Absolute retract involutions of Hilbert cubes: fixed point sets of infinite codimension

James West

ABSTRACT. Let $\alpha : Q \to Q$ be an involution of a Hilbert cube with fixed point set Q^α that has Property Z in Q. The first main result of this paper is Theorem 3.1: *Assume that (Q, α) is an absolute retract in the category of metric spaces with involutions and equivariant maps. If $T \subseteq Q$ is an equivariant retract of Q containing Q^α that is an inequivariant Z-set in Q, then for any equivariant retraction $r : Q \to T$, Q is equivariantly homeomorphic with the mapping cylinder $M(r; T)$ of r reduced at T.* The second main result is part of Theorem 3.3: *Q^α is an equivariant strong deformation retract of Q if and only if Q is equivariantly homeomorphic with $Q^\alpha \times \Pi_{i \geq 1} I_i$ equipped with the involution that reflects each interval coordinate I_i across its mid-point.*

CONTENTS

1. Introduction

Let $\alpha : Q \to Q$ be an involution on a space homeomorphic with the Hilbert cube $I^\infty = \Pi_{i \geq 1} I_i$, where $I_i = [0, 1]$. (These are precisely the compact metric absolute retracts X with the property that for each $n \geq 0$ each two maps $f, g : I^n \to X$ can be approximated arbitrarily closely by maps with disjoint images [T, M].) That the fixed point set Q^α need not be contractible or even an absolute neighborhood retract was remarked in [J] in light of an example in [F]. Another example may be constructed from the suspension of the complement of a 3-ball in the Poincaré sphere by taking products with I^∞ or itself. (Cf. [B], [Bo], page 124.) However, if Q is an absolute retract in the category $\mathcal{C}$ of metric spaces with involutions and equivariant maps, then it

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is easy to see that Q^α must be an inequivariant absolute retract. (It is also a standard result and easily seen in this case that the orbit space Q/α is an absolute retract, cf. [An]. This follows, for example, from Theorem 3.3 "(2) implies (1)" here and the inequivariant version of "(1) implies (2)," which may be found in [Bo].)

R.D. Anderson conjectured in 1966 that all involutions of Q with unique fixed points are topologically conjugate with the one $\beta : I^\infty \to I^\infty$ that reflects each coordinate I_i across its mid-point. Wong [Wo] showed that this is true if and only if the fixed point has a basis of invariant contractible neighborhoods. In [WWo], it is shown that Anderson's Conjecture is true if and only if Q/α is an absolute retract. (This was obtained independently by H. Hastings and was observed by R. Geoghegan to be equivalent to Q/α having the homotopy type of a CW complex.) An extension to actions of compact Lie groups on Hilbert cubes is given in [BeW].

It is the purpose of this paper to extend the results of [Wo] and [WWo] to the context that the Hilbert cube Q with involution α is an absolute retract in the category $\mathcal{C}$ and the fixed point set of α has Property Z in Q. (See Definition 2.1.) We prove (Theorem 3.3.) that *in this case, α is conjugate to the involution $\bar{\beta}$ of $Q^\alpha \times I^\infty$, where $\bar{\beta} = id \times \beta$.*

The primary difficulty overcome in [WWo] was to show that if the orbit space Q/α is an absolute retract, then there is an equivariant strong deformation retraction $F : Q \times I \to Q$ of Q to the fixed point that is isovariant on $Q \times [0, 1)$. Here, we are assuming that there is an equivariant strong deformation retraction $F : Q \times I \to Q$ of Q to the fixed point set. We prove Theorem 3.1: *Q is equivariantly homeomorphic with the relative mapping cylinder of the retraction $f_1 = F|_{Q \times \{1\}} : Q \to Q^\alpha$.*

We apply Theorem 3.1 to obtain Theorem 3.2: *Let (Q, α) and (L, γ) be Hilbert cubes with involutions that are absolute retracts for the category $\mathcal{C}$. If the fixed point sets Q^α and L^γ are homeomorphic and are (inequivariant) Z-sets, then Q and L are equivariantly homeomorphic.* Theorem 3.2 is the new part of Theorem 3.3.

In [W3], we apply Theorem 3.3 to invariant inclusion and growth hyperspaces $\mathcal{E}$ of Peano continua X to show that if $\alpha : X \to X$ is any involution of X with nowhere dense fixed point set, then the induced involution $\alpha_* : \mathcal{E} \to \mathcal{E}$ is conjugate with $id \times \beta : \mathcal{E}^{\alpha_*} \times I^\infty \to \mathcal{E}^{\alpha_*} \times I^\infty$ and that if $\mathcal{E} \setminus \{X\}$ is contractible, then α_* is conjugate with $id \times \beta : I^\infty \times I^\infty \to I^\infty \times I^\infty$. (For growth hyperspaces, it is necessary to require that X have no open set that is homeomorphic with the interval (0,1).)

The arguments of this paper all apply with the obvious changes to periodic homeomorphisms of prime period.

2. Preliminaries

This section collects definitions, notation, and basic technical results needed for the proof of the main theorems of the paper in the next section. The primary references for inequivariant Hilbert cube manifold theory are [C], [M], and [T]. The results in these references, particularly [M], often are stated for compact Hilbert cube manifolds, but extend to the non compact case with due care. (For example, "homotopies" must be replaced by "proper homotopies".) For us the most important example is the *Z-set Unknotting Theorem* which we state below as Theorem 2.2 for the convenience of the reader. It may easily be obtained from the compact version.

If K is a simplicial complex, then K^n is its n-skeleton and for a simplex σ of K $st(\sigma, K)$ is the closed star of v, i.e., the union of the closed simplices containing σ. See [D] for notation and basic results about covers, nerves, and refinements. The interior and closure of a set A will be denoted by $int(A)$ and $cl(A)$. Elements $(t_1, \dots)$ of I^∞ will be denoted by $\bar{t}$. The orbit space of α will be Q/α, and the orbit map will be written as q_α. The fixed point set Q^α of α will always be assumed to be an absolute retract and a Z-set in Q. We use the term *isotopy* to mean *ambient isotopy*, that is, a one-parameter family of (surjective) homeomorphisms.

Definition 2.1. A closed set A in an absolute neighborhood retract X has *Property Z* (is a *Z-set*) in X provided that for each open set $U \subseteq X$, the inclusion $U \setminus A \hookrightarrow U$ induces isomorphisms on all homotopy groups, or, equivalently, if the identity map of X can be approximated arbitrarily closely by maps into $X \setminus A$ ([H, M]). A is an *equivariant Z-set* if the identity of X may be approximated by equivariant maps into $X \setminus A$. Note that the entire fixed point set of an involution is never an equivariant Z-set unless it is empty.

Theorem 2.2 (Z-set Unknotting [C, M])**.** *Let A and B be homeomorphic Z-sets in a Hilbert cube manifold M. Suppose that $F : A \times I \to M$ is a proper homotopy between the inclusion of A and a homeomorphism h from A to B and that for each $a \in A$ there is given an open set U_a of M containing $F(\{a\} \times I)$. Then there is an isotopy G of M such that $g_0 = id$ and for all $a \in A$ $g_1(a) = h(a)$ and $G(\{a\} \times I) \subseteq U_a$. Moreover, G may be required to be stationary on the complement of $\bigcup\{U_a | a \in A\}$.*

Lemma 2.3. *Let Q be a space homeomorphic with the Hilbert cube, and let $X \subseteq Q$ be an absolute retract that has Property Z in Q. There is a homeomorphism $h : Q \to X \times I^\infty$ such that $h(X) = X \times \{\bar{0}\}$.*

Proof. As $X \times \{\bar{0}\}$ is a Z-set in $X \times I^\infty$, this is an immediate application of Toruńczyk's characterization of the Hilbert cube ([T, M]) and Theorem 2.2. □

Definition 2.4.

