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Expansive automorphisms on locally compact groups

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Abstract. We show that any connected locally compact group which admits an expansive automorphism is nilpotent. We also show that for any locally compact group G, an automorphism α of G is expansive if and only if for any α-invariant closed subgroup H which is either compact or normal, the restriction of α to H is expansive and the quotient map on G/H corresponding to α is expansive. We get a structure theorem for locally compact groups admitting expansive automorphisms. We prove that an automorphism of a non-discrete locally compact group can not be both distal and expansive.

Contents

1. Introduction

Let G be a locally compact (Hausdorff) group with the identity e. An automorphism α of G is said to be *expansive* if $\cap_{n\in\mathbb{Z}}\alpha^n(U)=\{e\}$ for some neighbourhood U of e; here U is called an *expansive neighbourhood* for α. Equivalently, α is expansive if there exists a neighbourhood V of e such that for every pair $x,y\in G$, $x\neq y$, there exists $n=n(x,y)\in\mathbb{Z}$, such that $\alpha^n(y^{-1}x)\notin V$. Expansive automorphisms on compact groups have been studied extensively and are well understood (see Lam [Lam70], Lawton [Law73], Kitchens and Schmidt [Kit87, KS89], Schmidt [Sch90, Sch95] and references cited therein). There has been some work on expansivity on connected solvable groups and Lie groups (see Eisenberg [Eis66], Aoki [Aok79] and Bhattacharya [Bha04]) and also on totally disconnected groups (see Willis [Wil14] and Glöckner and Raja [GR17]).

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The main aim of this paper is to study expansivity on general locally compact groups. Any connected locally compact solvable group admitting an expansive automorphism is nilpotent (cf. [Aok79], Theorem 1). We generalise this to all connected locally compact groups. (see Theorem 2.6). For a class of compact groups and that of totally disconnected locally compact groups, it is known that expansivity carries over to quotients modulo closed invariant normal subgroups (see Corollary 6.15 in [Sch95] and Theorem A in [GR17]). We generalise this to all locally compact groups (see Theorem 2.7). In addition, we show that the expansivity carries over to quotients modulo compact invariant (not necessarily normal) subgroups (see Theorem 2.5).

Let G be a locally compact group with an expansive automorphism α. If G is compact, then it has an open α-invariant subgroup H of finite index such that $\alpha|_H$, the restriction of α to H, is ergodic (see [Sch95]). The structure of compact groups admitting expansive automorphisms is well understood (See [Sch95]). If G is connected, Theorem 2.6 shows that G is nilpotent. If G is totally disconnected, the structure of such a pair (G, α) is studied in [GR17] and a structure theorem is obtained (cf. [GR17], Theorem B). We generalise the same to all locally compact groups G (see Theorem 2.8).

In the end, we show that an automorphism of a locally compact group can not be both distal and expansive unless the group is discrete (see Theorem 2.9).

A homeomorphism α of a topological (Hausdorff) space X is said to be *distal* if for every pair of distinct elements $x, y \in X$, the closure of $\{(\alpha^n(x), \alpha^n(y)) \mid n \in \mathbb{Z}\}$ in $X \times X$ does not intersect the diagonal $\{(g, g) \mid g \in X\}$. An automorphism α of a topological group G is distal if and only if the closure of $\{\alpha^n(x) \mid n \in \mathbb{Z}\}$ in G does not contain the identity e, for every $x \neq e$. Distal maps on compact spaces were introduced by David Hilbert. Distal automorphisms on locally compact groups have been studied extensively. We refer the reader to Raja and Shah [RS10, RS19], Shah [Sha12], and the references cited therein. Although we show that the distality and expansivity are mutually exclusive phenomena for a non-discrete locally compact group, they do satisfy some similar properties. It is easy to see that if the restriction of the automorphism α to the closed invariant subgroup and the corresponding map on the quotient have one of the properties, then so does α. It has also been shown that distality carries over to quotients modulo closed invariant subgroups which are either compact or normal (cf. [RS10] and [Sha12]).

2. Groups with expansive automorphisms

Any locally compact group G admitting an expansive automorphism α has a countable neighbourhood basis of the identity e given by $\{\cap_{n=-k}^{k} \alpha^n(U) \mid k \in \mathbb{N}\}$, where U is an expansive neighbourhood (of e) for α in G. Therefore, G is metrizable and it has a left invariant metric d compatible with the topology of G (see [HR79]). The definitions of expansivity given in the introduction are equivalent to the following definition in terms of the metric d: α is expansive if there exists an $\epsilon > 0$ such that for every pair $x, y \in G$, $x \neq y$, there exists $n = n(x,y) \in \mathbb{Z}$ such that $d(\alpha^n(x), \alpha^n(y)) > \epsilon$. (Note that this is how expansivity for a homeomorphism α on a metrizable space X is defined). It is well-known and easy to see that if α^n is expansive for some $n \in \mathbb{Z} \setminus \{0\}$, then α^n is expansive for every $n \in \mathbb{Z} \setminus \{0\}$.

If H is a closed subgroup of G, we consider the quotient space G/H, the set of left cosets $\{xH \mid x \in G\}$, with the usual quotient topology. For $\alpha \in \mathrm{Aut}(G)$ and a α-invariant closed subgroup H, we have the canonical map $\bar{\alpha}$ on G/H, defined as $\bar{\alpha}(xH) = \alpha(x)H$. It is a homeomorphism of G/H. We say that the map $\bar{\alpha}$ on G/H is *expansive* if there exists a neighbourhood U of H in G/H such that $\cap_{n\in\mathbb{Z}}\bar{\alpha}^n(U) = \{H\}$. Equivalently, there exists a neighbourhood V of H in G/H such that for every pair $x, y \in G$ with $xH \neq yH$, there exists $n = n(xH, yH) \in \mathbb{Z}$ such that $\bar{\alpha}^n(y^{-1}xH) \notin V$. This definition coincides with the definition given above for a group when H is normal in G and G/H is a group. If G is metrizable and if a closed subgroup H is normal or compact, then G/H is also metrizable with a G-invariant metric, say d, i.e. $d(xH, yH) = d(y^{-1}xH, H) = d(gxH, gyH)$ for all $g, x, y \in G$. For such a metrizable group G, the definition of expansivity on the quotient G/H is equivalent to the one given above in terms of the metric d if (the closed invariant subgroup) H is either normal or compact.

We first note the following well-known result which is easy to prove.

Lemma 2.1. *Let G be a locally compact group, $\alpha \in \mathrm{Aut}(G)$ and let H be a closed α-invariant subgroup of G. Then the following hold:*

(1) *If α is expansive, then $\alpha|_H$ is expansive.*
(2) *If $\alpha|_H$ is expansive and $\bar{\alpha}$ on G/H is expansive, then α is expansive.*

A Lie projective locally compact group is said to be *finite-dimensional* if it is a projective limit of Lie groups each of which has the same dimension. Any compact connected group admitting an expansive automorphism is abelian and finite-dimensional (see [Lam70], [Law73] and also Theorems 5.3 and 6.1 of [KS89]). An almost connected locally compact group G has the largest compact normal (characteristic) subgroup K such that G/K is a Lie group, hence G is finite-dimensional if K^0 is so, where K^0 is the connected component of the identity e in K. Therefore, Lemma 2.1(1) implies that such a G is finite-dimensional if it admits an expansive automorphism; (more generally, if K or K^0 admits an expansive automorphism).

