

Groupoid actions on C^* -correspondences

Valentin Deaconu

ABSTRACT. Let the groupoid G with unit space G^0 act via a representation ρ on a C^* -correspondence \mathcal{H} over the $C_0(G^0)$ -algebra A . By the universal property, G acts on the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ which becomes a $C_0(G^0)$ -algebra. The action of G commutes with the gauge action on $\mathcal{O}_{\mathcal{H}}$, therefore G acts also on the core algebra $\mathcal{O}_{\mathcal{H}}^{\mathbb{T}}$.

We study the crossed product $\mathcal{O}_{\mathcal{H}} \rtimes G$ and the fixed point algebra $\mathcal{O}_{\mathcal{H}}^G$ and obtain similar results as in [5], where G was a group. Under certain conditions, we prove that $\mathcal{O}_{\mathcal{H}} \rtimes G \cong \mathcal{O}_{\mathcal{H} \rtimes G}$, where $\mathcal{H} \rtimes G$ is the crossed product C^* -correspondence and that $\mathcal{O}_{\mathcal{H}}^G \cong \mathcal{O}_{\rho}$, where \mathcal{O}_{ρ} is the Doplicher-Roberts algebra defined using intertwiners.

The motivation of this paper comes from groupoid actions on graphs. Suppose G with compact isotropy acts on a discrete locally finite graph E with no sources. Since $C^*(G)$ is strongly Morita equivalent to a commutative C^* -algebra, we prove that the crossed product $C^*(E) \rtimes G$ is stably isomorphic to a graph algebra. We illustrate with some examples.

CONTENTS

1. Introduction	1020
2. Groupoid actions on spaces	1021
3. Groupoid actions on groupoids	1023
4. Groupoid actions on graphs	1024
5. Groupoid actions on C^* -algebras and C^* -correspondences	1027
6. Crossed products and Cuntz-Pimsner algebras	1029
7. Doplicher-Roberts algebras	1034
8. Compact isotropy groupoid actions on graphs	1035
References	1036

1. Introduction

Group actions provide important bridges between dynamical systems and C^* -algebras via the crossed product construction. In this paper, we consider

Received December 28, 2017.

2010 *Mathematics Subject Classification*. Primary 46L05.

Key words and phrases. C^* -correspondence; groupoid action; groupoid representation; graph algebra; Cuntz-Pimsner algebra.

The author would like to thank Alex Kumjian and Leonard Huang for helpful and illuminating discussions.

groupoids acting on fibered spaces and on directed graphs. Under certain conditions, these actions can be extended to the graph C^* -correspondence and we obtain interesting results about the associated Cuntz-Pimsner algebras.

Given a group G acting on a C^* -correspondence \mathcal{H} over a C^* -algebra A via $\rho : G \rightarrow \mathcal{L}_{\mathbb{C}}(\mathcal{H})$, in [5] we studied the fixed point algebra $\mathcal{O}_{\mathcal{H}}^G$ and the crossed product $\mathcal{O}_{\mathcal{H}} \rtimes G$, with applications to group actions on graphs. The Doplicher-Roberts algebra \mathcal{O}_{ρ} was defined from intertwiners (ρ^m, ρ^n) , where $\rho^n = \rho^{\otimes n}$ is the tensor power representation of G on the balanced tensor product $\mathcal{H}^{\otimes n}$. We proved that in certain cases \mathcal{O}_{ρ} is isomorphic to $\mathcal{O}_{\mathcal{H}}^G$ and strongly Morita equivalent to $\mathcal{O}_{\mathcal{H}} \rtimes G$.

In this paper, we obtain similar results for groupoid actions, under some extra assumptions. We first recall the machinery associated with groupoid actions on spaces, on other groupoids, on graphs, on $C_0(X)$ -algebras and on C^* -correspondences. If G is a locally compact amenable groupoid with Haar system acting on a C^* -correspondence \mathcal{H} and $J_{\mathcal{H}}$ is the Katsura ideal, we assume that $J_{\mathcal{H}} \rtimes G \cong J_{\mathcal{H} \rtimes G}$ in order to obtain the isomorphism $\mathcal{O}_{\mathcal{H}} \rtimes G \cong \mathcal{O}_{\mathcal{H} \rtimes G}$ as in [11]. We illustrate with some examples, including self-similar actions on the path space of a finite graph and actions on Hermitian vector bundles.

As an application, if G with compact isotropy groups acts on a discrete locally finite graph E with no sources, we prove that $C^*(E) \rtimes G$ is stably isomorphic to the C^* -algebra of a graph. Since the action of G commutes with the gauge action of \mathbb{T} on $C^*(E)$, the groupoid G also acts on the core AF-algebra $C^*(E)^{\mathbb{T}}$ and $C^*(E)^{\mathbb{T}} \rtimes G \cong (C^*(E) \rtimes G)^{\mathbb{T}}$ is an AF-algebra.

2. Groupoid actions on spaces

Let's assume that G is a second countable locally compact Hausdorff groupoid with unit space G^0 and range and source maps $r, s : G \rightarrow G^0$. The set of composable pairs is denoted G^2 and the set of $g \in G$ with $s(g) = u, r(g) = v$ is denoted G_u^v . We first recall the definition of a groupoid action on a space given in [24]:

Definition 2.1. A topological groupoid G is said to act (on the left) on a locally compact space X , if there are given a continuous, open surjection $p : X \rightarrow G^0$, called the anchor or momentum map, and a continuous map

$$G * X \rightarrow X, \quad \text{write } (g, x) \mapsto g \cdot x,$$

where

$$G * X = \{(g, x) \in G \times X \mid s(g) = p(x)\},$$

that satisfy

- i) $p(g \cdot x) = r(g)$ for all $(g, x) \in G * X$,
- ii) $(g_2, x) \in G * X, (g_1, g_2) \in G^2$ implies $(g_1 g_2, x), (g_1, g_2 \cdot x) \in G * X$ and $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$,

iii) $p(x) \cdot x = x$ for all $x \in X$.

We should mention that recently, many authors assume that $p : X \rightarrow G^0$ is not necessarily open (see [4] for example).

The action is free if $g \cdot x = x$ for some x implies $g = p(x)$. The set of fixed points is defined as

$$X^G = \{x \in X : g \cdot x = x \text{ for all } g \in G_{p(x)}^{p(x)}\},$$

where

$$G_u^u = \{g \in G : s(g) = r(g) = u\}$$

is the isotropy group at $u \in G^0$.

Remark 2.2. Note that the set X^G is fibered over G^0 via the restriction of the map p . It is the largest subset of X on which G acts trivially, in the sense that for any $u, v \in G^0$ and for any $g, h \in G_u^u$, the elements g, h induce the same maps between the fibers $p^{-1}(u)$ and $p^{-1}(v)$. If G has trivial isotropy, then $X^G = X$.

For $x \in X$, its stabilizer group is

$$G(x) = \{g \in G : g \cdot x = x\},$$

which is a subgroup of G_u^u where $u = p(x)$. The set of orbits

$$G * x = \{g \cdot x : g \in G, s(g) = p(x)\}$$

is denoted by X/G .

Remark 2.3. The fibered product $G * X$ has a natural structure of groupoid, called the semi-direct product or action groupoid and is denoted by $G \ltimes X$, where

$$(G \ltimes X)^2 = \{(g_1, x_1), (g_2, x_2) \mid x_1 = g_2 \cdot x_2\},$$

with operations

$$(g_1, g_2 \cdot x_2)(g_2, x_2) = (g_1 g_2, x_2), \quad (g, x)^{-1} = (g^{-1}, g \cdot x).$$

The source and range maps of $G \ltimes X$ are

$$s(g, x) = (s(g), x) = (p(x), x), \quad r(g, x) = (r(g), g \cdot x) = (p(g \cdot x), g \cdot x),$$

and the unit space $(G \ltimes X)^0$ may be identified with X via the map

$$i : X \rightarrow G \ltimes X, \quad i(x) = (p(x), x).$$

The projection map

$$\pi : G \ltimes X \rightarrow G, \quad \pi(g, x) = g$$

is a covering of groupoids, see [8].

Example 2.4. A groupoid G with open range and source maps acts on its unit space G^0 by $g \cdot s(g) = r(g)$. Notice that $g \cdot u = u$ for all $g \in G_u^u$, in particular $(G^0)^G = G^0$.

If there is only one orbit in G^0/G , the groupoid is called transitive. For example, let the group \mathbb{R} act on \mathbb{R}/\mathbb{Z} by translation. Then the action groupoid $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$ with unit space $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$ and isotropy \mathbb{Z} is transitive.

