{v_\mu v_\lambda}[u^n]\prod_{r=1}^{|\mu|}\left(\frac{u^r}{q^r-1}\right)^{n_r(\mu)}\prod_{r=1}^{|\lambda|}\left(\frac{u^r}{q^r+1}\right)^{n_r(\lambda)}\prod_{k=1}^{\infty}\frac{1}{1-u/q^{2k-1}}$$

as claimed. □

With the result of Theorem 3.5, we can now prove Theorem 1.6(ii), which gives explicit formulas for the stable mean values of the polynomial statistics $\binom{X}{\mu}\binom{Y}{\lambda}$ on $T(n,q)$, in the limit as n tends to infinity.

Proof of Theorem 1.6(ii). By Theorem 3.5,

$$\frac{1}{q^{2n^2}}\sum_{T \in T(n,q)} \binom{X}{\mu}\binom{Y}{\lambda}(T)$$

is equal to

$$\frac{|\mathrm{Sp}_{2n}(F_q)|}{q^{2n^2}\, v_\mu\, v_\lambda}[u^n]\prod_{r=1}^{|\mu|}\left(\frac{u^r}{q^r-1}\right)^{n_r(\mu)}\prod_{r=1}^{|\lambda|}\left(\frac{u^r}{q^r+1}\right)^{n_r(\lambda)}\prod_{k=1}^{\infty}\frac{1}{1-u/q^{2k-1}}.$$

Since $[u^n]f(u) = [u^n]\dfrac{f(uq)}{q^n}$ for any function f, this can be rewritten as

$$\frac{|\mathrm{Sp}_{2n}(F_q)|}{q^{2n^2+n}\, v_\mu\, v_\lambda}[u^n]\prod_{r=1}^{|\mu|}\left(\frac{u^r q^r}{q^r-1}\right)^{n_r(\mu)}\prod_{r=1}^{|\lambda|}\left(\frac{u^r q^r}{q^r+1}\right)^{n_r(\lambda)}\prod_{k=1}^{\infty}\frac{1}{1-u/q^{2k-2}},$$

which is equal to

$$(1-1/q^2)(1-1/q^4)\cdots(1-1/q^{2n})$$
$$\cdot \frac{1}{v_\mu\, v_\lambda}[u^n]\prod_{r=1}^{|\mu|}\left(\frac{u^r q^r}{q^r-1}\right)^{n_r(\mu)}\prod_{r=1}^{|\lambda|}\left(\frac{u^r q^r}{q^r+1}\right)^{n_r(\lambda)}\frac{1}{1-u}\prod_{k=1}^{\infty}\frac{1}{1-u/q^{2k}}.$$

Now applying Lemma 2.1 proves the theorem. □

With Theorem 1.6(ii) we recover the following four results from Table 1 of Jiménez Rolland–Wilson [JRW16].

Example 3.6 ([JRW16, Table 1])**.**

(a) $$\lim_{n\to\infty}\frac{1}{q^{2n^2}}\sum_{T\in T(n,q)}X_1(T)=\frac{q}{2(q-1)}.$$

(b) $$\lim_{n\to\infty}\frac{1}{q^{2n^2}}\sum_{T\in T(n,q)}[X_1(T)+Y_1(T)]=\frac{q}{2(q-1)}+\frac{q}{2(q+1)}=\frac{q^2}{q^2-1}.$$

(c) $$\lim_{n\to\infty}\frac{1}{q^{2n^2}}\sum_{T\in T(n,q)}\left[\binom{X_1(T)+Y_1(T)}{2}-(X_2(T)+Y_2(T))\right]$$
$$=\lim_{n\to\infty}\frac{1}{q^{2n^2}}\sum_{T\in T(n,q)}\left[\binom{X_1(T)}{2}+\binom{Y_1(T)}{2}+X_1(T)Y_1(T)\right.$$
$$\left.-(X_2(T)+Y_2(T))\right]$$
$$=\frac{1}{2}\left[\frac{q}{2(q-1)}\right]^2+\frac{1}{2}\left[\frac{q}{2(q+1)}\right]^2+\frac{q}{2(q-1)}\frac{q}{2(q+1)}$$
$$-\frac{q^2}{4(q^2-1)}-\frac{q^2}{4(q^2+1)}$$
$$=\frac{q^4}{(q^2-1)(q^4-1)}.$$