For a closed subgroup H of G, let H^0 denote the connected component of the identity e in H. It is a closed (normal) characteristic subgroup of H. For $x \in G$, let $\mathrm{inn}\, x$ denote the inner automorphism of G by the element x; i.e. $\mathrm{inn}\, x(g) = xgx^{-1}$, $g \in G$. Let $\mathrm{Inn}(G)$ denote the group of inner automorphisms of G. We first state a useful lemma which essentially follows from well-known results of Iwasawa [Iwa49] and a result in [Lam70].

Lemma 2.2. *Let G be a connected locally compact group. Let K be a compact normal subgroup of G. If K^0 is abelian, then K is abelian and central in G. In particular, if G, K or K^0 admits an expansive automorphism, then K is abelian and central in G.*

Proof. Suppose K^0 is abelian. Since G is connected and K^0 is abelian and normal in G, by Theorem 4 of [Iwa49], K^0 is central in G. By Theorem $1'$ of [Iwa49], $[\mathrm{Aut}(K)]^0 = [\mathrm{Inn}(K)]^0 = \{\mathrm{inn}\, k \mid k \in K^0\}$. As K^0 is central, $[\mathrm{Aut}(K)]^0$ is trivial. As G is connected, the restriction of any inner automorphism of G to K belongs to $[\mathrm{Aut}(K)]^0$ and hence it is trivial. This implies that K is central in G.

Let $\alpha \in \mathrm{Aut}(K)$ be expansive. Then K^0 is characteristic in K and $\alpha|_{K^0}$ is expansive (cf. Lemma 2.1(1)). We get that K^0 is abelian (cf. [Lam70], Corollary 3.3). Now it follows from the first assertion that K is central in G.

Let $\alpha \in \mathrm{Aut}(G)$ be expansive. Then so is $\alpha|_L$, where L is the largest compact normal subgroup of G. Arguing as above for L instead of K, we get that L is central in G. As $K \subset L$, K is also central in G. □

For a connected Lie group G, let $\mathcal{G}$ denote the Lie Algebra of G and let $\exp : \mathcal{G} \to G$ be the exponential map. There is a neighbourhood U of 0 in $\mathcal{G}$ such that $\exp|_U$ is a homeomorphism onto a neighbourhood of the identity e in G. For $\alpha \in \mathrm{Aut}(G)$, let $\mathrm{d}\alpha : \mathcal{G} \to \mathcal{G}$ be the Lie algebra automorphism such that $\exp \circ \mathrm{d}\alpha(X) = \alpha \circ \exp(X)$, $X \in \mathcal{G}$. We note the following which essentially follows from Theorem A and Propositions 2.1 and 2.3 of [Bha04].

Proposition 2.3. *Let G be a* (*nontrivial*) *connected Lie group and let $\alpha \in \mathrm{Aut}(G)$. Then the following are equivalent*:

(1) *α is expansive.*
(2) *$\mathrm{d}\alpha$ is expansive on the Lie algebra $\mathcal{G}$ of G.*
(3) *$\mathrm{d}\alpha$ does not have any eigenvalue of absolute value 1.*

In particular, if G admits an expansive automorphism, then it is nilpotent.

Proof. (1) $\Rightarrow$ (2) is proven in the proof of Theorem A of [Bha04] just by using the fact that one can choose a neighbourhood U of 0 in $\mathcal{G}$ such that $\exp(U)$ is an expansive neighbourhood for α. (2) $\Leftrightarrow$ (3) follows from Proposition 2.3 of [Bha04] (see also [Eis66]), and (3) $\Rightarrow$ (1) follows from Proposition 2.1 and Theorem A of [Bha04]. If G admits an expansive automorphism α, then $\mathrm{d}\alpha$ satisfies condition (3), and $\mathcal{G}$ is a nilpotent Lie algebra (see Exercise 21(b) among the exercises for Part I of [Bou89], §4, or Theorem 2 of [Jac55]), which in turn implies that G is nilpotent. □

We now focus on connected locally compact abelian groups. Following definitions and notations are standard; see [HM98]. Let G be a connected locally compact abelian group and let $\mathcal{L}(G)$ denote the space $\mathrm{Hom}(\mathbb{R}, G)$ of all continuous homomorphisms from $\mathbb{R}$ to G endowed with the topology of the uniform convergence on the compact subsets of $\mathbb{R}$. Then $\mathcal{L}(G)$ is a topological vector space with respect to pointwise addition and scalar multiplication (see Proposition 7.36 in [HM98]). Let the exponential map $\exp : \mathcal{L}(G) \to G$ be defined as $\exp(X) = X(1)$, $X \in \mathcal{L}(G)$. Then exp is continuous and it is a homomorphism; i.e. $\exp(X+Y) = X(1)Y(1)$. Also $\exp(tX) = X(t)$, $t \in \mathbb{R}$. For $\alpha \in \mathrm{Aut}(G)$, let $\mathrm{d}\alpha : \mathcal{L}(G) \to \mathcal{L}(G)$ be defined as $\mathrm{d}\alpha(X) = \alpha \circ X$, $X \in \mathcal{L}(G)$. Note that $\mathrm{d}\alpha$ defines a vector space isomorphism of $\mathcal{L}(G)$ and $\exp \circ \mathrm{d}\alpha = \alpha \circ \exp$.

As G is connected, abelian and locally compact, G is isomorphic to $\mathbb{R}^m \times K$, where K is the largest compact connected (abelian) subgroup of G, and $\mathcal{L}(G)$ is isomorphic to $\mathbb{R}^m \times \mathcal{L}(K)$. As K is connected and α-invariant, we get $\bar{\alpha}$, the automorphism of G/K corresponding to α. Moreover, $\mathrm{d}\alpha$ keeps $\mathcal{L}(K)$ invariant and we get $\overline{\mathrm{d}\alpha}$, the vector space isomorphism on $\mathcal{L}(G)/\mathcal{L}(K)$ corresponding to $\mathrm{d}\alpha$. Note that $\mathrm{d}\bar{\alpha} = \overline{\mathrm{d}\alpha}$ (under the isomorphism of $\mathcal{L}(G/K)$ and $\mathcal{L}(G)/\mathcal{L}(K)$).

