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Lacunary discrete spherical maximal functions

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Abstract. We prove new $\ell^p(\mathbb{Z}^d)$ bounds for discrete spherical averages in dimensions $d \geq 5$. We focus on the case of lacunary radii, first for general lacunary radii, and then for certain kinds of highly composite choices of radii. In particular, if $A_\lambda f$ is the spherical average of f over the discrete sphere of radius λ, we have

$$\bigl\lVert \sup_k |A_{\lambda_k} f| \bigr\rVert_{\ell^p(\mathbb{Z}^d)} \lesssim \lVert f \rVert_{\ell^p(\mathbb{Z}^d)}, \qquad \tfrac{d-2}{d-3} < p \leq \tfrac{d}{d-2},\ d \geq 5,$$

for any lacunary sets of integers $\{\lambda_k^2\}$. We follow a style of argument from our prior paper, addressing the full supremum. The relevant maximal operator is decomposed into several parts; each part requires only one endpoint estimate.

Contents

1. Introduction

We prove ℓ^p bounds for discrete spherical maximal operators, concentrating on variants of the lacunary versions of these operators. They have a surprising intricacy. For $\lambda^2 \in \mathbb{N}$, let s_λ be the cardinality of the number of $n \in \mathbb{Z}^d$ such that $|n|^2 = \lambda^2$. Define the spherical average of a function f on $\mathbb{Z}^d$ to be

$$A_\lambda f(x) = s_\lambda^{-1} \sum_{n \in \mathbb{Z}^d \,:\, |n|^2 = \lambda} f(x-n)$$

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We will always work in dimension $d \geq 5$, so that for any choice of $\lambda^2 \in \mathbb{N}$, one has $s_\lambda \simeq \lambda^{d-2}$. Define the maximal function $A_* f = \sup_\lambda A_\lambda f$, where f is non-negative and the supremum is over all λ for which the operator is defined. This operator was introduced by Magyar [15], and the ℓ^p bounds were proved by Magyar, Stein and Wainger [16]. Namely, this is a bounded operator on ℓ^p for $p > \frac{d}{d-2}$.

We address the discrete lacunary spherical maximal function. We say that a set of integers $\{\lambda_k^2 \,:\, k \geq 1\}$ is *lacunary* if $\lambda_{k+1}^2 \geq 2\lambda_k^2$ for all $k \in \mathbb{N}$. Let $A_{\mathrm{lac}} = \sup_{k \in \mathbb{Z}} A_{\lambda_k} f$. We will see that the choice of the λ_k have a strong impact on the results.

Theorem 1.1. *For $d \geq 5$, let $\{\lambda_k^2\}$ be any lacunary sequence of integers. The maximal operator A_{lac} maps $\ell^p(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d)$ for $p > \frac{d-2}{d-3}$.*

Our bound $\frac{d-2}{d-3}$ is smaller than the index $\frac{d}{d-2}$, for which the full supremum $A_* f$ is bounded [16]. Kevin Hughes [7] proved a version of the result above, for a very particular sequence of radii, and in dimension $d = 4$. In contrast to the continuous case, no such inequalities can hold close to ℓ^1. An example of Zienkiewicz [20] show that there are lacunary radii $\{\lambda_k\}$ for which the corresponding maximal operator A_{lac} is unbounded on ℓ^p, for $1 < p < \frac{d}{d-1}$. It is an interesting question to determine the best $p = p(d)$ for which any lacunary maximal function A_{lac} would be bounded on $\ell^p(\mathbb{Z}^d)$.

The Theorem above concerns classical type examples of radii. Brian Cook [5] has shown that for highly composite radii $\lambda_k^2 = 2^k!$, that the maximal function $\sup_k A_{\lambda_k} f$ is bounded on ℓ^p, for all $1 < p < \infty$. The Theorem below shows that this continues to hold for e.g. $\lambda_k^2 = [k^{\log\log k}]!$.

Theorem 1.2. *For $d \geq 5$, let μ_k be an increasing sequence of integers for which*

$$\lim_k \frac{\log \mu_k}{\log k} = \infty. \tag{1.1}$$

Then, for $\lambda_k^2 = \mu_k!$, the maximal function $\sup_k A_{\lambda_k} f$ maps $\ell^p(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d)$ for $1 < p < \infty$.

Our method of proof is inspired by a method of Bourgain [1], and its application to the discrete setting by Ionescu [8]. We used it for the full discrete spherical maximal operator of Magyar, Stein and Wainger in [10]. In particular, we proved an endpoint sparse bound in that setting.

These arguments are relatively easy. The maximal operators are treated as maximal multipliers. Each component of the decomposition of the multiplier needs only one estimate, either an ℓ^2 estimate, or an ℓ^1 estimate. As such, the argument can be used to simplify existing results, and simplify the search for new ones. We illustrate these ideas in a simple context in §2. The discrete lacunary theorem is proved in §3, and the highly composite case in § 4.

2. The continuous lacunary case

To illustrate the proof technique, we prove the classical results on the lacunary spherical averages on Euclidean spaces. Let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$, and let σ be the rotationally invariant probability measure on $\mathbb{S}^{d-1}$. Let

$$\mathcal{A}_\lambda f(x) = \int_{\mathbb{S}} f(x-y)\, d\sigma(y).$$

The key property of these averages that we will rely upon is the stationary decay estimate

$$|\widetilde{d\sigma}(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}}, \tag{2.1}$$

where the tilde represents the Fourier transform. We begin with this proposition.

Proposition 2.1. *For* $f = \mathbf{1}_F$ *and* $g = \mathbf{1}_G$ *supported on the unit cube in* $\mathbb{R}^d$*, there holds*

$$\langle \mathcal{A}_1 \mathbf{1}_F, \mathbf{1}_G \rangle \lesssim (|F| \cdot |G|)^{\frac{d}{d+1}}, \qquad F, G \subset [0,1]^d.$$

The inequality above is just a little weaker than the classical result of Littman [14] and Strichartz [19], that locally $\mathcal{A}_1$ maps $L^{\frac{d+1}{d}}$ into L^{d+1}. That inequality requires a sophisticated analytic interpolation argument.