(d) $$\lim_{n\to\infty}\frac{1}{q^{2n^2}}\sum_{T\in T(n,q)}[X_2(T)-Y_2(T)]=\frac{q^2}{4(q^2-1)}-\frac{q^2}{4(q^2+1)}$$
$$=\frac{q^2}{2(q^4-1)}.$$

3.1. Maximal tori in $\mathbf{Sp_{2n}}(\mathbb{C})$ and $\mathbf{SO_{2n+1}}(\mathbb{C})$. Next we follow Chen and Specter's approach [Che16, CheS16] and use our statistical computations on maximal tori in $\mathrm{Sp}_{2n}(\overline{F_q})$ and $\mathrm{SO}_{2n+1}(\overline{F_q})$ to obtain information about cohomology of the spaces of maximal tori $T(n,\mathbb{C})$ in the complex algebraic groups $\mathrm{Sp}_{2n}(\mathbb{C})$ and $\mathrm{SO}_{2n+1}(\mathbb{C})$. The bridge between the combinatorial and the topological data is given by Lehrer's Theorem 1.1(ii). In

the remainder of this section R_n^i always denotes the i^{th} graded piece of the complex coinvariant algebra R_n^* in type B/C. We first prove Theorem 1.8.

Proof of Theorem 1.8. Fix n and $\sigma \in B_n$. In Theorem 1.1(ii) let χ be the class function on B_n which is 1 on elements of type σ and 0 else. Since the centralizer size of σ is equal to

$$\prod_{r=1}^{n} X_r(\sigma)!Y_r(\sigma)!(2r)^{X_r(\sigma)+Y_r(\sigma)}$$

it follows that the number of maximal tori of type σ is equal to

$$q^{2n^2}\sum_{i=0}^{n^2}\chi_{R_n^i}(\sigma)q^{-i}\frac{1}{\prod_{r=1}^{n} X_r(\sigma)!Y_r(\sigma)!(2r)^{X_r(\sigma)+Y_r(\sigma)}}.$$

On the other hand, Lemma 3.1 indicates that the number of maximal tori of type σ is equal to

$$\frac{|\mathrm{Sp}_{2n}(F_q)|}{\prod_{r=1}^{n} X_r(\sigma)!Y_r(\sigma)!(2r)^{X_r(\sigma)+Y_r(\sigma)}\prod_{r=1}^{n}(q^r-1)^{X_r(\sigma)}(q^r+1)^{Y_r(\sigma)}}.$$

Thus

$$q^{2n^2}\sum_{i=0}^{n^2}\chi_{R_n^i}(\sigma)q^{-i} = \frac{|\mathrm{Sp}_{2n}(F_q)|}{\prod_{r=1}^{n}(q^r-1)^{X_r(\sigma)}(q^r+1)^{Y_r(\sigma)}}.$$

Since $|\mathrm{Sp}_{2n}(F_q)| = q^{2n^2+n}(1-1/q^2)(1-1/q^4)\cdots(1-1/q^{2n})$, it follows that

$$\sum_{i=0}^{n^2}\chi_{R_n^i}(\sigma)q^{-i} = \frac{(1-1/q^2)(1-1/q^4)\cdots(1-1/q^{2n})}{\prod_{r=1}^{n}(1-1/q^r)^{X_r(\sigma)}(1+1/q^r)^{Y_r(\sigma)}}.$$

Since this equality holds for any prime power q, we can set $z = 1/q$ to complete the proof. □

Now we prove Theorem 1.9, giving a generating function for the character polynomials associated to the cohomology groups of the generalized flag varieties in type B/C.