A transitive groupoid with discrete unit space is of the form $G^0 \times K \times G^0$ where K is the isotropy group.

3. Groupoid actions on groupoids

The notion of a groupoid action on another groupoid was defined in [3] in the algebraic case. In the topological context, we give the following definition (see [1], page 122).

Definition 3.1. We say that a topological groupoid G acts on another topological groupoid H if there are a continuous open surjection $p : H \rightarrow G^0$ and a continuous map $G * H \rightarrow H$, write $(g, h) \mapsto g \cdot h$ where

$$G * H = \{(g, h) \in G \times H \mid s(g) = p(h)\}$$

such that

- i) $p(g \cdot h) = r(g)$ for all $(g, h) \in G * H$,
- ii) $(g_2, h) \in G * H, (g_1, g_2) \in G^2$ implies $(g_1 g_2, h) \in G * H$ and

$$(g_1 g_2) \cdot h = g_1 \cdot (g_2 \cdot h),$$

- iii) $(h_1, h_2) \in H^2$ and $(g, h_1 h_2) \in G * H$ implies $(g, h_1), (g, h_2) \in G * H$ and

$$g \cdot (h_1 h_2) = (g \cdot h_1)(g \cdot h_2),$$

- iv) $p(h) \cdot h = h$ for all $h \in H$.

Remark 3.2. If G acts on H , then in particular G acts on the unit space H^0 via the restriction $p_0 := p|_{H^0} : H^0 \rightarrow G^0$ and we have $p = p_0 \circ r = p_0 \circ s$. Using the fact that $h = hs(h) = r(h)h$, we deduce that we also have $s(g \cdot h) = g \cdot s(h)$ and $r(g \cdot h) = g \cdot r(h)$.

For example, a Γ -sheaf as in [15] is given by an étale groupoid Γ acting on a group bundle A over Γ^0 . In this case p_0 is the identity.

Remark 3.3. If the groupoid G acts on H via p , then the triple (H, G^0, p) becomes a continuous field of groupoids in the terminology of [19], which determines a continuous C^* -bundle. If we do not assume p to be open, we just get an upper semicontinuous C^* -bundle.

When G^0 is discrete, the groupoid H is a disjoint union of groupoids $H_u = p^{-1}(u)$ for $u \in G^0$.

The fixed point groupoid H^G is defined as

$$H^G = \{h \in H : g \cdot h = h \text{ for all } g \in G_{p(h)}^{p(h)}\}.$$

Remark 3.4. If G is transitive, then H^G is non-empty if and only if $p : H \rightarrow G^0$ has invariant sections, i.e. continuous maps $\sigma : G^0 \rightarrow H^0$ such that $p(\sigma(u)) = u$ for all $u \in G^0$ and such that σ commutes with the actions of G on H and on G^0 , i.e. $g \cdot \sigma(u) = \sigma(g \cdot u)$.

Proof. Indeed, if σ is an invariant section, then for $u \in G^0$ and $h = \sigma(u) \in H^0$ we have $g \cdot h = g \cdot \sigma(u) = \sigma(u) = h$ for all $g \in G_u^u$. Conversely, given $a \in (H^G)^0$ with $p_0(a) = u$ we define $\sigma : G^0 \rightarrow H^0$ by $\sigma(v) = g \cdot a$ for some $g \in G_u^v$ and $g \cdot a$ is independent of the choice of g . \square

The crossed product groupoid $G \rtimes H$ is defined as follows. As a set, $G \rtimes H = G * H$ with multiplication

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, (g_2^{-1} \cdot h_1) h_2),$$

when this makes sense, i.e. when $r(g_2) = s(g_1) = p(h_1)$, $s(g_2) = p(h_2)$ and $g_2^{-1} \cdot s(h_1) = r(h_2)$. The inverse is given by

$$(g, h)^{-1} = (g^{-1}, g \cdot h^{-1}).$$

It is easy to check that the multiplication is associative and that

$$s(g, h) = (s(g), s(h)), \quad r(g, h) = (r(g), g \cdot r(h)).$$

The unit space of $G \rtimes H$ can be identified with H^0 , and we have an extension of groupoids

$$H \xrightarrow{i} G \rtimes H \xrightarrow{\pi} G,$$

where $i(h) = (p(h), h)$ and $\pi(g, h) = g$. The map π is a fibration of groupoids, see [8] or [4]. Indeed, given $g \in G$ and $u \in H^0 = (G \rtimes H)^0$ with $s(g) = p(u)$, choose $h \in H$ with $s(h) = u$. Then $(g, h) \in G \rtimes H$ satisfies $s(g, h) = s(h) = u$ and $\pi(g, h) = g$.

Remark 3.5. In [8] it was proved that a continuous open surjective homomorphism $\pi : G \rightarrow H$ of étale groupoids with amenable kernel gives rise to a Fell bundle \mathcal{E} over H such that $C_r^*(\mathcal{E})$ is isomorphic to $C_r^*(G)$. As was observed in [4], the Fell bundle is saturated only if π is a fibration. Unfortunately, this hypothesis was omitted in the main result of [8]. The authors of [4] also remove the condition that the groupoids are étale, considering the more general case of locally Hausdorff locally compact groupoids with Hausdorff unit spaces and with Haar systems.

In particular, when the groupoid G acts on H , we can apply the results of [4] to the fibration $G \rtimes H \xrightarrow{\pi} G$ and obtain a saturated Fell bundle \mathcal{E} over G such that

$$C_r^*(\mathcal{E}) \cong C_r^*(G \rtimes H) \cong C_r^*(H) \rtimes G.$$

4. Groupoid actions on graphs

Inspired from the notion of groupoid actions on groupoids, we define now the concept of groupoid actions on graphs. Let $E = (E^0, E^1, r, s)$ be a topological graph, i.e. E^0, E^1 are locally compact spaces with $r, s : E^1 \rightarrow E^0$ continuous maps and s a local homeomorphism.

Definition 4.1. We say that a topological groupoid G acts on E if G acts on both spaces E^0, E^1 in a compatible way. This means that there is a continuous open surjection $p : E^0 \rightarrow G^0$ such that $p \circ r = p \circ s : E^1 \rightarrow G^0$ and there are continuous maps $G * E^0 \rightarrow E^0, G * E^1 \rightarrow E^1$ such that the conditions in Definition 2.1 are satisfied and such that

$$s(g \cdot e) = g \cdot s(e), \quad r(g \cdot e) = g \cdot r(e)$$

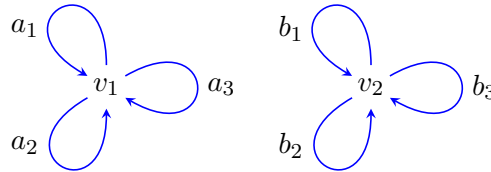
for $e \in E^1$ and $g \in G$.

Remark 4.2. Since p and s are open, it follows that $p \circ s : E^1 \rightarrow G^0$ is open. The action of G can be extended to the set of finite paths $E^* = \bigsqcup_{k \geq 0} E^k$, where E^k is the set of paths of length k , by

$$g \cdot (e_1 e_2 \cdots e_k) = (g \cdot e_1)(g \cdot e_2) \cdots (g \cdot e_k)$$

and similarly to the set of infinite paths E^∞ . If G^0 is discrete, note that since $p \circ r = p \circ s$, the graph E is a union of graphs E_u for $u \in G^0$.

Example 4.3. Let E be the graph



The transitive groupoid G with unit space $G^0 = E^0 = \{v_1, v_2\}$ and isotropy S_3 acts on E by permutations. In this case $(E^0)^G = E^0$ but $(E^1)^G = \emptyset$.

We recall now the definition of a self-similar action of a groupoid on the path space of a graph as in [18], which will provide more examples of groupoid actions on graphs.