(1) A *relative simplicial complex* is a metric space $Y = L \cup B$ where

(a) $L = Y \setminus B$ is a simplicial complex,

(b) the topology induced on L is the same as that given by the barycentric metric $(\rho(\Sigma t_i v_i, \Sigma s_i v_i) = \sqrt{\Sigma(t_i - s_i)^2})$,

(c) the diameters $dia(\sigma)$ of simplices go to zero uniformly as σ approaches B.

(2) A *relative Hilbert cube manifold* is a locally compact metric space $X = M \cup B$ where $M = X \setminus B$ is a Hilbert cube manifold and B is a closed subset of Y.

(3) The variable product $Y \times_\omega I$ is $\{(y, \omega(y) \cdot t) | (y, t) \in Y \times I\}$, where $\omega : Y \to [0, 1]$. The *variable product* $Y \times_\omega I^\infty$ is

$$\bigcup \{\{y\} \times \omega(y) \cdot I^\infty | y \in Y\}.$$

Here $s \cdot I^\infty$ denotes $\Pi_{i \geq 1} s \cdot I_i$.

(4) A *triangulation of X relative to B* of a relative Hilbert cube manifold $X = M \cup B$ is a homeomorphism of pairs $h : (Y \times_\omega I^\infty, B) \to (X, B)$ that is the identity on B, where $Y = L \cup B$ is a relative simplicial complex, L is locally finite, and $\omega : Y \to [0, 1]$ is a continuous mapping with $B = \omega^{-1}(0)$.

For a relative simplicial complex $Y = L \cup B$ and variable product $Y \times_\omega I^\infty$, where $B = \omega^{-1}(0)$, let $p_0 : Y \times_\omega I^\infty \to Y \times_0 I^\infty$ be the map $(y, z) \mapsto (y, \{\bar{0}\})$.

Lemma 2.5. *Let $X = M \cup B$ be a relative Hilbert cube manifold. Then:*

(1) *There is a relative triangulation $h : (Y \times_\omega I^\infty, B) \to (X, B)$, where $Y = L \cup B$ is a relative simplicial complex, L is locally finite, and $\omega^{-1}(0) = B$.*

(2) *If α is an involution of X and $B = X^\alpha$, then Y may be chosen to have an involution γ with $Y^\gamma = B$ that acts simplicially on L such that for each simplex σ of L, $\gamma(\sigma) \cap \sigma = \emptyset$ and*

$$h(\gamma(x), z) = \alpha(h(x, z)).$$

(3) *If $\epsilon : X \to [0, 1]$ is continuous with $B = \epsilon^{-1}(0)$, Y and h may be chosen so that for each $x \in X, d(h \circ p_0 \circ h^{-1}(x), x) \leq \epsilon(x)$.*

Proof. This follows from the Triangulation Theorem of [C] applied to the Hilbert cube manifold $(Q \setminus Q^\alpha)/\alpha$ and the variable product technique of [AS]. □

Lemma 2.6. *Let $\alpha : Q \to Q$ be an involution of a Hilbert cube with Q^α a Z-set in Q and an absolute retract. Let $Y = L \cup B$ be a relative simplicial complex and L locally finite. Let $\zeta : Y \to Y$ be an involution such that $B = Y^\zeta$ and ζ acts simplicially on L such that for each simplex σ of L, $\gamma(\sigma) \cap \sigma = \emptyset$. If $f : Y \to Q$ is an equivariant map and $\epsilon : Y \to [0, \infty)$ is a continuous function with $\epsilon(B) = 0$, then there is an equivariant map $g : Y \to Q$ that agrees with f on $\epsilon^{-1}(0)$, is isovariant on $\epsilon^{-1}((0, \infty))$, and is equivariantly ϵ-homotopic to f by a homotopy G that is isovariant on $\epsilon^{-1}((0, \infty)) \times [0, 1)$.*

Proof. Without loss of generality, we may assume that $(Q, Q^\alpha) = (I^\infty, P)$, where $P = \{0\} \times \Pi_{i \geq 2} I_i$ as follows. Let $\iota : (Q, Q^\alpha) \to (Q \times I^\infty, Q^\alpha \times I^\infty)$ by $x \mapsto (x, \bar{0})$. Set $\bar{\alpha} = \alpha \times id : Q \times I^\infty \to Q \times I^\infty$. As $Q^\alpha \times I^\infty$ is a Z-set and a Hilbert cube, there is a homeomorphism

$$h : (Q \times I^\infty, Q^\alpha \times I^\infty) \to (I^\infty, P).$$

Set $\alpha_1 = h \circ \bar{\alpha} \circ h^{-1}$ and let $c : Q \times I^\infty \to Q$ be projection. Let

$$f_1 = h \circ \iota \circ f : (Y, B) \to (I^\infty, P).$$

Now let $\epsilon_1 : Y \to [0, 1]$ be continuous with $\epsilon_1^{-1}(0) = \epsilon^{-1}(0)$ such that if $d(f_1(y), z) < \epsilon_1(y)$, then $d(f(y), c \circ h^{-1}(z)) < \epsilon(y)$. If $g_1 : (Y, B) \to (I^\infty, P)$ satisfies the conclusion of the Lemma with respect to α_1, f_1, and ϵ_1, then $g = c \circ h^{-1} \circ g_1$ satisfies the Lemma for f and ϵ.

We may also assume that ϵ is invariant, i.e., $\epsilon(y) = \epsilon(\zeta(y))$, and, by subdivision, that for $\sigma \in L$, $st(\sigma, L) \cap \zeta(st(\sigma, L)) = \emptyset$. Let $A = \epsilon^{-1}(0)$ and $C = f^{-1}(P)$. Let $U = Y \setminus A$ and let L_1 be a subdivision of U with $dia(st(v, L_1)) \to 0$ uniformly as $v \to A$. For $S \subseteq Q$, define $C^1(S) = conv(\alpha(conv(\alpha(S))))$ and $C^{n+1}(S) = C^1(C^n(S))$, where $conv(S)$ denotes the closed convex hull of S. Assure by subdivision that for each vertex v of L_1, $dia(C^{d(v)+1}(f(st(v, L_1)))) < \min\{\epsilon(y) \mid y \in st(v, L_1)\}$, where $d(v)$ is the dimension of $st(v, L_1)$. Denote the open t-neighborhood of a set S by $N_t(S)$. Let $\eta : L_1^0 \to (0, 1)$ be an invariant function such that $dia(C^{d(v)+1}(N_{\eta(v)}(f(st(v, L_1))))) < \min\{\epsilon(y) \mid y \in st(v, L_1)\}$. Let L_2 be the subcomplex of L_1 sent into $Q \setminus P$ by f and set $g|_{L_2} = f|_{L_2}$ and $G(y, t) = y$, if $y \in L_2$. Extend g to L_1 and G to $L_1 \times I$ as follows by induction on skeleta using convexity in Q and equivariance. For each vertex v of $L_1 \setminus L_2$, let $g(v) \in N_{\eta(v)}(f(v)) \setminus P$ and $g(\zeta(v)) = \alpha(g(v))$. For one, say v, of the pair $v, \zeta(v)$, define $G : \{v\} \times I \to [f(v), g(v)]$ to be the straight line homotopy, and extend over $(L_1)^0 \times I$ by equivariance. The induction hypothesis is that $G : L_1 \times \{0\} \cup ((L_1)^n \cup L_2) \times I \to Q$ is given satisfying:

(1) $G|_{L_1 \times 0 \cup L_2 \times I} = f \circ \pi$, where π is projection to L_1,
(2) $G^{-1}(P) = f^{-1}(P) \times \{0\}$, and
(3) if $v \in (L_1)^0$, then $G((st(v, L_1) \times I) \cap (L_1 \times \{0\} \cup ((L_1)^n \cup L_2) \times I)) \subseteq C^{n+1}(N_{\eta(v)}(f(st(v, L_1))))$.