Proof of Theorem 1.9. Fix a positive integer n and signed permutation $\sigma \in B_n$. Let Q_i be the character polynomial defined by the generating function

$$\sum_{i=0}^{\infty} Q_i(\sigma)t^i = \prod_{k=1}^{\infty}\frac{(1-t^{2k})}{(1-t^k)^{X_k(\sigma)}(1+t^k)^{Y_k(\sigma)}}.$$

Our goal is to show that the terms in the series

$$\sum_{i=0}^{n^2}\chi_{R_n^i}(\sigma)t^i - \sum_{i=0}^{\infty} Q_i(\sigma)t^i$$

vanish for $i \leq 2n+1$. By Theorem 1.8 this difference is equal to

$$\prod_{k=1}^{n} \frac{(1-t^{2k})}{(1-t^k)^{X_k(\sigma)}(1+t^k)^{Y_k(\sigma)}} - \prod_{k=1}^{\infty} \frac{(1-t^{2k})}{(1-t^k)^{X_k(\sigma)}(1+t^k)^{Y_k(\sigma)}}$$
$$= \prod_{k=1}^{n} \frac{(1-t^{2k})}{(1-t^k)^{X_k(\sigma)}(1+t^k)^{Y_k(\sigma)}} \left[1 - \prod_{k>n} \frac{(1-t^{2k})}{(1-t^k)^{X_k(\sigma)}(1+t^k)^{Y_k(\sigma)}}\right].$$

Since $\sigma \in B_n$, the class functions $X_k(\sigma)$ and $Y_k(\sigma)$ vanish when $k > n$, and so this series equals

$$\prod_{k=1}^{n} \frac{(1-t^{2k})}{(1-t^k)^{X_k(\sigma)}(1+t^k)^{Y_k(\sigma)}} \left[1 - \prod_{k>n} (1-t^{2k})\right].$$

The smallest power of t in this series is t^{2n+2}, which completes the proof. □

Remark 3.7 (The stable range in Theorem 1.9 is sharp)**.** The bound given in Theorem 1.9 is optimal: since for all $\sigma \in B_n$ the power series

$$\prod_{k=1}^{n} \frac{(1-t^{2k})}{(1-t^k)^{X_k(\sigma)}(1+t^k)^{Y_k(\sigma)}} \left[1 - \prod_{k>n} (1-t^{2k})\right]$$

appearing in the proof includes the term t^{2n+2}, the equality $\chi_{R_n^i} = Q_i$ holds only for $i \leq 2n+1$. We can see this concretely in small degrees: the spaces R_0^2 and R_1^4 are both zero, but by inspection the character polynomials Q_2 and Q_4 for the sequences $\{R_n^2\}_n$ and $\{R_n^4\}_n$ (given explicitly in Section 1.4) do not vanish identically on the groups B_0 and B_1, respectively.

Next we prove Theorem 1.13, which gives a double generating function for the twisted Betti numbers of $T(n, \mathbb{C})$.

Proof of Theorem 1.13. For q a prime power,

$$\sum_{n=0}^{\infty} \sum_{i=0}^{n^2} \frac{\beta_i(n)}{(1-1/q^2)\cdots(1-1/q^{2n})} q^{-i} u^n$$
$$= \sum_{n=0}^{\infty} \frac{1}{|\mathrm{Sp}_{2n}(F_q)|} \left[q^{2n^2} \sum_{i=0}^{n^2} \beta_i(n) q^{-i}\right] (uq)^n.$$

