New York Journal of Mathematics

New York J. Math. **24** (2018) 1068–1075.

Automatic continuity of $*$-representations for discrete twisted C^*-dynamical systems

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Abstract. In this paper, we prove that every $*$-representation for a discrete twisted C^*-dynamical system (G, A, α, ω) (on a C^*-algebra) is automatically contractive with respect to the L^1-norm on $C_c(G, A)$.

Contents

1. Introduction

Given a C^*-dynamical system $\mathscr{A} = (G, A, \alpha)$ with a Haar measure μ assumed on G, equip the vector space $C_c(G, A)$ of compactly-supported continuous A-valued functions on G with a $*$-algebraic structure by defining a convolution $\star_{\mathscr{A}}$ and an involution $^{*_{\mathscr{A}}}$ as follows:

$$\forall f, g \in C_c(G, A): \qquad f \star_{\mathscr{A}} g \overset{\text{df}}{=} \left\{ \begin{matrix} G & \to & A \\ x & \mapsto & \displaystyle\int_G f(y) \alpha_y\big(g\big(y^{-1}x\big)\big) \, \mathrm{d}\mu(y) \end{matrix} \right\};$$

$$f^{*_{\mathscr{A}}} \overset{\text{df}}{=} \left\{ \begin{matrix} G & \to & A \\ x & \mapsto & \Delta_G\big(x^{-1}\big) \cdot \alpha_x\Big(f\big(x^{-1}\big)^*\Big) \end{matrix} \right\}.$$

We may then turn $(C_c(G, A), \star_{\mathscr{A}}, ^{*_{\mathscr{A}}})$ into a normed $*$-algebra with a norm $\|\cdot\|_{\mathscr{A},1}$ on $C_c(G, A)$ defined by

$$\forall f \in C_c(G, A): \qquad \|f\|_{\mathscr{A},1} \overset{\text{df}}{=} \int_G \|f(x)\|_A \, \mathrm{d}\mu(x).$$

Received March 30, 2018.

2010 *Mathematics Subject Classification.* 22D15, 22D20, 22D25, 47L55.

Key words and phrases. twisted C^*-dynamical system, $*$-representation, multiplier $*$-representation, covariant representation, automatically contractive.

ISSN 1076-9803/2018

We call $\|\cdot\|_{\mathscr{A},1}$ the L^1-*norm* for $\mathscr{A}$.

A $*$-representation for $\mathscr{A}$ is now a pair (C,π) consisting of a C^*-algebra C and an algebraic $*$-homomorphism π from $(C_c(G,A),\star_{\mathscr{A}},{}^{*_{\mathscr{A}}})$ to C, and we may define the crossed-product C^*-algebra $C^*(\mathscr{A})$ as the completion of $(C_c(G,A),\star_{\mathscr{A}},{}^{*_{\mathscr{A}}})$ for a norm $\|\cdot\|_{\mathscr{A},\mathrm{u}}$ on $C_c(G,A)$ — called the *universal norm* for $\mathscr{A}$ — defined by

$$\forall f \in C_c(G,A):$$
$$\|f\|_{\mathscr{A},\mathrm{u}} \stackrel{\mathrm{df}}{=} \sup\left(\left\{ \|\pi(f)\|_C \;\middle|\; \begin{array}{c}(C,\pi) \text{ is a } *\text{-representation for } \mathscr{A} \text{ that is}\\ \text{contractive with respect to } \|\cdot\|_{\mathscr{A},1} \text{ and } \|\cdot\|_C\end{array} \right\}\right).$$

As far as we know, all treatments of crossed-product C^*-algebras (e.g. [3]) assume the contractivity condition in order to *enforce* that $\|\cdot\|_{\mathscr{A},\mathrm{u}}$ is actually well-defined.