For each n+1-simplex $\sigma \in L_1 \setminus L_2$, choose either σ or $\zeta(\sigma)$, say σ, and let $\hat{\sigma}$ denote the barycenter of $\sigma \times \{1\}$. Let $z \in conv(G(\partial\sigma \times \{1\}))$ and extend G over $\sigma \times I$ by sending the interval $[y, \hat{\sigma}]$ linearly to $[G(y), z]$, where y ranges over $\sigma \times \{0\} \cup \partial\sigma \times I$.

Extending to $\zeta(\sigma) \times I$ by equivariance gives $G(\zeta(\sigma) \times I) \subseteq (\alpha(G(\sigma)) \times I)$. The resulting G extends over A by $G(y, t) = f(y)$. $\square$

The next theorem collects technical results for reference in the proofs of Section 3.

Theorem 2.7. *Let* $\alpha : Q \to Q$ *be an involution on a space homeomorphic to the Hilbert cube such that* Q^α *is an absolute retract with Property Z in* Q. *Let* Y *be a locally compact separable metric space with involution* γ, *and let* $f, g : Y \to Q$ *be equivariant maps. For (2)-(6), assume that* (Q, α) *is an absolute retract in the category* $\mathcal{C}$. *Then the following hold:*

  (1) *For every continuous* $\epsilon : Y \to [0,1]$ *with* $Y^\gamma \subseteq \epsilon^{-1}(0)$ *there is an equivariant map* $\theta : Y \to Q$ *agreeing with* f *on* $\epsilon^{-1}(0)$ *and isovariant on* $\epsilon^{-1}((0,1])$ *such that* $d(\theta(y), f(y)) \leq \epsilon(y)$ *for all* $y \in Y$.
  (2) *There is an equivariant homotopy* $F : Y \times I \to Q$ *from* f *to* g. *For every open cover* $\mathcal{U}$ *of* Q *there is an open cover* $\mathcal{V}$ *of* Q *such that if for each* $y \in Y$, *there is a* $V \in \mathcal{V}$ *containing* $f(y)$ *and* $g(y)$ *then* F *may be required to be stationary on* $\{y \in Y | f(y) = g(y)\}$ *and have the property that for each* $y \in Y$ *there is a* $U \in \mathcal{U}$ *containing* $F(\{y\} \times I)$.
  (3) *The map* θ *of* (1) *may be chosen to be equivariantly homotopic to* f *by a homotopy* Θ *that is:*
   (a) *stationary on* $\epsilon^{-1}(0)$,
   (b) *isovariant on* $(Y \setminus \epsilon^{-1}(0)) \times [0,1)$,
   (c) *satisfies dia* $(\Theta(\{y\} \times I)) \leq \epsilon(y)$ *for each* $y \in Y$.
  (4) *If* f *and* g *are isovariant, then the homotopy* F *of* (2) *may be required to be isovariant. If* $f = g$ *on* Y^γ *and* $\mathcal{U}$ *is an open cover of* $Q \setminus Q^\alpha$ *then we may choose* $\mathcal{V}$ *to be an open cover of* $Q \setminus Q^\alpha$ *such that if for each* $y \in Y$ *there is a* $V \in \mathcal{V}$ *containing* $f(y)$ *and* $g(y)$, *then the homotopy* $F : Y \setminus Y^\gamma \times I \to Q$ *obtained by restricting* (2) *to* $Y \setminus Y^\gamma$ *extends to a homotopy* $F' : Y \times I \to Q$ *that is stationary on* Y^γ.
  (5) *If* f *and* g *are isovariant embeddings, then the homotopy* F' *of* (4) *may be required to embed* $Y \times \{t\}$ *for each* t. *If additionally* Y *is compact, then* F' *may be required to embed* $(Y \setminus Y^\gamma) \times [a,b]$ *as an equivariant Z-set in* $Q \setminus Q^\alpha$ *for each* $[a,b] \subseteq (0,1)$.
  (6) *Assume that* Y *is compact, that* f *and* g *are equivariant embeddings, that* $f = g$ *on* Y^γ, *and that* $f(Y \setminus Y^\gamma)$ *and* $g(Y \setminus Y^\gamma)$ *are equivariant Z-sets of* $Q \setminus Q^\alpha$. *Let* F *be an equivariant homotopy between* f *and* g *that is stationary on* Y^γ. *If* $\mathcal{U}$ *is an open cover of* $Q \setminus Q^\alpha$ *such that for each* $y \in Y \setminus Y^\gamma$ *there is a* $U \in \mathcal{U}$ *containing* $F(\{y\} \times I)$, *then there is an equivariant isotopy* G *of* Q *from the identity to a homeomorphism* g_1 *such that* $g_1 \circ f = g$. *The isotopy* G *may be required to be stationary on* Q^α *and on the complement of any open set containing the image of* F *and to have the property that for each* $y \in Y \setminus Y^\gamma$ *there is a* $U \in \mathcal{U}$ *containing* $F(\{y\} \times I) \cup G(\{y\} \times I)$.

Proof. Without loss of generality, we may assume that ϵ is invariant. Triangulate the relative Hilbert cube manifold Q/α as $K \times_\omega I^\infty \cup Q^\alpha$, and let $p : \tilde{K} \to K$ denote the universal (2-fold) covering if K. Then $\tilde{K} \times_\tau I^\infty \cup Q^\alpha$ triangulates Q as a relative Hilbert cube manifold, where $\tau = \omega \circ p$, and $\alpha = g \times id$ on $Q \setminus Q\alpha^\alpha$, where $g : \tilde{K} \to \tilde{K}$ is the deck transformation. Define

$\kappa : K \times I^\infty \to K \times_\omega I^\infty$ and $\tilde{\kappa} : \tilde{K} \times I^\infty \to \tilde{K} \times_\tau I^\infty$ by $(x, \bar{t}) \mapsto (x, \omega(x) \cdot \bar{t})$ and $(x, \bar{t}) \mapsto (x, \tau(x) \cdot \bar{t})$, respectively. Choose an invariant metric d for Q such that for each $x \in \tilde{K} \times \Pi_{i=1}^{n-1} I_i$, the diameter of $\tilde{\kappa}(\{x\} \times \Pi_{i \geq n} I_i)$ is less than 2^{1-n}.

To prove conclusion (1), first assume that $Y^\gamma = \epsilon^{-1}(0)$. Define

$$Z \subseteq \tilde{K} \times_\tau I^\infty \times [0, \infty)$$

by

$$Z = \tilde{K} \times \{\bar{0}\} \times [0, \infty) \cup \bigcup_{n \geq 1} \tilde{\kappa}(\tilde{K} \times I^n \times \{(0, 0, \dots)\}) \times [n, \infty).$$

Let d' be the metric for $Q \times R$ that is the sum of d and the usual distance in R. Triangulate Z by an invariant simplicial complex, also denoted by Z, such that:

(a) Each simplex of Z lies in $\tilde{\kappa}(\sigma) \times [0, \infty)$ for some simplex σ of $\tilde{K}$.

(b) If $z \in \tilde{\kappa}(\tilde{K} \times I^n \times \{(0, 0, \dots)\}) \times [n, \infty)$, then $dia(st(z, Z)) < 2^{-n}$.

Let $\psi : I^\infty \times [0, \infty) \to I^\infty$ be given by $\psi(q, t) = (q_1, \dots, q_{n-1}, s \cdot q_n, 0, 0, \dots)$, where $t \in [n, n+1)$ and $s = t - n$, and let

$$\bar{\psi} : \tilde{K} \times I^\infty \times [0, \infty) \to \tilde{K} \times I^\infty \times [0, \infty)$$

by $\bar{\psi}(x, q, t) = (x, \psi(q, t), t)$. Then $\hat{\psi} = (\tilde{\kappa} \times id) \circ \bar{\psi} \circ (\tilde{\kappa} \times id)^{-1}$ carries $\tilde{K} \times_\tau I^\infty \times [0, \infty)$ equivariantly into Z, and $d'((x, q, t), \hat{\psi}(x, q, t)) \leq 2^{1-n}$ if $n < t$. Let $\pi : Z \to \tilde{K} \times_\tau I^\infty$ be projection. Define $h : Y \setminus f^{-1}(Q^\alpha) \to Z$ by $h(y) = (f(y), \lambda(y))$, where $\lambda(y) = 4 - log_2(\epsilon(y))$. Then $d'(\hat{\psi} \circ h(y), h(y)) < \frac{\epsilon(y)}{4}$. Also, if x and $\hat{\psi} \circ h(y)$ are in $st(z, Z)$ for some vertex z of Z, then $d(x, \hat{\psi} \circ h(y)) < \frac{\epsilon(y)}{4}$.