Definition 4.4. (Self-similar actions) Suppose E is a finite graph with no sources. Let vE^* denote the set of finite paths ending at v . We define the graph

$$T_E = \bigsqcup_{v \in E^0} vE^*$$

to be the union of rooted trees (also called forest) with the set of vertices $T_E^0 = E^*$ and with the set of edges

$$T_E^1 = \{(\mu, \mu e) : \mu \in E^*, e \in E^1, s(\mu) = r(e)\}.$$

The set $\text{Iso}(E^*)$ of partial isomorphisms $vE^* \rightarrow wE^*$ for $v, w \in E^0$ becomes a groupoid. A *self-similar action* of a groupoid G with $G^0 = E^0$ on E^*

is given by a homomorphism $G \rightarrow \text{Iso}(E^*)$ such that for every $g \in G$ and $e \in s(g)E^1$ there exists a unique $h \in G$ denoted also by $g|_e$ such that

$$g \cdot (e\mu) = (g \cdot e)(h \cdot \mu) \text{ for all } \mu \in s(e)E^*.$$

Proposition 4.5. *A self-similar action of a groupoid G on the path space E^* of a finite graph E as above determines an action of G on the graph T_E as in Definition 4.1.*

Proof. Indeed, the vertex space T_E^0 is fibered over $G^0 = E^0$ via the map $\mu \mapsto r(\mu)$. For $(\mu, \mu e) \in T_E^1$ we set $s(\mu, \mu e) = \mu e$ and $r(\mu, \mu e) = \mu$. Since $r(\mu e) = r(\mu)$, the edge space T_E^1 is also fibered over G^0 . The action of G on T_E^1 is given by

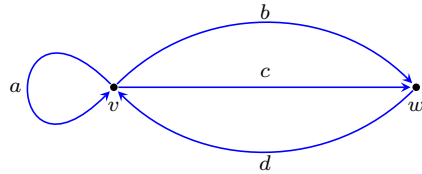
$$g \cdot (\mu, \mu e) = (g \cdot \mu, g \cdot (\mu e)) \text{ when } s(g) = r(\mu).$$

Since

$$\begin{aligned} s(g \cdot (\mu, \mu e)) &= s(g \cdot \mu, g \cdot (\mu e)) = g \cdot (\mu e) = g \cdot s(\mu, \mu e), \\ r(g \cdot (\mu, \mu e)) &= r(g \cdot \mu, g \cdot (\mu e)) = g \cdot \mu = g \cdot r(\mu, \mu e), \end{aligned}$$

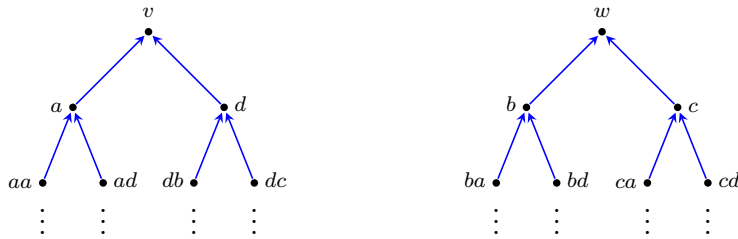
the actions on T_E^0 and T_E^1 are compatible. □

Example 4.6. Let E be the graph



with forest

T_E



Consider the groupoid G with unit space $G^0 = \{v, w\}$ and generators g, h where $g \in G_v^w, h \in G_w^v$ such that

$$\begin{aligned} g \cdot a &= c, \quad g|_a = v, \quad g \cdot d = b, \quad g|_d = h, \\ h \cdot b &= a, \quad h|_b = v, \quad h \cdot c = d, \quad h|_c = g. \end{aligned}$$

These conditions are also presented as

$$g \cdot a\mu = c\mu, \quad g \cdot d\mu = b(h \cdot \mu), \quad h \cdot (b\mu) = a\mu, \quad h \cdot c\mu = d(g \cdot \mu),$$

and they determine uniquely an action of $G = \langle g, h \rangle$ on the graph T_E .

Proposition 4.7. *Suppose E is a discrete locally finite graph with no sources. If the groupoid G acts on E via a map $p : E^0 \rightarrow G^0$, then G acts on the path groupoid \mathcal{G}_E of E .*

Proof. Recall that

$$\mathcal{G}_E = \{(x, k, y) \in E^\infty \times \mathbb{Z} \times E^\infty : \exists N \text{ with } x_i = y_{i-k} \text{ for } i \geq N\}$$

with operations

$$(x, m, y) \cdot (y, n, z) = (x, m + n, z), \quad (x, k, y)^{-1} = (y, -k, x)$$

and that the unit space \mathcal{G}_E^0 can be identified with E^∞ . There is a map $\pi_0 : E^\infty \rightarrow G^0$ induced by $p : E^0 \rightarrow G^0$ and for $g \in G$ and $(x, k, y) \in \mathcal{G}_E$ with $s(g) = \pi_0(x) = \pi_0(y)$ we define

$$g \cdot (x, k, y) = (g \cdot x, k, g \cdot y).$$

It is routine to check that this action satisfies all the properties of Definition 3.1. Continuity is proved using cylinder sets. In particular, since E^0 and G^0 are discrete, the groupoid \mathcal{G}_E is a disjoint union of groupoids \mathcal{G}_{E_u} for $u \in G^0$. \square

5. Groupoid actions on C^* -algebras and C^* -correspondences

Let X be a locally compact Hausdorff space. Recall that a C^* -algebra A is a $C_0(X)$ -algebra if there is a non-degenerate homomorphism $\varphi : C_0(X) \rightarrow ZM(A)$, where $ZM(A)$ denotes the center of the multiplier algebra of A . This means that $\overline{\varphi(C_0(X))A} = A$, i.e. there is an approximate unit $\{f_i\}$ in $C_0(X)$ such that $\lim_i \|\varphi(f_i)a - a\| = 0$ for all $a \in A$. Sometimes we write fa instead of $\varphi(f)a$.

Given a $C_0(X)$ -algebra A , for each $x \in X$ we can define the fiber A_x as $A/\overline{I_x A}$ where

$$I_x = \{f \in C_0(X) : f(x) = 0\}.$$

Denote the quotient map $A \rightarrow A_x$ by π_x . It is known that the canonical map

$$A \rightarrow \prod_{x \in X} A_x, \quad a \mapsto (\pi_x(a))_{x \in X}$$

is injective and that the fibers A_x give rise to an upper semicontinuous C^* -bundle \mathcal{A} over X such that $A \cong \Gamma_0(\mathcal{A})$, the algebra of continuous sections vanishing at infinity, see Theorem C.26 in [29].

Definition 5.1. We say that the topological groupoid G with unit space G^0 acts on a C^* -algebra A if A is a $C_0(G^0)$ -algebra and for each $g \in G$ there is a $*$ -isomorphism $\alpha_g : A_{s(g)} \rightarrow A_{r(g)}$ such that if $(g_1, g_2) \in G^2$ we have $\alpha_{g_1 g_2} = \alpha_{g_1} \circ \alpha_{g_2}$ and for a fixed a , the map $g \mapsto \alpha_g(a)$ is norm continuous. We also write $g \cdot a$ for $\alpha_g(a)$.

Remark 5.2. An action of a groupoid G on a C^* -algebra A can be understood as an isomorphism of $C_0(G)$ -algebras $\alpha : s^*A \rightarrow r^*A$ such that $\alpha_{g_1g_2} = \alpha_{g_1} \circ \alpha_{g_2}$ for all $(g_1, g_2) \in G^2$. Here the pull backs s^*A and r^*A become $C_0(G)$ -algebras such that $(s^*A)_g = A_{s(g)}$ and $(r^*A)_g = A_{r(g)}$. It can also be understood as a functor from the small category G to the category of C^* -algebras as in [22].

Remark 5.3. If the groupoid G with Haar system λ acts on A , then one can define the crossed product $A \rtimes G$ and the reduced crossed product $A \rtimes_r G$ by completing the $*$ -algebra $\Gamma_c(G, r^*A)$ of continuous sections with compact support in the appropriate norms (see [25] for example).

Recall that for $f, f' \in \Gamma_c(G, r^*A)$, we define

$$f * f'(g) := \int_G f(h)\alpha_h(f'(h^{-1}g))d\lambda^{r(g)}(h), \quad f^*(g) = \alpha_g(f(g^{-1})^*).$$

The fixed point algebra A^G is defined as the $C_0(G^0)$ -algebra with fibers

$$A_x^G = \{a \in A_x : g \cdot a = a \text{ for all } g \in C_x^x\}.$$

Example 5.4. If the groupoid G acts on the locally compact space X , then $C_0(X)$ becomes in a natural way a $C_0(G^0)$ -algebra and G acts on $C_0(X)$ by $(g \cdot f)(x) = f(g^{-1} \cdot x)$ such that $C_0(X) \rtimes G \cong C^*(G \ltimes X)$ and $C_0(X)^G = C_0(X/G)$.

Example 5.5. Groupoid actions on elementary C^* -bundles over G^0 satisfying Fell's condition appear in the context of defining the Brauer group $\text{Br}(G)$, see [16].

Example 5.6. If the groupoid G with discrete unit space acts on the groupoid H , then G acts on $C^*(H)$, which is a $C_0(G^0)$ -algebra in a natural way.

Indeed, the map $p_0 : H^0 \rightarrow G^0$ determines a homomorphism $\varphi : C_0(G^0) \rightarrow ZM(C^*(H))$ and the fibers of $C^*(H)$ are $C^*(H_u)$ for $u \in G^0$.