We can therefore ask: Is a $*$-representation for a C^*-dynamical system automatically contractive with respect to $\|\cdot\|_{\mathscr{A},1}$ and $\|\cdot\|_C$? We know of no counterexamples, and we have been unable to find anything relevant to this problem in the literature.

We hope to advertise the problem by showing that every $*$-representation for a discrete C^*-dynamical system is automatically contractive. Actually, we will prove a stronger result: Every $*$-representation for a discrete *twisted* C^*-dynamical system is automatically contractive.

For every C^*-algebra A, we will adopt the following notation:

- $\mathrm{Aut}(A)$ denotes the group of $*$-automorphisms on A.
- $M(A)$ denotes the multiplier C^*-algebra of A.
- $\mathcal{U}(A)$ denotes the group of unitary elements of A.

2. Twisted C^*-dynamical systems

Definition 2.1 ([1])**.** A *twisted C^*-dynamical system* is a 4-tuple (G,A,α,ω) with the following properties:

(1) G is a locally compact Hausdorff topological group, with a Haar measure μ on G tacitly assumed.
(2) A is a C^*-algebra.
(3) α is a strongly continuous map from G to $\mathrm{Aut}(A)$, i.e.,
$$\begin{Bmatrix} G & \to & A \\ x & \mapsto & \alpha_x(a) \end{Bmatrix}$$
is a continuous map for each $a \in A$.
(4) ω is a strictly continuous map from $G\times G$ to $\mathcal{U}(M(A))$, i.e.,
$$\begin{Bmatrix} G\times G & \to & A \\ (x,y) & \mapsto & a\omega(x,y) \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} G\times G & \to & A \\ (x,y) & \mapsto & \omega(x,y)a \end{Bmatrix}$$
are continuous maps for each $a \in A$.
(5) $\alpha_e = \mathrm{Id}_A$, and $\omega(e,r) = 1_{M(A)} = \omega(r,e)$ for all $r \in G$.

(6) $\overline{\alpha}_r \circ \overline{\alpha}_s = \mathrm{Ad}(\omega(r,s)) \circ \overline{\alpha}_{rs}$ for all $r,s \in G$, i.e.,

$$\forall m \in M(A): \quad \overline{\alpha}_r(\overline{\alpha}_s(m)) = \omega(r,s)\overline{\alpha}_{rs}(m)\omega(r,s)^*.$$

Here, $\overline{\alpha}$ denotes the map from G to $\mathrm{Aut}(M(A))$ that assigns to each $r \in G$ the unique $*$-automorphism on $M(A)$ that extends α_r.

(7) $\overline{\alpha}_r(\omega(s,t))\omega(r,st) = \omega(r,s)\omega(rs,t)$ for all $r,s,t \in G$.

If G is discrete, then we call (G,A,α,ω) a *discrete twisted C^*-dynamical system.*

For the rest of this paper, $\mathscr{A} = (G,A,\alpha,\omega)$ is a twisted C^*-dynamical system.

Remark 2.1. Our definition of a twisted C^*-dynamical system differs from that in [1], which merely assumes that $\alpha : G \to \mathrm{Aut}(A)$ is strongly Borel-measurable and $\omega : G \times G \to \mathcal{U}(M(A))$ is strictly Borel-measurable. Such generality is not needed in our setting because we are only interested in continuous maps.

Definition 2.2 ([1])**.** Define a convolution $\star_{\mathscr{A}}$ and an involution $^{*_{\mathscr{A}}}$ on $C_c(G,A)$ by

$$\forall f,g \in C_c(G,A):$$

$$f \star_{\mathscr{A}} g \stackrel{\mathrm{df}}{=} \left\{ \begin{matrix} G & \to & A \\ x & \mapsto & \displaystyle\int_G f(y)\alpha_y\big(g(y^{-1}x)\big)\omega\big(y,y^{-1}x\big) \ \mathrm{d}\mu(y) \end{matrix} \right\};$$

$$f^{*_{\mathscr{A}}} \stackrel{\mathrm{df}}{=} \left\{ \begin{matrix} G & \to & A \\ x & \mapsto & \Delta_G\big(x^{-1}\big) \cdot \omega\big(x,x^{-1}\big)^* \alpha_x\Big(f\big(x^{-1}\big)^*\Big) \end{matrix} \right\}.$$

Note: $(C_c(G,A), \star_{\mathscr{A}}, ^{*_{\mathscr{A}}})$ is thus a $*$-algebra.