Proof. The proof proceeds by this supplementary procedure. For integers N, we estimate $\mathcal{A}_1 f \le M_1 + M_2$, where

$$\|M_1\|_\infty \le N|F|, \qquad \|M_2\|_2 \le N^{-\frac{d-1}{2}} |F|^{1/2}. \tag{2.2}$$

With this established, we have

$$\langle \mathcal{A}_1 \mathbf{1}_F, \mathbf{1}_G \rangle \le N|F| \cdot |G| + N^{-\frac{d-1}{2}} \big[|F| \cdot |G| \big]^{1/2}$$

Optimizing the right hand side over N proves the proposition. We omit the details.

It remains to construct M_1 and M_2. Let φ be a non-negative Schwartz function, with integral one, and compact spatial support. Likewise, set $\varphi_t(x) = t^{-d}\varphi(x/t)$. Then, $M_1 = \varphi_{1/N} * \mathcal{A}_1 f$. This is convolution of f against a uniform probability measure supported on an annulus around the unit sphere of width $1/N$. So it is clear that M_1 satisfies the first estimate in (2.2), and the second estimate (2.2) for M_2 follows from (2.1). (This proof is known to experts in the subject.) □

The next argument addresses the lacunary spherical maximal function.

Theorem 2.2. *Let* $\{\lambda_k\} \subset (0,\infty)$ *be a lacunary sequence of reals. Then, there holds*

$$\|\sup_k \mathcal{A}_{\lambda_k} f\|_p \lesssim \|f\|_p, \qquad 1 < p < \infty. \tag{2.3}$$

Proof. The inequality in (2.3) is elementary for $p = 2$. And we take it for granted, while noting that a certain quantification of this familiar argument will appear below. It remains to prove the inequality for $1 < p < 2$. We aim to prove the restricted weak type estimate

$$\big\langle \sup_k \mathcal{A}_{\lambda_k} f, g \big\rangle \lesssim |F|^{1/p} |G|^{1/p'}, \tag{2.4}$$

where $f = \mathbf{1}_F$ and $g = \mathbf{1}_G$. Note that the L^2 inequality implies this for $|G| \le |F|$. So we assume the converse below.

We set up a supplementary objective. For sets $F \subset \mathbb{R}^d$ of finite measure, choices of $1 < p < 2$, and all integers N, we can write $\sup_k \mathcal{A}_{\lambda_k} f \le M_1 + M_2$,

$$\|M_1\| \lesssim (\log N) |F|^{1/p}, \tag{2.5}$$

$$\|M_2\|_2 \lesssim N^{-\frac{d-1}{2}} |F|^{1/2}. \tag{2.6}$$

We have

$$\begin{aligned} \big\langle \sup_k \mathcal{A}_{\lambda_k} f, g \big\rangle &\lesssim \langle M_1, \mathbf{1}_G \rangle + \langle M_2, \mathbf{1}_G \rangle \\ &\lesssim (\log N) |F|^{1/p} |G|^{(p-1)/p} + N^{-\frac{d-1}{2}} |F|^{1/2} |G|^{1/2}. \end{aligned}$$

Recalling that $|G| > |F|$, we can optimize this over N, and then let p tend to one to complete the proof of (2.4). We omit the details, except to say that the restriction to indicators is very useful at this point.

We turn to the construction of M_1 and M_2. Using the same notation is in the proof of Proposition 2.1, set

$$M_1 = \sup_k \varphi_{\lambda_k / N} * \mathcal{A}_{\lambda_k} f.$$

This defines M_2 implicitly. The stationary decay estimate (2.1) and a standard square function argument combine in a familiar way to prove (2.6).

$$\begin{aligned} \|M_2\|_2^2 &\lesssim \sum_k \|\varphi_{\lambda_k/N} * \mathcal{A}_{\lambda_k} f - \mathcal{A}_{\lambda_k} f\|_2^2 \\ &\lesssim \|f\|_2^2 \sup_\xi \sum_k |\widetilde{\varphi}(\lambda_k \xi) - 1|^2 \cdot |\widetilde{d\sigma}(\xi)|^2 \\ &\lesssim N^{1-d} |F|. \end{aligned}$$

Note that this argument is a certain quantification of the standard square function proof of the boundedness of the lacunary spherical maximal operator on L^2.

For (2.5), namely the control of M_1, we show that the maximal function $B_N f = \sup_k \varphi_{\lambda_k/N} * \mathcal{A}_{\lambda_k} f$ satisfies a strong type L^p bound smaller than $\log N$.

Now, it is clear that B_N is a bounded operator on L^2. One can approach the L^p bounds for $1 < p < 2$ directly, using a bit of Calderón-Zygmund theory. We use duality, however. This requires that we linearize the maximal operator $B_N f$, which is done as follows.

For any collection of pairwise disjoint subsets of $\mathbb{R}^d$ denoted by $\{S_k \,:\, k \in \mathbb{Z}\}$, we can form the linear operator

$$Tf = \sum_k \mathbf{1}_{S_k} \varphi_{\lambda_k/N} * \mathcal{A}_{\lambda_k} f.$$

This is bounded on L^2, independently of the selection of the sets S_k. We show that T^* maps L^∞ into BMO with norm at most $\log N$. By interpolation and duality, we see that (2.5) holds.

