Groups with no coarse embeddings into hyperbolic groups

David Hume and Alessandro Sisto

ABSTRACT. We introduce an obstruction to the existence of a coarse embedding of a given group or space into a hyperbolic group, or more generally into a hyperbolic graph of bounded degree. The condition we consider is "admitting exponentially many fat bigons", and it is preserved by a coarse embedding between graphs with bounded degree. Groups with exponential growth and linear divergence (such as direct products of two groups one of which has exponential growth, solvable groups that are not virtually nilpotent, and uniform higher-rank lattices) have this property and hyperbolic graphs do not, so the former cannot be coarsely embedded into the latter. Other examples include certain lacunary hyperbolic and certain small cancellation groups.

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1. Introduction

Hyperbolic groups have been at the heart of geometric group theory since Gromov's seminal paper [Gro87] and are still of vital importance to the present day. They are among the best understood classes of groups with a large, diverse and ever-expanding literature. Despite this it is not at all

well understood which finitely generated groups may appear as subgroups of hyperbolic groups. One algebraic obstruction is admitting a Baumslag–Solitar subgroup $BS(m,n) = \langle a, b \, | \, b^{-1}a^m b a^{-n} \rangle$ with $|m|, |n| \geq 1$. The goal of this paper is to consider a more geometric obstruction. To do this, we consider every finitely generated group as a Cayley graph with respect to some finite symmetric generating set, and consider every graph as a metric space with the shortest path metric.

In the same way that one may view the existence of a quasi-isometry $q : H \to G$ between finitely generated groups as the natural geometric generalization of the algebraic statement "H and G are (abstractly) commensurable", we will consider the existence of a coarse embedding $\phi : H \to G$ as the comparable generalization of the statement "H is virtually isomorphic to a subgroup of G". In both cases the algebraic statement is known to be stronger than the geometric one: all Baumslag–Solitar groups $BS(m,n)$ with $1 < |m| < |n|$ are quasi-isometric, but, for example, $BS(2,p)$ and $BS(2,q)$ are not commensurable whenever p, q are distinct odd primes [Why01]; while $\mathbb{Z}^2$ is never a subgroup of a hyperbolic group, but $\mathbb{R}^2$ coarsely embeds into real hyperbolic 3-space (as a horosphere) and hence into the fundamental group of any closed hyperbolic 3-manifold. A much more remarkable statement can be found in [AoDT16]: any infinite order element of a finitely generated subgroup H of a finitely generated hyperbolic group G has stable orbits in any Cayley graph of H.

Coarse embeddings of groups into other spaces (particularly certain Banach spaces) are also highly sought, since groups admitting such an embedding satisfy the Novikov and coarse Baum–Connes conjectures [Yu00, KaY06].

There are few invariants which can provide a general geometric obstruction to a coarse embedding, of which the most commonly studied are growth and asymptotic dimension. More recently constructed obstructions include separation and Poincaré profiles, as well as properties of L^p-cohomology related to sphere packings [BeST12, HMT17, Pan16].

Our main result is as follows:

Theorem 1.1. *Let G be a group admitting exponentially many fat bigons. Then G does not coarsely embed into any hyperbolic graph of bounded degree.*

Since groups which are hyperbolic relative to virtually nilpotent subgroups coarsely embed into hyperbolic graphs of bounded degree [DaY05], we can also deduce that no group admitting exponentially many fat bigons is a subgroup of such a relatively hyperbolic group.

The exact definition of admitting exponentially many fat bigons is given in §2. Here we will just focus on examples, our main source of which is the following proposition.

Proposition 1.2. *Any finitely generated group with exponential growth and linear divergence admits exponentially many fat bigons.*

We recall the definition of divergence in Section 3. Examples of groups with linear divergence include direct products of infinite groups, groups with infinite center, groups satisfying a law (e.g., solvable groups) that are not virtually cyclic [DrS05], all uniform [KlL97] and many nonuniform [DrMS10, CDG10, LB15] higher-rank lattices. Since the lamplighter $\mathbb{Z}_2 \wr \mathbb{Z}$ admits exponentially many fat bigons, the same is true for every wreath product $H \wr G$ where H is nontrivial and G is infinite, since $\mathbb{Z}_2 \wr \mathbb{Z}$ quasi-isometrically embeds into all such groups. For an infinite finitely generated group, linear divergence is equivalent to the statement that no asymptotic cone contains a cut-point by results in [DrMS10], see Remark 3.2.

Proposition 1.2 has a number of interesting consequences.