Definition 2.3. A *$*$-representation* for $\mathscr{A}$ is a pair (C,π), where C is a C^*-algebra and π is an algebraic $*$-homomorphism from $(C_c(G,A), \star_{\mathscr{A}}, ^{*_{\mathscr{A}}})$ to C.

3. The main result

For the rest of this paper, $\mathscr{A}$ is a discrete twisted C^*-dynamical system. The goal of this section is to establish the main result, stated as follows.

Theorem 3.1. *A $*$-representation (C,π) for $\mathscr{A}$ is automatically contractive with respect to $\|\cdot\|_{\mathscr{A},1}$ and $\|\cdot\|_C$.*

By Haar's Theorem, the only Haar measures on G are positive scalar multiples of the counting measure c. For $k \in \mathbb{R}_{>0}$, the measure $k \cdot \mathsf{c}$ gives rise to the following rules for convolution and involution via Definition 2.2:

$$\forall f,g \in C_c(G,A):$$

$$f \star_{\mathscr{A}} g \stackrel{\text{df}}{=} \begin{Bmatrix} G & \to & A \\ x & \mapsto & k \cdot \sum_{y \in G} f(y) \alpha_y\big(g\big(y^{-1}x\big)\big) \omega\big(y, y^{-1}x\big) \end{Bmatrix};$$

$$f^{*_{\mathscr{A}}} \stackrel{\text{df}}{=} \begin{Bmatrix} G & \to & A \\ x & \mapsto & \omega\big(x, x^{-1}\big)^* \alpha_x\Big(f\big(x^{-1}\big)^*\Big) \end{Bmatrix}.$$

Note that because G is discrete, it is unimodular, i.e., $\Delta_G \equiv 1$.

It can be easily shown that different $(C_c(G,A), \star_{\mathscr{A}}, {}^{*_{\mathscr{A}}})$, equipped with different Haar measures on G, are all $*$-isomorphic. We thus only need to prove Theorem 3.1 for the case $k = 1$, i.e., for the counting measure c, which we henceforth assume.

Before proving Theorem 3.1, we require a definition and a lemma.

Definition 3.1. For each $a \in A$ and $r \in G$, define the function $a \bullet \delta_r$ in $C_c(G,A)$ by

$$\forall x \in G: \qquad (a \bullet \delta_r)(x) \stackrel{\text{df}}{=} \begin{cases} a & \text{if } x = r; \\ 0_A & \text{if } x \in G \setminus \{r\}. \end{cases}$$

Lemma 3.1. *The following identities hold:*

(1) $(a \bullet \delta_r) \star_{\mathscr{A}} (b \bullet \delta_s) = a \alpha_r(b) \omega(r,s) \bullet \delta_{rs}$ *for all* $a, b \in A$ *and* $r, s \in G$.
(2) $(a \bullet \delta_r)^{*_{\mathscr{A}}} = \omega\big(r^{-1}, r\big)^* \alpha_{r^{-1}}(a^*) \bullet \delta_{r^{-1}}$ *for all* $a \in A$ *and* $r \in G$.
(3) $(a \bullet \delta_e) \star_{\mathscr{A}} (b \bullet \delta_e) = ab \bullet \delta_e$ *for all* $a, b \in A$.
(4) $(a \bullet \delta_e)^{*_{\mathscr{A}}} = a^* \bullet \delta_e$ *for all* $a \in A$.