Let A be a $C_0(X)$ -algebra and let \mathcal{H} be a Hilbert A -module. We define the fibers $\mathcal{H}_x := \mathcal{H} \otimes_A A_x$, for each $x \in X$. Then \mathcal{H}_x becomes a Hilbert A_x -module with the usual operations. The set $\text{Iso}(\mathcal{H})$ of \mathbb{C} -linear isomorphisms $\mathcal{H}_x \rightarrow \mathcal{H}_y$ becomes a groupoid with unit space X , sometimes called the frame groupoid.

Remark 5.7. For $T \in \mathcal{L}_A(\mathcal{H})$, let $T_x \in \mathcal{L}_{A_x}(\mathcal{H}_x)$ be $T \otimes 1_{A_x}$, where 1_{A_x} is the identity map. This gives a map $\mathcal{L}_A(\mathcal{H}) \rightarrow \mathcal{L}_{A_x}(\mathcal{H}_x)$ and T is totally determined by the family $(T_x)_{x \in X}$. The identification $\mathcal{H} = \mathcal{H}A$ defines a homomorphism $C_0(X) \rightarrow Z\mathcal{L}_A(\mathcal{H})$ which takes a function $f \in C_0(X)$ into the operator $\xi \mapsto \xi f$. Since $\overline{\mathcal{H}C_0(X)} = \mathcal{H}$, this homomorphism induces a structure of $C_0(X)$ -algebra on $\mathcal{K}_A(\mathcal{H})$ with fibers $\mathcal{K}_{A_x}(\mathcal{H}_x)$. In general, $\overline{C_0(X)\mathcal{L}_A(\mathcal{H})} \neq \mathcal{L}_A(\mathcal{H})$, so that $\mathcal{L}_A(\mathcal{H})$ may not be a $C_0(X)$ -algebra.

Definition 5.8. Let A, B be $C_0(X)$ -algebras and let \mathcal{H} be a Hilbert A -module. We say that \mathcal{H} is a $C_0(X)$ - C^* -correspondence from B to A if there

is a $*$ -homomorphism $\phi : B \rightarrow \mathcal{L}_A(\mathcal{H})$ such that

$$\phi(fb)(\xi a) = \phi(b)(\xi fa) \text{ for all } f \in C_0(X), a \in A, b \in B, \xi \in \mathcal{H}.$$

Sometimes we write $b\xi$ for $\phi(b)\xi$.

Remark 5.9. Since $\phi(\overline{I_x B}) \subset \mathcal{H}(\overline{I_x A})$, the homomorphism ϕ decomposes into a family of homomorphisms $\phi_x : B_x \rightarrow \mathcal{L}_{A_x}(\mathcal{H}_x)$ and each fiber \mathcal{H}_x becomes a B_x - A_x C^* -correspondence, see Definition 4.1 in [21].

Definition 5.10. Let G be a topological groupoid acting on the $C_0(G^0)$ -algebras A and B . We say that G acts on the $C_0(G^0)$ - C^* -correspondence \mathcal{H} from B to A if there is a homomorphism $\rho : G \rightarrow \text{Iso}(\mathcal{H})$ such that $g \mapsto \rho(g)\xi$ is norm continuous and such that the following compatibility conditions are satisfied:

$$\langle \rho(g)\xi, \rho(g)\eta \rangle_{r(g)} = g \cdot \langle \xi, \eta \rangle_{s(g)},$$

for $\xi, \eta \in \mathcal{H}_{s(g)}$ and

$$\rho(g)(\xi a) = (\rho(g)\xi)(g \cdot a), \quad \rho(g)(b\xi) = (g \cdot b)(\rho(g)\xi)$$

for $\xi \in \mathcal{H}_{s(g)}$, $a \in A_{s(g)}$ and $b \in B_{s(g)}$.

Remark 5.11. The action of G on \mathcal{H} induces a $*$ -isomorphism of $\mathcal{L}_A(\mathcal{H})$ given by

$$(g \cdot T)(\xi) = \rho(g)T(\rho(g^{-1})\xi)$$

where $T \in \mathcal{L}_{A_{s(g)}}(\mathcal{H}_{s(g)})$ and an action of G on the $C_0(G^0)$ -algebra $\mathcal{K}_A(\mathcal{H})$ via

$$g \cdot \theta_{\xi, \eta} = \theta_{\rho(g)\xi, \rho(g)\eta},$$

where for $\xi, \eta \in \mathcal{H}_{s(g)}$ we have $\theta_{\xi, \eta}(\zeta) = \xi \langle \eta, \zeta \rangle$.

It is easy to check that the left multiplication $\phi : B \rightarrow \mathcal{L}_A(\mathcal{H})$ is G -equivariant. Indeed,

$$(g \cdot \phi(b))(\xi) = \rho(g)(\phi(b)(\rho(g^{-1})\xi)) = \phi(g \cdot b)(\rho(g)\rho(g^{-1})\xi) = \phi(g \cdot b)(\xi).$$

Example 5.12. If G is a locally compact groupoid with Haar system $\{\lambda^u\}_{u \in G^0}$, then G acts on the C^* -correspondence over $C_0(G^0)$ with fibers $L^2(G^u, \lambda^u)$ via the left regular representation.

6. Crossed products and Cuntz-Pimsner algebras

In this section we assume that G is a locally compact groupoid with Haar system $\{\lambda^x\}_{x \in G^0}$. We make some extra assumptions in order to obtain similar results as in [11] for the case of amenable groupoid actions on C^* -correspondences. We begin with the definition of the crossed product C^* -correspondence, given in the following Proposition.

Proposition 6.1. *Suppose the locally compact groupoid G with Haar system acts on the $C_0(G^0)$ - C^* -correspondence \mathcal{H} as in Definition 5.10. Then the completion of $\Gamma_c(G, r^*\mathcal{H})$, the continuous sections with compact support, becomes a C^* -correspondence $\mathcal{H} \rtimes G$ from $B \rtimes G$ to $A \rtimes G$ called the crossed product, using the inner product*

$$\langle \xi, \eta \rangle(g) = \int_G h \cdot \langle \xi(h^{-1}), \eta(h^{-1}g) \rangle_{s(h)} d\lambda^{r(g)}(h),$$

and the multiplications

$$(\xi f_1)(g) = \int_G \xi(h) h \cdot (f_1(h^{-1}g)) d\lambda^{r(g)}(h),$$

$$(f_2 \xi)(g) = \int_G f_2(h) \rho(h) \xi(h^{-1}g) d\lambda^{r(g)}(h),$$

where $\xi, \eta \in \Gamma_c(G, r^*\mathcal{H})$, $f_1 \in \Gamma_c(G, r^*A)$ and $f_2 \in \Gamma_c(G, r^*B)$

Proof. The verification of the Hilbert module axioms for $\mathcal{H} \rtimes G$ are routine calculations. For example $\langle \xi, \eta f \rangle = \langle \xi, \eta \rangle f$ because

$$\begin{aligned} \langle \xi, \eta f \rangle(g) &= \int_G h \cdot \langle \xi(h^{-1}), (\eta f)(h^{-1}g) \rangle_{s(h)} d\lambda^{r(g)}(h) = \\ &= \int_G h \cdot \langle \xi(h^{-1}), \int_G \eta(t) t \cdot (f(t^{-1}h^{-1}g)) d\lambda^{s(h)}(t) \rangle_{s(h)} d\lambda^{r(g)}(h) \end{aligned}$$

and

$$\begin{aligned} ((\xi, \eta) f)(g) &= \int_G \langle \xi, \eta \rangle(t) t \cdot (f(t^{-1}g)) d\lambda^{r(g)}(t) = \\ &= \int_G \left(\int_G h \cdot \langle \xi(h^{-1}), \eta(h^{-1}t) \rangle_{s(h)} d\lambda^{r(g)}(h) \right) t \cdot (f(t^{-1}g)) d\lambda^{r(g)}(t) = \\ &= \int_G h \cdot \langle \xi(h^{-1}), \int_G \eta(h^{-1}t)(h^{-1}t) \cdot f(t^{-1}g) d\lambda^{r(g)}(t) \rangle_{s(h)} d\lambda^{r(g)}(h) = \\ &= \int_G h \cdot \langle \xi(h^{-1}), \int_G \eta(u) u \cdot f(u^{-1}h^{-1}g) d\lambda^{s(h)}(u) \rangle_{s(h)} d\lambda^{r(g)}(h). \end{aligned}$$

The left multiplication provides a $*$ -homomorphism

$$B \rtimes G \rightarrow \mathcal{L}_{A \rtimes G}(\mathcal{H} \rtimes G),$$

so $\mathcal{H} \rtimes G$ becomes a C^* -correspondence from $B \rtimes G$ to $A \rtimes G$. \square

Lemma 6.2. *If the groupoid G acts on a $C_0(G^0)$ - C^* -correspondence \mathcal{H} over the $C_0(G^0)$ -algebra A , then there is an isomorphism*

$$\kappa : \mathcal{K}_{A \rtimes G}(\mathcal{H} \rtimes G) \rightarrow \mathcal{K}_A(\mathcal{H}) \rtimes G$$

given by

$$\kappa(\theta_{\xi, \eta})(u) = \int_G \theta_{\xi(h), \rho(u)\eta(u^{-1}h)} d\lambda^{r(u)}(h),$$

where $\xi, \eta \in \Gamma_c(G, r^*\mathcal{H})$.