To verify our BMO claim we need to show this: For $\phi \in L^\infty$, and cube Q, there is a constant μ so that

$$\int_Q |T^*\phi - \mu|^2 \lesssim (\log N)^2 \|\phi\|_\infty^2 |Q|. \tag{2.7}$$

Split T^* into three parts, T_0^*, T_1^*, T_2^*, where

$$T_0^*\phi = \sum_{k \,:\, \lambda_k < \ell Q} \varphi_{\lambda_k/N} * \mathcal{A}_{\lambda_k}(\mathbf{1}_{S_k}\phi),$$
$$T_2^*\phi = \sum_{k \,:\, \ell Q < \lambda_k/N} \varphi_{\lambda_k/N} * \mathcal{A}_{\lambda_k}(\mathbf{1}_{S_k}\phi),$$

This defines T_1^* implicitly. Define $\mu = T_2^*\phi(x_Q)$, where x_Q is the center of Q. Straight forward kernel estimates and lacunarity of λ_k show that

$$\sup_{x\in Q} |T_2^*\phi(x) - \mu| \lesssim \|\phi\|_\infty.$$

For T_0^*, we have the L^2 bound for T^* which implies

$$\int_Q |T_0^*\phi|^2 \, dx = \int_Q |T_0^*(\phi \mathbf{1}_{2Q})|^2 \, dx \lesssim \|\phi\|_\infty^2 |Q|.$$

That leaves T_1^*, but it is the sum of at most $\log N$ functions each bounded by $\|\phi\|_\infty$. Thus, (2.7) follows.

□

We make these additional remarks on this method of proof used in this paper.

1. The fine analysis of the L^1 endpoint of the continuous lacunary spherical maximal function is still an open question [18, 4]. It would be interesting to know if this technique can simplify those arguments.
2. For the local maximal operator $\sup_{1\le\lambda\le 2} \mathcal{A}_\lambda f$, considered by Schlag [17], there is an elegant proof of the L^p improving estimates along these lines of this section, given by Sanghyuk Lee [13]. The latter argument can be modified in an interesting way to prove sparse variants for the Stein maximal operator, giving certain improvements over the sparse bounds of [11].
3. Likewise, the ℓ^1 endpoint cases are of interest in the discrete case. Can one show that for the maximal functions M in Theorem 1.2, that they map $\ell \log \ell$ into weak ℓ^1?

(4) The two proofs can be combined to prove a restricted weak type sparse bound for the lacunary spherical maximal function at the point $(\frac{d+1}{d}, \frac{d+1}{d})$. This is an interesting extension of the sparse bounds proved in [11]. We leave the details to the reader.

(5) The main results of [10] prove sparse bounds for the Magyar Stein Wainger discrete spherical maximal function. Those inequalities can be combined with Theorem 1.1 and Theorem 1.2 to give novel sparse bounds for these operators. These in turn imply novel weighted inequalities, which we leave to the interested reader. However, in the special case of Theorem 1.1, one can prove additional sparse bounds. We do not purse these details here.

We thank the referee for encouraging us to include this section in the paper.

3. General lacunary sequences

The key Lemma is the restricted type estimate below.

Lemma 3.1. *Let* λ_k^2 *be a lacunary set of integers. For a finitely supported function* $f = \mathbf{1}_F$*, and function* $\tau : \mathbb{Z}^d \to \{\lambda_k\}$*, there holds*

$$\|A_\tau f\|_p \lesssim |F|^{1/p}, \qquad \tfrac{d-2}{d-3} < p < 2. \tag{3.1}$$

We will use the stopping time τ to simplify notation throughout. We turn to the proof. It suffices to show that for all integers N, we can decompose $A_\tau f \le M_1 + M_2$ with

$$\|M_1\|_{1+\epsilon} \lesssim N\|f\|_{1+\epsilon}, \qquad \|M_2\|_2 \lesssim N^{-\frac{4-d}{2}}\|f\|_2. \tag{3.2}$$

Above, implied constants depend upon $0 < \epsilon < 1$, but we do not make this explicit here, nor at any point of the paper. Optimizing over N proves (3.1).

Both M_1 and M_2 have several parts. The first part of M_1 is $M_{1,1} = \mathbf{1}_{\tau \le N} A_{\lambda_k} f$. It trivially satisfies the first half of (3.2).

Recall the decomposition of $A_\lambda f$ from Magyar, Stein and Wainger [16]. We have the decomposition below, in which upper case letters denote a convolution operator, and lower case letters denote the corresponding multiplier. Let $e(x) = e^{2\pi i x}$ and for integers q, $e_q(x) = e(x/q)$.

$$A_\lambda f = C_\lambda f + E_\lambda f, \tag{3.3}$$

$$C_\lambda f = \sum_{1 \le \lambda \le q} \sum_{a \in \mathbb{Z}_q^\times} e_q(-\lambda^2 a) C_\lambda^{a/q} f,$$

$$c_\lambda^{a/q}(\xi) = \widehat{C_\lambda^{a/q}}(\xi) = \sum_{\ell \in \mathbb{Z}_q^d} G(a, \ell, q)\widetilde{\psi}_q(\xi - \ell/q)\widetilde{d\sigma_\lambda}(\xi - \ell/q) \tag{3.4}$$

$$G(a, \ell, q) = q^{-d} \sum_{n \in \mathbb{Z}_q^d} e_q(|n|^2 a + n \cdot \ell).$$

The term $G(a,\ell,q)$ is a normalized Gauss sum. Above, a is in the multiplicative group $\mathbb{Z}_q^\times$. Recall that

$$|G(a,\ell,q)| \lesssim q^{-d/2}, \qquad \gcd(a,\ell,q)=1. \tag{3.5}$$

In (3.4), the hat indicates the Fourier transform on $\mathbb{Z}^d$, and the notation identifies the operator $C_\lambda^{a/q}$, and the kernel. All our operators are convolution operators or maximal operators formed from the same. The function ψ is a radial Schwartz function on $\mathbb{R}^d$ which satisfies

$$\mathbf{1}_{|\xi|\le 1/2} \le \widetilde{\psi}(\xi) \le \mathbf{1}_{|\xi|\le 1}. \tag{3.6}$$