Corollary 1.3. *Let G be a virtually solvable finitely generated group. Then G coarsely embeds in some hyperbolic group if and only if G is virtually nilpotent.*

Proof. By Assouad's Theorem [Ass82], every virtually nilpotent group can be coarsely embedded into some $\mathbb{R}^n$, and $\mathbb{R}^n$ embeds into $\mathbb{H}^{n+1}$ (as a horosphere) and hence into some hyperbolic group.

If G is not virtually nilpotent, then it has exponential growth [Mil68, Wol68]. Also, it has linear divergence (since its asymptotic cones do not have cut-points by [DrS05, Corollary 6.9]) and hence it cannot embed into any hyperbolic group by Proposition 1.2 and Theorem 1.1. $\qquad\square$

Corollary 1.4. *Let $m, n \in \mathbb{Z}$ with $|m| \leq |n|$. Then $BS(m, n)$ coarsely embeds into a hyperbolic group if and only if $m = 0$ or $|n| \leq 1$.*

Proof. If $m = 0$ or $|m| = |n| = 1$ then $BS(m, n)$ is virtually free or virtually abelian, so either is a hyperbolic group or coarsely embeds into one.

If $|m| = 1$ and $|n| > 1$ then $BS(m, n)$ is solvable with exponential growth so does not coarsely embed in a hyperbolic space by Corollary 1.3. When $1 < |m| < |n|$, $BS(1, 2)$ coarsely embeds into $BS(m, n)$, since $BS(1, 2)$ is isomorphic to a subgroup of $BS(2, 4)$ (via the map $a \mapsto a^2,\ b \mapsto b$) which is quasi-isometric to $BS(m, n)$ [Why01, Theorem 0.1]. It remains to check the case $1 < |m| = |n|$. In this case $BS(m, n)$ has a finite index subgroup isomorphic to $\mathbb{Z} \times F_n$ [Why01, Theorem 0.1] which has exponential growth and linear divergence, so we are done by Proposition 1.2. $\qquad\square$

Proposition 1.2 will follow from the more general Proposition 3.4. From this stronger statement and [OOS09, Theorem 6.1] one could deduce that the lacunary hyperbolic groups with "slow nonlinear divergence" constructed in [OOS09, Section 6] do not coarsely embed into any hyperbolic group, and in particular they are not subgroups of any hyperbolic group. However, at the time of writing the proof of [OOS09, Theorem 6.1] relies on [DrMS10, Theorem 2.1], but there is no such theorem in the published version, and no

result from which the property needed in [OOS09, Theorem 6.1] obviously follows.[1]

Additionally, in Proposition 3.5, we give a criterion for a $C'(\frac{1}{6})$ small cancellation group to have exponentially many fat bigons. This can be used to give an explicit example of a small cancellation group that does not coarsely embed in, and in particular is not a subgroup of any hyperbolic group. This is in contrast with the $C(6)$ small cancellation subgroups of hyperbolic groups constructed by Kapovich–Wise [KaW01].

We finish with two natural questions.

Question 1.5. Which (infinitely presented) small cancellation groups admit a coarse embedding into some hyperbolic group? Which are subgroups of some hyperbolic group?

Question 1.6. Does every amenable group that admits a coarse embedding into some hyperbolic group have polynomial growth?

2. Fat bigons

Given a metric space (X, d) $r > 0$ and $x \in X$ we denote by $B_r(x)$ the closed ball of radius r centred at x, and given a subset Y of X we denote the closed r-neighborhood of Y in X by $N_r(Y) = \{x \in X \mid d(x, Y) \leq r\}$. In this paper all graphs are assumed to be connected, have uniformly bounded degree, and be equipped with the shortest path metric.

Definition 2.1. Let X be a metric graph, with base vertex x_0, and let x be a vertex. Given $L \geq 1$, $s, C \geq 0$, an (L, s, C)-*bigon* at x is given by two paths α_1, α_2 from x_0 to x with the following properties:

(1) $l(\alpha_i) \leq L d(x_0, x)$.
(2) For $B = N_C(\{x_0, x\})$, we have $d(\alpha_1 - B, \alpha_2 - B) > s$.

Denote by $\mathcal{B}^X(L, s, C)$ the set of vertices x so that there exists an (L, s, C)-*bigon* at x.

Definition 2.2. Let X be a graph, with basepoint x_0. We say that X has *exponentially many fat bigons* (at x_0) if there exist constants $s_0, c, L > 1$ such that for every $s \geq s_0$ there exists a C so that the the function

$$g(n) = |\mathcal{B}^X(L, s, C) \cap B_n(x_0)|$$

is bounded from below by c^n for infinitely many $n \in \mathbb{N}$.