By Theorem 1.1(ii), this is

$$\sum_{n=0}^{\infty} \frac{1}{|\mathrm{Sp}_{2n}(F_q)|} \left[\sum_{T \in T(n,q)} \binom{X}{\mu} \binom{Y}{\lambda}(T)\right] (uq)^n$$

which (by evaluating the series in Theorem 3.5 at uq) equals

$$\frac{1}{v_\mu v_\lambda} \prod_{r=1}^{|\mu|} \left(\frac{u^r q^r}{q^r - 1}\right)^{n_r(\mu)} \prod_{r=1}^{|\lambda|} \left(\frac{u^r q^r}{q^r + 1}\right)^{n_r(\lambda)} \prod_{r=1}^{\infty} \frac{1}{1 - u/q^{2r-2}}.$$

Since the equality holds for all prime powers q, the equality also holds when q^{-1} is replaced by a formal variable z. □

We now prove Corollary 1.14, a generating function for the stable twisted Betti numbers associated to the character polynomial $\binom{X}{\mu}\binom{Y}{\lambda}$. We are grateful to Weiyan Chen for help with filling in the details of this proof.

Proof of Corollary 1.14. Fix a double partition (μ, λ), and let

$$\beta_i(n) = \left\langle \binom{X}{\mu}\binom{Y}{\lambda}, R_n^i \right\rangle_{B_n}.$$

For each $n \geq 0$, define the power series $f_n(z)$ by

$$f_n(z) := \sum_{i=0}^{n^2} \frac{\beta_i(n) z^i}{(1-z^2)(1-z^4)\cdots(1-z^{2n})}.$$

By Theorem 1.13, we have equality

$$\begin{aligned}\sum_{n=0}^{\infty} f_n(z)u^n &= \sum_{n=0}^{\infty}\sum_{i=0}^{n^2} \frac{\beta_i(n) z^i u^n}{(1-z^2)(1-z^4)\cdots(1-z^{2n})} \\ &= \frac{1}{v_\mu\, v_\lambda} \prod_{r=1}^{|\mu|} \left(\frac{u^r}{1-z^r}\right)^{n_r(\mu)} \prod_{r=1}^{|\lambda|} \left(\frac{u^r}{1+z^r}\right)^{n_r(\lambda)} \prod_{r=1}^{\infty} \frac{1}{1-uz^{2r-2}}.\end{aligned}$$

Let $f(z)$ denote the pointwise limit of the sequence $\{f_n(z)\}$ on the open unit disk $\{z \in \mathbb{C} \,:\, |z| < 1\}$. Then for fixed z with $|z| < 1$,

$$\begin{aligned} f(z) &= \lim_{n\to\infty} [u^n] \sum_{n=0}^{\infty} f_n(z) u^n \\ &= \lim_{n\to\infty} [u^n] \frac{1}{v_\mu\, v_\lambda} \prod_{r=1}^{|\mu|} \left(\frac{u^r}{1-z^r}\right)^{n_r(\mu)} \prod_{r=1}^{|\lambda|} \left(\frac{u^r}{1+z^r}\right)^{n_r(\lambda)} \prod_{r=1}^{\infty} \frac{1}{1-uz^{2r-2}}.\end{aligned}$$

By Lemma 2.1 this pointwise limit is

$$f(z) = \frac{1}{v_\mu\, v_\lambda} \prod_{r=1}^{|\mu|} \left(\frac{1}{1-z^r}\right)^{n_r(\mu)} \prod_{r=1}^{|\lambda|} \left(\frac{1}{1+z^r}\right)^{n_r(\lambda)} \prod_{r=1}^{\infty} \frac{1}{1-z^{2r}}.$$

Next, recall that by Wilson [Wil14, Corollary 6.5] and Jiménez Rolland–Wilson [JRW16, Proposition 3.1], for each fixed i the sequence $\beta_i(n)$ is eventually constant, with stable value $\beta_i := \lim_{n\to\infty} \beta_i(n)$; see Corollary 1.12. It follows that the sequence $\{f_n(z)\}$ also converges as a sequence of formal power series, that is, for each $i \geq 0$ the coefficient $[z^i]f_n(z)$ is eventually constant in n. Its limit, the power series $g(z)$ with $[z^i]g(z) = \lim_{n\to\infty}[z^i]f_n(z)$, is

$$g(z) = \lim_{n\to\infty}^{z\text{-adic}} \sum_{i=0}^{n^2} \frac{\beta_i(n) z^i}{(1-z^2)(1-z^4)\cdots(1-z^{2n})} = \sum_{i=0}^{\infty} \frac{\beta_i z^i}{\prod_{r=1}^{\infty}(1-z^{2r})}.$$

