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Algebraic nonhyperbolicity of hyperkähler manifolds with Picard rank greater than one

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Abstract. A projective manifold is algebraically hyperbolic if the degree of any curve is bounded from above by its genus times a constant, which is independent from the curve. This is a property which follows from Kobayashi hyperbolicity. We prove that hyperkähler manifolds are not algebraically hyperbolic when the Picard rank is at least 3, or if the Picard rank is 2 and the SYZ conjecture on existence of Lagrangian fibrations is true. We also prove that if the automorphism group of a hyperkähler manifold is infinite then it is algebraically nonhyperbolic.

Contents

1. Introduction

In [V2] M. Verbitsky proved that all hyperkähler manifolds are Kobayashi nonhyperbolic. It is interesting to inquire if projective hyperkähler manifolds are also algebraically nonhyperbolic (Definition 2.5). For a given projective manifold algebraic nonhyperbolicity implies Kobayashi nonhyperbolicity. We prove algebraic nonhyperbolicity for projective hyperkähler manifolds with infinite group of automorphisms.

Theorem 1.1. *Let M be a projective hyperkähler manifold with infinite automorphism group. Then M is algebraically nonhyperbolic.*

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If a projective hyperkähler manifold has Picard rank at least three, we show that it is algebraically nonhyperbolic. For the case when the Picard rank equals to two we need an extra assumption in order to prove algebraic nonhyperbolicity. The SYZ conjecture states that a nef parabolic line bundle on a hyperkähler manifold gives rise to a Lagrangian fibration (Conjecture 2.4).

Theorem 1.2. *Let M be a projective hyperkähler manifold with Picard rank ρ. Assume that either $\rho > 2$, or $\rho = 2$ and the SYZ conjecture holds. Then M is algebraically nonhyperbolic.*

2. Basic notions

Definition 2.1. A *hyperkähler manifold of maximal holonomy* (or *irreducible holomorphic symplectic*) manifold M is a compact complex Kähler manifold with $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}\sigma$, where σ is everywhere nondegenerate. From now on we would tacitly assume that hyperkähler manifolds are of maximal holonomy.

Due to results of Matsushita, holomorphic maps from hyperkähler manifolds are quite restricted.

Theorem 2.2 (Matsushita, [Mat])**.** *Let M be a hyperkähler manifold and $f\colon M \to B$ a proper surjective morphism with a smooth base B. Assume that f has connected fibers and $0 < \dim B < \dim M$. Then f is Lagrangian and $\dim_{\mathbb{C}} B = n$, where $\dim_{\mathbb{C}} M = 2n$.*

Following Theorem 2.2, we call the surjective morphism $f\colon M \to B$ a *Lagrangian fibration* on the hyperkähler manifold M. A dominant map $f\colon M \dashrightarrow B$ is a *rational Lagrangian fibration* if there exists a birational map $\varphi\colon M \dashrightarrow M'$ between hyperkähler manifolds such that the composition $f \circ \varphi^{-1}\colon M' \to B$ is a Lagrangian fibration. J.-M. Hwang proved that if the base B of a hyperkähler Lagrangian fibration is smooth, then $B \cong \mathbb{P}^n$ (see [Hw]).

Definition 2.3. Given a hyperkähler manifold M, there is a nondegenerate primitive form q on $H^2(M, \mathbb{Z})$, called the *Beauville–Bogomolov–Fujiki form* (or the *"BBF form"* for short), of signature $(3, b_2 - 3)$, and satisfying the *Fujiki relation*

$$\int_M \alpha^{2n} = c \cdot q(\alpha)^n \qquad \text{for } \alpha \in H^2(M, \mathbb{Z}),$$

with $c > 0$ a constant depending on the topological type of M. This form generalizes the intersection pairing on K3 surfaces. A detailed description of the form can be found in [Be], [Bog] and [F].

Notice that given a Lagrangian fibration $f\colon M \to \mathbb{P}^n$, if h is the hyperplane class on $\mathbb{P}^n$, and $\alpha = f^*h$, then α belongs to the birational Kähler cone of M and $q(\alpha) = 0$. The following SYZ conjecture states that the converse is also true.