The function $\widetilde{\psi}_q(\xi)=\widetilde{\psi}(q\xi)$. The uniform measure on the sphere of radius λ is denoted by $d\sigma_\lambda$ and $\widetilde{d\sigma_\lambda}$ denotes its Fourier transform computed on $\mathbb{R}^d$. The standard stationary phase estimate is

$$|\widetilde{d\sigma_1}(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}}. \tag{3.7}$$

We have this estimate, stronger than what we need, from [16, Prop. 4.1]: For all $\Lambda \ge 1$,

$$\Big\| \sup_{\Lambda\le\lambda\le 2\Lambda} |E_\lambda \cdot| \Big\|_{2\to 2} \lesssim \Lambda^{\frac{4-d}{2}}. \tag{3.8}$$

Our first contribution to M_2 is $M_{2,1}=|E_\tau f|$. This clearly satisfies the second half of (3.2).

It remains to bound $C_\tau f$, requiring further contributions to M_1 and M_2. Recall the estimate below, which is a result of Magyar, Stein and Wainger [16, Prop. 3.1].

$$\Big\|\sup_{\lambda>q} |C_\lambda^{a/q} f|\Big\|_2 \lesssim q^{-\frac{d}{2}} \|f\|_2.$$

It follows that

$$\sum_{q>N} \sum_{a\in\mathbb{Z}_q^\times} \|C_\tau^{a/q} f\|_2 \lesssim N^{-\frac{d-4}{2}} \|f\|_2. \tag{3.9}$$

Our second contribution to M_2 is therefore

$$M_{2,2} = \sum_{N<q\le\lambda} \sum_{a\in\mathbb{Z}_q^\times} |C_\tau^{a/q} f|.$$

We are left with the term below, which will be controlled with further contributions to M_1 and M_2.

$$\sum_{1\le q\le N} \sum_{a\in\mathbb{Z}_q^\times} C_\tau^{a/q} f$$

Decompose $C_\lambda^{a/q} = C_{\lambda,1}^{a/q} + C_{\lambda,2}^{a/q}$ where we modify the definition of $c_\lambda^{a/q}$ in (3.4) as follows.

$$c_{\lambda,1}^{a/q}(\xi) = \sum_{\ell\in\mathbb{Z}^d} G(a,\ell,q) \widetilde{\psi}_{\lambda/N}(\xi-\ell/q) \widetilde{d\sigma_\lambda}(\xi-\ell/q).$$

The last contribution to M_2 is

$$M_{2,2} = \Big| \sum_{1\le q\le N} C^{a/q}_{\tau,2} f \Big|.$$

When considering $C^{a/q}_{\tau,2}$, the difference $\widetilde{\psi}_q(\xi) - \widetilde{\psi}_{\lambda/N}(\xi)$ arises. But this is zero if $|\xi| < N/2\lambda$. Using the Gauss sum estimate (3.5) and the stationary decay estimate (3.7), we have

$$\begin{aligned} \|M_{2,2}\|_2^2 &\le \sum_{k>N} \Big\| \sum_{1\le q\le N} \sum_{a\in\mathbb{Z}_q^\times} C^{a/q}_{\lambda_k,2} f \Big\|_2^2 \\ &\le N \sum_{1\le q\le N} \sum_{k>N} \sum_{a\in\mathbb{Z}_q^\times} q \| C^{a/q}_{\lambda_k,2} f\|_2^2 \\ &\le N^{2-d} \sum_{1\le q\le N} q^{2-d} \lesssim N^{2-d}. \end{aligned}$$

This is smaller than required.

The principle point is the control of

$$M_{1,2,\tau} f = \sum_{1\le q\le N} \sum_{a\in\mathbb{Z}_q^\times} C^{a/q}_{\tau,1} f,$$

and here we adopt our notation for operators. In particular, we examine the kernel for the convolution operator $M_{1,2,\lambda}$. By a well known computation, (See [8, pg. 1415], [7, (42)], or the detailed argument in [12, Lemma 2.13].)

$$M_{1,2,\lambda}(n) = K_\lambda(n) \cdot \mathsf{C}_N(\lambda^2 - |n|^2), \tag{3.10}$$

$$\text{where} \quad K_\lambda(n) = \psi_{\lambda/N} * d\sigma_\lambda(n), \tag{3.11}$$

$$\text{and} \quad \mathsf{C}_N(n) = \sum_{1\le q\le N} \mathsf{c}_q(n) = \sum_{1\le n\le N} \sum_{a\in\mathbb{Z}_q^\times} e_q(am).$$

The terms c_q are Ramanujan sums, well-known for having more than square root cancellation. We need a further quantification of this fact. We find this result in a paper by Bourgain [2, (3.43), page 126] and will give a short proof for completeness. (Also see [9].) We remark that the main result of [3] gives a precise asymptotic for the expression below for $j = 2$. In particular, this result shows that the inequality below is sharp, up the ϵ dependence.

Lemma 3.2. *Given $\epsilon > 0$ and integer j, the inequality below holds for all integers $M > Q^j$.*

$$\Big[\frac{1}{M} \sum_{n\le M} \Big[\sum_{q\le Q} |\mathsf{c}_q(n)| \Big]^j \Big]^{1/j} \lesssim Q^{1+\epsilon}. \tag{3.12}$$

We postpone the proof of this fact to the end of this section. We also need

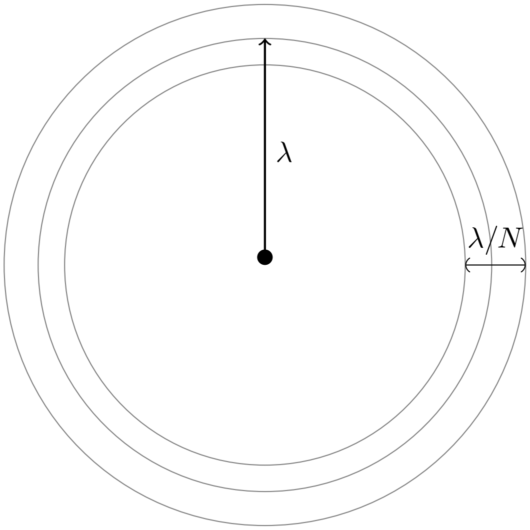

FIGURE 1. A sketch to indicate the estimates (3.13). The convolution $d\sigma_\lambda * \psi_{\lambda/N}$ is essentially supported in an annulus around a sphere of radius λ of width about λ/N.