To complete the proof, we wish to equate the pointwise limit $f(z)$ and the formal power series limit $g(z)$ of the sequence $\{f_n(z)\}$. (These limits need not be equal in general, even when both limits exist and are analytic functions on the unit disk.) In order to do this, we will prove that the sequence $\{f_n(z)\}$ converges *uniformly* to $f(z)$ on the closed disk $\left\{z \in \mathbb{C} \,:\, |z| \leq \frac{1}{2}\right\}$, and then equality $f(z) = g(z)$ follows from an application of the Cauchy integral formula.

To show uniform convergence, by Dini's theorem, it suffices to show that at each point z in the closed disk, the sequence $\{f_n(z)\}$ is monotone increasing. But

$$\sum_{n=0}^{\infty} \big(f_n(z) - f_{n-1}(z)\big)u^n = (1-u)\sum_{n=0}^{\infty} f_n(z)u^n$$
$$= \frac{1}{v_\mu\, v_\lambda} \prod_{r=1}^{|\mu|} \left(\frac{u^r}{1-z^r}\right)^{n_r(\mu)} \prod_{r=1}^{|\lambda|} \left(\frac{u^r}{1+z^r}\right)^{n_r(\lambda)} \prod_{r=1}^{\infty} \frac{1}{1-uz^{2r}},$$

and by inspection, for each $n \geq 0$ and z with $|z| \leq \frac{1}{2}$, the coefficient $\big(f_n(z) - f_{n-1}(z)\big)$ of u^n is positive as desired. We can therefore equate $f(z) = g(z)$, and we conclude

$$\sum_{i=0}^{\infty} \beta_i z^i = \frac{1}{v_\mu\, v_\lambda} \prod_{r=1}^{|\mu|} \left(\frac{1}{1-z^r}\right)^{n_r(\mu)} \prod_{r=1}^{|\lambda|} \left(\frac{1}{1+z^r}\right)^{n_r(\lambda)}$$

as claimed. □

Remark 3.8. Corollary 1.14 can also be proven using Theorem 1.6 and results of Jiménez Rolland and Wilson [JRW16]. Specifically, combining Theorem 1.6(ii) and Lehrer's identity Theorem 1.1(ii), we obtain the formula

$$\lim_{n\to\infty} \sum_{i=0}^{n^2} q^{-i}\beta_i(n) = \frac{1}{v_\mu v_\lambda} \prod_{r=1}^{|\mu|} \left(\frac{q^r}{q^r-1}\right)^{n_r(\mu)} \prod_{r=1}^{|\lambda|} \left(\frac{q^r}{q^r+1}\right)^{n_r(\lambda)}.$$

By [JRW16, Theorem 4.3] the left-hand side of this formula is equal to

$$\sum_{i=0}^{\infty} \lim_{n\to\infty} q^{-i}\beta_i(n) = \sum_{i=0}^{\infty} q^{-i}\beta_i,$$

and so by the substitution $z = \frac{1}{q}$ we conclude Corollary 1.14. The proof of Corollary 1.14 given above avoids the nontrivial combinatorial and analytic work performed in [JRW16].

Using the generating function from Corollary 1.14, we now prove Corollary 1.15, on linear recurrence relations satisfied by the stable twisted Betti numbers.