Let $\mathcal{O} = \{int(st(z, Z)) \mid z \in Z^0\}$. Then α acts on $\mathcal{O}$ without fixed points. Define $\mathcal{U} = \{(\hat{\psi} \circ h)^{-1}(O) \mid O \in \mathcal{O}\}$. Then $\mathcal{U}$ also inherits an involution from Z. Let $\mathcal{V}$ be a star-finite open refinement of $\mathcal{U}$ by pre-compact open sets of $Y \setminus (f^{-1}(Q^\alpha))$, and let $\mathcal{N}(\mathcal{U})$, and $\mathcal{N}(\mathcal{V})$ be their nerves with induced involutions. Let $\iota : \mathcal{V} \to \mathcal{U}$ be an equivariant choice function with $V \subseteq \iota(V)$, and let ι also denote the simplicial map $\iota : \mathcal{N}(\mathcal{V}) \to \mathcal{N}(\mathcal{U})$ that this defines by the vertex map $v_V \mapsto v_{\iota(V)}$. Additionally, let $\mu : \mathcal{N}(\mathcal{U}) \to Z$ be the simplicial map given by $v_U \mapsto z_U$, where $U = (\hat{\psi} \circ h)^{-1}(int(st(z_U, Z)))$. Choose an equivariant partition of unity $\Phi = \{\phi_V | V \in \mathcal{V}\}$ on $Y \setminus (f^{-1}(Q^\alpha)$ subordinate to $\mathcal{V}$ with $\phi_V^{-1}(0, 1]) = V$ for all V, and let $\hat{b} : Y \setminus f^{-1}(Q^\alpha) \to \mathcal{N}(\mathcal{V})$ be the associated (equivariant) barycentric map. Now if $y \in V$, then $\mu \circ \iota \circ \hat{b}(y) \in \mu \circ \iota(st(v_V, \mathcal{N}(\mathcal{V}))) \subseteq \mu(st(v_{\iota(V)}, \mathcal{N}(\mathcal{U}))) = \mu(st(v_U, \mathcal{N}(U) = st(z_U, Z)$, where $U = (\hat{\psi} \circ h)^{-1}(O)$ and $O = int(st(z_U, Z))$. On the other hand, $y \in V \subseteq U = (\hat{\psi} \circ h)^{-1}(O) = (\hat{\psi} \circ h)^{-1}(st(z_U, Z))$, so $\hat{\psi} \circ h(y) \in st(z_U, Z)$. Thus, $d(\pi \circ \mu \circ \iota \circ \hat{b}(y), f(y)) = d(\pi \circ \mu \circ \iota \circ \hat{b}(y), \pi \circ h(y)) \leq d'(\mu \circ \iota \circ \hat{b}(y), \circ h(y)) \leq d'(\mu \circ \iota \circ \hat{b}(y), \hat{\psi} \circ h(y)) + d'(\hat{\psi} \circ h(y), h(y)) < \frac{\epsilon(y)}{2}$.

Let $\hat{Y} = \mathcal{N}(\mathcal{V}) \cup f^{-1}(Q^\alpha)$ be the indicated relative simplicial complex, extend $\hat{b}$ to Y by the identity, and let $\zeta : \hat{Y} \to Q$ be equal to $\pi \circ \mu \circ \iota$ on $\mathcal{N}(\mathcal{V})$ and to f on $f^{-1}(Q^\alpha)$.

Set $A = f^{-1}(Q^\alpha) \setminus Y^\gamma$. We are now ready to move $\zeta|_A$ off Q^α. Let $\delta : \hat{Y} \to [0,1]$ be a continuous map satisfying $\delta \circ \zeta(y) \leq \epsilon(y)/2$ and $\delta^{-1}(0) = Y^\gamma$. By Lemma 2.6, there is an equivariant map $g : \hat{Y} \to Q$ agreeing with f on Y^γ that is isovariant on $\delta^{-1}((0,1])$ and satisfies $d(g(x), \zeta(x)) < \delta(x)$ for all $x \in \hat{Y}$. Set $\theta = g \circ \hat{b}$.

Now for $y \in Y \setminus f^{-1}(Q^\alpha)$,

$$
\begin{aligned}
d(f(y), \theta(y)) &\leq d(f(y), \zeta(\hat{b}(y))) + d(\zeta(\hat{b}(y)), g(\zeta(\hat{b}(y))) \\
&= d(f(y), \pi \circ \mu \circ \iota \circ \hat{b}(y)) + d(\zeta(\hat{b}(y)), g(y)) \\
&< \frac{\epsilon(y)}{2} + \frac{\epsilon(y)}{2} = \epsilon(y).
\end{aligned}
$$

For $y \in A$, $d(f(y), \theta(y)) = d(\zeta(y), g(y)) < \frac{\epsilon(y)}{2}$.

If $\epsilon^{-1}(0) \neq Y^\gamma$, let $Y_1 = Y \setminus (\epsilon^{-1}(0) \setminus Y^\gamma)$ and apply the previous construction to (Y_1, Y^γ) and ϵ.

Conclusion (2) is proved by following the inequivariant argument. Let d be an invariant metric for Q, and let $h : Q \to E$ be the embedding of Q in the Banach space of continuous functions $\phi : Q \to R$ using the correspondence $x \mapsto d(., x)$. The linear map $T(\phi) = \phi \circ \alpha$ makes h equivariant, and the result follows from an equivariant retraction $r : E \to h(Q)$. (Cf. [J], [Bo].)

To prove (3), replace ϵ in the case $Y^\gamma = \epsilon^{-1}(0)$ in the proof of (1) by $\epsilon_1 \leq \epsilon$ with $Y^\gamma = \epsilon_1^{-1}(0)$ and sufficiently small that there is an equivariant homotopy $F : Y \times [0,1] \to Q$ from θ to f that is stationary on Y^γ and such that for each $y \in Y$, $dia(F(\{y\} \times I)) \leq \epsilon(y)/3$. Now let $\epsilon_2 : Y \times I \to [0,1]$ be an invariant map with $\epsilon_2^{-1}(0) = Y^\gamma \times [0,1] \cup Y \times \{0,1\}$ and for each $(y,t) \in Y \times [0,1]$, $\epsilon_2(y,t) \leq \epsilon(y)/3$. Applying (1) to $Y \times [0,1], F$, and ϵ_2, we obtain the desired homotopy Θ from θ to f.

Conclusion (4) is proved by using (2) to get an equivariant homotopy F from f to g and then applying (3) to $F|_{Y \times (0,1)}$ as done in the proof of (3) to obtain an isovariant map that extends to $Y \times I$. Conclusion (5) may be obtained using the infinite product structure of I^∞ in a relative triangulation of (Q, Q^α).

To prove conclusion (6) we can reduce to the inequivariant case. Let $\mathcal{U}_1 = \{U \in \mathcal{U} | U \supseteq F(\{y\} \times I)$ for some $y \in Y \setminus Y^\gamma\}$. By refining $\mathcal{U}$, we may assume that the union of the elements of $\mathcal{U}_1$ lies in any neighborhood of $F(Y \setminus Y^\gamma \times I)$. Given such a $\mathcal{U}$, we may by (1), (5), and Lemma 2.5 replace F by a homotopy Λ satisfying:

(a) Λ is isovariant.
(b) $\lambda_i = f_i$, for $i = 0, 1$.
(c) Λ embeds $(Y \setminus Y^\gamma) \times I$ as an equivariant Z-set of $Q \setminus Q^\alpha$.