If G is a (linear) Lie group, then $\mathcal{L}(G)$ coincides with the Lie algebra $\mathcal{G}$ of G. In case G is compact, $\mathcal{L}(G)$ is isomorphic to $\mathrm{Hom}(\hat{G}, \mathbb{R})$, where $\hat{G}$ is the character group of G. Suppose K as above is finite-dimensional, then $\mathcal{L}(G) = \mathbb{R}^m \times \mathcal{L}(K)$ is also finite-dimensional and it is isomorphic to $\mathbb{R}^{m+n}$, where $\mathcal{L}(K)$ is isomorphic to $\mathbb{R}^n$. Moreover, the kernel of exp is contained in $\mathcal{L}(K)$. If K (and hence G) is a (linear) Lie group, then exp is a local isomorphism, i.e. there exists a neighbourhood U of 0 in $\mathcal{L}(G)$ (resp. V of e in G) such that $\exp : U \to V$ is a homeomorphism. If G is isomorphic to $\mathbb{R}^n$ (i.e. if K is trivial), then exp is a vector space isomorphism. Note that G/K, being a finite-dimensional real vector space, is isomorphic to $\mathcal{L}(G/K)$ under the exponential map. We refer the reader to Ch. 7 of [HM98] for more details. (We use same notations for the exponential map on $\mathcal{L}(G)$ and also on the Lie algebra $\mathcal{G}$ of a Lie group G. Similarly, we use the same notation for the corresponding vector space isomorphism on $\mathcal{L}(G)$ as well as for the Lie algebra automorphism when it is induced by an automorphism of a group or a Lie group.)

As noted earlier, any connected locally compact abelian group G admitting an expansive automorphism is finite-dimensional, and hence, so is $\mathcal{L}(G)$. Therefore, we can discuss the expansivity of the corresponding map on $\mathcal{L}(G)$ in the following lemma which will be useful for the proof of Theorem 2.5.

Lemma 2.4. *Let G be a connected locally compact abelian group and let $\alpha \in \mathrm{Aut}(G)$ be expansive. Let K be the largest compact (normal) subgroup of G and let $\bar{\alpha}$ be the corresponding automorphism on G/K. Then $\mathrm{d}\alpha$ on $\mathcal{L}(G)$ and $\bar{\alpha}$ on G/K are expansive.*

Proof. Here, $G = \mathbb{R}^m \times K$ for some $m \in \mathbb{N} \cup \{0\}$, where K is the largest compact (connected) abelian characteristic subgroup as in the hypothesis. As α is expansive, K as well as G is finite-dimensional. Here, $\mathcal{L}(G) = \mathbb{R}^{m+n}$, where $n = \dim K$ and $\mathcal{L}(K) = \mathbb{R}^n$. Let V be an expansive neighbourhood of e in G for α. Let $\exp : \mathcal{L}(G) \to G$ be as above. As $\mathcal{L}(G)$ is a finite-dimensional real vector space, there is a vector space norm on it. As exp is continuous, we can choose $r > 0$ such that for the open neighbourhood $U = \{X \in \mathcal{L}(G) \mid \|X\| < r\}$ of 0 in $\mathcal{L}(G)$, we have $\exp(U) \subset V$. Observe that $tU \subset U$ for all $t \in [-1, 1]$. We show that U is an expansive neighbourhood of 0 in $\mathcal{L}(G)$ for $\mathrm{d}\alpha$. Let $X \in U$ be such that $\mathrm{d}\alpha^n(X) \in U$ for all $n \in \mathbb{Z}$. Then for all $t \in [-1, 1]$ and $n \in \mathbb{Z}$, $\mathrm{d}\alpha^n(tX) = t\,\mathrm{d}\alpha^n(X) \in U$, and hence, $\exp(\mathrm{d}\alpha^n(tX)) = \alpha^n(\exp(tX)) \in V$. As V is expansive for α, $\exp(tX) = X(t) = e$ for all $t \in [-1, 1]$. Since X is a (real) one parameter subgroup, the preceding assertion implies that $X(t) = e$ for all $t \in \mathbb{R}$, and hence, $X = 0$. Therefore, U is an expansive neighbourhood of 0 for $\mathrm{d}\alpha$, and hence, $\mathrm{d}\alpha$ is expansive on $\mathcal{L}(G)$.

By Proposition 2.3 of [Bha04], the eigenvalues of $\mathrm{d}\alpha$ do not have absolute value 1. As α keeps K invariant, $\mathrm{d}\alpha$ keeps $\mathcal{L}(K)$ invariant. Let $\overline{\mathrm{d}\alpha}$ be the map corresponding to $\mathrm{d}\alpha$ on $\mathcal{L}(G)/\mathcal{L}(K)$. As noted earlier, $\mathrm{d}\bar{\alpha} = \overline{\mathrm{d}\alpha}$. Therefore, the eigenvalues of $\overline{\mathrm{d}\alpha}$, and hence of $\mathrm{d}\bar{\alpha}$, do not have absolute value 1. By Proposition 2.3 of [Bha04], $\overline{\mathrm{d}\alpha}$, and hence, $\mathrm{d}\bar{\alpha}$ is expansive. Since G/K is a finite-dimensional real vector space, the exponential map from $\mathcal{L}(G)/\mathcal{L}(K)$ to G/K is a vector space isomorphism and it is easy to see that the preceding assertion implies that $\bar{\alpha}$ is expansive (see also Proposition 2.3). □

Remark. It follows from Theorems 8.20 and 8.22 of [HM98] that for a connected finite-dimensional abelian group G, $\exp : \mathcal{L}(G) \to G$ is injective on a small neighbourhood of 0, i.e. given a neighbourhood V of e in G, there exists a neighbourhood U of 0 in $\mathcal{L}(G)$ such that $\exp(U) \subset V$ and $\exp|_U$ is injective. Hence, for an expansive $\alpha \in \mathrm{Aut}(G)$, if V is an expansive neighbourhood for α, then it is easy to see using the injectivity of $\exp|_U$ that U is an expansive neighbourhood for $\mathrm{d}\alpha$. This provides an alternative proof for the first part of the assertion in Lemma 2.4.

Now we prove that expansivity carries over to quotients modulo compact invariant (not necessarily normal) subgroups.

Theorem 2.5. *Let G be a locally compact group and let $\alpha \in \mathrm{Aut}(G)$. Let K be a compact α-invariant subgroup of G and let $\bar{\alpha} : G/K \to G/K$ be the map corresponding to α. If α is expansive, then so is $\bar{\alpha}$; equivalently there exists an open set O containing K in G such that $\cap_{n\in\mathbb{Z}}\alpha^n(O) = K$.*

Proof of Theorem 2.5 for the case when K is normal. Here G/K is a group and $\bar{\alpha} \in \mathrm{Aut}(G/K)$.

Step 1: Suppose K is central in G. Let W be an expansive neighbourhood of the identity e in G for α, i.e. $\cap_{n\in\mathbb{Z}}\alpha^n(W) = \{e\}$. Let V be an open symmetric relatively compact neighbourhood of e in G such that $\overline{V}^4 \subset W$. Let $A = \cap_{n\in\mathbb{Z}}[\alpha^n(V)K] = \{x \in V \mid \alpha^n(x) \in VK \text{ for all } n \in \mathbb{Z}\}K$. Then $K \subset A \subset VK$ and $\alpha(A) = A = AK$. If $x, y \in A$, then as K is central in