Proof. It suffices to prove (1) and (2), because (3) and (4) are simply direct consequences.[1]

Let $a, b \in A$ and $r, s \in G$. Then we have for all $x \in G$ that

$$\begin{aligned}
[(a \bullet \delta_r) \star_{\mathscr{A}} (b \bullet \delta_s)](x) &= \sum_{y \in G} (a \bullet \delta_r)(y) \alpha_y\big((b \bullet \delta_s)\big(y^{-1}x\big)\big) \omega\big(y, y^{-1}x\big) \\
&= a \alpha_r\big((b \bullet \delta_s)\big(r^{-1}x\big)\big) \omega\big(r, r^{-1}x\big) \\
&= \begin{cases} a \alpha_r(b) \omega(r,s) & \text{if } x = rs; \\ 0_A & \text{if } x \in G \setminus \{rs\} \end{cases} \\
&= [a \alpha_r(b) \omega(r,s) \bullet \delta_{rs}](x); \\
(a \bullet \delta_r)^{*_{\mathscr{A}}}(x) &= \omega\big(x, x^{-1}\big)^* \alpha_x\Big((a \bullet \delta_r)\big(x^{-1}\big)^*\Big) \\
&= \begin{cases} \omega\big(r^{-1}, r\big)^* \alpha_{r^{-1}}(a^*) & \text{if } x = r^{-1}; \\ 0_A & \text{if } x \in G \setminus \big\{r^{-1}\big\} \end{cases} \\
&= \Big[\omega\big(r^{-1}, r\big)^* \alpha_{r^{-1}}(a^*) \bullet \delta_{r^{-1}}\Big](x).
\end{aligned}$$

This completes the proof. □

[1]To prove (3) and (4), we require the normalizations in Property (5) of Definition 2.1.

Proof of Theorem 3.1. We will omit $\mathscr{A}$ as a subscript to ease notation.

Let $a \in A$ and $r \in G$. By Lemma 3.1, we have

$$\begin{aligned}(a \bullet \delta_r)^* \star (a \bullet \delta_r) &= \left[\omega\left(r^{-1}, r\right)^* \alpha_{r^{-1}}(a^*) \bullet \delta_{r^{-1}}\right] \star (a \bullet \delta_r) \\ &= \omega\left(r^{-1}, r\right)^* \alpha_{r^{-1}}(a^*) \alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right) \bullet \delta_{r^{-1}r} \\ &= \omega\left(r^{-1}, r\right)^* \alpha_{r^{-1}}(a^*) \alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right) \bullet \delta_e \\ &= \left[\omega\left(r^{-1}, r\right)^* \alpha_{r^{-1}}(a^*) \bullet \delta_e\right] \star \left[\alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right) \bullet \delta_e\right] \\ &= \left[\omega\left(r^{-1}, r\right)^* \alpha_{r^{-1}}(a)^* \bullet \delta_e\right] \star \left[\alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right) \bullet \delta_e\right] \\ &= \left(\left[\alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right)\right]^* \bullet \delta_e\right) \star \left[\alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right) \bullet \delta_e\right] \\ &= \left[\alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right) \bullet \delta_e\right]^* \star \left[\alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right) \bullet \delta_e\right].\end{aligned}$$

Applying π to both ends and then using the C^*-norm identity yields

$$\|\pi(a \bullet \delta_r)\|_C = \left\|\pi\left(\alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right) \bullet \delta_e\right)\right\|_C. \tag{3.1}$$