(d) for each $y \in Y \setminus Y^\gamma$, $\Lambda(\{y\} \times [\frac{i-1}{2}, \frac{i}{2}])$ contains no orbit of α, for $i = 1, 2$.

(e) For each $y \in Y \setminus Y^\gamma$, there is a $U(y) \in \mathcal{U}$ containing $F(\{y\} \times I) \cup \Lambda(\{y\} \times I)$.

Now let $\chi : Y \to (0, 1)$ be an invariant map such that the open $\chi(y)$-neighborhoods $V_i(y)$ of $q_\alpha(\Lambda(\{y\} \times [\frac{i-1}{2}, \frac{i}{2}]))$ are evenly covered by the restriction of q_α to $Q \setminus Q^\alpha$, for $i = 1, 2$. Let $\tilde{V}_1(y)$ be the lift of $V_1(y)$ containing y, and let $\tilde{V}_2(y)$ be the lift of $V_2(y)$ containing $\lambda_{1/2}(y)$ and that for some $U \in \mathcal{U}$, U contains the lift of $V_1(y)$ containing y and the lift of $V_2(y)$ containing $F(y, \frac{1}{2}))$. Require that $U(y) \supseteq F(\{y\} \times I) \cup \tilde{V}_1(y) \cup \tilde{V}_2(y)$.

Passing to $(Q \setminus Q^\alpha)/\alpha$, we may by Theorem 2.2 choose isotopies H^1 and H^2 of $(Q \setminus Q^\alpha)/\alpha$ such that:

(i) h_0^i is the identity, $i = 1, 2$.

(ii) For each $y \in Y \setminus Y^\gamma$, $h_1^1 \circ q_\alpha \circ f(y) = q_\alpha \circ \lambda_{1/2}(y)$.

(iii) For each $y \in Y \setminus Y^\gamma$, $h_1^2 \circ q_\alpha \circ \lambda_{1/2}(y) = q_\alpha \circ \lambda_1(y) = q_\alpha \circ g(y)$.

(iv) for each $y \in Y \setminus Y^\gamma$, $H^i(\{q_\alpha(f(y))\} \times [\frac{i-1}{2}, \frac{i}{2}]) \subseteq V_i(y)$, for $i = 1, 2$.

(v) If H^i is not stationary on $x \in Q$, then for some $y \in Y \setminus Y^\gamma$,

$$H^1(\{x\} \times I) \cup H^2(\{h_1^1(x)\} \times I) \subseteq V_1(y) \cup V_2(y).$$

As q_α restricted to $Q \setminus Q^\alpha$ is a covering map, we may let G^i be the lift of H^i, $i = 1, 2$, with g_0^i the identity of $Q \setminus Q^\alpha$, and set

$$G(x, t) = \begin{cases} g_{2t}^1(x), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ g_{2t-1}^2 \circ g_1^1(x), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

The extension of G to Q that is stationary on Q^α satisfies (6). $\qquad \square$

Corollary 2.8. *If $\alpha : Q \to Q$ is an absolute retract in $\mathcal{C}$, then there is an equivariant strong deformation retraction $F : Q \times I \to Q$ of Q to Q^α that is isovariant on $Q \times [0, 1)$.*

Definition 2.9.

(1) A *collaring* of a closed subset B of a topological space Y is an embedding $c : B \times I \to Y$ with image a closed neighborhood S of B such that for $b \in B$ $c(b, 0) = b$ and $c(B \times \{1\})$ is the boundary of S in Y. The restriction $c|_{B \times [0,1)}$ is called an *open collaring* of B. The image of a closed (respectively, open) collaring is a *closed* (respectively, *open*) collar of B. We always require that the boundary of a closed collar S be itself collared in $Y \setminus c(B \times [0, 1))$.

(2) The *mapping cylinder* $M(f)$ of a map $f : Y \to Z$ is the quotient space $(Y \times I \cup Z)/\sim$, where $(y, 0) \sim f(y)$. The inclusion $\iota : Y \to M(f)$ is induced by $y \mapsto (y, 1)$.

(3) If $A \subseteq Y$ is closed, the *relative mapping cylinder of f reduced at A*, $M(f; A)$, is $M(f)/\cong$, where $(a, t) \cong f(a)$ for each $(a, t) \in A \times I$.

Lemma 2.10. *Let A_3 be a Hilbert cube manifold with a fixed point free involution γ. Let $A_1 \subseteq A_2 \subseteq A_3$ be invariant sub-Hilbert cube manifolds with A_i a Z- set in A_{i+1}, $i = 1, 2$, and let $j : A_1 \to A_2$ be the inclusion.*

(1) *A_i has equivariant closed collars in A_j if $i < j$.*

(2) *There are equivariant closed collars B_i of A_i in A_{i+1}, $i = 1, 2$, and B_3 of A_1 in the interior of B_2 such that $B_3 \cap A_2 = B_1$ and the boundary of each is equivariantly collared (on both sides).*

(3) *If $c : A_1 \times I \to A_2$ is an equivariant closed collar there is a homeomorphism $f : A_2 \to M(j)$ supported in the union of $c(A_1 \times I)$ and an equivariant collar of $c(A_1 \times \{1\})$ in $A_2 \setminus c(A_1 \times [0, 1))$ such that for $x \in A_1$, $f(x) = (x, 1) \in M(j)$, $f(c(x, 1)) = x \in A_2$, and f moves points along collar curves in A_1 and in $M(\iota)$.*

(4) *Let $c_1 : A_1 \times [0, 1] \to A_2 \times [0, 1)$ be an equivariant closed collar of $A_1 \times \{0\}$ in $A_2 \times I$. For $a \in A_1$, let a' denote $\iota(a) = (a, 1) \in M(j)$. Then there is an equivariant homeomorphism*

$$f : A_2 \times I \setminus c_1(A_1 \times [0, 1)) \to M(j)$$

with $f(c_1(a, 1)) = a'$ and $f(A_2 \times \{1\}) = A_2$. The restriction $f|_{A_2 \times \{1\}}$ may be required to be as close to the projection as desired.

Proof. (1) Let $q : A_3 \to A_3/\gamma$ be the orbit mapping. As γ is without fixed points, q is a covering map, so A_i/γ is a Hilbert cube manifold, and A_i/γ is a Z-set in A_j/γ. Therefore it is locally collared in A_j/γ as each point of A_i/γ has a neighborhood U homeomorphic to an open subset of I^∞ by a homeomorphism ϕ carrying $U \cap A_i/\gamma$ to $\phi(U) \cap P$. Now M. Brown's collaring theorem [Br], provides an open collar. Lifting to Y gives a collar of Z, and restricting to $A_i \times [0, 1/2]$ provides the closed collar.

(2) Let $c_i : A_i \times I \to A_{i+1}$, $i = 0, 1$, be equivariant closed collarings and consider $c_2 : A_1 \times I \times I \to A_3$ given by $c_2(x, s, t) = c_1(c_0(x, s), t/2)$. Since $A_1 \times (I \times \{1\} \cup \{1\} \times I)$ is an equivariant Z-set in $A_1 \times I \times I$ equivariantly homeomorphic to $A_1 \times \{1\}$ there is an equivariant homeomorphism

$$f : A_1 \times I \to A_1 \times I \times I$$

with $f(A_1 \times \{0\}) = A_1 \times \{(0, 0)\}$ and $f(A_1 \times \{1\}) = A_1 \times (I \times \{1\} \cup \{1\} \times I)$. Then $c_3 \circ f : A_1 \times I \to A_3$ is the desired collaring.