G, $\alpha^n(xyx^{-1}y^{-1}) \in V^4 \subset W$, for all $n \in \mathbb{Z}$ and hence, $xyx^{-1}y^{-1} = e$ since W is an expansive neighbourhood for α. This implies that the elements of A, and hence, $\overline{A}$ commute. Let H be the closed subgroup generated by $\overline{A}$. Since $\overline{A} \subset \overline{V}K$ is compact, H is compactly generated. Moreover, H is abelian and locally compact. Therefore, H is isomorphic to $\mathbb{R}^d \times \mathbb{Z}^k \times C$ and $H^0 = \mathbb{R}^d \times C^0$, where $C \subset H$, C is compact and $d, k \in \mathbb{N} \cup \{0\}$. Since $K \subset A$, $K \subset C$. Since $\alpha(A) = A$, H is α-invariant and $\alpha|_H$ is expansive (cf. Lemma 2.1(1)). Note that $\alpha(C) = C$ and $\alpha(C^0) = C^0$, as C is the largest compact (normal) subgroup of H. Since C is compact and abelian, the restriction of $\bar{\alpha}$ to C/K is expansive (cf. [Sch90], Corollary 3.11). As H^0C/C is isomorphic to H^0/C^0, the corresponding action of α on H^0C/C is expansive by Lemma 2.4. As H^0C/C is isomorphic to $(H^0C/K)/(C/K)$, by Lemma 2.1(2) we get that the restriction of $\bar{\alpha}$ to H^0C/K is expansive. As H^0C is open in H, the restriction of $\bar{\alpha}$ to H/K is also expansive. Let $\pi : G \to G/K$ be the natural projection. Let $V' \subset V$ be a neighbourhood of e in G such that $\pi(V') \cap (H/K)$ is an expansive neighbourhood for the restriction of $\bar{\alpha}$ to H/K.

Now we show that $\pi(V')$ is an expansive neighbourhood of $\pi(e)$ for $\bar{\alpha}$ in G/K. Let $x \in V'$ be such that $\pi(x) \in \cap_{n\in\mathbb{Z}}\bar{\alpha}^n(\pi(V'))$. Then $x \in \cap_{n\in\mathbb{Z}}[\alpha^n(V')K] \subset A$. Therefore, $x \in V' \cap A \subset V' \cap H$. As $\pi(V') \cap (H/K)$ is an expansive neighbourhood for the restriction of $\bar{\alpha}$ to H/K, we have that $\pi(x) \in \pi(K)$, and hence, $x \in K$. This implies that $\pi(V')$ is an expansive neighbourhood of $\pi(e)$ for $\bar{\alpha}$ in G/K; i.e. $\bar{\alpha}$ is expansive on G/K.

Step 2: Now suppose K is normal but not central in G. If G is compact, then the assertion follows from Corollary 6.15 of [Sch95]. Let L be the largest compact normal subgroup of G^0. Since $\alpha|_{G^0}$ is expansive, by Lemma 2.2, L is central in G^0. Here, L is a compact characteristic, and hence, α-invariant normal subgroup in G. Therefore, KL is a compact normal α-invariant subgroup of G. As $G/(KL)$ is isomorphic to $(G/K)/((KL)/K)$ and the restriction of $\bar{\alpha}$ to $(KL)/K$ is expansive (cf. [Sch95], Corollary 6.15), by Lemma 2.1(2) it is enough to prove that the automorphism of $G/(KL)$ corresponding to α is expansive. Therefore, replacing K by KL if necessary, we may assume that $L \subset K$, i.e. $G^0 \cap K = L$. Since L is central in G^0, it follows from the assertion in Step 1 that the automorphism of G^0/L corresponding to $\alpha|_{G^0}$ is expansive. As $(G^0K)/K$ is isomorphic to G^0/L, the preceding assertion implies that the restriction of $\bar{\alpha}$ to $(G^0K)/K$ is expansive.

As G/G^0 is totally disconnected, it admits an open compact subgroup. Therefore, G admits an open almost connected subgroup, say M'. Let $M = M'K$. It is an open almost connected subgroup containing K. Let W be an expansive neighbourhood of the identity e in G for α, with an additional property that $W \subset M$. As M is Lie projective, there exists a compact subgroup $C' \subset W$ such that C' is normal in M and M/C' is a Lie group. Let $C = C'L$. Then C is a closed normal subgroup in M and M/C is a Lie group. As $K \subset M$ and M is open, $G^0 \subset M$ and both K and G^0 normalise C. Now CG^0 is an open almost connected subgroup, $C \cap G^0 = L$ and $(CG^0)/L = C/L \times G^0/L$. As K normalises C as well as G^0, we get that CKG^0 is an open subgroup, CG^0 has finite index in CKG^0. Moreover, C (resp. CK) is the largest compact normal subgroup of CG^0 (resp. CKG^0) and $C \cap G^0 = L = K \cap G^0$.

As the restriction of $\bar{\alpha}$ to G^0K/K is expansive, we can choose an open symmetric relatively compact neighbourhood U of e such that $U \subset CG^0 \cap W$ and the image of $U \cap G^0K$ in $(G^0K)/K$ is an expansive neighbourhood for the restriction of $\bar{\alpha}$ to $(G^0K)/K$. Let $B = \cap_{n\in\mathbb{Z}}[\alpha^n(U)K]$. Here $K \subset B \subset UK \subset CKG^0$. If $B = K$, then the image of U in G/K is an expansive neighbourhood for $\bar{\alpha}$. Now suppose $B \neq K$. From the choice of U, it follows that $B \cap G^0K = K$.

Let H' be the closed subgroup generated by B in G. Then H' is α-invariant and $K \subset H'$. We first show that the restriction of $\bar{\alpha}$ to H'/K is expansive. Note that $H' \subset CKG^0$; where CKG^0 is an open almost connected subgroup of G. Let H'' be the closure of $H'G^0$. Then $H'' \subset CKG^0$ and H'' is an almost connected α-invariant subgroup of G. Let E be the largest compact normal subgroup of H''. Then E is characteristic in H'', $K \subset E$ and $H'' \cap CK \subset E$. Therefore, $H'' = (H'' \cap CK)G^0 \subset EG^0$. This implies that $H'' = EG^0$ and, $H''/K = E/K \times (G^0K)/K$ as $E \cap G^0 = L = K \cap G^0$. Note that E is α-invariant and $\alpha|_E$ is expansive. As E is compact, by Corollary 6.15 of [Sch95], the restriction of $\bar{\alpha}$ to E/K is expansive. As the restriction of $\bar{\alpha}$ to $(G^0K)/K$ is also expansive, it follows that the restriction of $\bar{\alpha}$ to H''/K is expansive. Since $H' \subset H''$, by Lemma 2.1 (1) the restriction of $\bar{\alpha}$ to H'/K is expansive.

Let $U' \subset U$ be a neighbourhood of the identity e in G such that the image of $U' \cap H'$ in H'/K is an expansive neighbourhood for the restriction of $\bar{\alpha}$ to H'/K. Concluding the argument as in Step 1, replacing V', A and H by U', B and H' respectively, it is easy to deduce that the image of U' in G/K is an expansive neighbourhood for $\bar{\alpha}$. This completes the proof for the case when K is normal. □

We will complete the proof of Theorem 2.5 after the next result. An invertible linear map on $\mathbb{R}^n$ is expansive if it does not have any eigenvalue of absolute value 1 (cf. Proposition 2.3). For expansive automorphisms on non-abelian nilpotent Lie groups, see an example in §3 of [Bha04]. There are also examples of compact connected abelian finite-dimensional groups (which are not Lie groups) admitting expansive automorphisms (cf. see [Sch95]). The following theorem shows that there is no connected locally compact non-nilpotent group which admits expansive automorphisms.