Proof. Indeed, since $\theta_{\xi,\eta}(\zeta) = \xi \langle \eta, \zeta \rangle$ for $\zeta \in \Gamma_c(G, r^*\mathcal{H})$, we have

$$\begin{aligned} (\theta_{\xi,\eta}(\zeta))(g) &= \int_G \xi(h)h \cdot (\langle \eta, \zeta \rangle(h^{-1}g))d\lambda^{r(g)}(h) = \\ &= \int_G \xi(h)h \cdot \left(\int_G t \cdot \langle \eta(t^{-1}), \zeta(t^{-1}h^{-1}g) \rangle_{s(t)}d\lambda^{s(h)}(t) \right) d\lambda^{r(g)}(h) = \\ &= \int_G \xi(h) \int_G \langle \rho(u)\eta(u^{-1}h), \rho(u)\zeta(u^{-1}g) \rangle_{r(u)}d\lambda^{r(g)}(u)d\lambda^{r(g)}(h) = \\ &= \int_G \left(\int_G \theta_{\xi(h),\rho(u)\eta(u^{-1}h)}d\lambda^{r(u)}(h) \right) (\rho(u)\zeta(u^{-1}g))d\lambda^{r(g)}(u) \end{aligned}$$

and the map $u \mapsto \int_G \theta_{\xi(h),\rho(u)\eta(u^{-1}h)}d\lambda^{r(u)}(h)$ belongs to $\Gamma_c(G, r^*\mathcal{K}_A(\mathcal{H}))$.

For $\theta_{\xi,\eta} \in \mathcal{K}_{A \rtimes G}(\mathcal{H} \rtimes G)$ a rank one operator with $\xi, \eta \in \Gamma_c(G, r^*\mathcal{H})$ we define

$$\kappa(\theta_{\xi,\eta})(u) = \int_G \theta_{\xi(h),\rho(u)\eta(u^{-1}h)}d\lambda^{r(u)}(h),$$

which extends to a linear bijection $\kappa : \mathcal{K}_{A \rtimes G}(\mathcal{H} \rtimes G) \rightarrow \mathcal{K}_A(\mathcal{H}) \rtimes G$. Since $\theta_{\xi,\eta}^* = \theta_{\eta,\xi}$ and $\theta_{\xi_1,\eta_1}\theta_{\xi_2,\eta_2} = \theta_{\xi_1 \langle \eta_1, \xi_2 \rangle, \eta_2}$ a computation shows that

$$\kappa(\theta_{\xi,\eta}^*) = \kappa(\theta_{\eta,\xi})^*, \quad \kappa(\theta_{\xi_1,\eta_1}\theta_{\xi_2,\eta_2}) = \kappa(\theta_{\xi_1,\eta_1})\kappa(\theta_{\xi_2,\eta_2})$$

and κ is a $*$ -isomorphism. □

Theorem 6.3. *Let G be a locally compact groupoid with Haar system that acts on a $C_0(G^0)$ - C^* -correspondence \mathcal{H} over the $C_0(G^0)$ -algebra A .*

Then the Katsura ideal $J_{\mathcal{H}}$ is G -invariant and there is an action of G on the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ which becomes a $C_0(G^0)$ -algebra with fibers $\mathcal{O}_{\mathcal{H}_x}$ where $x \in G^0$.

Proof. If $\phi : A \rightarrow \mathcal{L}_A(\mathcal{H})$ defines the left action, recall that the Katsura ideal $J_{\mathcal{H}}$ is equal to $\phi^{-1}(\mathcal{K}_A(\mathcal{H})) \cap (\ker \phi)^\perp$. Since ϕ is G -equivariant, both ideals $\phi^{-1}(\mathcal{K}_A(\mathcal{H}))$ and $(\ker \phi)^\perp$ are G -invariant, hence $J_{\mathcal{H}}$ is G -invariant.

A representation of \mathcal{H} in a $C_0(G^0)$ -algebra B is a pair (π, t) where $\pi : A \rightarrow B$ is a $C_0(G^0)$ -homomorphism and $t : \mathcal{H} \rightarrow B$ is linear such that we have

$$t(a \cdot \xi) = \pi(a)t(\xi), \quad \pi(\langle \xi, \eta \rangle) = t(\xi)^*t(\eta)$$

for all $a \in A$ and $\xi, \eta \in \mathcal{H}$. Note that in particular $t(\xi f) = t(\xi)f$ for $\xi \in \mathcal{H}$ and $f \in C_0(G^0)$.

Moreover, if $t^{(1)} : \mathcal{K}_A(\mathcal{H}) \rightarrow B$ is given by

$$t^{(1)}(\theta_{\xi,\eta}) = t(\xi)t(\eta)^*,$$

then (π, t) is covariant if

$$t^{(1)}(\phi(a)) = \pi(a) \quad \text{for all } a \in J_{\mathcal{H}}.$$

The Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ is generated by a universal covariant representation $(\pi_A, t_{\mathcal{H}})$ in the sense that for any other covariant representation (π, t) in a $C_0(G^0)$ -algebra B there is a unique $*$ -homomorphism $\pi \times t : \mathcal{O}_{\mathcal{H}} \rightarrow B$ such that

$$(\pi \times t) \circ \pi_A = \pi \text{ and } (\pi \times t) \circ t_{\mathcal{H}} = t.$$

Since A is a $C_0(G^0)$ -algebra and each \mathcal{H}_x is a C^* -correspondence over A_x , using Remark 5.7 it follows that $\mathcal{O}_{\mathcal{H}}$ becomes a $C_0(G^0)$ -algebra with fibers $\mathcal{O}_{\mathcal{H}_x}$. The non-degenerate homomorphism $C_0(G^0) \rightarrow ZM(\mathcal{O}_{\mathcal{H}})$ is defined using the universal property and is given by

$$f \cdot \pi_A(a) = \pi_A(fa), \quad f \cdot t_{\mathcal{H}}(\xi) = t_{\mathcal{H}}(\xi f) \text{ for } f \in C_0(G^0), a \in A, \xi \in \mathcal{H}.$$

The action of G on the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ is defined using

$$g \cdot \pi_A(a) = \pi_A(g \cdot a) \text{ for } a \in A_{s(g)},$$

$$g \cdot t_{\mathcal{H}}(\xi) = t_{\mathcal{H}}(\rho(g)\xi) \text{ for } \xi \in \mathcal{H}_{s(g)}.$$

In particular, the ideal $J_{\mathcal{H}}$ is a $C_0(G^0)$ -algebra with fibers $J_{\mathcal{H}_x}$. □

Theorem 6.4. *Let G be a locally compact amenable groupoid with Haar system that acts on a $C_0(G^0)$ - C^* -correspondence \mathcal{H} over the $C_0(G^0)$ -algebra A . Assume that $J_{\mathcal{H} \rtimes G} \cong J_{\mathcal{H}} \rtimes G$. Then there are maps*

$$\mu : A \rtimes G \rightarrow \mathcal{O}_{\mathcal{H}} \rtimes G, \quad \mu(f)(g) = \pi_A(f(g)) \text{ for } f \in \Gamma_c(G, r^*A)$$

and

$$\tau : \Gamma_c(G, r^*\mathcal{H}) \rightarrow \mathcal{O}_{\mathcal{H}} \rtimes G, \quad \tau(\xi)(g) = t_{\mathcal{H}}(\xi(g))$$

which induce an isomorphism

$$\mathcal{O}_{\mathcal{H} \rtimes G} \cong \mathcal{O}_{\mathcal{H}} \rtimes G.$$

Proof. First, let's verify that $\tau(\xi)^* \tau(\eta) = \mu(\langle \xi, \eta \rangle)$. Indeed,

$$\begin{aligned} \tau(\xi)^* \tau(\eta)(g) &= \int_G \tau(\xi)^*(h) h \cdot (\tau(\eta)(h^{-1}g)) d\lambda^{r(g)}(h) = \\ &= \int_G h \cdot ((\tau(\xi)(h^{-1}))^* \tau(\eta)(h^{-1}g)) d\lambda^{r(g)}(h) = \\ &= \int_G h \cdot (t_{\mathcal{H}}(\xi(h^{-1}))^* t_{\mathcal{H}}(\eta(h^{-1}g))) d\lambda^{r(g)}(h) = \\ &= \int_G h \cdot \pi_A(\langle \xi(h^{-1}), \eta(h^{-1}g) \rangle_{s(h)}) d\lambda^{r(g)}(h) = \\ &= \pi_A \left(\int_G h \cdot \langle \xi(h^{-1}), \eta(h^{-1}g) \rangle_{s(h)} d\lambda^{r(g)}(h) \right) = \mu(\langle \xi, \eta \rangle)(g), \end{aligned}$$

and τ extends to $\mathcal{H} \rtimes G$.