Proof of Corollary 1.15. A general character polynomial P may be written as a linear combination of character polynomials of the form $\binom{X}{\mu}\binom{Y}{\lambda}$

with $|\mu| + |\lambda| \leq \deg(P)$. Hence, its generating function can be written as a rational function with denominator given by the common denominator of the generating functions

$$\frac{1}{v_\mu v_\lambda} \prod_{r=1}^{|\mu|} \left(\frac{1}{1-z^r}\right)^{n_r(\mu)} \prod_{r=1}^{|\lambda|} \left(\frac{1}{1+z^r}\right)^{n_r(\lambda)} \qquad |\mu| + |\lambda| \leq \deg(P).$$

A coarse bound on the degree of this denominator follows from the observation that, for each $1 \leq r \leq \deg(P)$, the corresponding factors $(1-z^r)$ and $(1+z^r)$ each appear in this common denominator with multiplicities at most $\frac{\deg(P)}{r}$, and hence together contribute at most $2r\left(\frac{\deg(P)}{r}\right) = 2\deg(P)$ to its degree. This gives a total degree of at most $2\deg(P)^2$. □

The form of the generating functions in Corollary 1.14 also implies that the stable twisted Betti numbers β_d will be quasipolynomial, as in Corollary 1.17.

Proof of Corollary 1.17. Stanley [Sta97, Proposition 4.4.1] proves that an integer function $p(t)$ is quasipolynomial with period M if and only if the associated generating function $\sum_{n\geq 0}^{\infty} p(n)x^n$ is a rational function $A(x)/B(x)$ (reduced to lowest terms) such that all roots of $B(x)$ are M^{th} roots of unity, and $\deg(A) <\deg(B)$. Moreover, the degree of $p(t)$ is strictly bounded above by the largest multiplicity of a root of $B(x)$.

If the character polynomial P has constant term zero, we may write P as a linear combination of character polynomials $\binom{X}{\mu}\binom{Y}{\lambda}$ with $|\mu| + |\lambda| > 0$. Then by Corollary 1.14, the generating function for the associated stable Betti numbers is given by a linear combination of functions of the form

$$\frac{1}{v_\mu v_\lambda} \prod_{i=1}^{|\mu|} \left(\frac{1}{1-z^i}\right)^{n_i(\mu)} \prod_{i=1}^{|\lambda|} \left(\frac{1}{1+z^i}\right)^{n_i(\lambda)}$$

such that $0 < |\mu|+|\lambda| \leq \deg(P)$. Since the denominators $(1-z^i)$ and $(1+z^i)$ have positive degree and all roots are $(2i)^{th}$ roots of unity, we conclude that these stable Betti numbers are quasipolynomial, and have a quasiperiod bounded by the least common multiple of the numbers $2, 4, \ldots 2\deg(P)$. Moreover, in the common denominator for this generating function the maximum multiplicity of any root is $\deg(P)$, and so the quasipolynomial has degree at most $\deg(P) - 1$ as claimed.

Now suppose that P has constant term c. Only the 0^{th} graded piece R^0_n of the coinvariant algebra contains the trivial B_n representation, hence the stable Betti numbers for P and $(P - c)$ will be the same for all $i > 0$. This concludes the proof. □

3.2. Examples of stable twisted Betti numbers in type B/C. We end the paper using Corollary 1.14 to compute some examples of stable Betti numbers. Let $\mathbb{C}^n$ denote the canonical n-dimensional B_n-representation by

signed permutation matrices; this is the irreducible representation associated to the double partition $((n-1),(1))$. Below we describe combinatorial properties of twisted Betti numbers with coefficients in $\mathbb{C}^n$, and some symmetric and exterior powers of $\mathbb{C}^n$ in small degree.

Example 3.9. (Example: $\mathbb{C}^n$).