Theorem 2.6. *Any connected locally compact group admitting an expansive automorphism is nilpotent.*

Proof. Let G be a connected locally compact group and let $\alpha \in \mathrm{Aut}(G)$ be expansive. Let K be the largest compact normal subgroup of G. By Lemma 2.2, K is abelian and central in G. Moreover, we get from Theorem 2.5 for the normal case (proven above) that the automorphism on G/K corresponding to α is expansive. Note that G is nilpotent if G/K is so. Therefore, it is enough if we assume that G is a connected Lie group without any nontrivial compact normal subgroup. Now the assertion follows from Proposition 2.3. □

For a compact group G and $\alpha \in \mathrm{Aut}(G)$, we say that (G, α) satisfies the *descending chain condition* if for every sequence $G \supset G_1 \supset \cdots \supset G_k \supset \dots$ of closed α-invariant subgroups, there exists $N \in \mathbb{N}$ such that $G_k = G_N$ for all $k \geq N$. If a compact group G admits an expansive automorphism α, then (G, α) satisfies the descending chain condition; the converse holds in the special case when G is totally disconnected (cf. [KS89], Theorem 5.2).

Proof of Theorem 2.5 for the general case. Here, the compact group K is not assumed to be normal in G. As in the hypothesis, $\alpha \in \mathrm{Aut}(G)$ is expansive and K is α-invariant. We want to show that the α-action on G/K is expansive. By Lemma 2.1(1), $\alpha|_{G^0}$ is expansive and hence, G^0 is nilpotent by Theorem 2.6. Observe that G^0 has the largest compact normal subgroup, say L such that G^0/L is a connected nilpotent group without any nontrivial compact normal subgroups. Therefore, the center Z of G^0/L is connected and simply connected and hence G^0/L itself is simply connected (see Lemma 3.6.4 of [Var84] and its proof). In particular, G^0/L has no nontrivial compact subgroups and hence, any compact subgroup of G^0 is contained in L. Note that L is characteristic in G^0 and hence, α-invariant and normal in G. By Lemma 2.2 of [Iwa49], L^0 is abelian, and hence, by Lemma 2.2, L is central in G^0. Here, $K \cap G^0 = K \cap L$ is central in G^0. Also, LK is an α-invariant compact subgroup of G. Observe that $L/(L \cap K)$ is isomorphic to $(LK)/K$ under the natural isomorphism, say φ defined as $\varphi(x(L \cap K)) = xK$, $x \in L$. Also, $\varphi(\alpha(x)(L \cap K)) = \alpha(x)K$ $x \in L$. Therefore, the α-action on $(LK)/K$ is expansive if and only if the α-action on $L/(L \cap K)$ is expansive. Since α is expansive and L is abelian, by Lemma 3.11 of [Sch90], the α-action on $L/(L \cap K)$ is expansive. Therefore, the α-action on $(LK)/K$ is also expansive. Now, to prove that the α-action on G/K is expansive, one can easily see that it is enough to prove that the α-action on $G/(LK)$ is expansive.

Note that $G/(LK)$ is isomorphic to $(G/L)/((LK)/L)$ and, as L is normal, from the proof of the normal case above, the α-action on G/L is expansive. Therefore, replacing G by G/L and LK by $(LK)/L$, without loss of any generality, we may assume that L is trivial. Now G^0 is a connected nilpotent Lie group without any nontrivial compact subgroups. In particular, $K \cap G^0$ is trivial and K is totally disconnected.

Let U be an open relatively compact expansive neighbourhood of e in G for α. We first construct a sequence $\{C_n\}$ of compact totally disconnected groups contained in U with certain properties. Let V be an open relatively compact neighbourhood of e in G such that $V^2 \subset U$. Let $\pi : G \to G/G^0$ be the natural projection. Since G/G^0 is totally disconnected, it admits a neighbourhood basis of compact open subgroups. Let B be a compact open subgroup in G/G^0 and let $B' = \cap_{k\in K}\pi(k)B\pi(k)^{-1}$. As K is compact, so is $\pi(K)$, and hence B' is a compact open subgroup in G/G^0 (this follows from Theorem 4.9 of [HR79]). Let $H' = \pi^{-1}(B')$. Then H' is an open subgroup normalised by elements of K, and H'/G^0 is compact. Let $H = KH'$. Then H is an open subgroup in G and H/G^0 is compact. Therefore, H is Lie projective. As α is expansive, we have that G is metrizable, and hence, so is H. Therefore, H admits a sequence of compact normal subgroups $\{C_n\}_{n\in\mathbb{N}}$ such that $C_1 \subset V \cap H$, $C_{n+1} \subset C_n$, H/C_n is a Lie group with finitely many connected components, $n \in \mathbb{N}$, and $\cap_n C_n = \{e\}$. In particular, for each $n \in \mathbb{N}$, C_nG^0 is an open subgroup in H, and hence, in G. As $KG^0 \subset H$, elements of KG^0 normalise C_n, $n \in \mathbb{N}$. Moreover, $C_1 \cap G^0 = \{e\}$ as G^0 has no nontrivial compact subgroups. Therefore, C_n is totally disconnected and $C_nG^0 = C_n \times G_0$, $n \in \mathbb{N}$.

We choose a neighbourhood basis $\{W_n\}_{n\in\mathbb{N}}$ of the identity e in G^0 such that $W_1 \subset V$, $W_{n+1} \subset W_n$ and $U_n = C_n \times W_n \subset V^2 \subset U$, $n \in \mathbb{N}$. Here, $\{U_n\}_{n\in\mathbb{N}}$ forms a neighbourhood basis of the identity e in G such that $U_{n+1} \subset U_n$ and $W_n = U_n \cap G^0$, $n \in \mathbb{N}$.

Fix any $n \in \mathbb{N}$. Suppose $x \in U_n$ is such that $\alpha^m(x) \in U_nK$, for all $m \in \mathbb{Z}$. We show that $x \in C_n$ and $\alpha^m(x) \in C_nK$ for all $m \in \mathbb{Z}$. As $U_n = C_n \times W_n$, we have that $x = wc = cw$ for some $c \in C_n$ and $w \in W_n$. Now, for $m \in \mathbb{Z}$, $\alpha^m(x) = c_mw_mk_m = \alpha^m(c)\alpha^m(w) = \alpha^m(w)\alpha^m(c)$, where $c_m \in C_n$, $w_m \in W_n$, $k_m \in K$, $c_0 = c$, $w_0 = w$ and $k_0 = e$. Let $m \in \mathbb{Z}$ be fixed. As both C_n and $\alpha^m(C_n)$ centralise G^0, we get that

$$\alpha^m(c^{-1})c_mk_m = w_m^{-1}\alpha^m(w) = c_mk_m\alpha^m(c^{-1}).$$

Recall that C_n is normalised by K, and hence, C_nK is a compact subgroup. Also, $\alpha^m(C_n)$ is a compact subgroup and $\alpha^m(c^{-1})$ and c_mk_m commute with each other, hence we get that $w_m^{-1}\alpha^m(w) = \alpha^m(c^{-1})c_mk_m$ generates a compact subgroup contained in $G^0 \cap \alpha^m(C_n)C_nK$. As G^0 has no nontrivial compact subgroup, we get that $w_m^{-1}\alpha^m(w) = e$, and hence, that $\alpha^m(w) = w_m \in W_n \subset U$. Since this holds for all $m \in \mathbb{Z}$ and since U is expansive for α, we get that $w = e = w_m$, $m \in \mathbb{Z}$, and hence, $x = c \in C_n$. Now $\alpha^m(x) = c_mk_m \in C_nK$, for all $m \in \mathbb{Z}$. Let $C'_n = \{c \in C_nK \mid \alpha^m(c) \in C_nK \text{ for all } m \in \mathbb{Z}\}$. Then $x \in C'_n$.