Similarly, we have that $\tau(f\xi) = \mu(f)\tau(\xi)$, hence the pair (μ, τ) is a representation of $\mathcal{H} \rtimes G$ in $\mathcal{O}_{\mathcal{H}} \rtimes G$.

Since $J_{\mathcal{H} \rtimes G} \cong J_{\mathcal{H}} \rtimes G$, it follows as in Corollary 2.9 of [11] that (μ, τ) is covariant. Indeed, since $\tau^{(1)}(\theta_{\xi, \eta})(g) = t_{\mathcal{H}}^{(1)}(\kappa(\theta_{\xi, \eta})(g))$ where κ is as in Lemma 6.2, for $b \in J_{\mathcal{H} \rtimes G}$ we have

$$\tau^{(1)}(\kappa^{-1}(\psi(b)))(g) = t_{\mathcal{H}}^{(1)}(\psi(b)(g)) = t_{\mathcal{H}}^{(1)}(\phi(b(g))) = \pi_A(b(g)) = \mu(b)(g),$$

where $\psi : A \rtimes G \rightarrow \mathcal{L}_{A \rtimes G}(\mathcal{H} \rtimes G)$ is induced by $\phi : A \rightarrow \mathcal{L}_A(\mathcal{H})$. The pair (μ, τ) induces a $*$ -homomorphism

$$\mu \times \tau : \mathcal{O}_{\mathcal{H} \rtimes G} \rightarrow \mathcal{O}_{\mathcal{H}} \rtimes G.$$

Since the images of μ and τ generate $\mathcal{O}_{\mathcal{H}} \rtimes G$, the map μ is injective, and there exists a gauge action on $\mathcal{O}_{\mathcal{H}} \rtimes G$, by Theorem 6.4 in [14] it follows that the map $\mu \times \tau$ is an isomorphism. \square

Example 6.5. Let G be a groupoid with compact unit space X . Then G acts on $C(X)$ in the usual way, $(gf)(x) = f(g^{-1}x)$. A (locally trivial) complex vector bundle $p : \mathcal{E} \rightarrow X$ is called a G -bundle if G acts on \mathcal{E} by linear maps $\mathcal{E}_{s(g)} \rightarrow \mathcal{E}_{r(g)}$, where $\mathcal{E}_x = p^{-1}(x)$. The groupoid G acts on $\Gamma(\mathcal{E})$, the space of continuous sections $\xi : X \rightarrow \mathcal{E}$, by

$$(g\xi)(x) = g(\xi(g^{-1}x)).$$

If \mathcal{E} is Hermitian and G acts by isometries, then the space $\Gamma(\mathcal{E})$ has a natural structure of C^* -correspondence over $C(X)$, where the left and right multiplications are given by

$$(f\xi)(x) = (\xi f)(x) = f(x)\xi(x).$$

Since

$$\begin{aligned} g(f\xi)(x) &= g(f\xi)(g^{-1}x) = g(f(g^{-1}x)\xi(g^{-1}x)) = \\ &= f(g^{-1}x)g\xi(g^{-1}x) = (gf)(g\xi)(x), \end{aligned}$$

the groupoid G acts on the Cuntz-Pimsner algebra $\mathcal{O}_{\Gamma(\mathcal{E})}$ which is a continuous field of Cuntz algebras (see [28]). It would be interesting to determine $\mathcal{O}_{\Gamma(\mathcal{E})} \rtimes G$ in some particular cases.

Corollary 6.6. *If the groupoid G acts on a graph E as in Definition 4.1, then G acts on the graph C^* -correspondence \mathcal{H}_E and on the graph algebra $C^*(E) = \mathcal{O}_{\mathcal{H}_E}$ which becomes a $C_0(G^0)$ -algebra. The gauge action on $C^*(E)$ commutes with the action of G and therefore the action of G on $C^*(E)$ restricts to an action of G on the core algebra $C^*(E)^{\mathbb{T}}$.*

Proof. Indeed, the action of G on E determines an action on $C_0(E^0)$ and an action on $C_c(E^1)$ via

$$(g \cdot f)(x) = f(g^{-1}x), \quad \rho(g)\xi(e) = \xi(g^{-1}e).$$

These actions satisfy the compatibility conditions in Definition 5.10 for $A = B = C_0(E^0)$ and for $\mathcal{H} = \mathcal{H}_E$. We use Theorem 6.3 to get an action of G on $C^*(E)$, which becomes a $C_0(G^0)$ -algebra.

Recall that the gauge action γ is defined by

$$\gamma(z)f = f \text{ and } \gamma(z)\xi = z\xi \text{ for } f \in C_0(E^0), \xi \in C_c(E^1), z \in \mathbb{T}.$$

The gauge action is extended to $C^*(E)$ using the universal property and we have

$$\rho(g)(z\xi) = z\rho(g)\xi,$$

so the action of G can be restricted to the fixed point algebra $C^*(E)^\mathbb{T}$. \square

Example 6.7. The transitive groupoid G in Example 4.3 acts on the graph C^* -correspondence $\mathcal{H} = \mathbb{C}^3 \oplus \mathbb{C}^3$ over $A = \mathbb{C} \oplus \mathbb{C}$. It follows that G acts on the graph algebra $\mathcal{O}_{\mathcal{H}} \cong \mathcal{O}_3 \oplus \mathcal{O}_3$ and on $\mathcal{O}_{\mathcal{H}}^\mathbb{T} \cong M_{3^\infty} \oplus M_{3^\infty}$. The fixed point algebra $\mathcal{O}_{\mathcal{H}}^G$ is isomorphic with $\mathcal{O}_3^{S_3} \oplus \mathcal{O}_3^{S_3}$.

Example 6.8. A self-similar action as in Definition 4.4 determines an action of G on the graph C^* -algebra $C^*(T_E)$. Since $C^*(T_E)$ is strongly Morita equivalent with $C_0(\partial T_E)$ in an equivariant way (see section 4 in [17]), it follows that $C^*(T_E) \rtimes G$ is strongly Morita equivalent with $C_0(\partial T_E) \rtimes G$. Note that T_E is a union of trees which in general are not the universal cover of the graph E .

7. Doplicher-Roberts algebras

The Doplicher-Roberts algebras (denoted by \mathcal{O}_G in [9]) were introduced to construct a new duality theory for compact Lie groups G which strengthens the Tannaka-Krein duality. Let \mathcal{T}_G denote the representation category whose objects are tensor powers of the n -dimensional representation ρ of G defined by the inclusion $G \subseteq U(n)$ in some unitary group $U(n)$ and whose arrows are the intertwiners. The C^* -algebra \mathcal{O}_G is identified in [9] with the fixed point algebra \mathcal{O}_n^G , where \mathcal{O}_n is the Cuntz algebra. If σ_G denotes the restriction to \mathcal{O}_G of the canonical endomorphism of the Cuntz algebra, then \mathcal{T}_G can be reconstructed from the pair $(\mathcal{O}_G, \sigma_G)$. Subsequently, Doplicher-Roberts algebras were associated to any object ρ in a strict tensor C^* -category, see [10].

Consider now a groupoid G acting on a C^* -correspondence \mathcal{H} over the $C_0(G^0)$ -algebra A as in Definition 5.10 via the homomorphism $\rho : G \rightarrow \text{Iso}(\mathcal{H})$. Since the balanced tensor power $\mathcal{H}^{\otimes n}$ is fibered over G^0 with fibers $\mathcal{H}_x^{\otimes n}$, it follows that $\mathcal{K}_A(\mathcal{H}^{\otimes n}, \mathcal{H}^{\otimes m})$ has fibers $\mathcal{K}_{A_x}(\mathcal{H}_x^{\otimes n}, \mathcal{H}_x^{\otimes m})$.