We say that X has *no fat bigons* (at x_0) if for every L there exists s so that for every C we have that $\mathcal{B}^X(L, s, C)$ is a bounded subset of X.

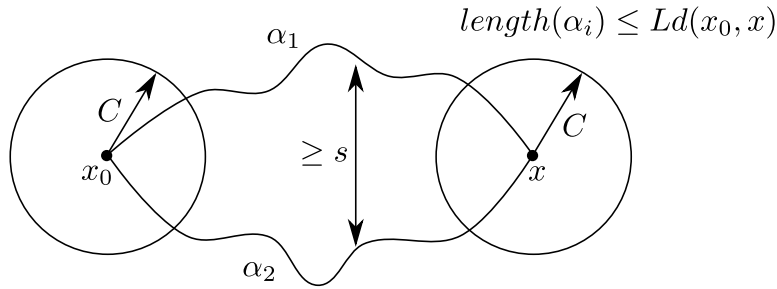[1]We are grateful to the referee for pointing this out to us.

FIGURE 1. An (L, s, C)-bigon at x. The two paths connect the basepoint x_0 to some x, stay far from each other in the middle and are not too long.

These two definitions are basepoint invariant (see Lemmas 2.3 and 2.5), so we may just talk about a graph having exponentially many (or no) fat bigons without specifying x_0.

Lemma 2.3. *Let X be a graph with no fat bigons at x_0, then for any $x_0' \in X$, X has no fat bigons at x_0'.*

Proof. If $d(x_0, x) \geq d(x_0', x_0) = d$ then an (L, s, C)-bigon at x (with respect to the basepoint x_0') can be extended by a geodesic from x_0' to x_0 to form a $(K, s, C + d)$-bigon at x (with respect to the basepoint x_0), where K will be determined shortly. Since the sides of the original bigon had length at most $Ld(x_0', x)$, the sides of the new bigon have length at most

$$Ld(x_0', x) + d(x_0', x_0) \leq Ld(x_0, x) + (L + 1)d(x_0', x_0) \leq (2L + 1)d(x_0, x),$$

so we may choose $K = 2L + 1$.

Suppose for a contradiction that there exists some $L \geq 1$ such that for all $s > 0$ there is a $C > 0$ and infinitely many (L, s, C)-bigons at $y_1, y_2, \ldots$ (based at x_0'). Since X has uniformly bounded degree, infinitely many of the y_i satisfy $d(x_0, y_i) \geq d(x_0', x_0)$, so there are infinitely many $(2L + 1, s, C + d)$-bigons based at x_0. This is a contradiction. $\square$

Having no fat bigons is a strong negation of having exponentially many bigons. Our goal for this section is the following:

Theorem 2.4. *Let X be a graph with exponentially many fat bigons. Then X does not coarsely embed into any hyperbolic graph of uniformly bounded degree.*

For notational purposes let us recall the definition of a coarse embedding. Given two graphs X, Y, with vertex sets VX, VY respectively, a *coarse embedding* of X into Y is a map $f : VX \to VY$, a constant $K \geq 1$ and a function $\rho : \mathbb{N} \to \mathbb{N}$ such that $\rho(n) \to \infty$ as $n \to \infty$ and

$$(1) \qquad \rho(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq Kd_X(x, y).$$

The proof of Theorem 2.4 is given as a pair of lemmas.

Lemma 2.5. *Let X, Y be bounded degree graphs. If f is a coarse embedding of X into Y and X has exponentially many fat bigons at x_0, then Y has exponentially many fat bigons at $y_0 = f(x_0)$. In particular, having exponentially many fat bigons does not depend on the choice of a basepoint x_0.*

The idea of proof is that, despite the fact that the distance from the basepoint of a point of X could decrease drastically after applying a coarse embedding, this cannot happen for too many points because the growth of Y is (at most) exponential. More specifically, there must be many points x so that there is a fat bigon at x and the distance from the basepoint of Y to $f(x)$ is linear in the distance from the basepoint of X to x. For such x, there is a fat bigon at $f(x)$ (with slightly worse constants).