For each $n \in \mathbb{N}$, C'_n is a closed (compact) subgroup of C_nK, $K \subset C'_{n+1} \subset C'_n \subset C'_1$, $\alpha(C'_n) = C'_n$, and $\cap_{n\in\mathbb{N}}C'_n = K$. As $\alpha|_{C'_1}$ is expansive, it satisfies the descending chain condition (cf. [KS89], Theorem 5.2), and hence there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $C'_n = C'_N$. Therefore, $C'_N = \cap_n C'_n =$

K. This implies that if $x \in U_N = C_N \times W_N$ is such that $\alpha^m(x) \in U_N K$ for all $m \in \mathbb{Z}$, then $x \in C'_N = K$. This shows that the α-action on G/K is expansive. □

The following theorem generalises Theorem A of [GR17], which is for totally disconnected locally compact groups, to all locally compact groups.

Theorem 2.7. *Let G be a locally compact group and let $\alpha \in \mathrm{Aut}(G)$. Let H be a closed normal α-invariant subgroup of G and let $\bar{\alpha}$ be the automorphism of G/H corresponding to α. Then α is expansive if and only if $\alpha|_H$ and $\bar{\alpha}$ are expansive.*

Proof. The 'if' statement follows from Lemma 2.1(2). Now suppose α is expansive. Let H be a closed normal α-invariant subgroup of G. Then $\alpha|_H$ is expansive by Lemma 2.1(1). We show that $\bar{\alpha}$ on G/H is expansive. If H is compact or G is totally disconnected, then the assertion follows from Theorem 2.5 above or Theorem A of [GR17] respectively. Let K be the largest compact normal subgroup of G^0. Then K is characteristic in G and HK is a closed normal α-invariant subgroup. Note that $(HK)/H$ is isomorphic to $K/(K \cap H)$. Since the α-action on K is expansive, so is the corresponding action on $K/(K \cap H)$ (cf. [Sch95], Corollary 6.15). This implies that the α-action on HK/H is expansive. By Lemma 2.1(2), it is enough to show that the α-action on G/HK is expansive, i.e. we may assume that $K \subset H$. Moreover, G/H is isomorphic to $(G/K)/(H/K)$ and from Theorem 2.5, the α-action on G/K is expansive. Replacing G by G/K and H by H/K, we may assume that G^0 has no nontrivial compact normal subgroup. As α is expansive, by Theorem 2.6, G^0 is nilpotent, and hence, a simply connected nilpotent Lie group.

Suppose H is connected. Then $H \subset G^0$ and G^0/H is a connected (nilpotent) Lie group. Let $\mathcal{G}$ (resp. $\mathcal{H}$) denote the Lie algebra of G^0 (resp. H) and let $\alpha_0 = \alpha|_{G^0}$. By Proposition 2.3, the eigenvalues of $\mathrm{d}\alpha_0$ on $\mathcal{G}$ do not have absolute value 1. The same holds for the eigenvalues of $\mathrm{d}\bar{\alpha}_0$, the corresponding map on the Lie algebra $\mathcal{G}/\mathcal{H}$ of G^0/H. By Proposition 2.3, $\bar{\alpha}_0$, the restriction of $\bar{\alpha}$ to G^0/H is expansive.

Let U be an expansive open symmetric relatively compact neighbourhood of the identity e in G for α. As G has an open Lie projective subgroup, there exists a compact subgroup C normalised by G^0 such that $C \subset U$ and CG^0 is open in G. As G^0 has no nontrivial compact subgroup, $C \cap G^0 = \{e\}$, i.e. C is totally disconnected, and $CG^0 = C \times G^0$, which is open in G. Replacing U by a smaller open symmetric neighbourhood of e, we may assume that $U = C \times W$, where $W = U \cap G^0 = W^{-1}$ is open in G^0, and that the image of W in G^0/H is an expansive neighbourhood for $\bar{\alpha}_0$.

Let $x \in U$ be such that $\alpha^n(x) \in UH$ for all $n \in \mathbb{Z}$. Then $x = cw = wc$, for some $c \in C$ and $w \in W$. As H is connected, $UH = CWH \subset CG^0$, and we have

$$\alpha^n(c)\alpha^n(w) = \alpha^n(w)\alpha^n(c) = \alpha^n(x) = c_n w_n h_n \in C \times G^0,$$

where $c_n \in C$, $w_n \in W$ and $h_n \in H \subset G^0$ for all $n \in \mathbb{Z}$. We can assume that $c_0 = c$, $w_0 = w$ and $h_0 = e$. Fix any $n \in \mathbb{Z}$. Using the fact that both C and $\alpha^n(C)$ centralise G^0, we have that $\alpha^n(c)$ and c_n commute with each other, and we get that $c_n^{-1}\alpha^n(c) = \alpha^n(w^{-1})w_n h_n$ generates a compact subgroup in $C\alpha^n(C) \cap G^0$. As G^0 has no nontrivial compact subgroup, we get that $\alpha^n(c) = c_n$. Since this holds for all $n \in \mathbb{Z}$ and U is expansive for α, we get that $c = e = c_n$, $n \in \mathbb{Z}$. Hence $x = w \in W \subset G^0$ and $\alpha^n(x) = w_n h_n \in WH$, $n \in \mathbb{Z}$. This implies that $x \in H$, as the image of W in G^0/H is an expansive neighbourhood for $\bar{\alpha}_0$, the restriction of $\bar{\alpha}$ to G^0/H. This proves that $\bar{\alpha}$ is expansive if H is connected.

Now suppose H is not connected. Observe that H^0 is α-invariant and normal in G and we have that the α-action on G/H^0 is expansive. As H^0 is a connected normal subgroup in the simply connected nilpotent group G^0, we get that G^0/H^0 is a simply connected nilpotent group (cf. [Hoc65], Ch. XII, Theorem 1.2). In particular, G^0/H^0 has no nontrivial compact subgroup. Since G/H is isomorphic to $(G/H^0)/(H/H^0)$, we may replace G by G/H^0 and H by H/H^0 and assume that $H^0 = \{e\}$ (i.e. H is totally disconnected) and that G^0 has no nontrivial compact subgroup.