Proposition 3.3. *For the kernel K_λ defined in* (3.11)*, we have this maximal inequality, valid for any lacunary choice of radii $\{\lambda_k\}$.*

$$\Big\| \sup_{k>N} K_{\lambda_k} * g \Big\|_p \lesssim \|g\|_p, \qquad 1 < p < 2. \tag{3.13}$$

Proof. This follows by comparison to lacunary averages on $\mathbb{R}^d$, which we can do since the inner and outer radii compare favorably, as indicated in Figure 1. Let us elaborate. Consider $1 \ll M \ll \lambda$, with $\lambda/M \gg 1$. The annulus $\mathrm{Ann}(M,\lambda) = \{x \in \mathbb{R}^d \ : \ |\|x\| - \lambda| < \lambda/M\}$. Then, the volume of the annulus is comparable to λ^d/M. And, the number of lattice points is,

$$\begin{aligned} \left|\mathbb{Z}^d \cap \mathrm{Ann}(M,\lambda)\right| &= \sum_{\mu^2 \in \mathbb{N} \,:\, \lambda(1-1/M) \le \mu \le \lambda(1+1/M)} |\{n \in \mathbb{Z}^d \ : \ |n| = \mu\}| \\ &\simeq \sum_{\mu^2 \in \mathbb{N} \,:\, \lambda(1-1/M) \le \mu \le \lambda(1+1/M)} \mu^{d-2} \simeq |\mathrm{Ann}(M,\lambda)|. \end{aligned}$$

The last equivalence holds as we are summing over approximately λ^2/M values of μ. In dimension $d \ge 5$, we always have a good estimate for the number of lattice points on a sphere.

□

Let us give the proof of $\|M_{1,2,\tau} f\|_{1+\epsilon} \lesssim N^{1+\epsilon}\|f\|_{1+\epsilon}$, as required for (3.2). We can estimate $M_{1,2,\tau} f$ from the kernel estimate (3.10). We use Hölder's inequality for a large even integer j, and fixed λ_k

$$\begin{aligned}\Big|\sum_{n\in\mathbb{Z}^d} & K_\lambda(n)\mathsf{C}_N(\lambda^2-|n|^2)f(x-n)\Big| \\ & \le \big[K_\lambda * |f|^{j'}(x)\big]^{1/j'} \times \Big[\sum_{n\in\mathbb{Z}^d} K_\lambda(n)|\mathsf{C}_N(\lambda^2-|n|^2)|^j\Big]^{1/j} \\ & := \Psi_{1,\lambda}f\cdot\Psi_{2,\lambda}. \end{aligned} \tag{3.14}$$

We pick $j\simeq 10/\epsilon$, and claim that

$$\big\|\sup_k \Psi_{1,\lambda_k}f\big\|_p \lesssim |F|^{1/p}, \qquad \sup_{k>N}\Psi_{2,\lambda_k}\lesssim N^{1+\epsilon}. \tag{3.15}$$

Indeed, we have $1<j'<p$. Therefore, we can use (3.13) to verify the first claim in (3.15).

Concerning the second term in (3.14), we turn to Lemma 3.2, and argue that

$$\sup_{k>N}\Psi_{2,\lambda_k}\lesssim N^{1+\epsilon}$$

from which (3.15) follows. Apply Lemma 3.2 with $Q=N$,

$$\Big[\frac{1}{M}\sum_{|n|\le M}|\mathsf{C}_N(n)|^j\Big]^{1/j}\lesssim N^{1+\epsilon}, \qquad M>M_0>N^{p'}.$$

The following extension holds: Let ζ be monotone smooth non-negative decreasing function, constant on $[0,M_0]$, with L^1 norm one. We then have

$$\Big[\sum_{n=0}^{\infty}|\mathsf{C}_N(n)|^j\zeta(n)\Big]^{1/j}\lesssim N^{1+\epsilon}. \tag{3.16}$$

This follows by a standard convexity argument, based on the identity

$$\zeta(x) = -\int_0^\infty \tfrac{1}{t}\mathbf{1}_{[0,t]}(x)\cdot t\zeta'(t)\,dt, \qquad x>0.$$

Recall that $k\ge N$, so that $\lambda_k>2^N>M_0$. And, we can write

$$\begin{aligned}\Psi_{2,\lambda_k}^j &\lesssim \sum_{r=0}^{\infty}|\mathsf{C}_N(\lambda^2-r)|^j r^{\frac{d-2}{2}}\psi_{\lambda_k/N^\beta}*d\sigma_{\lambda_k}(0,\dots,0,\sqrt{r}) \\ &= \sum_{r=0}^{\infty}|\mathsf{C}_N(\lambda^2-r)|^j\psi(\lambda_k,r).\end{aligned}$$

The inequality (3.16) shows that this last term is uniformly bounded by N^j, since $\beta=\frac{d-2}{d-1}<1$. To see this, consider first the case of $r\ge\lambda^2$. By inspection,

$$\sup_{|\lambda_k-|x||<\lambda_k/N^\beta}\psi_{\lambda_k/N^\beta}*d\sigma_{\lambda_k}(x)\lesssim\frac{N^\beta}{\lambda_k^d}.$$

The left-hand side is essentially constant on the annulus around the sphere of λ_k of width λ_k/N^β, and has total integral one. It follows that $\psi(\lambda_k, r)$ is essentially constant on the same region, and

$$\sup_{|\lambda_k-\sqrt{r}|<\lambda_k/N^\beta} \psi(\lambda_k, r) \lesssim \frac{N^\beta}{\lambda_k}.$$

And, $\int_0^\infty \psi(\lambda_k, r)\, dr \lesssim 1$ by construction. The case of $0 < r < \lambda^2$ is entirely similar.