(3) Let C be an equivariant collar of $c(A_1 \times \{1\})$ in $A_2 \setminus c(A_1 \times [0, 1))$, and let D denote $A_2 \setminus (C \cup c(A_1 \times [0, 1)))$. Then $M(j) \setminus D$ is a collar of the subset $A_0 = A_1 \times \{0\}$ of $M(j)$.

(4) Let k denote the inclusion $A_1 \to A_1 \times \{0\} \hookrightarrow A_2 \times I$. By (3), there is an equivariant homeomorphism $g : A_2 \times I \to M(k)$ such that

$$g(A_2 \times I \setminus c_1(A_1 \times [0, 1))) = A_2 \times I,$$

$g(A_1 \times \{0\})$ is the copy of A_1 that is the domain end of $M(k)$, and g is the identity on $A_2 \times \{1\}$. Because $A_1 \times \{0\}$ is a Z-set in $A_2 \times \{0\}$, Theorem 2.7(5)

and (6) provide an equivariant homeomorphism $h : A_2 \times I \to A_2 \times \{0\}$ that is the identity on $A_1 \times \{0\}$ and approximates the projection map

$$A_2 \times \{1\} \to A_2 \times \{0\}$$

arbitrarily closely. Now (3) extends h to a homeomorphism $\bar{h} : M(k) \to M(j)$. Then $f = \bar{h} \circ g$ is the desired homeomorphism. $\square$

3. Main theorems

Theorem 3.1. *Suppose that $\alpha : Q \to Q$ is an involution on a Hilbert cube that is an absolute retract in $\mathcal{C}$. If $T \subseteq Q$ is an equivariant retract of Q containing Q^α that is an inequivariant Z-set in Q, then for any equivariant retraction $r : Q \to T$, Q is equivariantly homeomorphic with the mapping cylinder $M(r; T)$ of r reduced at T.*

Proof. Let $\tau : Q \to [0, 1]$ be an invariant mapping with $\tau^{-1}(0) = T$. Using Lemma 2.5 and refactoring I^∞ as $I^\infty \times I$, we obtain an equivariant homeomorphism $h_1 : Q \to Q \times_\tau I$ that is the identity on T. The map $h_2 : M(id; T) \to Q \times_\tau I$ given by $(q, t) \mapsto (q, \tau(q) \cdot t)$ is an equivariant homeomorphism, showing that Q is equivariantly homeomorphic with $M(id; T)$. It suffices to show that there is an equivariant map g of $M(id; T)$ onto itself that is one-to-one on $M(id; T) \setminus Q \times \{0\}$ such that $g(q, 0) = r(q)$. This is because the quotient maps

$$q_0 : Q \times I \to Q \times_\tau I \xrightarrow{k} M(id; T) \xrightarrow{g} M(id; T),$$

$$q_1 : Q \times I \to M(g \circ \iota; T),$$

have the same sets of point inverses, where $k = h_2^{-1}$ and $\iota : Q \to Q \times \{0\}$ is $q \mapsto (q, 0)$.

To simplify notation, let $Y = M(id; T)$ and $A = Q \times \{0\} \subseteq M(id; T)$. For $y = (q, t) \in Y$ let $s \cdot y = (q, s \cdot t)$. Let $s \cdot Y = \{s \cdot y | y \in Y\}$.

Let d be an invariant metric for Y such that in $Y \setminus T$, $d(s \cdot y, t \cdot y) = |s - t|$.

By Theorem 2.7(2), (1), and (5), and using the fact that $T \setminus Q^\alpha$ is an equivariant Z-set of $Q \setminus Q^\alpha$, there is an equivariant homotopy $F : Q \times I \to Q$ from id to r that is stationary on T, is isovariant on $Q \times [0, 1)$, embeds $Q \setminus T \times (0, 1)$, and embeds $Q \setminus T \times [a, b]$ in $Q \setminus T$ as an equivariant Z-set of $Q \setminus T$ when $0 < a < b < 1$.

Let $F' : A \times I \to A$ be given by $F'((q, 0), s) = (F(q, s), 0)$. Select a sequence $0 < \delta(1) < \delta(2) < \ldots < 1$, such that for each $a \in A$,

$$dia(f'_{\delta(1)} \circ f'_{\delta(2)} \circ \cdots \circ f'_{\delta(n-1)}(\{a\} \times [\delta(n), 1])) < 2^{-n}.$$

Let $\zeta_0 : A \to A$ be the identity, and for $n \geq 1$ set $\zeta_n = f'_{\delta(1)} \circ f'_{\delta(2)} \circ \cdots \circ f'_{\delta(n)}$. For $n \geq 2$, define homotopies $\Lambda^n : \zeta_{n-2}(A) \times I \to \zeta_{n-2}(A)$ by

$$\Lambda^n(\zeta_{n-1}(a), s) = \begin{cases} \zeta_{n-2} \circ f'_{\delta(n-1)+2(1-\delta(n-1))s}(a), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ \zeta_{n-1} \circ f'_{\delta(n)+2(1-\delta(n))(1-s)}(a), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

Now Λ^n is stationary on T, $\lambda_0^n = id$, and $\lambda_1^n \circ \zeta_{n-1} = \zeta_n$. Moreover, for each $a \in A$, $dia(\Lambda^n(\zeta_{n-1}(a) \times I)) \le 2^{1-n} + 2^{-n} = 3 \cdot 2^{-n}$. Applying Theorem 2.7 and using the fact that $T \setminus Q^\alpha$ is an equivariant Z-set in $Q \setminus Q^\alpha$, hence in $\zeta_n(A)$ for each n, we may adjust Λ^n to an equivariant embedding Φ^n that equals Λ^n on $\zeta_{n-1}(A) \times \{0, 1\}$ and has the same properties. Theorem 2.7 allows us to extend Φ^n to an isotopy $\Psi^n : \zeta_{n-2}(A) \times I \to \zeta_{n-2}(A)$ satisfying the following:

(1) Ψ^n is stationary on T.
(2) ψ_0^n is the identity of $\zeta_{n-2}(A)$.
(3) $\psi_1^n \circ \zeta_{n-1} = \zeta_n$.
(4) $dia(\Psi^n(\{x\} \times I)) < 3 \cdot 2^{-n}$ for each $x \in \zeta_{n-2}(A)$.

Note that for $n \ge 2$, $\zeta_n = \psi_1^n \circ \psi_1^{n-1} \circ \cdots \circ \psi_1^2$.

Since Ψ^n is an isotopy of $\zeta_{n-2}(A)$ that is stationary on Q^α and since $\zeta_{n-2}(A) \setminus Q^\alpha$ is an equivariant Z-set in $Y \setminus Q^\alpha$, we may extend the Ψ^n's inductively to isotopies $\Theta^n : Y \times I \to Y$ satisfying:

(a) $dia(\Theta^n(\{y\} \times I)) < 3 \cdot 2^n$ for each $y \in Y$.
(b) Θ^n is stationary on $\theta_1^{n-1} \circ \theta_1^{n-2} \circ \cdots \circ \theta_1^2(Y \setminus c_n \cdot Y)$, where $c_n < min\{c_{n-1}, 2^{-n}\}$ is such that if $d(x, y) < c_n$ then
$$d(\theta^{n-1} \circ \cdots \circ \theta_1^2(x), \theta^{n-1} \circ \cdots \circ \theta_1^2(y)) < 2^{-n}.$$

Consider the equivariant homeomorphisms g_n of Y given by
$$g_n = \theta_1^n \circ \cdots \circ \theta_1^2.$$

Then for $y \in Y$, if $g_n(y) \ne g_{n-1}(y)$, we have
$$\begin{aligned} d(g_n(y), g_{n-1}(y)) &\le d(g_n(y), g_n(0 \cdot y)) + d(g_n(0 \cdot y), g_{n-1}(0 \cdot y)) \\ &\quad + d(g_{n-1}(0 \cdot y), g_{n-1}(y)) \\ &\le d(g_n(y), g_n(0 \cdot y)) + d(\zeta_n(0 \cdot y), \zeta_{n-1}(0 \cdot y)) + 2^{-n} \\ &\le d(g_n(y), g_n(0 \cdot y)) + 3 \cdot 2^{-n} + 2^{-n} \\ &\le 3 \cdot 2^{-n} + 3 \cdot 2^{-n} + 2^{-n} \\ &= 7 \cdot 2^{-n}. \end{aligned}$$