By Lemma 3.1 again, we have

$$\begin{aligned}&\left[\alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right) \bullet \delta_e\right] \star \left[\alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right) \bullet \delta_e\right]^* \\ &= \left[\alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right) \bullet \delta_e\right] \star \left(\left[\alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right)\right]^* \bullet \delta_e\right) \\ &= \left[\alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right) \bullet \delta_e\right] \star \left[\omega\left(r^{-1}, r\right)^* \alpha_{r^{-1}}(a)^* \bullet \delta_e\right] \\ &= \alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right) \omega\left(r^{-1}, r\right)^* \alpha_{r^{-1}}(a)^* \bullet \delta_e \\ &= \alpha_{r^{-1}}(a) \alpha_{r^{-1}}(a)^* \bullet \delta_e \qquad \left(\text{As } \omega\left(r^{-1}, r\right) \text{ is unitary.}\right) \\ &= [\alpha_{r^{-1}}(a) \bullet \delta_e] \star [\alpha_{r^{-1}}(a)^* \bullet \delta_e] \\ &= [\alpha_{r^{-1}}(a) \bullet \delta_e] \star [\alpha_{r^{-1}}(a) \bullet \delta_e]^*.\end{aligned}$$

Applying π to both ends and then using the C^*-norm identity again yields

$$\left\|\pi\left(\alpha_{r^{-1}}(a) \omega\left(r^{-1}, r\right) \bullet \delta_e\right)\right\|_C = \|\pi(\alpha_{r^{-1}}(a) \bullet \delta_e)\|_C. \tag{3.2}$$

As $a \in A$ and $r \in G$ are arbitrary, we see from (3.1) and (3.2) that

$$\forall a \in A, \ \forall r \in G: \quad \|\pi(a \bullet \delta_r)\|_C = \|\pi(\alpha_{r^{-1}}(a) \bullet \delta_e)\|_C. \tag{3.3}$$

Now, define for each $r \in G$ a linear map $\rho_r : A \to C_c(G, A)$ by

$$\forall a \in A: \quad \rho_r(a) \stackrel{\text{df}}{=} \alpha_{r^{-1}}(a) \bullet \delta_e.$$

Notice that Lemma 3.1 also gives us the following relations:

$$\begin{aligned}\forall a, b \in A, \ \forall r \in G: \quad \rho_r(ab) &= \alpha_{r^{-1}}(ab) \bullet \delta_e \\ &= \alpha_{r^{-1}}(a) \alpha_{r^{-1}}(b) \bullet \delta_e \\ &= [\alpha_{r^{-1}}(a) \bullet \delta_e] \star [\alpha_{r^{-1}}(b) \bullet \delta_e] \\ &= \rho_r(a) \star \rho_r(b); \\ \rho_r(a^*) &= \alpha_{r^{-1}}(a^*) \bullet \delta_e\end{aligned}$$

$$= \alpha_{r^{-1}}(a)^* \bullet \delta_e$$
$$= \rho_r(a)^*.$$

Hence, ρ_r is an algebraic $*$-homomorphism from A to $(C_c(G,A),\star,^*)$ for all $r \in G$, which makes $\pi \circ \rho_r$ an algebraic $*$-homomorphism from A to C. As any algebraic $*$-homomorphism from one C^*-algebra to another is already contractive, we have

$$\forall a \in A,\ \forall r \in G: \quad \|\pi(\alpha_{r^{-1}}(a) \bullet \delta_e)\|_C = \|(\pi \circ \rho_r)(a)\|_C \leq \|a\|_A. \tag{3.4}$$

Combining (3.3) and (3.4) thus gives us

$$\forall a \in A,\ \forall r \in G: \quad \|\pi(a \bullet \delta_r)\|_C \leq \|a\|_A. \tag{3.5}$$

Finally, let $f \in C_c(G,A)$. Then

$$\begin{aligned}
\|\pi(f)\|_C &= \left\|\pi\left(\sum_{r \in G} f(r) \bullet \delta_r\right)\right\|_C \\
&= \left\|\sum_{r \in G} \pi(f(r) \bullet \delta_r)\right\|_C \\
&\leq \sum_{r \in G} \|\pi(f(r) \bullet \delta_r)\|_C \\
&\leq \sum_{r \in G} \|f(r)\|_A \qquad \text{(By (3.5).)} \\
&= \|f\|_1.
\end{aligned}$$