Conjecture 2.4 (Tyurin, Bogomolov, Hassett–Tschinkel, Huybrechts, Sawon)**.** *If L is a line bundle on a hyperkähler manifold M with $q(L) = 0$, and such that $c_1(L)$ belongs to the birational Kähler cone of M, then L defines a rational Lagrangian fibration.*

For more reference on this conjecture, please see [HT], [Saw], [Hu3] and [V1].

This conjecture is known for deformations of Hilbert schemes of points on K3 surfaces (Bayer–Macrì [BM]; Markman [Mar]), and for deformations of the generalized Kummer varieties $K_n(A)$ (Yoshioka [Y]).

In [V2] M. Verbitsky proved that all hyperkähler manifolds are Kobayashi nonhyperbolic. In [KLV] together with S. Lu we proved that the Kobayashi pseudometric vanishes identically for K3 surfaces and for hyperkähler manifolds deformation equivalent to Lagrangian fibrations under some mild assumptions. In [De] Demailly introduced the following notion.

Definition 2.5. A projective manifold M is *algebraically hyperbolic* if for any Hermitian metric h on M there exists a constant $A > 0$ such that for any holomorphic map $\varphi\colon C \to M$ from a curve of genus g to M we have that $2g - 2 \geqslant A \int_C \varphi^*\omega_h$, where ω_h is the Kähler form of h.

In this paper all varieties we consider are smooth and projective. For projective varieties, Kobayashi hyperbolicity implies algebraic hyperbolicity ([De]). Here we explore nonhyperbolic properties of projective hyperkähler manifolds. Algebraic nonhyperbolicity implies Kobayashi nonhyperbolicity.

3. Main results

Proposition 3.1. *Let M be a hyperkähler manifold admitting a (rational) Lagrangian fibration. Then M is algebraically nonhyperbolic.*

Proof. We use the fact that the fibers of a Lagrangian fibrations are abelian varieties ([Mat]). The isogeny self-maps on an abelian variety provide curves of fixed genus and arbirary large degrees, and therefore they are algebraically nonhyperbolic.

An alternative way of proving this proposition is by using the following result whose proof was suggested by Prof. K. Oguiso.

Lemma 3.2. *If a hyperkähler manifold M admits a Lagrangian fibration, then there exists a rational curve on M.*

Indeed, in [HwO] J.-M. Hwang and K. Oguiso give a Kodaira-type classification of the general singular fibers of a holomorphic Lagrangian fibration. All of the general singular fibers are covered by rational curves. The locus of singular fibers is nonempty (e.g., Proposition 4.1 in [Hw]), and therefore there is a rational curve on M.

According to Lemma 3.2, M contains a rational curve, and therefore, M is algebraically nonhyperbolic. This finishes the proof of Proposition 3.1. □

Lemma 3.3. *Let M be a projective hyperkähler manifold with infinite automorphism group Γ. Consider the natural map*

$$f : \Gamma \longrightarrow \operatorname{Aut}(H^{1,1}(M)).$$

Then the elements of the Kähler cone have infinite orbits with respect to $f(\Gamma)$.

Proof. See the discussion in Section 2 of [O2]. □

Lemma 3.4. *Let M be a projective hyperkähler manifold, and Γ its automorphism group. Consider the natural map*

$$g : \Gamma \longrightarrow \operatorname{Aut}(H^2_{tr}(M)) \times \operatorname{Aut}(H^{1,1}(M)).$$

Then $g(\Gamma)$ is finite in the first component $\operatorname{Aut}(H^2_{tr}(M))$.

Proof. This has been proven by Oguiso, see [O1]. The idea is that the BBF form restricted to the transcendental part $H^2_{tr}(M)$ is of K3-type. Then we can apply Zarhin's theorem (Theorem 1.1.1 in [Z]) to deduce that

$$g(\Gamma) \subset \operatorname{Aut}(H^2_{tr}(M))$$

is finite. □

Theorem 3.5. *Let M be a projective hyperkähler manifold with infinite automorphism group. Then M is algebraically nonhyperbolic.*

Proof. For any Kähler class w on M, its $f(\Gamma)$-orbit is infinite by Lemma 3.3. Fix a polarization w on M with normalization $q(w) = 1$. For a given constant $C > 0$ consider the set

$$\mathcal{D}_C = \{x \in H^{1,1}(M, \mathbb{Z}) \ \mid \ q(x) \geqslant 0, \quad q(x, w) \leqslant C\}.$$