Proof. Let f be a coarse embedding of X into Y and let K, ρ satisfy (1). Fix r such that $\rho(r) > 0$ and let $\Delta_X, \Delta_Y \geq 2$ be the maximal vertex degrees of X, Y respectively. By assumption there exist constants $d, L > 1$ such that for all $s \geq s_0$ there is a constant C such that

$$|\mathcal{B}^X(L, s, C) \cap B_n^X(x_0)| \geq d^n$$

holds for all n in an infinite subset $I \subseteq \mathbb{N}$. Define

$$\mathcal{A}_\epsilon = \{x \in \mathcal{B}^X(L, s, C) : d(y_0, f(x)) > \epsilon d(x_0, x)\}$$

and notice that

$$|\mathcal{A}_\epsilon \cap B_n^X(x_0)| \geq (\Delta_X)^{-(r+1)}|\mathcal{B}^X(L, s, C) \cap B_n^X(x_0)| - \Delta_Y^{\epsilon n + 1}$$

since $|f^{-1}(y)| \leq \Delta_X^{r+1}$ for any vertex $y \in VY$ and any ball of radius ϵn in Y contains at most $\Delta_Y^{\epsilon n + 1}$ vertices. Choose $\epsilon > 0$ sufficiently small that

$$(\Delta_X)^{-(r+1)}d^n - \Delta_Y^{\epsilon n + 1} \geq \left(\frac{1+d}{2}\right)^n$$

holds for all n sufficient large. Fix such an $\epsilon > 0$ and set $\mathcal{A} = \mathcal{A}_\epsilon$.

Claim. $f(\mathcal{A} \cap B_n^X(x_0)) \subset \mathcal{B}^Y(KL\epsilon^{-1}, \rho(s) - 2K, KC + K) \cap B_{Kn}^Y(y_0)$.

Proof of Claim. Let $x \in \mathcal{A} \cap B_n^X(x_0)$. Since f is K-Lipschitz and $f(x_0) = y_0$, we have $f(x) \in B_{Kn}^Y(y_0)$.

If α_1, α_2 form a (L, s, C)-bigon at x, we can apply f to the vertices of the α_i and connect consecutive points by geodesics in Y, thereby obtaining new paths α_1', α_2' from y_0 to $f(x)$. The length of α_i' is at most K times the length of α_i, and hence $|\alpha_i'| \leq KL\epsilon^{-1}d_Y(y_0, f(x))$.

Given two vertices $v_1' \in \alpha_1'$, $v_2' \in \alpha_2'$ not in $N_{KC+K}(\{y_0, f(x)\})$ there are vertices $v_i \in \alpha_i'$ and $w_i \in \alpha_i$ such that $d_Y(v_i, v_i') \leq K$, $f(w_i) = v_i$ and $w_i \notin N_C(\{x_0, x\})$ for $i = 1, 2$. Hence $d_X(w_1, w_2) > s$, so $d_Y(v_1, v_2) \geq \rho(s)$ by assumption and $d_Y(v_1', v_2') \geq \rho(s) - 2K$, as required. $\square$

Since $\left|f^{-1}(v)\right| \leq (\Delta_X + 1)^r$ for each $v \in VY$, we see that

$$\left|\mathcal{B}^Y(LK\epsilon^{-1}, \rho(s) - K, KC + K) \cap B^Y_{Kn}(y_0)\right| \geq (\Delta_X + 1)^{-r}\left(\frac{1+d}{2}\right)^n$$

holds for all sufficiently large $n \in I$. This easily implies that Y has exponentially many fat bigons. $\qquad\square$

Lemma 2.6. *Let X be a δ-hyperbolic graph. Then X has no fat bigons.*

We recall that X is δ-hyperbolic if, for any geodesic triangle with sides $\gamma_1, \gamma_2, \gamma_3$, we have $\gamma_1 \subseteq N_\delta(\gamma_2 \cup \gamma_3)$.

The idea of proof is the following. Suppose we have two paths connecting the same pair of points which stay far from each other in the middle. Then any point on a geodesic connecting the endpoints can be close to at most one of the paths. Hence, one of the two paths stays far from at least "half" of the geodesic. Travelling far from a geodesic in a hyperbolic space is expensive, hence the path that stays far from half of the geodesic is long. More precisely, we consider disjoint balls along the geodesic, and count how many are avoided by each path.

Proof. We start with two easy facts about hyperbolic spaces. Several forms of the first one are well-known, and some version of the second one is used in [Sis16, Lemma 4.2]. From now on we assume $\delta \geq 1$.