Therefore the sequence g_n is uniformly Cauchy and converges to a surjective equivariant mapping $g : Y \to Y$ that is injective on $Y \setminus A$. On A, we have
$$g(x) = \lim g_n(x) = \lim \zeta_n(x) = \lim f'_{\delta(1)} \circ f'_{\delta(2)} \circ \cdots \circ f'_{\delta(n)}(x) = f'_1(x) = r(x).$$

Therefore, g is the desired mapping demonstrating that Q is equivariantly homeomorphic with $M(r; T)$. $\square$

Theorem 3.2. *Let (Q, α) and (L, γ) be Hilbert cubes with involutions that are absolute retracts for the category $\mathcal{C}$. If the fixed point sets Q^α and L^γ are homeomorphic and are (inequivariant) Z-sets, then Q and L are equivariantly homeomorphic.*

Proof. This proof proceeds by using infinite (relative) mapping cylinders to obtain the necessary control at the fixed point sets. First we set up the notation we shall need. By Theorem 3.1, we may assume that

$$Q = M(f_\alpha; Q^\alpha) = A \times I/ \sim,$$

where A is a Hilbert cube with an involution α' having fixed point set Q^α and $f_\alpha : A \to Q^\alpha$ is an equivariant retraction to the fixed point set of α'.

Similarly, we may assume that $L = M(f_\gamma; L^\gamma) = B \times I/ \approx$, where B is a Hilbert cube with an involution γ' and fixed point set L^γ. Denote $A \times \{s\} \subseteq Q$ by A_s, $B \times \{s\}$ by B_s, $A \times [0, s]/ \sim$ by Q_s, and $B \times [0, s]/ \approx$ by L_s. So $A_0 = Q_0 = Q^\alpha$ and $B_0 = L_0 = L^\gamma$. Let $\omega_s^Q : Q \to Q$ and $\omega_s^L : L \to L$ by $(x, t) \mapsto (x, st)$. To reduce the multiple subscripts, we shall set $A(n) = A_{2^{-n}}, B(n) = B_{2^{-n}}, Q(n) = Q_{2^{-n}}$, and $L(n) = L_{2^{-n}}$.

Now $Q(n) \setminus \text{int } Q(n+1)$ is equivariantly homeomorphic with

$$M_n^Q = M \left(\omega_{1/2}^Q|_{A(n)} : A(n) \to A(n+1); Q^\alpha \right),$$

so Q is equivariantly homeomorphic with the infinite relative mapping cylinder $M_\infty^Q = \bigcup_n M_n^Q$ (with appropriate scaling of the mapping cylinders' interval coordinates.) Similarly, $L(n) \setminus \text{int} (L(n+1)$ is equivariantly homeomorphic with $M_n^L = M(\omega_{1/2}^L|_{B(n)} : B(n) \to B(n+1); L^\gamma)$, and L is equivariantly homeomorphic with the infinite reduced mapping cylinder $M_\infty^L = \bigcup_n M_n^L$.

By applying Theorem 2.7 to an equivariant homotopy equivalence that restricts to a homeomorphism of Q^α onto L^γ, we get an isovariant homotopy equivalence, $h : Q \to L$ that restricts to a homeomorphism of Q^α onto L^γ. Then we may adjust h by sliding along mapping cylinder coordinates in L so that $h(A(n)) \subseteq B(n)$ and $h(Q(n)) \setminus int(Q(n+1)) \subseteq L(n) \setminus int(L(n+1))$ for each n. Applying Theorem 2.7, we may further adjust h so that

    (1) h is an equivariant embedding, and $h(Q \setminus Q^\alpha)$ is an equivariant Z-set in $L \setminus L^\gamma$,

    (2) $A(n) = h^{-1}(B(n))$,

    (3) $h(A(n) \setminus Q^\alpha)$ is an equivariant Z-set in $B(n) \setminus L^\gamma$, and

    (4) $h(Q(n) \setminus (int(Q(n+1)) \cup Q^\alpha))$ is an equivariant Z-set in $L(n) \setminus (int(L(n+1)) \cup L^\gamma)$.

As (Q, α) is an absolute retract in $\mathcal{C}$, there is an equivariant retraction $\rho : L \to h(Q)$. By Theorem 2.7, we may assume that ρ is isovariant. By transferring the mapping cylinder structure of Q to $h(Q)$ and sliding along the maping cylinder coordinates, we may assume that $\rho^{-1}(h(A(n))) = B(n)$ and that $\rho^{-1}(h(Q(n) \setminus Q(n+1))) = L(n) \setminus L(n+1)$. Moreover, ρ is, by Theorem 2.7 and sliding, isovariantly homotopic to the identity by a homotopy Ξ that is stationary on $h(Q)$ and such that $\Xi(B(n) \times I) \subseteq B(n)$ and $\Xi(L(n) \setminus int(L(n+1)) \times I) \subseteq L(n) \setminus int(L(n+1))$.

By Lemma 2.10, there is an equivariant closed collar neighborhood $C = c((Q \setminus Q^\alpha) \times I)$ of $h(Q \setminus Q^\alpha)$ in $L \setminus L^\gamma$. Theorem 2.7 allows us to require

that c restrict to closed collar neighborhoods of $h(A(n) \setminus Q^\alpha)$ in $B(n) \setminus L^\gamma$ and of $h(Q(n) \setminus (int(Q(n+1)) \cup Q^\alpha))$ in $L(n) \setminus (int(L(n+1)) \cup L^\gamma)$.

By shortening the collar lines as $n \to \infty$, we may extend c to the fixed point sets as an equivariant embedding of the reduced mapping cylinder $M(id : Q \to Q; Q^\alpha)$.

Let $D = \bigcup_n B(n)$. Using Theorem 2.7, approximate $\rho|_D$ by an isovariant embedding $\zeta : D \to c(\bigcup_n (A(n) \setminus Q^\alpha) \times [0, \frac{1}{2})) \cup L^\gamma$ that is the identity on each $h(A(n))$. Let ι_n denote the inclusion $c((A(n) \setminus Q^\alpha) \times [0, \frac{1}{2})) \cup L^\gamma \to B(n)$, and set $\zeta_n = \iota_n \circ \zeta|_{B(n)}$.

Define $\eta_n = \zeta_{n+1} \circ \omega^L_{1/2}|_{B(n)}$, and let Z_∞ denote the infinite mapping cylinder $\bigcup_n M(\eta_n; L^\gamma)$. Next, reduce each of the mapping cylinders $M(\eta_n; L^\gamma)$ at the complement, $B'(n)$, of the collar $c((A(n) \setminus Q^\alpha) \times [0, 1))$ in $B(n)$. We thus obtain the infinite reduced mapping cylinder $\hat{Z}_\infty = \bigcup_n M(\eta_n; B'(n))$.

We now produce equivariant homeomorphisms

$$\kappa : M^L_\infty \to Z_\infty, \quad \mu : Z_\infty \to \hat{Z}_\infty, \quad \text{and} \quad \chi : M^Q_\infty \to \hat{Z}_\infty,$$

completing the proof.