Notice that $\mathcal{D}_C$ is compact. Indeed, $y = x - q(x, w)w$ is orthogonal to w with respect to the BBF form q. The quadratic form q is of type $(1, b_2 - 1)$ on $H^{1,1}(M, \mathbb{Z})$ and since $q(w) > 0$, the restriction $q|_{w^\perp}$ is negative-definite. A direct computation shows that

$$q(y) = q(x) - 2q(x, w)^2 + q(x, w)^2 q(w) = q(x) - q(x, w)^2 \geqslant -C^2.$$

The set $\mathcal{D}_C$ is equivalent to the set of elements $\{y \in w^\perp | q(y) \geqslant -C^2\}$, which is compact because $q|_{w^\perp}$ is negative-definite. Since the set $\mathcal{D}_C$ is compact, $\sup_{x\in\Gamma\cdot\eta} \deg x = \infty$, which means there is a class of a curve η with $q(\eta) > 0$. However, all curves in the orbit $\Gamma\cdot\eta$ have constant genus. Since their degrees could be arbitrarily high, then M is algebraically nonhyperbolic. □

Lemma 3.6. *Let M be a hyperkähler manifold such that the positive cone does not coincide with the Kähler cone. Then M contains a rational curve.*

Proof. This is a classical result that Boucksom and Huybrechts knew in the early 2000's [Bou, Hu2]. □

Theorem 3.7. *Let M be a hyperkähler manifold with Picard rank ρ. Assume that either $\rho > 2$ or $\rho = 2$ and the SYZ conjecture holds. Then M is algebraically nonhyperbolic.*

Proof. Notice that the Hodge lattice $H^{1,1}(M,\mathbb{Z})$ of a hyperkähler manifold has signature $(1,k)$. Therefore, for $\rho \geqslant 2$, the Hodge lattice contains a vector with positive square, and M is projective ([Hu1]). First, consider the case when $\rho > 2$. If the Kähler cone coincides with the positive cone, then the automorphism group $\mathrm{Aut}(M)$ is commensurable with the group of isometries $SO(H^2(M,\mathbb{Z}))$ (Theorem 2.17 in [AV]) preserving the Hodge type. By Lemma 3.4, this group is commensurable with the group of isometries of the Hodge lattice $H^{1,1}(M,\mathbb{Z})$. By Borel and Harish-Chandra's theorem ([BorHC]), if $\rho > 2$, any arithmetic subgroup of $SO(1,\rho-1)$ is a lattice. However, Borel density theorem implies that any lattice in a noncompact simple Lie group is Zariski dense ([Bor]). Therefore, for $\rho > 2$, $SO(H^{1,1}(M,\mathbb{Z}))$ is infinite. In this case $\mathrm{Aut}(M)$ is also infinite and we can apply Theorem 3.5. On the other hand, if the Kähler cone does not coincide with the positive cone, then by Lemma 3.6 there is a rational curve on M. Therefore, M is algebraically nonhyperbolic.

Now let $\rho = 2$. Assume the positive cone and the Kähler cone coincide. If there is no $\eta \in H^{1,1}(M,\mathbb{Z})$ with $q(\eta) = 0$, then by Theorem 87 in [Di], $SO(H^{1,1}(M,\mathbb{Z}))$ is isomorphic to $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Therefore, both $SO(H^{1,1}(M,\mathbb{Z}))$ and $\mathrm{Aut}(M)$ are infinite and we can apply Theorem 3.5. If there is $\eta \in H^{1,1}(M,\mathbb{Z})$ with $q(\eta) = 0$, then the SYZ conjecture implies that η defines a rational fibration on M and we could apply Proposition 3.1. If $\rho = 2$ and the positive and the Kähler cones are different (i.e., the positive cone is divided into Kähler chambers), then there is a nef class $\eta \in H^{1,1}(M,\mathbb{Z})$ with $q(\eta) = 0$. Since we assumed that the SYZ conjecture holds, the class η defines a Lagrangian fibration on M. Applying Proposition 3.1 we conclude that M is algebraically nonhyperbolic. □

Remark 3.8. We conjecture that all projective hyperkähler manifolds are algebraically nonhyperbolic. However, our proof fails for manifolds with Picard rank 1.

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