Claim 1. Let $x, y, p, q \in X$, and let p', q' on a geodesic $[x, y]$ satisfy

$$d(p, p') = d(p, [x, y]) \quad \text{and} \quad d(q, q') = d(q, [x, y]).$$

Then $d(p', q') \leq d(p, q) + 8\delta$. Moreover, any geodesic $[p, q]$ passes 2δ-close to any point m on the subgeodesic $[p', q']$ of $[x, y]$ satisfying

$$d(p', m), d(q', m) > 2\delta.$$

Proof of Claim 1. Let α_p, α_q be geodesics from p to p' and from q to q' respectively. Consider the geodesic quadrangle whose sides are α_p, α_q and geodesics $[p, q]$ and $[p', q']$. Since X is δ-hyperbolic, we have

$$[p', q'] \subseteq N_{2\delta}(\alpha_p \cup \alpha_q \cup [p, q]).^2$$

Moreover, any point r on α_p within distance 2δ of $[x, y]$ must be in $B_{2\delta}(p')$, and similarly for α_q: to see this, let r' on $[x, y]$ satisfy $d(r, r') \leq 2\delta$. Then $d(p, r') \leq d(p, r) + d(r, r') \leq (d(p, p') - d(p', r)) + 2\delta$. Since by our choice of p' we must have $d(p, p') \leq d(p, r')$, this yields $d(p', r) \leq 2\delta$, as required. Therefore, every point on $[p', q']$ is either in $B_{2\delta}(p')$, $B_{2\delta}(q')$ or in the 2δ-neighbourhood of $[p, q]$, proving the "moreover" part of the Claim. Continuing, we see that either $d(p', q') \leq 4\delta$ (in which case $d(p', q') \leq d(p, q) + 8\delta$ clearly holds), or, for every $\epsilon > 0$ there exist points $a, b \in [p', q']$ such that

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[2] This is a standard fact about quadrangles in hyperbolic spaces, proven by cutting it into two triangles using a diagonal.

$2\delta < d(a, p') \le 2\delta + \epsilon$, $2\delta < d(b, q') \le 2\delta + \epsilon$ and points $c, d \in [p, q]$ such that $d(a, c), d(b, d) \le 2\delta$. Thus

$$d(p', q') \le d(a, b) + 4\delta + 2\epsilon \le d(c, d) + 8\delta + 2\epsilon \le d(p, q) + 8\delta + 2\epsilon.$$

As this holds for all $\epsilon > 0$, the result follows. $\qquad\square$

Claim 2. There exist $\epsilon, s_0 > 0$ so that for each $s \ge s_0$ the following holds. Let $x, y \in X$ and let $B_1, \ldots, B_k$ be disjoint balls of radius s centred on a geodesic $[x, y]$, with centre at distance at least s from $\{x, y\}$. Let α be a path from x to y that avoids all B_i. Then $l(\alpha) \ge k \cdot (1 + \epsilon)^s$.

Proof of Claim 2. Recall that we always assume $\delta \ge 1$. Let us now set $s_0 = 50\delta$.

Let m_i be the centre of the ball B_i. The B_i define disjoint subgeodesics $I_i = [c_i, d_i]$ of $[x, y]$ with midpoint m_i of length $2s$, and up to reindexing we can assume that, in the natural order $\prec$ on $[x, y]$ defined by $a \prec b$ whenever $d(x, a) < d(x, b)$, we have $c_i \prec d_i$ and $d_i \prec c_{i+1}$.

Let $\pi : X \to [x, y]$ be a closest-point projection, meaning a map satisfying $d(p, \pi(p)) = d(p, [x, y])$ for all $p \in X$. By Claim 1 $d(\pi(p), \pi(q)) \le d(p, q) + 8\delta$ for all $p, q \in X$, so if $d(p, q) \le 1$, then $d(\pi(p), \pi(q)) \le 9\delta$. Using this, one can construct disjoint subpaths $\alpha_1, \ldots, \alpha_k$ of α such that the closest point projection of the endpoints a_i, b_i of α_i onto $[x, y]$ are contained in I_i and $d(\pi(a_i), c_i) \le 9\delta, d(\pi(b_i), d_i) \le 9\delta$ in the following way: let a_i be the first vertex of α that projects inside I_i, let b_i' the first vertex of α so that $d_i \prec \pi(b_i')$, and define b_i to be the vertex immediately preceding b_i'. Now set α_i to be the corresponding subpath of α from a_i to b_i: it is immediate that the required properties are satisfied.