Proof of Lemma 3.2. We will marshal four facts. First, $n \to \mathsf{c}_q(n)$ is q-periodic, and bounded by q. Moreover, we have the bound $|\mathsf{c}_q(n)| \le (q, n)$. To see this, recall that if q is a power of a prime p, we have

$$\mathsf{c}_{p^k}(n) = \begin{cases} 0 & p^{k-1} \nmid n \\ -p^{k-1} & p^{k-1} \mid n, p^k \nmid n \\ p^k(1-1/p) & p^k \mid n \end{cases}$$

We see that the conclusion holds in this case. The general case follows since $\mathsf{c}_q(n)$ is multiplicative in q.

Second, for $\vec{q} = (q_1, \dots, q_j) \in [1, Q]^k$, let $\mathcal{L}(\vec{q})$ be the least common multiple of $q_1, \dots, q_k$. Observe that $n \to \prod_{i=}^{j} \mathsf{c}_{q_j}(n)$ is periodic with period $\mathcal{L}(\vec{q})$. This, with the condition that $M > Q^j$, implies that

$$\frac{1}{M} \sum_{n \le M} \prod_{i=1}^{j} |\mathsf{c}_{q_j}(n)| \le \frac{2}{\mathcal{L}(\vec{q})} \sum_{n \le \mathcal{L}(\vec{q})} \prod_{i=1}^{j} (q_j, n). \tag{3.17}$$

Third, for all $\epsilon > 0$, uniformly in $\vec{q} \in [1, Q]^k$,

$$\sum_{n \le \mathcal{L}(\vec{q})} \prod_{i=1}^{k} (q_j, n) \lesssim Q^{k+\epsilon}. \tag{3.18}$$

To see this, begin with the case of $q = p^x$, for prime p and $x \ge 1$. For integers k,

$$\sum_{n \le p^x} (p^x, n)^k \lesssim p^{xk+\epsilon},$$

as is easy to check. We need an extension of this. Let $x_1, \dots, x_t$ be distinct integers, and let $k_1, \dots, k_t$ be integers. There holds

$$\sum_{n \le p^{x_1}} \prod_{s=1}^{t} (p^{x_s}, n)^k \lesssim p^{\sum_{s=1}^{t} x_s k_s + \epsilon}. \tag{3.19}$$

where above we assume that $x_1 > x_2 > \cdots > x_t$. As $n \to (p^{x_s}, n)^k$ is periodic with period p^{x_s}, one has

$$\sum_{n \le p^x} \prod_{s=1}^{t} (p^{x_s}, n)^k = \prod_{s=1}^{t} \sum_{n \le p^{x_s}} (p^{x_s}, n)^k$$

and the claim follows.

Turning to a vector $\vec{q}$, write the prime factorization of $\mathcal{L}(\vec{q}) = p_1^{x_1} \cdots p_t^{x_t}$. Write each $q_j = \prod_{s=1}^t p_s^{y_s}$, where $0 \le y_s \le x_s$. Then, for appropriate integers k_y, we have

$$\prod_{i=1}^{k} (q_j, n) = \prod_{s=1}^{t} \prod_{y=1}^{x_s} (p_s^y, n)^{k_y}.$$

One must note that $\prod_{y=1}^{x_s} (p_s^y)^{k_y} \le Q^k$. Again appealing to periodicity and using (3.19), we can then write

$$\begin{aligned} \sum_{n \le \mathcal{L}(\vec{q})} \prod_{i=1}^{k} (q_j, n) &= \sum_{n \le \mathcal{L}(\vec{q})} \prod_{s=1}^{t} \prod_{y=1}^{x_s} (p_s^y, n)^{k_y} \\ &= \prod_{s=1}^{t} \sum_{n \le p^{x_s}} \prod_{y=1}^{x_s} (p_s^y, n)^{k_y} \lesssim \prod_{s=1}^{t} p_s^{\epsilon + \sum_{y=1}^{x_s} y \cdot k_y} \lesssim Q^{\epsilon + k}. \end{aligned}$$

Fourth, we have the inequality below, valid for all $\epsilon > 0$

$$\sum_{\vec{q} \in [1,Q]^j} \frac{1}{\mathcal{L}(\vec{q})} \lesssim Q^{\epsilon}. \tag{3.20}$$

Appealing to the divisor function $d(r) = \sum_{q \le r : q | r} 1$, and the estimate $d(r) \lesssim r^{\epsilon}$, we have

$$\sum_{\vec{q} \in [1,Q]^j} \frac{1}{\mathcal{L}(\vec{q})} \le \sum_{q \le Q^j} \frac{d(q)^j}{q} \lesssim Q^{\epsilon j}.$$

As $\epsilon > 0$ is arbitrary, we are finished.

We turn to the main line of the argument. Estimate

$$\begin{aligned} \frac{1}{M} \sum_{n \le M} \Big[\sum_{q \le Q} |c_q(n)| \Big]^j &= \frac{1}{M} \sum_{n \le M} \sum_{\vec{q} \in [1,Q]^j} \prod_{i=}^{j} |\mathsf{c}_{q_j}(n)| \\ &\overset{(3.17)}{\lesssim} \sum_{\vec{q} \in [1,Q]^j} \sum_{n \le \mathcal{L}(\vec{q})} \prod_{i=}^{j} (q_j, n) \\ &\overset{(3.18)}{\lesssim} \sum_{\vec{q} \in [1,Q]^j} \frac{Q^{\epsilon + j}}{\mathcal{L}(\vec{q})} \overset{(3.20)}{\lesssim} Q^{2\epsilon + j}. \end{aligned}$$

This is our bound (3.12).