Therefore, π is automatically contractive with respect to $\|\cdot\|_1$ and $\|\cdot\|_C$. □

Let us now describe an important corollary of Theorem 3.1. Recall that a *covariant representation* of $\mathscr{A}$ is defined as a triple (ρ, u, C) with the following properties:

(1) C is a C^*-algebra.
(2) ρ is a non-degenerate $*$-homomorphism from A to $M(C)$.
(3) u is a function from G to $\mathcal{U}(M(C))$.
(4) $\rho(\alpha_r(a)) = u(r)\rho(a)u(r)^*$ for all $r \in G$ and $a \in A$.
(5) $u(r)u(s) = \overline{\rho}(\omega(r,s))u(rs)$ for all $r, s \in G$.

These properties allow us to define an associated $*$-homomorphism $\Pi_{\rho,u,C}$ from $(C_c(G,A),\star_{\mathscr{A}},^{*_{\mathscr{A}}})$ to $M(C)$ by

$$\forall f \in C_c(G,A): \quad \Pi_{\rho,u,C}(f) \stackrel{\text{df}}{=} \sum_{r \in G} \rho(f(r))u(r).$$

It is well-known that $(M(C), \Pi_{\rho,u,C})$ is a non-degenerate $*$-representation for $\mathscr{A}$.

Conversely, every non-degenerate $*$-representation for $\mathscr{A}$ having the form $(M(C), \pi)$ for some C^*-algebra C (called a *multiplier $*$-representation* for $\mathscr{A}$), that is assumed to be bounded with respect to $\|\cdot\|_{\mathscr{A},1}$ and $\|\cdot\|_{M(C)}$, arises from a covariant representation of $\mathscr{A}$ in the manner above. Theorem 3.1 says that the boundedness assumption is unnecessary, so we obtain the following algebraic result.

Corollary 3.1. *There is a one-to-one correspondence between the class of covariant representations for $\mathscr{A}$ and the class of multiplier $*$-representations for $\mathscr{A}$.*

Covariant representations play a significant role in the theory of twisted crossed products (see [2]), which suggests the usefulness of Corollary 3.1.

Remark 3.1. If π is merely an algebraic homomorphism that does not respect the involution $^{*_{\mathscr{A}}}$, then continuity may fail spectacularly. Consider $\mathscr{A} = \left(\mathbb{Z}, \mathbb{C}, \alpha^0, \omega^0\right)$, where α^0 and ω^0 denote, respectively, the trivial action and the trivial multiplier. Suppose that $\mathbb{Z}$ is equipped with the counting measure. Then the map

$$\begin{Bmatrix} C_c(\mathbb{Z}) & \to & \mathbb{C} \\ f & \mapsto & \displaystyle\sum_{n \in \mathbb{Z}} f(n) e^n \end{Bmatrix}$$

is an unbounded algebraic homomorphism from $(C_c(\mathbb{Z}), \star_{\mathscr{A}})$ to $\mathbb{C}$, because $\|\delta_n\|_{\mathscr{A},1} = 1$ for all $n \in \mathbb{Z}$ but $\lim\limits_{n \to \infty} e^n = \infty$. It clearly *does not* respect the involution $^{*_{\mathscr{A}}}$.

4. Conclusions

The proof of the main result does not apply to other classes of locally compact Hausdorff groups, such as the abelian ones or the compact ones. One might work first on group C^*-algebras instead of more general twisted C^*-dynamical systems. Hopefully, the Peter-Weyl Theorem for compact groups and the Fourier transform for abelian groups could find a use, as they exploit the structure of these groups.

Acknowledgments

The author thanks Judith Packer and Dana Williams for their extremely helpful comments on this topic. The deepest appreciation, however, goes to the anonymous referee, who offered valuable suggestions on how to improve the exposition in this paper.

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This paper is available via http://nyjm.albany.edu/j/2018/24-51.html.