As noted earlier, we can choose a compact totally disconnected subgroup C of G and an expansive open symmetric relatively compact neighbourhood U of the identity e in G such that $C \times G^0$ is open and $U = C \times W$, where $W = U \cap G^0$ is an open symmetric relatively compact neighbourhood of e in G^0. Observe that CH is a closed subgroup. Note that CH/H, being isomorphic to $C/(C \cap H)$, is totally disconnected as C is so. Therefore, $(CH)^0 \subset H$ and hence, $(CH)^0 = H^0 = \{e\}$ as H is totally disconnected. Therefore, CH is also totally disconnected.

Let $C_H = \{c \in CH \mid \alpha^n(c) \in CH \text{ for all } n \in \mathbb{Z}\}$. Then C_H is a closed α-invariant group and $H \subset C_H$ is co-compact. As CH is totally disconnected, so is C_H. By Theorem A in [GR17], the restriction of $\bar{\alpha}$ to C_H/H is expansive. Let U' be a neighbourhood of the identity e in G such that $U'U' \subset U$ and the image of $U'H \cap C_H$ in C_H/H is an expansive neighbourhood for the restriction of $\bar{\alpha}$ to C_H/H. As CH is totally disconnected, replacing U' by a smaller neighbourhood of e if necessary, we may assume that $U'U' \cap CH$ is contained in a compact open subgroup, say P of CH.

Since U' is open and α and α^{-1} are continuous, we may choose an open symmetric (relatively compact) neighbourhood V' of the identity e in G such that $V'\alpha(V')\alpha^{-1}(V') \subset U'$. Let $W' = V' \cap G^0$. Then $W' \subset W$ and W' is an open symmetric (relatively compact) neighbourhood of the identity e in G^0. As C is a compact totally disconnected group and $V' \cap C$ is open in C, there exists a compact subgroup, say C', which is contained in $V' \cap C$ and it is open in C.

Let $V = C' \times W'$. Here, V is an open symmetric neighbourhood of the identity e in G and $V\alpha(V)\alpha^{-1}(V) \subset U'U' \subset U$. Also, $V\alpha(V)\alpha^{-1}(V) \cap CH \subset U'U' \cap CH \subset P$, where P, chosen as above, is an open compact subgroup of

CH. Now $W'\alpha(W')\alpha^{-1}(W')\cap CH \subset G^0\cap P=\{e\}$, as G^0 has no nontrivial compact subgroup. We also have that $C'\alpha(C')\alpha^{-1}(C') \subset U = C\times W$, and hence, that for any $x\in C'$, $\alpha(x)=cw=wc$ for some $c\in C$ and $w\in W\subset G^0$. This implies that $c^{-1}\alpha(x)=\alpha(x)c^{-1}$ generates a compact group in $C\alpha(C')\cap G^0$. As G^0 has no nontrivial compact subgroup, we get that $c^{-1}\alpha(x)=e$. This shows that $\alpha(C')\subset C$ and $C'\alpha(C')\subset C$. Similarly, we get that $C'\alpha^{-1}(C')\subset C$.

Let $x\in V$ be such that $\alpha^m(x)\in VH$ for all $m\in\mathbb{Z}$. Then

$$\alpha^m(x)=c_m w_m h_m,\ \text{where } c_m\in C',\ w_m\in W' \text{ and } h_m\in H,$$

for all $m\in\mathbb{Z}$. As $V=C'\times W'$, we have $x=cw=wc$ for some $c\in C'$ and $w\in W'$, and we may assume that $c_0=c$, $w_0=w$ and $h_0=e$.

We show that $\alpha^m(w)=w_m$ and $\alpha^m(c)\in c_mH$ for all $m\in\mathbb{N}$ by induction. For $m=1$, $\alpha(x)=\alpha(c)\alpha(w)=c_1w_1h_1$, and hence, $c_1^{-1}\alpha(c)=w_1h_1\alpha(w^{-1})=w_1\alpha(w^{-1})h_1'$ for some $h_1'\in H$; here h_1' exists as H is normal. Therefore, $w_1\alpha(w^{-1})\in W'\alpha(W')\cap C'\alpha(C')H\subset W'\alpha(W')\cap CH=\{e\}$, as $C'\alpha(C')\subset C$; and hence, $\alpha(w)=w_1$. Now $\alpha(c)=c_1h_1'$, i.e. $\alpha(c)\in c_1H$. This concludes the base case of the induction.

For a fixed $k\in\mathbb{N}$, suppose $\alpha^k(w)=w_k$ and $\alpha^k(c)\in c_kH$. We have

$$\alpha^{k+1}(x)=\alpha(\alpha^k(c)\alpha^k(w))=\alpha(c_k)\alpha(w_k)\alpha(h_k)=c_{k+1}w_{k+1}h_{k+1}.$$

This implies that $\alpha(c_k)\alpha(w_k)=c_{k+1}w_{k+1}h'$, where $h'=h_{k+1}\alpha(h_k^{-1})\in H$. Arguing as above for c_k, w_k instead of c, w and, c_{k+1}, w_{k+1}, h' instead of c_1, w_1, h_1 respectively, we get that $\alpha(w_k)=w_{k+1}$ and $\alpha(c_k)\in c_{k+1}H$. Using the induction hypothesis for k, we have $\alpha^{k+1}(w)=\alpha(w_k)=w_{k+1}$ and $\alpha^{k+1}(c)\in\alpha(c_k)H=c_{k+1}H$. This proves the statement for all $m\in\mathbb{N}$ by induction. Replacing α by α^{-1} and using the facts that $C'\alpha^{-1}(C')\subset C$ and $W'\alpha^{-1}(W')\cap CH=\{e\}$, we get that $\alpha^{-m}(w)=w_{-m}$ and $\alpha^{-m}(c)\in c_{-m}H$ for all $m\in\mathbb{N}$. Now $\alpha^m(w)=w_m\in W'\subset U$, $m\in\mathbb{Z}$, and U is expansive for α, it follows that $w=e$, and hence, $x=c\in C'$ and $\alpha^m(x)\in C'H$, for all $m\in\mathbb{Z}$. Since $C'\subset C$, we have that $x\in C'\cap C_H$ and $\alpha^m(x)\in C'H\cap C_H$ for all $m\in\mathbb{Z}$. As $C'\subset V\subset U'$, and the image of $U'H\cap C_H$ in C_H/H is expansive for the restriction of $\bar\alpha$ to C_H/H, we get that $x\in H$. This implies that the image of VH in G/H is an expansive neighbourhood for $\bar\alpha$. Therefore, $\bar\alpha$ is expansive. □

Note that Theorem 2.5 and Lemma 2.1 imply that Theorem 2.7 also holds when H is a compact (not necessarily normal) subgroup.

We have shown in Theorem 2.6 that a connected locally compact group admitting expansive automorphisms is nilpotent. In [GR17], a structure theorem for totally disconnected locally compact groups admitting expansive automorphisms is obtained (cf. [GR17], Theorem B). The following theorem generalises the same to all locally compact groups. A locally compact group is said to be *topologically perfect* if its commutator subgroup is dense in the whole group.