Notice that $d(\pi(a_i), m_i), d(\pi(b_i), m_i) \ge s - 9\delta > 2\delta$. Hence, Claim 1 implies that, for each i, there exists n_i on a geodesic $[a_i, b_i]$ with $d(m_i, n_i) \le 2\delta$. In particular, α_i avoids a ball of radius $s - 2\delta$ around n_i. Notice also that $l(\alpha_i) \ge 1$ because the distance between its endpoints a_i, b_i is larger than 1 (a_i, b_i are outside $B_s(m_i)$ but any geodesic connecting them passes 2δ-close to m_i, and $s - 2\delta \ge 1$).

Applying [BrH99, Proposition III.H.1.6] (and taking the exponential on both sides of the inequality that it provides), we get $l(\alpha_i) \ge 2^{(s-2\delta-1)/\delta}$. There exists $\epsilon > 0$ so that the right hand side is $\ge (1+\epsilon)^s$, for each $s \ge 50\delta$.

Since the α_i are disjoint subpaths of α, we have $l(\alpha) \ge k \cdot (1 + \epsilon)^s$, as required. $\qquad\square$

The final claim is the following.

Claim 3. For every $L \ge 1$, there exists s large enough so that for every C there exists n with the following property: Let $x, y \in X$ with $d(x, y) \ge n$. Let α_1, α_2 be paths from x to y so that $d(\alpha_1 - B, \alpha_2 - B) \ge s$, for $B = B_C(x_0) \cup B_C(x)$. Then either $l(\alpha_1) \ge Ld(x, y)$ or $l(\alpha_2) \ge Ld(x, y)$.

Proof of Claim 3. Fix s_0, ϵ as in Claim 2. Up to increasing s_0, we can assume that for every C there exists $n = n(C)$ so that $t \geq n$ implies

$$k(t)(1 + \epsilon)^{s_0} \geq Lt,$$

where $k(t) = \lfloor (t - 2C - 2s_0)/6s_0 \rfloor$. Let $s = 2s_0$, fix any C and let $n = n(C)$.

Suppose $d(x, y) \geq n$. We can find $2k(d(x, y))$ disjoint balls B_i of radius s_0 whose centres lie on $[x, y]$ at distance at least $C + s_0$ from the endpoints. At most one of α_1, α_2 can intersect any given B_i and hence, up to swapping indices we can assume that α_1 avoids at least $k(d(x, y))$ of the B_i. By Claim 2, the length of α_1 is at least $Ld(x, y)$, as required. $\qquad\square$

Claim 3 clearly implies that X has no fat bigons. $\qquad\square$

3. Groups with fat bigons

Let X be a Cayley graph of a finitely generated one-ended group. Following [DrMS10] we define the divergence of a pair of points $a, b \in X$ relative to a point $c \notin \{a, b\}$ as the length of the shortest path from a to b avoiding a ball around c of radius $\frac{1}{2}d(c, \{a, b\}) - 2$. Such paths always exist by [DrMS10, Lemma 3.4 (p. 2496)]. The divergence of a pair a, b, $\mathrm{Div}(a, b)$ is the supremum of the divergences of a, b relative to all $c \in X \setminus \{a, b\}$.

The divergence of X is given by $\mathrm{Div}_X(n) = \max \{ \mathrm{Div}(a, b) \mid d(a, b) \leq n \}$.

Remark 3.1. Up to the usual notion of equivalence of functions, the parameters $\delta_0 = \frac{1}{2}$ and $\gamma_0 = 2$ in the definition of divergence function can be replaced by any positive constants $\delta \leq \delta_0$ and $\gamma \geq \gamma_0$ by [DrMS10, Lemma 3.11.(c) (p. 2500)], where δ_0 and γ_0 for a one-ended Cayley graph are provided by [DrMS10, Lemma 3.4 (p. 2496)].

Remark 3.2. When combined with the fact that the parameters in the divergence function can be changed as discussed above, [DrMS10, Lemma 3.17.(ii) (p. 2502)] states that if an infinite group has no asymptotic cone with a cut-point, then the divergence function of (any Cayley graph of) the group is linear. The proof of [DrMS10, Lemma 3.17.(ii)] uses one unstated hypothesis, which is that the parameters δ, γ should be chosen so that the divergence function only takes finite values (see [DrMS10, Remark 3.5 (p. 2497)] for a discussion of the issue of finiteness of the divergence function). The extra hypothesis is used in the following way. Assuming that the divergence function (with certain parameters) is superlinear, the authors find a sequence of triples of points a_n, b_n, c_n that witness superlinearity of a suitable variation of the divergence function. They set $d_n = d(a_n, b_n)$, and consider an asymptotic cone with (d_n) as scaling factor. However, if the divergence function takes infinite values, then it might happen that the sequence (d_n) is bounded (since then the variation of the divergence function would also take infinite values), in which case the asymptotic cone is not well-defined. With the additional hypothesis, this problem does not arise.