Character polynomial:

$$\begin{aligned} P^{\mathbb{C}^n} =& X_1 - Y_1 \\ =& \binom{X}{\square} - \binom{Y}{\square}. \end{aligned}$$

Betti numbers:

$$\begin{aligned} \sum_{i=0}^{\infty} \beta_i z^i =& \frac{z}{(1-z)(1+z)} \\ =& z + z^3 + z^5 + z^7 + z^9 + \cdots + z^{2d+1} + \cdots . \end{aligned}$$

Recurrence: $\beta_d = \beta_{d-2}$ for $d \geq 3$.

Quasipolynomiality: For $d \geq 0$, $\beta_d = \begin{cases} 0 & d \equiv 0 \pmod 2 \\ 1 & d \equiv 1 \pmod 2. \end{cases}$

Example 3.10. ($\bigwedge^2 \mathbb{C}^n$).

Character polynomial:

$$\begin{aligned} P^{\bigwedge^2 \mathbb{C}^n} =& \binom{X_1}{2} + \binom{Y_1}{2} - X_2 + Y_2 - X_1 Y_1 \\ =& \binom{X}{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + \binom{Y}{\begin{smallmatrix}\square\\\square\end{smallmatrix}} - \binom{X}{\square\square} + \binom{Y}{\square\square} - \binom{X}{\square}\binom{Y}{\square}. \end{aligned}$$

Betti numbers:

$$\begin{aligned} \sum_{i=0}^{\infty} \beta_i z^i =& \frac{z^4}{(1-z^2)^2(1+z^2)} \\ =& z^4 + z^6 + 2z^8 + 2z^{10} + 3z^{12} + 3z^{14} + \cdots + \left\lfloor \frac{d}{2} \right\rfloor z^{2d} + \cdots . \end{aligned}$$

Recurrence: $\beta_d = \beta_{d-2} + \beta_{d-4} - \beta_{d-6}$ for $d \geq 6$.

Quasipolynomiality: For $d \geq 0$,

$$\beta_d = \begin{cases} \frac{d}{4} & d \equiv 0 \pmod 4 \\ \frac{d-2}{4} & d \equiv 2 \pmod 4 \\ 0 & d \equiv 1, 3 \pmod 4. \end{cases}$$

Example 3.11. $(\bigwedge^3 \mathbb{C}^n)$.

Character polynomial:

$$\binom{X_1}{3} - \binom{Y_1}{3} + X_1\binom{Y_1}{2} - Y_1\binom{X_1}{2} \\ - X_1X_2 + X_2Y_1 + Y_2X_1 - Y_1Y_2 + X_3 - Y_3.$$

Betti numbers:

$$\sum_{i=0}^{\infty} \beta_i z^i = \frac{z^9}{(1-z)^2(1+z)^2(1+z^2)(1-z^3)(1+z^3)} \\ = z^9 + z^{11} + 2z^{13} + 3z^{15} + 4z^{17} + 5z^{19} + 7z^{21} + 8z^{23} \\ + 10z^{25} + 12z^{27} + 14z^{29} + 16z^{31} + 19z^{33} + 21z^{35} \\ + 24z^{37} + 27z^{39} + 30z^{41} + 33z^{43} + \cdots.$$

Recurrence:

$$\beta_d = \beta_{d-2} + \beta_{d-4} - \beta_{d-8} - \beta_{d-10} + \beta_{d-12} \qquad \text{for } d \geq 12.$$

Quasipolynomiality: For $d \geq 0$,

$$\beta_d = \begin{cases} \frac{d^2}{48} - \frac{d}{8} + \frac{5}{48} & d \equiv 1, 5 \pmod{12} \\ \frac{d^2}{48} - \frac{d}{8} + \frac{9}{48} & d \equiv 3 \pmod{12} \\ \frac{d^2}{48} - \frac{d}{8} - \frac{7}{48} & d \equiv 7, 11 \pmod{12} \\ \frac{d^2}{48} - \frac{d}{8} + \frac{21}{48} & d \equiv 9 \pmod{12} \\ 0 & d \equiv 0, 2, 4, 6, 8, 10 \pmod{12}. \end{cases}$$

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