Theorem 2.8. *Let G be a locally compact group and let $\alpha \in Aut(G)$ be expansive. Then there exist finitely many α-invariant closed subgroups*

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{e\}$$

of G such that G_j is normal in G_{j-1} for $j \in \{1, ..., n\}$ and each of the quotient groups G_{j-1}/G_j is discrete, abelian or topologically perfect. Moreover, one can choose $\{G_j\}$ in such a way that every α_j-invariant closed normal subgroup of G_{j-1}/G_j is discrete or open, where $\alpha_j : G_{j-1}/G_j \to G_{j-1}/G_j$ is defined as $gG_j \mapsto \alpha(g)G_j$ for all $g \in G_{j-1}$, for all j.

Proof. Let $\bar{\alpha} : G/G^0 \to G/G^0$ be the automorphism of G/G^0 corresponding to α. Then by Theorem 2.7, $\bar{\alpha}$ is expansive. As G/G^0 is totally disconnected, by Theorem B of [GR17], assertions hold for G/G^0 and $\bar{\alpha}$. Hence, it is enough to show that assertions hold for a connected group G. We first produce a finite sequence of closed normal subgroups satisfying the first assertion. By Theorem 2.6, G is a (connected) nilpotent group. Let K be the largest compact normal subgroup of G. Then K is characteristic and G/K is a simply connected nilpotent Lie group. As observed in the proof of Theorem 2.6, K is central in G. Moreover, K is connected as G is connected, nilpotent and Lie projective and the largest compact normal subgroup of a connected nilpotent Lie group is connected (see Lemma 3.6.4 of [Var84] and its proof). Observe that G has a central series of connected characteristic subgroups $G^{(1)} = \overline{[G, G]}$, the closure of the commutator subgroup of G, and $G^{(m+1)} = \overline{[G, G^{(m)}]}$ such that for some $k \in \mathbb{N}$, $G^{(k-1)}$ is nontrivial and $G^{(k)}$ is trivial. We choose $G_0 = G$, $G_m = G^{(m)}K$, $m \in \{1, \ldots, k-1\}$, and if $G_{k-1} = K$, then we choose $G_k = \{e\}$, otherwise we choose $G_k = K$ and $G_{k+1} = \{e\}$. We have that G has a finite sequence of closed connected characteristic (normal) decreasing subgroups $\{G_m\}$ whose successive quotients, being simply connected and abelian, are isomorphic to $\mathbb{R}^{n_m}$ (for some $n_m \in \mathbb{N}$) except for the last one, which is equal to K, a compact connected abelian group. Therefore, the first assertion in the theorem holds. Now we expand this finite sequence to a possibly larger finite sequence of closed normal subgroups for which the second assertion in the theorem also holds. As G_{m-1}/G_m is central in G/G_m for all m, we have that any subgroup of G_{m-1}, which contains G_m, is normal in the connected group G. Since the automorphism corresponding to α on each G_{m-1}/G_m is expansive (cf. Theorem 2.7), it is enough to assume that $G = \mathbb{R}^n$ or $G = K$, a compact connected abelian group.

Now suppose $G = \mathbb{R}^n$ and α is an invertible linear map. We take a sequence of α-invariant subspaces $\{V_j\}$ such that $V_0 = G$ and for $j \geq 1$, if $V_{j-1} \neq \{0\}$, then we choose V_j as any proper (closed) α-invariant subspace in V_{j-1} of maximum possible dimension; it is possible to choose such a subspace because $\dim(V_{j-1}) \leq n$. It follows that there is no other proper α-invariant subspace of V_{j-1} which contains V_j. This implies that $\dim V_j < \dim V_{j-1}$ unless $V_{j-1} = \{0\}$. Therefore, there exists k such that $V_k = \{0\}$, and hence, the sequence $\{V_j\}$ is finite. If $\alpha_j : V_{j-1}/V_j \to V_{j-1}/V_j$ is the natural quotient map defined from the restriction of α to V_{j-1}, then from the choice of $\{V_j\}$ as above, any α_j-invariant subgroup in V_{j-1}/V_j is either discrete or the whole of V_{j-1}/V_j. So far the expansivity of α on the connected group is used only to ascertain that it is nilpotent, the assertions in the theorem would follow for any (not necessarily expansive) automorphism on a simply connected nilpotent group.

Now suppose $G = K$, a compact connected abelian group. Then G is finite-dimensional. Let $G_0 = G$ and for $j \geq 1$, if $G_{j-1} \neq \{e\}$, then we choose G_j to be any proper closed connected (compact) α-invariant subgroup in G_{j-1} of maximum possible dimension. As G is connected and finite-dimensional, it is possible to choose such a sequence $\{G_j\}$. Note that since G is Lie projective and finite-dimensional, for any two closed connected subgroups $H_1 \subset H_2$ in G, either $H_1 = H_2$ or $\dim H_1 < \dim H_2$. It follows that there is no other proper closed connected α-invariant subgroup of G_{j-1} containing G_j. As (G, α) satisfies the descending chain condition (or as G is finite-dimensional), we have that there exists k such that $G_k = \{e\}$. For $\alpha_j : G_{j-1}/G_j \to G_{j-1}/G_j$, we have that α_j is expansive (cf. [Sch90], Corollary 3.11), and hence, $(G_{j-1}/G_j, \alpha_j)$ satisfies the descending chain condition for each j (cf. [KS89], Theorem 5.2). Moreover, each G_{j-1}/G_j is connected and finite-dimensional. Due to the choice of G_j, we have that any proper closed α_j-invariant subgroup in G_{j-1}/G_j is totally disconnected, and hence, it is finite (cf. [Jaw12], Propositions 6.2 and 6.4). This completes the proof. □

Recall that an automorphism α of a locally compact group G is distal if the closure of $\{\alpha^n(x) \mid n \in \mathbb{Z}\}$ does not contain the identity e for every $x \neq e$.

Theorem 2.9. *Let G be a locally compact group and let $\alpha \in \mathrm{Aut}(G)$. Then G is discrete if and only if α is both expansive and distal.*

Proof. The 'only if' statement is obvious. Now suppose α is both expansive and distal. Suppose, in the first case, that G is a connected Lie group. As α is expansive, by Theorem 2.6, G is nilpotent. Moreover, either G is trivial or $\mathrm{d}\alpha$, the corresponding Lie algebra automorphism on the Lie algebra of G, does not have any eigenvalue of absolute value 1 (cf. Proposition 2.3). On the other hand, since α is distal, all the eigenvalues of $\mathrm{d}\alpha$ have absolute value 1 (cf. [Abe79, Abe81]). This implies that G is trivial.

Now suppose that G is a compact group. Since α is expansive, (G, α) satisfies the descending chain condition (cf. [KS89], Theorem 5.2). This, together with the fact that α is distal, implies that G is a Lie group (cf. [RS19], Lemma 2.4).

We now turn to the case when G is a general locally compact group. Let K be the largest compact normal subgroup of G^0. Then K is characteristic in G, G^0/K is a Lie group and $\alpha|_K$ is expansive as well as distal. It follows from above that K is a Lie group. Hence G^0 itself is a Lie group. As

$\alpha|_{G^0}$ is expansive as well as distal, we get that G^0 is trivial, and hence, G is totally disconnected. As α is distal, by Proposition 2.1 of [JR07], G has a neighbourhood basis of compact open α-invariant subgroups. As α is expansive, it leads to a contradiction unless G is discrete. □

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