This does not affect the consequence that the divergence function of a one-ended group *with the parameters given above* is linear if no asymptotic cone contains cut-points, since this extra hypothesis holds for one-ended Cayley graphs by [DrMS10, Lemma 3.4].

The remaining case is that of a multi-ended group, but such groups always have cut-points in their asymptotic cones. This can be shown either by noticing that [DrMS10, Lemma 3.14 (p. 2501)] (with $\delta = 0$) applies, or by using the fact that multi-ended groups admit nontrivial splittings over a finite subgroup ($G = A *_C B$ or $G = A*_C$ with finite C) [Sta68, Sta71], so they are relatively hyperbolic with proper peripheral subgroups. Therefore all their asymptotic cones are nontrivially tree-graded, so admit cut-points [DrS05, Theorem 1.11].

The converse statement that if a group has linear divergence then its asymptotic cones do not have cut-points is [DrMS10, Lemma 3.17.(i) (p. 2502)].

3.1. Linear divergence groups.

Proposition 3.3. *Let X be the Cayley graph of a finitely generated one-ended group and fix the vertex 1 as a basepoint. If*

$$\mathrm{Div}_X(20d(1,x)) \leq Dd(1,x)$$

for some $D \geq 1$ and $x \in VX$, then for every $s \geq 1$ there exists a $(20D, s, 2s)$-bigon at x.

Proof. We need a simple lemma about the geometry of Cayley graphs of infinite groups first.

Claim. For any s the following holds. Let $[p, q]$ be a geodesic in X. Then there exists a geodesic ray β starting at p so that for each $w \in \beta$ either $d(w, p) \leq 2s$ or $d(w, [p, q]) > s$.

Proof. There exists a bi-infinite geodesic γ through p.[3] We claim that we can choose β to be one of the two rays starting at p and contained in γ. If not there exist $w_1, w_2 \in \gamma$ on opposite sides of p so that $\ell_i = d(w_i, p) > 2s$ but $d(w_i, x_i) \leq s$ for some $x_i \in [p, q]$. Without loss, we assume $\ell_1 \leq \ell_2$. Notice that $d(x_1, x_2) = |d(x_2, p) - d(x_1, p)| \leq \ell_2 - \ell_1 + 2s$.

Hence, $\ell_1 + \ell_2 = d(w_1, w_2) \leq 2s + d(x_1, x_2) \leq 2s + (\ell_2 - \ell_1) + 2s$, from which we deduce $\ell_1 \leq 2s$, a contradiction. $\square$

Let $x \in VX$ satisfy $\mathrm{Div}_X(20d(1,x)) \leq Dd(1,x)$; let us construct a $(20D, s, 2s)$-bigon at x. If $d(1, x) \leq 4s$ we can just take $\alpha_1 = \alpha_2$ to be any geodesic from 1 to x, so assume that this is not the case. Let α_1 be any geodesic from 1 to x. Using the claim, let β, β' be geodesic rays starting at 1 and x respectively so that for every w on either β or β' at distance larger than $2s$ from the starting point we have $d(w, [1, x]) > s$. We can form α_2 by concatenating

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[3]For an outline of a proof of this well-known fact see [dlH00, Exercise IV.A.12].

- a sub-geodesic of β of length $10d(1, x)$, from 1 to a vertex y,
- a path of length at most $Dd(1, x)$ that avoids $N_s([1, x])$ connecting y to a vertex $y' \in \beta'$, and
- a sub-geodesic of β' from y' to x.

More precisely, we take y' at distance $9d(1, x)$ from x. To construct the path from the second item we use the divergence bound, applied with $a = y, b = y'$ and $c = 1$, to obtain a suitable path avoiding the ball B of radius $\frac{1}{2}d(1, \{y, y'\}) - 2$, centred at 1. It suffices to prove that $N_s([1, x]) \subseteq B$. To see this, first of all notice that $N_s([1, x]) \subseteq B_{d(1,x)+s}(1)$ so that it suffices to prove $d(1, x) + s \leq \frac{1}{2}d(1, \{y, y'\}) - 2$. We have $d(1, y) = 10d(1, x)$ and

$$d(1, y') \geq 9d(1, x) - d(1, x) = 8d(1, x).$$

Hence we need $d(1, x) + s \leq 4d(1, x) - 2$, which holds since we are assuming $d(1, x) > 4s$ and $s \geq 1$.