Let $E = c(\bigcup_n ((A(n) \setminus Q^\alpha) \times I)) \cup L^\gamma$, and choose an isovariant homotopy $\Theta : D \times I \to E$ from $\rho|_D$ to ζ that is stationary on L^γ. Because $B(n+1) \setminus L^\gamma$ is an equivariant Z-set in $M^L_{n+1} \setminus L^\gamma$, the homotopy $\Lambda : (D \setminus L^\gamma) \times I \to D \setminus L^\gamma$ from the identity to ζ defined by

$$\Lambda_s(x) = \begin{cases} \xi_{1-2s}, & \text{if } 0 \le t \le 1/2, \\ \theta_{2s-1}, & \text{if } 1/2 \le t \le 1, \end{cases}$$

may by Theorem 2.7 be approximated in $\bigcup_n (L(n) \setminus int(L_{3 \cdot 2^{-n-2}})) \setminus L^\gamma$ by an isotopy Φ of with $\phi_0 = id$ and $\phi_1|_D = \zeta$ that is stationary on $\bigcup_n B_{3 \cdot 2^{-n-2}} \setminus L^\gamma$ and extends over L^γ to an isotopy stationary on it.

For each n, ϕ_1 defines, by restriction and reparameterization of the interval coordinates, a homeomorphism κ_{n+1} of $L_{3 \cdot 2^{-n-2}} \setminus int(L_{3 \cdot 2^{-n-3}})$ onto $\frac{1}{2} \cdot M(\eta_n; L^\gamma) \cup M(\eta_{n+1}; L^\gamma) \setminus int(\frac{1}{2} \cdot M(\eta_{n+1}; L^\gamma))$. To obtain a formula, we use (y, s) to denote the mapping cylinder coordinates in L and $[y, s]$ to denote mapping cylinder coordinates in $M(\eta_n; L^\gamma)$ and in $M(\eta_{n+1}; L^\gamma)$. Thus, for $(y, t) \in L_{3 \cdot 2^{-n-2}} \setminus (int(L_{3 \cdot 2^{-n-3}}) \cup L^\gamma)$, if $2^{-n-1} \le t \le 3 \cdot 2^{-n-2}$, $\kappa_{n+1}(x, t) = [x, 2^{n+2}t - 1] \in M(\eta_n; L^\gamma)$, and if $3 \cdot 2^{-n-3} \le t \le 2^{-n-1}$ and $\phi_1(x, t) = (y, u)$, then $\kappa_{n+1}(x, t) = [y, 2^{n+2}u - 1] \in M(\eta_{n+1})$. Taking these simultaneously gives us the equivariant homeomorphism $\kappa : M^L_\infty \to Z_\infty$.

Now $\eta_n(B(n)) \setminus L^\gamma = \zeta_{n+1}(B(n+1)) \setminus L^\gamma$ is an equivariant Z-set in $L(n+1) \setminus (L_{3 \cdot 2^{-n-3}} \cup L^\gamma)$ and thus is collared in it. It is also a subset of the (relative) interior in $B(n+1)$ of the collar $c((A(n+1) \setminus Q^\alpha) \times [0, \frac{1}{2}])$ of $h(A(n+1) \setminus Q^\alpha)$. For each n, choose a collar K_n of $\zeta_n(B(n) \setminus L^\gamma)$ in $M(\eta_n; L^\gamma)$ that is contained in $c((A(n) \setminus Q^\alpha) \times [0, \frac{1}{2})) \times (0, 1]$, where $(0, 1]$ is the mapping cylinder coordinate of $M(\eta_n; L^\gamma)$. Now, by sliding down the mapping cylinder lines of $M(\eta_n; L^\gamma)$ and into K_{n+1}, we obtain a homeomorphism μ_n of $M(\eta_n; L^\gamma) \cup M(\eta_{n+1}; L^\gamma)$ onto $M(\eta_n; B'(n)) \cup M(\eta_{n+1}; L^\gamma)$

that is the identity on $(c((A(n) \setminus Q^\alpha) \times [0, \frac{1}{2}]) \times I) \cup (M(\eta_{n+1}; L^\gamma) \setminus K_{n+1})$. Thus, μ_{n+1} is the identity on $M(\eta_{n+1}; L^\gamma) \cap \mu_n(M(\eta_n))$. Then $\bigcup_n \mu_n$ defines an equivariant homeomorphism $\mu : Z_\infty \to \bigcup_n M(\eta_n; B'(n)) = \hat{Z}_\infty$.

Analogously, let $\psi_n : c(A_{2^{-n}} \setminus Q^\alpha \times I) \cup L^\gamma \to c(A_{2^{-n-1}} \setminus Q^\alpha \times I) \cup L^\gamma$ be the restriction of η_n. Letting

$$T_\infty = \bigcup_n M(\psi_n; L^\gamma) \quad \text{and} \quad \hat{T}_\infty = \bigcup_n M(\psi_n; c(A_{2^{-n}} \setminus Q^\alpha \times \{1\} \cup L^\gamma)),$$

we find that there is an equivariant homeomorphism $\nu : T_\infty \to \hat{T}_\infty$, which equals $\hat{Z}_\infty$.

Next, we observe that as ψ_n is the restriction of $\eta \circ \omega^L_{1/2}$, if we set

$$\psi = \bigcup_n \psi_n : E \to E,$$

then ψ is isovariantly homotopic to $c \circ \omega^Q_{1/2} \circ c^{-1}|_E$ in E. This is because E is an isovariant retract of D. (To see this, observe that $C \cup L^\gamma$ is equivariantly homeomorphic with $M(id : Q \to Q; Q^\gamma)$, which is equivariantly homeomorphic with Q, so the same process that produced ρ produces an isovariant retraction $\hat{\rho} : L \to C \cup L^\gamma$ that restricts to a retraction of D onto E.)

By Theorem 2.7, ψ is equivariantly homotopic to $c \circ \omega^Q_{1/2} \circ c^{-1}|_E$ in E. Let

$$S_\infty = \bigcup_n M \left(c \circ \omega^Q_{1/2} \circ c^{-1}|_{c((A(n) \setminus Q^\alpha) \times I) \cup L^\gamma}; L^\gamma \right).$$

Then by an argument completely analogous to those above, there is an equivariant homeomorphism $\sigma : S_\infty \to T_\infty$. Let $\tau : M^Q_\infty \to S_\infty$ be an equivariant homeomorphism, and set $\chi = \nu \circ \sigma \circ \tau : M^Q_\infty \to \hat{T}_\infty$. Now $\chi^{-1} \circ \mu \circ \kappa : M^L_\infty \to M^Q_\infty$, and thus L is equivariantly homeomorphic with Q. $\square$

Theorem 3.3. *Let α be an involution of a Hilbert cube Q with fixed point set Q^α an absolute retract that has Property Z in Q. Then the following are equivalent:*

(1) *Q^α is an equivariant strong deformation retract of Q.*
(2) *Q with the involution α is an absolute retract in the category $\mathcal{C}$ of metric spaces with involutions and equivariant maps.*
(3) *There is an equivariant homeomorphism $h : Q \to Q^\alpha \times I^\infty$, with the involution $\bar{\beta} : Q^\alpha \times I^\infty \to Q^\alpha \times I^\infty$ given by*

$$(x, t_1, t_2, \dots) \mapsto (x, 1 - t_1, 1 - t_2, \dots).$$

Proof. That (1) implies (2) is a straightforward exercise. It is a special case of the equivariant version of Lemma (9.9) of Chapter V of [Bo], reformulated slightly here as follows. "Let Y be a compact metric space. Suppose that $X \subseteq Y$ is closed and an absolute retract and that $Y \setminus X$ is an absolute neighborhood retract. Then Y is an absolute neighborhood retract provided

that X is a deformation retract of a neighborhood in Y." To see that (3) implies (1), note that I^∞ is β-equivariantly contractible to $\{\bar{0}\}$, so $Q^\alpha \times I^\infty$ deformation retracts $\bar{\beta}$- equivariantly to its fixed point set $Q^\alpha \times \{\bar{0}\}$, and Q deformation retracts to Q^α. Now Theorem 3.2 shows that (2) implies (3), since $Q^\alpha \times I^\infty$ is a Hilbert cube ([M, T]) and $Q^\alpha \times I^\infty$ is an absolute retract in $\mathcal{C}$. $\square$

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(James West) DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14053-4201, USA
`west@math.cornell.edu`