□

4. The highly composite case

We follow the lines of the previous argument, but the underlying details are substantially different, as we are modifying Cook's argument [5], also see [6]. The essential features are due to Cook. We hope that this way of presenting the proof makes the argument more accessible.

The point is to show that for any $0 < \epsilon < 1$, and $f = \mathbf{1}_F$, a finitely supported function, stopping time $\tau : \mathbb{Z}^d \to \{\lambda_k\}$, and any integer N, we can choose M_1 and M_2 so that $A_\tau f \leq M_1 + M_2$ where

$$\|M_1\|_p \lesssim N^\epsilon |F|^{1/p}, \tag{4.1}$$

$$\|M_2\|_2 \lesssim N^{-\frac{4-d}{2}} |F|^{1/2}. \tag{4.2}$$

The implied constants depend upon $\epsilon > 0$. This proves our Theorem 1.2.

Fix $\epsilon > 0$. It suffices to prove (4.1) and (4.2) for sufficiently large $N > N_0$. Recall that $\lambda_k^2 = \mu_k!$. By our key assumption (1.1), namely that μ_k grows faster than any polynomial, there is a choice of N_0 so that for all $N > N_0$, we have $\mu_{[N^\epsilon]} > N^3$. For these integers, the first contribution to M_1 is $M_{1,1} f = \mathbf{1}_{\tau \leq \lambda_{[N^\epsilon]}} A_\tau f$. This clearly satisfies (4.1). We can assume that $\tau > \lambda_{[N^\epsilon]}$ below.

The decomposition of the averages A_{λ_k} is different from that in (3.3). Modify the definition in (3.4) as follows. Set $Q = N!$, and define

$$b_\lambda(\xi) = \sum_{0 \leq a < Q} \sum_{\ell \in \mathbb{Z}_Q^d} G(a, \ell, Q) \widetilde{\psi}_{2Q}(\xi - \ell/Q) \widetilde{d\sigma_\lambda}(\xi - \ell/Q). \tag{4.3}$$

Note that this is a very big sum. In particular it is typical to restrict Gauss sums $G(a, \ell, Q)$ to the case where $\gcd(a, \ell, Q) = 1$, but we are not doing this here. Our second contribution to M_2 is $M_{2,2} f = |B_\tau f - A_\tau f|$. Here, we are adopting our conventions about operators and their multipliers.

Lemma 4.1. *We have the estimate* $\|M_{2,2} f\|_2 \lesssim N^{\frac{4-d}{2}} |F|^{1/2}$.

Proof. The difference $M_{2,2} f$ is split into several terms. Using the expansion of A_λ from (3.3), the expansion is

$$\begin{aligned} M_{2,2} f \leq |E_\tau f| &+ \sum_{q > N} \sum_{a \in \mathbb{Z}_q^\times} |C_\tau^{a/q} f| \\ &+ \Big| B_\tau f - \sum_{1 \leq q \leq N} \sum_{a \in \mathbb{Z}_q^\times} e_q(-\tau^2 a) C_\tau^{a/q} f \Big|. \end{aligned} \tag{4.4}$$

We bound the ℓ^2 norm of each of these terms in order.

The first term on the right is bounded by appeal to (3.8). The second term on the right is bounded by appeal to (3.9). Thus, it is the third term (4.4) that is crucial. We have this critical point about the term $e_q(-\tau^2 a)$ appearing in (4.4). The stopping time τ takes values in $\{\lambda_k : k > N^\epsilon\}$. The highly composite nature of the λ_k shows that $e_q(-\lambda_k^2 a) \equiv 1$, for $k > N^\epsilon$,

$1 \leq q \leq N$, and $a \in \mathbb{Z}_q^\times$. (Indeed, this is the crucial simplifying feature of the highly composite case.) And so the term in (4.4) is

$$B_\tau f - \sum_{1 \leq q \leq N} \sum_{a \in \mathbb{Z}_q^\times} C_\tau^{a/q} f.$$

For a fixed value of τ, the multiplier above is

$$\begin{aligned} \sum_{0 \leq a' < Q} \sum_{\ell' \in \mathbb{Z}_Q^d} & G(a', \ell', Q) \widetilde{\psi}_{2Q}(\xi - \ell'/Q) \widetilde{d\sigma_\lambda}(\xi - \ell'/Q) \\ & - \sum_{1 \leq q \leq N} \sum_{a \in \mathbb{Z}_q^\times} G(a, \ell, q) \widetilde{\psi}_q(\xi - \ell/q) \widetilde{d\sigma_\lambda}(\xi - \ell/q). \end{aligned} \tag{4.5}$$

Recall the following basic property of Gauss sums. For a', ℓ', Q as above, we have

$$G(a', \ell', Q) = G(a'/\rho, \ell'/\rho, Q/\rho), \qquad \rho = \rho_{a',\ell'} = \gcd(a', \ell', Q). \tag{4.6}$$

It follows that the difference (4.5) splits naturally between the two cases when for fixed a', ℓ' we have Q/ρ being either strictly bigger than N or less than or equal to N.