The length of α_2 is at most $10d(1, x) + 9d(1, x) + Dd(1, x) \leq 20Dd(1, x)$. Thus, we have constructed a $(20D, s, 2s)$-bigon at x. $\qquad\square$

We now prove the following proposition, which is a generalisation of Proposition 1.2.

Proposition 3.4. *Let G be a finitely generated group with exponential growth. If there exists a constant C such that $\mathrm{Div}(n) \leq Cn$ for infinitely many $n \in \mathbb{N}$, then G has exponentially many fat bigons.*

Proof. If G has exponential growth and $\mathrm{Div}(n) \leq Cn$ for all n in an infinite subset $I \subseteq \mathbb{N}$, then there exists some $D \geq 1$ such that $\mathrm{Div}(20m) \leq Dm$ holds for all m such that there is an $n \in I$ satisfying $\lfloor \frac{n}{40} \rfloor \leq m \leq \lfloor \frac{n}{20} \rfloor$. Hence, for every x in $B_{\lfloor \frac{n}{20} \rfloor}(1) \setminus B_{\lfloor \frac{n}{40} \rfloor}(1)$ and every $s \geq 1$, there is a $(20D, s, 2s)$-bigon at x, by Proposition 3.3. Thus, for each $n \in I$,

$$|\mathcal{B}^G(20D, s, 2s) \cap B_n(1)| \geq |B_{\lfloor \frac{n}{80} \rfloor}(1)|,$$

since $B_{\lfloor \frac{n}{20} \rfloor}(1) \setminus B_{\lfloor \frac{n}{40} \rfloor}(1)$ contains a ball of radius $\lfloor \frac{n}{80} \rfloor$. Since G has exponential growth, $\lim_{n \to \infty} |B_n|^{\frac{1}{n}} = 1 + \epsilon > 1$, so there can only be finitely many n such that $|B_n| \leq (1 + \frac{\epsilon}{2})^n$. Hence, $|B_{\lfloor \frac{n}{80} \rfloor}(1)| \geq (1 + \frac{\epsilon}{2})^{\lfloor \frac{n}{80} \rfloor}$ for all sufficiently large $n \in I$. Thus, G has exponentially many fat bigons. $\qquad\square$

3.2. More fat bigons. Relations in a $C'(\frac{1}{6})$ small cancellation group define isometrically embedded cyclic subgraphs in the appropriate Cayley graph (cf. [LS01, Gro03]), so are natural examples of fat bigons. Therefore we obviously have the following:

Proposition 3.5. *Let G be a group which admits a $C'(\frac{1}{6})$ small cancellation presentation $G = \langle S \,|\, R \rangle$, where each $r \in R$ is cyclically reduced and no word in R can be obtained from any other via cyclic conjugation and/or inversion. If there are constants $d > 1$, $C \geq 0$ and an infinite subset $I \subseteq \mathbb{N}$ such that for each $n \in I$, $|\{r \in R \mid n - C \leq |r| \leq n\}| > d^n$, then $X = Cay(G, S)$ admits exponentially many fat bigons.*

One way to build such a collection of relations is as follows. Set

$$S = \{a, b, c\}.$$

For each nontrivial word $w = F(a, b)$, define $r_w = cwc^2w\ldots c^{24}w$. The longest piece is of the form $c^{22}wc^{23}$ which has length $|w| + 45$, while

$$|r_w| = \binom{24}{2} + 24|w| = 300 + 24|w| > 6(45 + |w|).$$

The collection $R = \{r_w \mid w \in F(a, b)\}$ satisfies the hypotheses of the above proposition with $d = 3$, $C = 0$ and $I = \{24n + 300 \mid n \in \mathbb{N}\}$. If desired, we can ensure the group we construct is lacunary hyperbolic by instead taking $R = \{r_w \mid w \in F(a, b),\ |w| \in I\}$ for some suitably sparse infinite subset $I \subseteq \mathbb{N}$. By excluding any relators r_w where w is a proper power, we can ensure that every nontrivial element of G is stable [ArCGH16] yielding examples of small cancellation groups which cannot be subgroups of hyperbolic groups and where this statement does not follow from [AoDT16].

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(David Hume) Mathematical Institute, University of Oxford, Woodstock Road, Oxford OX2 6GG, UK
david.hume@maths.ox.ac.uk

(Alessandro Sisto) Department of Mathematics, ETH Zurich, 8092 Zurich, Switzerland
sisto@math.ethz.ch