Consider the tensor powers $\rho^n : G \rightarrow \text{Iso}(\mathcal{H}^{\otimes n})$ and define the set of intertwiners (ρ^m, ρ^n) with fibers

$$(\rho^m, \rho^n)_x = \{T \in \mathcal{K}_{A_x}(\mathcal{H}_x^{\otimes n}, \mathcal{H}_x^{\otimes m}) \mid T\rho^n(g) = \rho^m(g)T\}.$$

We identify (ρ^m, ρ^n) with a subset of (ρ^{m+r}, ρ^{n+r}) via $T \mapsto T \otimes I_r$, where $I_r : \mathcal{H}^{\otimes r} \rightarrow \mathcal{H}^{\otimes r}$ is the identity map. After this identification, it follows that the linear span ${}^0\mathcal{O}_\rho$ of $\bigcup_{m,n \geq 0} (\rho^m, \rho^n)$ has a natural multiplication given by

composition: if $S \in (\rho^m, \rho^n)$ and $T \in (\rho^p, \rho^q)$, then the product ST is

$$(S \otimes I_{p-n}) \circ T \in (\rho^{m+p-n}, \rho^q) \text{ if } p \geq n,$$

or

$$S \circ (T \otimes I_{n-p}) \in (\rho^m, \rho^{q+n-p}) \text{ if } p < n.$$

The adjoint of $T \in (\rho^m, \rho^n)$ is $T^* \in (\rho^n, \rho^m)$.

From Theorem 4.2 in [10], it follows that the C^* -closure of ${}^0\mathcal{O}_\rho$ is well defined, obtaining the Doplicher-Roberts algebra \mathcal{O}_ρ associated to the homomorphism $\rho : G \rightarrow \text{Iso}(\mathcal{H})$.

Theorem 7.1. *Let \mathcal{H} be a $C_0(X)$ - C^* -correspondence over a $C_0(X)$ -algebra A which is full and finite projective, and assume that the left multiplication $A \rightarrow \mathcal{L}_A(\mathcal{H})$ is injective. If G is a groupoid with $G^0 = X$ acting on A and on \mathcal{H} via $\rho : G \rightarrow \text{Iso}(\mathcal{H})$ as in Definition 5.10, then the Doplicher-Roberts algebra \mathcal{O}_ρ is isomorphic to the fixed point algebra $\mathcal{O}_\mathcal{H}^G$.*

Proof. Since \mathcal{H} is finite projective, it is known that $\mathcal{L}_A(\mathcal{H}) \cong \mathcal{K}_A(\mathcal{H})$. Moreover, the Cuntz-Pimsner algebra $\mathcal{O}_\mathcal{H}$ is isomorphic to the C^* -algebra generated by the span of $\bigcup_{m,n \geq 0} \mathcal{K}_A(\mathcal{H}^{\otimes n}, \mathcal{H}^{\otimes m})$ after we identify T with $T \otimes I$, (see Proposition 2.5 in [12]). This isomorphism preserves the $C_0(X)$ -algebra structures.

The groupoid G acts on $\mathcal{K}_A(\mathcal{H}^{\otimes n}, \mathcal{H}^{\otimes m})$ by

$$(g \cdot T)(\xi) = \rho^m(g)T(\rho^n(g^{-1})\xi)$$

and the fixed point algebra is (ρ^m, ρ^n) . Indeed, for a fixed $x \in X$ and $g \in G_x^x$ we have $g \cdot T = T$ if and only if $T\rho^n(g) = \rho^m(g)T$.

It follows that ${}^0\mathcal{O}_\rho \subseteq \mathcal{O}_\mathcal{H}$ and that \mathcal{O}_ρ is isomorphic to $\mathcal{O}_\mathcal{H}^G$. □

Corollary 7.2. *Let E be a topological graph such that \mathcal{H}_E is full and finite projective and the left multiplication of $C_0(E^0)$ is injective. If G is a groupoid acting on E as in Definition 4.1 inducing a homomorphism $\rho : G \rightarrow \text{Iso}(\mathcal{H}_E)$, then $\mathcal{O}_\rho \cong C^*(E)^G$.*

Example 7.3. The Doplicher-Roberts algebra for the groupoid action in Example 4.3 is isomorphic with $\mathcal{O}_3^{S_3} \oplus \mathcal{O}_3^{S_3}$.

8. Compact isotropy groupoid actions on graphs

Suppose the groupoid G with compact isotropy groups acts on a discrete locally finite graph E . It follows that G^0 is also discrete. It is known that in this case $C_0(E^0) \rtimes G$ is strongly Morita equivalent with a commutative C^* -algebra $C_0(X)$ with X at most countable.

If we denote by $\{p_x\}_{x \in X}$ the minimal projections in $C_0(X)$, recall that the isomorphism classes of separable nondegenerate C^* -correspondences \mathcal{H} over $C_0(X)$ determine a discrete graph. More precisely, the $*$ -homomorphism $\phi : C_0(X) \rightarrow \mathcal{L}(\mathcal{H})$ will define an incidence matrix $(a_{xy})_{x,y \in X}$ where $a_{xy} = \dim \phi(p_x)\mathcal{H}p_y$. (see Theorem 1.1 in [13]).

We recall now the following result from [23]:

Lemma 8.1. *Suppose C and D are strongly Morita equivalent C^* -algebras with C - D imprimitivity bimodule \mathcal{Z} .*

If \mathcal{H} is a C^ -correspondence over C , then $\mathcal{N} = \mathcal{Z}^* \otimes_C \mathcal{H} \otimes_C \mathcal{Z}$ is a C^* -correspondence over D such that $\mathcal{O}_{\mathcal{H}}$ and $\mathcal{O}_{\mathcal{N}}$ are strongly Morita equivalent.*

Theorem 8.2. *Let E be a locally finite discrete graph with no sources and let G be a groupoid with compact isotropy groups acting on E . Then the crossed product $C^*(E) \rtimes G$ is strongly Morita equivalent to a graph C^* -algebra, where the number of vertices is the cardinality of the spectrum of $C_0(E^0) \rtimes G$.*

Proof. The idea is to decompose the C^* -correspondence $\mathcal{H}_E \rtimes G$ over the C^* -algebra $C_0(E^0) \rtimes G$, which is strongly Morita equivalent with a commutative C^* -algebra.

Let $C_0(E^0) \rtimes G \cong \bigoplus_{i=1}^N M_{n(i)}$, where $N \in \mathbb{N} \cup \{\infty\}$ and $M_{n(i)}$ denotes the set of $n(i) \times n(i)$ matrix algebras. Consider now the graph with N vertices and at each vertex v_i we assign the C^* -algebra $M_{n(i)}$. If p_i is the unit in $M_{n(i)}$, whenever $p_i(\mathcal{H}_E \rtimes G)p_j \neq 0$, we decompose this as a direct sum of minimal $M_{n(i)}-M_{n(j)}$ C^* -correspondences. Since $\mathbb{C}-\mathbb{C}$ C^* -correspondences are Hilbert spaces, a minimal $M_{n(i)}-M_{n(j)}$ C^* -correspondence is of the form $M_{n(i),n(j)}$, the set of rectangular matrices with $n(i)$ rows and $n(j)$ columns, with the obvious bimodule structure and inner product. This decomposition determines the number of edges between v_j and v_i .

Since in this case $J_{\mathcal{H}_E} = C_0(E^0)$ and $J_{\mathcal{H}_E \rtimes G} \cong J_{\mathcal{H}_E \times G}$, we deduce from Theorem 6.4 that

$$\mathcal{O}_{\mathcal{H}_E \times G} \cong \mathcal{O}_{\mathcal{H}_E} \rtimes G \cong C^*(E) \rtimes G.$$

It follows that $C^*(E) \rtimes G$ is isomorphic to the C^* -algebra of a graph of (minimal) C^* -correspondences (see [6]), hence strongly Morita equivalent to a graph C^* -algebra.