In the case of $Q/\rho \leq N$, define

$$t_{a',\lambda}(\xi) = \sum_{\substack{\ell' \in \mathbb{Z}_Q^d \\ Q/\rho_{a',\ell'} \leq N}} G(a', \ell', Q) \{ \widetilde{\psi}_{2Q}(\xi - \ell'/Q) - \widetilde{\psi}_{Q/\rho}(\xi - \ell'/Q) \} \widetilde{d\sigma_\lambda}(\xi - \ell/q).$$

Notice that the difference $\{\widetilde{\psi}_{2Q}(\xi) - \widetilde{\psi}_{Q/\rho}(\xi)\}$ is zero for $|\xi| < (4Q)^{-1}$. We have by a square function argument and the stationary phase estimate (3.7),

$$\sum_{k > N^\epsilon} \|T_{a',\lambda_k} f\|_2^2 \lesssim \|f\|_2^2 \sum_{k > N^\epsilon} (Q/\lambda_k)^{1-d} \lesssim Q^{2(1-d)} |F|,$$

since we have $\mu_{[N^\epsilon]} > N^3$, and so $\lambda_k \geq N^3!$, while $Q = N!$. This is summed over $0 \leq a' < Q$ to give a smaller estimate than claimed.

In the case of $Q/\rho > N$, a modification of the argument that leads to (3.9) will complete the proof. Fix $q > N$, and set

$$s_\lambda(\xi) = \sum_{a' \in \mathbb{Z}_Q^d} \sum_{\substack{\ell' \in \mathbb{Z}_Q^d \\ Q/\rho_{a',\ell'} = q}} G(a', \ell', Q) \widetilde{\psi}_{2Q}(\xi - \ell'/Q) \widetilde{d\sigma_\lambda}(\xi - \ell/q).$$

This differs from $\sum_{a \in \mathbb{Z}_q} c_\tau^{a/q}$ by only the cut-off term $\widetilde{\psi}_{2Q}(\cdot)$. This is however a trivial term, due to our growth condition on λ_k and the stationary decay estimate (3.7). Note that from the Gauss sum estimate (4.6), and an easy square function argument, and (3.5), we have

$$\|S_\tau f\|_2 \lesssim q^{1-\frac{d}{2}} \|f\|_2 + \sum_{a \in \mathbb{Z}_q} \|C_\tau^{a/q} f\|_2.$$

But then, we can complete the proof from (3.9). And the proof is finished. □

It remains to consider $M_{1,2}f = |B_\tau f|$, where $B_\lambda f$ is defined in (4.3). We show that it satisfies the ℓ^p estimate (4.1), using a variant of the factorization argument of Magyar, Stein and Wainger [16]. The factorization is given by $B_\lambda = T_\lambda \circ U$, where the multipliers for these operators are given by

$$t_\lambda(\xi) = \sum_{0 \le a < Q} \sum_{\ell \in \mathbb{Z}_Q^d} \widetilde{\psi}_{2Q}(\xi - \ell/Q)\widetilde{d\sigma_\lambda}(\xi - \ell/Q),$$

$$\text{and} \qquad u(\xi) = \sum_{0 \le a < Q} \sum_{\ell \in \mathbb{Z}_Q^d} G(a, \ell, Q)\widetilde{\psi}_Q(\xi - \ell/Q).$$

Namely, the multiplier t_λ is $1/Q$-periodic, and has the spherical part of the multiplier. All the Gauss sum terms are in $u(\xi)$. The fact that $B_\lambda = T_\lambda \circ U$ follows from choice of ψ in (3.6).

Concerning the maximal operator $T_\tau \phi$, we can appeal to the transference result of [16, Prop 2.1] to bound ℓ^p norms of this maximal operator. Since the lacunary spherical maximal function is bounded on all $L^p(\mathbb{R}^d)$, we conclude that

$$\|T_\tau \phi\|_{\ell^p} \lesssim \|\phi\|_{\ell^p}, \qquad 1 < p < \infty.$$

Apply this with $\phi = Uf$. It remains to see that Uf is bounded in the same range. But this is the proposition below, which concludes the proof of (4.1), and hence the proof of Theorem 1.2.

Proposition 4.2. *For $1 \le p \le 2$, we have $\|Uf\|_p \lesssim \|f\|_p$.*

Proof. The ℓ^2 estimate follows Plancherel and $\|u\|_\infty \lesssim 1$. It remains to verify the ℓ^1 estimate. But, that amounts to the estimate $\|U\|_1 = \sum_m |U(m)| \lesssim 1$. And so we compute

$$\begin{aligned} U(-m) &= \int_{\mathbb{T}^d} u(\xi) e^{-im\cdot\xi}\, d\xi \\ &= \sum_{0 \le a < Q} \sum_{\ell \in \mathbb{Z}_Q^d} G(a, \ell, Q) \int_{\mathbb{T}^d} \widetilde{\psi}_Q(\xi - \ell/Q) e^{-im\cdot\xi}\, d\xi \\ &= \psi_Q(m) \sum_{0 \le a < Q} \sum_{\ell \in \mathbb{Z}_Q^d} G(a, \ell, Q) e^{-im\cdot\ell/Q} \\ &= \frac{\psi_Q(m)}{Q^d} \sum_{0 \le a < Q} \sum_{n \in \mathbb{Z}_Q^d} \sum_{\ell \in \mathbb{Z}_Q^d} e_Q(a|n|^2 + (n - m)\cdot \ell) \\ &= \frac{\psi_Q(m)}{Q^{d-1}} \sum_{n \in \mathbb{Z}_Q^d} \sum_{\ell \in \mathbb{Z}_Q^d} e_Q((n - m)\cdot \ell)\delta_{\{|n|^2 \equiv 0 \mod Q\}} \\ &= Q\psi_Q(m)\delta_{\{|m|^2 \equiv 0 \mod Q\}}. \end{aligned}$$

And, then, recalling (3.6), it follows that

$$\|U\|_1 = \sum_m |U(m)| \lesssim Q^{1-d} \sum_{|m|\leq Q} \delta_{\{|m|^2 \equiv 0 \mod Q\}}$$
$$\lesssim Q^{1-d} \sum_{j=1}^{Q} |jQ|^{\frac{d-2}{2}} \lesssim Q^{-d/2} \sum_{j=1}^{Q} j^{\frac{d}{2}-1} \lesssim 1.$$

□

A more general version of this last lemma is proved in [6, Lemma 15].

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