Note that in this case $C^*(E)$ is the C^* -algebra of a groupoid \mathcal{G}_E and that $C^*(E) \rtimes G = C^*(\mathcal{G}_E \rtimes G)$

□

Corollary 8.3. *If G acts on E as above, then G acts on the AF-core $C^*(E)^\mathbb{T}$ and $C^*(E)^\mathbb{T} \rtimes G \cong (C^*(E) \rtimes G)^\mathbb{T}$ is an AF-algebra.*

References

- [1] ANANTHARAMAN-DELAROCHE, CLAIRE; RENAULT, JEAN. Amenable groupoids. Monographies de L'Enseignement Mathematique, 36. *L'Enseignement Mathématique, Geneva*, 2000. 196 pp. ISBN: 2-940264-01-5. MR1799683, Zbl 0960.43003. 1023

- [2] BLANCHARD, ÉTIENNE. Déformations de C^* -algèbres de Hopf. *Bull. Soc. Math. France* **124** (1996), no. 1, 141–215. MR1395009, Zbl 0851.46040, doi:10.24033/bsmf.2278.
- [3] BROWN, RONALD. Groupoids as coefficients. *Proc. London Math. Soc.* (3) **25** (1972), 413–426. MR0311744, Zbl 0245.20045, doi:10.1112/plms/s3-25.3.413. 1023
- [4] BUSS, ALCIDES; MEYER, RALF. Iterated crossed products for groupoid fibrations. Preprint, 2016. arXiv:1604.02015. 1022, 1024
- [5] DEACONU, VALENTIN. Group actions on graphs and C^* -correspondences. *Houston J. Math.* **44** (2018), no. 1, 147–168. MR3796442, Zbl 06903038, arXiv:1410.3846. 1020, 1021
- [6] DEACONU, VALENTIN; KUMJIAN, ALEX; PASK, DAVID; SIMS, AIDAN. Graphs of C^* -correspondences and Fell bundles. *Indiana Univ. Math. J.* **59** (2010), no. 5, 1687–1735. MR2865427, Zbl 1232.46052, arXiv:0901.0032, doi:10.1512/iumj.2010.59.3893. 1036
- [7] DEACONU, VALENTIN; KUMJIAN, ALEX; QUIGG, JOHN. Group actions on topological graphs. *Ergodic Theory Dynam. Systems* **32** (2012), no. 5, 1527–1566. MR2974209, Zbl 1322.46043, arXiv:1007.2616, doi:10.1017/S014338571100040X.
- [8] DEACONU, VALENTIN; KUMJIAN, ALEX; RAMAZAN, BIRANT. Fell bundles associated to groupoid morphisms. *Math. Scand.* **102** (2008), no. 2, 305–319. MR2437192, Zbl 1173.46034, arXiv:math/0612746, doi:10.7146/math.scand.a-15064. 1022, 1024
- [9] DOPLICHER, SERGIO; ROBERTS, JOHN E. Duals of compact Lie groups realized in the Cuntz algebras and their actions on C^* -algebras. *J. Funct. Anal.* **74** (1987), no. 1, 96–120. MR0901232, Zbl 0619.46053, doi:10.1016/0022-1236(87)90040-1. 1034
- [10] DOPLICHER, SERGIO; ROBERTS, JOHN E. A new duality theory for compact groups. *Invent. Math.* **98** (1989), no. 1, 157–218. MR1010160, Zbl 0691.22002, doi:10.1007/BF01388849. 1034, 1035
- [11] HAO, GAI; NG, CHI-KEUNG. Crossed products of C^* -correspondences by amenable group actions. *J. Math. Anal. Appl.* **345** (2008), no. 2, 702–707. MR2429169, Zbl 1153.46040, doi:10.1016/j.jmaa.2008.04.058. 1021, 1029, 1033
- [12] KAJIWARA, TSUYOSHI; PINZARI, CLAUDIA; WATATANI, YASUO. Ideal structure and simplicity of the C^* -algebras generated by Hilbert bimodules. *J. Funct. Anal.* **159** (1998), no. 2, 295–322. MR1658088, Zbl 0942.46035, arXiv:math/9802067, doi:10.1006/jfan.1998.3306. 1035
- [13] KALISZEWSKI, STEVE; PATANI, NURA; QUIGG, JOHN. Characterizing graph C^* -correspondences. *Houston J. Math.* **38** (2012), no. 3, 751–759. MR2970656, Zbl 1262.46041, arXiv:0906.3311. 1035
- [14] KATSURA, TAKESHI. On C^* -algebras associated with C^* -correspondences. *J. Funct. Anal.* **217** (2004), no. 2, 366–401. MR2102572, Zbl 1067.46054, arXiv:math/0309088, doi:10.1016/j.jfa.2004.03.010. 1033
- [15] KUMJIAN, ALEX. On equivariant sheaf cohomology and elementary C^* -bundles. *J. Operator Theory* **20** (1988), no. 2, 207–240. MR1004121, Zbl 0692.46066. 1023
- [16] KUMJIAN, ALEX; MUHLY, PAUL S.; RENAULT, JEAN N.; WILLIAMS, DANA P. The Brauer group of a locally compact groupoid. *Amer. J. Math.* **120** (1998), no. 5, 901–954. MR1646047, Zbl 0916.46050, doi:10.1353/ajm.1998.0040. 1028
- [17] KUMJIAN, ALEX; PASK, DAVID. C^* -algebras of directed graphs and group actions. *Ergodic Theory Dynam. Systems* **19** (1999), no. 6, 1503–1519. MR1738948, Zbl 0949.46034, doi:10.1017/S0143385799151940. 1034
- [18] LACA, MARCELO; RAEBURN, IAIN; RAMAGGE, JACQUI; WHITTAKER, MICHAEL F. Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs. *Adv. Math.* **331** (2018), 268–325. MR3804678, Zbl 1392.37007, doi:10.1016/j.aim.2018.03.030. 1025

- [19] LANDSMAN, N. P. KLAAS; RAMAZAN, BIRANT. Quantization of Poisson algebras associated to Lie algebroids. *Groupoids in Analysis, geometry, and physics* (Boulder, CO, 1999), 159–192, Contemp. Math., 282, Amer. Math. Soc., Providence, RI, 2001. MR1855249, Zbl 1013.46053, doi: 10.1090/conm/282. 1023
- [20] LE GALL, PIERRE-YVES. Théorie de Kasparov équivariante et groupoïdes. Thèse de Doctorat, Université Paris VII, 1994.
- [21] LE GALL, PIERRE-YVES. Théorie de Kasparov équivariante et groupoïdes. I. *K-Theory* **16** (1999), no. 4, 361–390. MR1686846, Zbl 0932.19004, doi: 10.1023/A:1007707525423. 1029
- [22] MITCHENER, PAUL D. C^* -categories, groupoid actions, equivariant KK -theory, and the Baum–Connes conjecture. *J. Funct. Anal.* **214** (2004), no. 1, 1–39. MR2079884, Zbl 1054.19002, arXiv:math/0204291, doi: 10.1016/j.jfa.2004.04.016. 1028
- [23] MUHLY, PAUL S.; PASK, DAVID; TOMFORDE, MARK. Strong shift equivalence of C^* -correspondences. *Israel J. Math.* **167** (2008), 315–346. MR2448028, Zbl 1220.46036, arXiv:math/0508059, doi: 10.1007/s11856-008-1051-9. 1036
- [24] MUHLY, PAUL S.; RENAULT, JEAN, N.; WILLIAMS, DANA P. Equivalence and isomorphism for groupoid C^* -algebras. *J. Operator Theory* **17** (1987), no. 1, 3–22. MR0873460, Zbl 0645.46040. 1021
- [25] MUHLY, PAUL S.; WILLIAMS, DANA P. Renault’s equivalence theorem for groupoid crossed products. New York Journal of Mathematics. NYJM Monographs, 3. State University of New York, University at Albany, Albany, NY, 2008. 87 pp. MR2547343, Zbl 1191.46055, arXiv:/0707.3566. 1028
- [26] PIMSNER, MICHAEL V. A class of C^* -algebras generalizing both Cuntz–Krieger algebras and crossed products by \mathbb{Z} . *Free probability theory* (Waterloo, ON, 1995), 189–212, Fields Inst. Commun., 12, Amer. Math. Soc., Providence, RI, 1997. MR1426840, Zbl 0871.46028.
- [27] RENAULT, JEAN. Groupoid cocycles and derivations. *Ann. Funct. Anal.* **3** (2012), no. 2, 1–20. MR2948385, Zbl 1257.22005, arXiv:1201.4599, doi: 10.15352/afa/1399899929.
- [28] VASSELLI, EZIO. The C^* -algebra of a vector bundle and fields of Cuntz algebras. *J. Funct. Anal.* **222** (2005), no. 2, 491–502. MR2132397, Zbl 1081.46048, arXiv:math/0404166, doi: 10.1016/j.jfa.2004.09.004. 1033
- [29] WILLIAMS, DANA P. Crossed-products of C^* -algebras. Mathematical Surveys and Monographs, 134. American Mathematical Society, Providence, RI, 2007. xvi+528 pp. ISBN: 978-0-8218-4242-3; 0-8218-4242-0. MR2288954, Zbl 1119.46002. 1027

(Valentin Deaconu) DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEVADA, RENO, NV 89557-0084, USA.

vdeaconu@unr.edu

This paper is available via <http://nyjm.albany.edu/j/2018/24-48.html>.